The $W_{k,p}$–Continuity of the Schrödinger Wave Operators on the Line *

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Abstract

We prove that the wave operatos for the Schrödinger equation on the line are continuous on the Sobolev spaces $W_{k,p}, 1 < p < \infty$. Moreover, if the potential is *exceptional* and $a := \lim_{x\to-\infty} f_1(x,0) = 1$, where $f_1(x,0)$ is a Jost solution at zero energy, the wave operators are continuous on $W_{k,1}$ and on $W_{k,\infty}$.

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1 Introduction

In this paper we consider the one-dimensional Schrödinger equation

$$i\frac{\partial}{\partial t}u(t,x) = (-\frac{\partial^2}{\partial x^2} + V(x))u(t,x),\ u(0,x) = \phi(x), \tag{1.1}$$

with $t, x \in \mathbf{R}$. Below we always assume that $V \in L^1_1$, where for any $\gamma \in \mathbf{R}$ we denote by L^1_γ the space of all complex–valued measurable functions, ϕ, defined on $\mathbf{R}$ such that,

$$\|\phi\|_{L^1_\gamma} := \int |\phi(x)|(1+|x|)^\gamma\, dx\ < \infty. \tag{1.2}$$

L^1_γ is a Banach space with the norm (1.2). Under this condition (see [4] and [26]) the differential expression $\tau := -\frac{d^2}{dx^2} + V(x)$ is essentially self–adjoint on the domain

$$D(\tau) := \{\phi \in L^2_C : \phi\, \text{and} \frac{d}{dx}\phi \text{ are absolutely continuous and } \tau\phi \in L^2\}, \tag{1.3}$$

where we denote by L^2_C the set of all functions in L^2 that have compact support. We designate by H the unique self–adjoint realization of τ. It is known (see [4] for example) that H has a finite number of negative eigenvalues, that it has no positive or zero eigenvalues, that it has no singular–continuous spectrum and that the absolutely continuous spectrum of H is $[0, \infty)$. Let H_0 denotes the unique self-adjoint realization of $-\frac{d^2}{dx^2}$, with domain the Sobolev space $W_{2,2}$. See Section 2 for the definition of the spaces that we use. The wave operators are given by:

$$W_\pm := s - \lim_{t\to\pm\infty} e^{itH}\, e^{-itH_0}. \tag{1.4}$$

It is proven in [19] that the limits in (1.4) exit in the strong topology in L^2 and that $\text{Range}\, W_\pm = \mathcal{H}_c(H)$, where $\mathcal{H}_c(H)$ denotes the subspace of continuity of H. The adjoints of the $W_\pm$ are given by

$$W^*_\pm = s - \lim_{t\to\pm\infty} e^{itH_0} e^{-itH}\, P_c, \tag{1.5}$$

where P_c denotes the orthogonal projector onto the subspace of continuity of H.

The problem that we address on this paper is the continuity of the $W_\pm$ and of the $W^*_\pm$ on the Sobolev spaces $W_{k,p}$. This question has been studied in a series of papers by Yajima [28], [29], [30], [31] and [32] in the case of Schrödinger operators on $\mathbf{R}^n, n \geq 3$. He proved that the $W_\pm$ and the $W^*_\pm$ have bounded extensions to operators on $W_{k,p}, k = 0, 1, \cdots, 1 \leq p \leq \infty$, under appropriate conditions on the regularity and on the decay of V and assuming that zero is neither an eigenvalue nor a resonance (half–bound state) for H. We prove a related result in the case $n = 1$. As we show below, in the one-dimensional case $W_\pm$

and $W_\pm^*$ extend to bounded operators on $W_{k,p}$, $1 < p < \infty$, but in the general case they do not extend to bounded operators on $W_{k,1}$ and on $W_{k,\infty}$. The reason why we have a more singular behaviour in the one-dimensional case is that the low-energy parts of $W_\pm$ and of $W_\pm^*$ contain terms with the Hilbert transform (see the end of the Introduction). If the potential is *exceptional* and $a := \lim_{x\to -\infty} f_1(x,0) = 1$, the terms that contain the Hilbert transform are not present and $W_\pm, W_\pm^*$ extend to bounded operators on $W_{k,1}$ and on $W_{k,\infty}$.

Before we state our main theorem we introduce some standard notation. For u, v any pair of solutions to the stationary Schrödinger equation

$$-\frac{d^2}{dx^2}u + Vu = k^2 u, k \in \mathbf{C}, \tag{1.6}$$

let $[u, v]$ denotes the Wronskian of u and v :

$$[u, v] := \left(\frac{d}{dx}u\right)v - u\frac{d}{dx}v. \tag{1.7}$$

Let $f_j(x,k), j = 1, 2, \Im k \geq 0$, be the Jost solutions to (1.6) (see [5], [6], [4], [3], and Section 2 below). A potential V is said to be *generic* if $[f_1(x,0), f_2(x,0)] \neq 0$ and V is said to be *exceptional* if $[f_1(x,0), f_2(x,0)] = 0$. If V is *exceptional* there is a bounded solution (a half–bound state, or a zero energy resonance) to (1.6) with $k = 0$. Note that the trivial potential, $V = 0$, is *exceptional*. For these definitions and related issues see [15]. For $l = 0, 1, \cdots$, we denote $V^{(l)} := \frac{d^l}{dx^l}V(x)$. Note that $V^{(0)} = V$. Our main result is the following theorem.

THEOREM 1.1. *Suppose that $V \in L^1_\gamma$, where in the* generic case $\gamma > 3/2$ *and in the* exceptional case $\gamma > 5/2$, *and that for some $k = 1, 2, \cdots$, $V^{(l)} \in L^1$, for $l = 0, 1, 2, \cdots, k-1$. Then $W_\pm$ and $W_\pm^*$ originally defined on $W_{k,p} \cap L^2, 1 \leq p \leq \infty$, have extensions to bounded operators on $W_{k,p}$, $1 < p < \infty$. Moreover, there are constants $C_p, 1 < p < \infty$, such that:*

$$\|W_\pm f\|_{k,p} \leq C_p \|f\|_{k,p}; \ \left\|W_\pm^* f\right\|_{k,p} \leq C_p \|f\|_{k,p}, f \in W_{k,p} \cap L^2, 1 < p < \infty. \tag{1.8}$$

Furthermore, if V is exceptional *and $a := \lim_{x\to -\infty} f_1(x,0) = 1$, $W_\pm$ and $W_\pm^*$ have extensions to bounded operators on $W_{k,1}$ and to bounded operators on $W_{k,\infty}$, and there are constants C_1 and C_∞ such that (1.8) holds for $p = 1$ and $p = \infty$.*

Our proof of Theorem 1.1 is quite different from the proof of Yajima [28], [29], [30], [31]. The main difference between the one-dimensional case and $n \geq 3$ is the low–energy behaviour . This can be seen by looking to the behaviour of the resolvent $(H_0 - z)^{-1}$. In the case $n = 1$ the resolvent has a singularity of the type $1/\sqrt{z}$ as $z \to 0$, whereas for $n \geq 3$ the resolvent is regular as $z \to 0$. We base our analysis of the low-energy behaviour of the spectral measure of H on the generalized Fourier maps that are constructed from the scattering solutions $\Psi_{\pm}(x, k)$. The crucial issue here is that for $n = 1$ the scattering solutions are given in terms of the Jost solutions as follows

$$\Psi_+(x,k) := \frac{1}{\sqrt{2\pi}} \begin{cases} T(k) f_1(x,k), & k \geq 0, \\ T(-k) f_2(x,-k), & k < 0, \end{cases} \tag{1.9}$$

and $\Psi_-(x,k) := \overline{\Psi_+(x,-k)}$, where $T(k)$ is the transmission coefficient. The f_j are solutions to Volterra integral equations. They are obtained by iteration as uniformly convergent series (see [5], [6],[4], [3] and Section 2). This fact and ordinary differential equations methods allows us (see [25]) to analyse the low–energy behaviour of the spectral measure of H in a detailed way. We prove Theorems 1.1 in Section 2 using these results.

An important property of the wave operators is that they intertwine between the continuous part of H and H_0, i.e., $H\, P_c = W_{\pm}\, H_0\, W_{\pm}^*$ and $H_0 = W_{\pm}^*\, H P_c\, W_{\pm}$. It follows that for any Borel function$f$:

$$f(H)\, P_c = W_{\pm}\, f(H_0)\, W_{\pm}^*; \;\; f(H_0) = W_{\pm}^* f(H)\, P_c\, W_{\pm}, \tag{1.10}$$

where $f(H)$ and $f(H_0)$ are defined by functional calculus. For X, Y, spaces as above, we denote by $\mathcal{B}\,(X, Y)$ the space of all bounded operators from X into Y and by $\mathcal{B}\,(X)$ the space of all bounded operators on X.

Theorem 1.1 and equations (1.10) imply that $f(H_0)$ and $f(H)P_c$ have equivalent operator norms from $W_{l,p}$ into $W_{\acute{l},\acute{p}}, 0 \leq l, \acute{l} \leq k, 1 < p, \acute{p} < \infty$. More precisely,

COROLLARY 1.2. *Suppose that the assumptions of Theorem 1.1 are satisfied. Then for any* $0 \leq l, \acute{l} \leq k, 1 < p, \acute{p} < \infty$, *there is a constant* C *such that for all Borel functions* f

$$\frac{1}{C} \left\| f(H_0) \right\|_{\mathcal{B}\left(W_{l,p}, W_{\acute{l},\acute{p}}\right)} \leq \left\| f(H) P_c \right\|_{\mathcal{B}\left(W_{l,p}, W_{\acute{l},\acute{p}}\right)} \leq C \left\| f(H_0) \right\|_{\mathcal{B}\left(W_{l,p}, W_{\acute{l},\acute{p}}\right)}. \tag{1.11}$$

Moreover, if V *is* exceptional *and* $a = 1$ *we can take* $1 \leq p, \acute{p} \leq \infty$.

For related results see [10] and [11]. The crucial point of Corollary 1.2 is that C is independent of f. This allows to obtain extensions to the case when $V \neq 0$, of space-time estimates that are known in the case $V = 0$ for the Schrödinger equation as well as for other evolution equations like the Klein–Gordon equation and the wave equation. Examples are $L^p - L^{\acute{p}}$ and Strichartz's estimates and also Fourier multiplier theorems for the generalized Fourier transforms $F_{\pm}$. For a discussion of these applications see [28], [29], [30], [31] and [32]. For the use of these espace-time estimates in nonlinear analysis see [22], [14], [12], [20], [16], [23], [2], [7], [9], [8], [24] , and [25]. In particular, Corollary 1.2 gives us a new proof, in the case $1 < p \leq 2$, of the $L^p - L^{\acute{p}}$ estimate for the Schrödinger equation (1.1) that we obtained in [25].

We prove in Proposition 2.5 in Section 2 that given $\chi \in C^\infty$ with $\chi(x) = 0, x \leq 0$, and $\chi(x) = 1, x \geq 1$, and given $\Psi \in C_0^\infty$ with $\Psi(k) = 1, |k| \leq \sqrt{k_1}$, for some $|k_1| > 0$, we can decompose $W_{\pm}$ as follows:

$$W_{\pm} = W_{\pm,r} \pm i\chi(x) f_1(x,0) \frac{\mathcal{H}C_3}{2} \left[T(0) - 1 + R_1(0)P\right] \pm i(1-\chi(x)) f_2(x,0) \frac{\mathcal{H}C_3}{2} \left[1 - T(0) - R_2(0)P\right], \quad (1.12)$$

where (the regular parts) $W_{\pm,r}, W^*_{\pm,r}$ are bounded operators on $W_{k,p}$, $1 \leq p \leq \infty$, and $C_3 := F^{-1}\Psi(k^2)F$, with F the Fourier transform. Moreover, $R_j(k), j = 1,2$, are the reflection coefficients (see (2.7)), and $(Pf)(x) := f(-x)$. For any f in the Schwartz space, $\mathcal{S}$, we denote by $\mathcal{H}f$ the Hilbert transform of f:

$$(\mathcal{H}f)(x) := \frac{1}{\pi} \text{P.V.} \int \frac{f(x-y)}{y} \, dy, \quad (1.13)$$

where P.V. denotes the principal value of the integral. As is well known (see [21],[18]) the Hilbert transform extends to a bounded operator on $\mathcal{B}(W_{k,p})$, $k = 0,1,\cdots, 1 < p < \infty$, on $\mathcal{B}(W_{k,1}, L^1_{k,w})$ and on $\mathcal{B}(W_{k,\infty}, \text{BMO}_k), k = 0,1,\cdots,$ where $L^1_{k,w}$ denotes the space of functions that together with all its derivatives of order up to k are in L^1-weak, and BMO_k designates the space of all functions that together with all its derivatives of order up to k are in the space, BMO, of functions of bounded mean oscillation. It follows from (1.12) that $W_{\pm}$ extends to a bounded operator from $W_{k,1}$ into $L^1_{k,w}$ and that $W^*_{\pm}$ extend to bounded operators on $\mathcal{B}(W_{k,1}, L^1_{k,w})$ and on $\mathcal{B}(W_{k,\infty}, \text{BMO}_k)$.

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2 The Proofs

We first introduce some standard notation. By $W_{k,p}, k = 0,1,\cdots, 1 \leq p \leq \infty$, we denote the Sobolev space [1] of all functions on L^p such that all its derivatives of order up to k are functions in L^p. By $\|\cdot\|_{k,p}$ we denote the norm in $W_{k,p}$. Clearly, $W_{0,p} = L^p$. By L^1_{loc} we denote the space of all functions that are integrable over every compact set in $\mathbf{R}$. By $\mathcal{L}^2_\alpha, \alpha \geq 0$, we denote the potential spaces [21]. By L^1_w we denote the weak L^1 space [17] of all complex–valued, measurable functions on $\mathbf{R}$ such that $m(x \in \mathbf{R} : |f(x)| > t) \leq C/t$, $t > 0$, where $m(\cdot)$ denotes the Lebesgue measure. For $f \in L^1_w$ let us denote

$$\|f\|_{1,w} := \sup_{t>0} \; t\, m(x \in \mathbf{R} : |f(x)| > t). \quad (2.1)$$

Note that $\|\cdot\|_{1,w}$ is not a norm [17].

By $L^1_{k,w}, k = 0, 1, \cdots$, we denote the space of all $f \in L^1_w$ such that $\frac{d^l}{dx^l} f(x) \in L^1_w, 0 \leq l \leq k$. For $f \in L^1_{k,w}$ we denote

$$\|f\|_{1,k,w} := \sum_{l=0}^{k} \left\| \frac{d^l}{dx^l} f(x) \right\|_{1,w}. \tag{2.2}$$

The space of functions of bounded mean oscillation is designated by BMO [18]. By $\mathrm{BMO}_k, k = 0, 1, \cdots$, we denote the space of all functions $f \in \mathrm{BMO}$ such that $\frac{d^l}{dx^l} f(x) \in \mathrm{BMO}$ with norm

$$\|f\|_{\mathrm{BMO},k} := \sum_{l=0}^{k} \left\| \frac{d^l}{dx^l} f(x) \right\|_{\mathrm{BMO}}. \tag{2.3}$$

For k any complex number we denote by $\Re k$ and $\Im k$, respectively, the real and the imaginary parts of k. The Jost solutions [4] , $f_j(x,k), j = 1, 2, \Im k \geq 0$, are solutions to the stationary Schrödinger equation (1.6) that satisfy: $f_1(x,k) \approx e^{ikx}, x \to \infty$, and $f_2(x,k) \approx e^{-ikx}, x \to -\infty$. To construct the Jost solutions let us define $m_1(x,k) := e^{-ikx} f_1(x,k)$ and $m_2(x,k) := e^{ikx} f_2(x,k)$. The m_j are the unique solutions to the following Volterra integral equations:

$$m_1(x,k) = 1 + \int_x^{\infty} D_k(y-x) V(y) m_1(y,k) dy; \ m_2(x,k) = 1 + \int_{-\infty}^{x} D_k(x-y) V(y) m_2(y,k) dy, \tag{2.4}$$

$$D_k(x) := \int_0^x e^{2iky} dy = \begin{cases} \frac{1}{2ik}\left(e^{2ikx} - 1\right), & k \neq 0, \\ x, & k = 0. \end{cases} \tag{2.5}$$

It is proven in [4] that for each fixed $x \in \mathbf{R}$ the $m_j(x,k)$ are analytic in k for $\Im k > 0$ and continuous for $\Im k \geq 0$ and that

$$|m_1(x,k) - 1| \leq C \frac{1 + \max(-x, 0)}{1 + |k|}; \ |m_2(x,k) - 1| \leq C \frac{1 + \max(x, 0)}{1 + |k|}. \tag{2.6}$$

Moreover [4], the Jost solutions are independent solutions to (1.6) for $k \neq 0$ and there are unique functions $T(k)$ and $R_j(k), j = 1, 2$, such that

$$f_2(x,k) = \frac{R_1(k)}{T(k)} f_1(x,k) + \frac{1}{T(k)} f_1(x,-k); \ f_1(x,k) = \frac{R_2(k)}{T(k)} f_2(x,k) + \frac{1}{T(k)} f_2(x,-k), \tag{2.7}$$

for $k \in \mathbf{R} \setminus 0$. The function $T(k) f_1(x,k)$ describes scattering from left to right of a plane wave e^{ikx} and $T(k) f_2(x,k)$ describes scattering from right to left of a plane wave e^{-ikx}. The function $T(k)$ is the transmission coefficient, $R_2(k)$ is the reflection coefficient from left to right and $R_1(k)$ is the reflection coefficient from right to left. The relations (2.7) are expressed in terms of the $m_j(x,k)$ as follows:

$$T(k) m_2(x,k) = R_1(k) e^{2ikx} m_1(x,k) + m_1(x,-k); \ T(k) m_1(x,k) = R_2(k) e^{-2ikx} m_2(x,k) + m_2(x,-k). \tag{2.8}$$

Moreover, $T(k) \neq 0$ for $k \neq 0$, and $T(k)$ is continuous for $\Im k \geq 0, k \neq i\beta_l, \beta_l > 0, l = 1, 2, \cdots N$. The numbers $-\beta_l^2, l = 1, 2, \cdots N$ are the simple eigenvalues of H. The $R_j(k)$ are continuous for $k \in \mathbf{R}$. Moreover (see [4] or [25]),

$$\left|\dot{R}_j(k)\right| \leq \frac{C}{|k|}; \ \left|\dot{T}(k)\right| \leq \frac{C}{|k|}, \tag{2.9}$$

where we denote $\dot{f}(k) := \frac{d}{dk} f(k)$. Furthermore,

$$T(k) = 1 + O\left(\frac{1}{|k|}\right); \ R_j(k) = O\left(\frac{1}{|k|}\right), \ |k| \to \infty. \tag{2.10}$$

The behaviour as $k \to 0$ is as follows:

(a) In the *generic case*

$$T(k) = \alpha k + o(k), \alpha \neq 0, k \to 0, \ \Im k \geq 0; \ R_1(0) = R_2(0) = -1. \tag{2.11}$$

(b) In the *exceptional case*

$$T(k) = \frac{2a}{1+a^2} + o(1), \ k \to 0, \Im k \geq 0; \ R_1(k) = \frac{1-a^2}{1+a^2} + o(1); \ R_2(k) = \frac{a^2-1}{1+a^2} + o(1), k \to 0, \ k \in \mathbf{R}, \tag{2.12}$$

where $a := \lim_{x\to -\infty} f_1(x, 0) \neq 0$. For the results above about $T(k)$ and $R_j(k)$ see [4], [15] and [13]. In particular for the continuity of $T(k)$ and of $R_j(k)$ as $k \to 0$ in the *exceptional case* for $V \in L^1_1$ see [13]. In [25] we have proven the following theorem.

THEOREM 2.1. *Assume that $V \in L^1_\gamma$.*

(a) If V is generic *and $1 \leq \gamma \leq 2$, then*

$$\left|\dot{T}(k)\right| \leq C\left(1 + |k|\right)^{-1}, \Im k \geq 0; \ R_j(k_1) - R_j(k_2) = \begin{cases} o\left(|k_1 - k_2|^{\gamma-1}\right), & 1 \leq \gamma < 2, \\ O\left(|k_1 - k_2|\right), & \gamma = 2, \end{cases} \tag{2.13}$$

as $k_1 - k_2 \to 0$.

(b) If V is exceptional *and $2 \leq \gamma \leq 3$, then*

$$\left|\dot{T}(k)\right| \leq C\frac{|k|^{\gamma-3}}{(1+|k|)^{\gamma-2}}; \ T(k) - T(0) = O(|k|), R_j(k) - R_j(0) = O(|k|), k \to 0, \tag{2.14}$$

$$R_j(k_1) - R_j(k_2) = O\left(|k_1 - k_2|^{\gamma-2}\right), k_1 - k_2 \to 0. \tag{2.15}$$

The results on the spectral theorem for H that we state below follow from the Weyl–Kodaira–Titchmarsh theory. See [4] for example. For a version of the Weyl–Kodaira–Titchmarsh theory adapted to our situation see Appendix 1 of [27] and also the proof of Theorem 6.1 in page 78 of [27]. For every $\phi \in \mathcal{H}_c(H)$ the following limits exist in the strong topology in L^2:

$$\hat{\phi}_\pm(k) := s - \lim_{N\to\infty} \int_{-N}^{N} \overline{\Psi_\pm(x,k)}\phi(x)\, dx. \tag{2.16}$$

Moreover, the operators:

$$\left(F_\pm\phi\right)(k) := \hat{\phi}_\pm(k), \tag{2.17}$$

are unitary from $\mathcal{H}_c(H)$ onto L^2. Furthermore, the adjoint operators are given by

$$\left(F_\pm^*\phi\right)(x) = s - \lim_{N\to\infty} \int_{-N}^{N} \Psi_\pm(x,k)\phi(k)\, dk. \tag{2.18}$$

The operators $F_\pm^* F_\pm$ are equal to the orthogonal projector onto $\mathcal{H}_c(H)$, and

$$HP_c = F_\pm^*\, k^2\, F_\pm. \tag{2.19}$$

Finally, by the stationary formulas for the wave operators (see equation 12.7.5 of [19])

$$W_\pm = F_\pm^*\, F, \tag{2.20}$$

where F denotes the Fourier transform as a unitary operator in L^2:

$$F(\phi)(k) := s - \lim_{N\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{-ikx}\phi(x)\ dx, \tag{2.21}$$

for every $\phi \in L^2$. We use below the notation: $\hat{\phi}(k) := F(\phi)(k)$.

Formula (2.20) is the starting point of our proof of Theorem 1.1 . We first prepare some results that we need. Since $m_1(x,k) - 1$ belongs to the Hardy class it can be written as follows (see [4])

$$m_1(x,k) = 1 + \int_0^\infty B_1(x,y)e^{2iky}\, dy \tag{2.22}$$

where for each fixed $x \in \mathbf{R}$, $B_1(x,\cdot) \in L^2(0,\infty)$. Moreover (see Lemma 3 of [4]),

$$|B_1(x,y)| \le e^{\gamma_1(x)}\eta_1(x+y), x \in \mathbf{R},\ y > 0; \gamma_1(x) := \int_x^\infty (t-x)|V(t)|\, dt; \eta_1(x) := \int_x^\infty |V(t)|\, dt. \tag{2.23}$$

Similarly,

$$m_2(x,k) = 1 + \int_{-\infty}^{x} B_2(x,y)e^{-2iky}\, dy, \tag{2.24}$$

where for each fixed $x \in \mathbf{R}$, $B_2(x,\cdot) \in L^2(-\infty,0)$ and,

$$|B_2(x,y)| \le e^{\gamma_2(x)}\eta_2(x+y),\ x \in \mathbf{R},\ y < 0; \gamma_2(x) := \int_{-\infty}^{x} (x-t)|V(t)|\, dt; \eta_2(x) := \int_{-\infty}^{x} |V(t)|\, dt. \tag{2.25}$$

We denote:

$$g_k(x) := \sum_{l=0}^{k-1} \left| V^{(l)}(x) \right| ; h_{j,k}(x) := \eta_j(x) + g_k(x),\ j = 1, 2, \tag{2.26}$$

$$Q_{1,k}(x,y) := 2^{k-1} \int_0^y g_k(x+y-z)\, dz; Q_{2,k}(x,y) := 2^{k-1} \int_y^0 g_k(x+y-z)\, dz. \tag{2.27}$$

LEMMA 2.2. *Suppose that $V \in L_1^1$ and that for some $k \geq 1$, $g_k \in L^1_{loc}$. Then for $l = 1, 2, \cdots, k$*

$$\left| \frac{d^l}{dx^l} B_j(x,y) \right| \leq h_{j,k}(x+y)\, e^{Q_{j,k}(x,y)}. \tag{2.28}$$

Proof: We give the proof for $B_1(x,y)$. The case of $B_2(x,y)$ follows similarly. It is proven in Lemma 3 of [4] that:

$$B_1(x,y) = \sum_{n=0}^{\infty} K_n(x,y), \tag{2.29}$$

where the series converges absolutely and

$$K_0(x,y) := \int_{x+y}^{\infty} V(t)\, dt;\ K_{n+1} := \int_0^y dz \int_{x+y-z}^{\infty} dt\ V(t)\, K_n(t,z)\,, n = 0, 1, \cdots. \tag{2.30}$$

We claim that for $1 \leq l \leq k$:

$$\left| K_n^{(l)}(x,y) \right| \leq \frac{2^{n(k-1)}}{n!} h_{1,k}(x+y) \left(\int_0^y dz\, g_k(x+y-z) \right)^n. \tag{2.31}$$

It follows from (2.30) that this is true for $n = 0$. Assuming that (2.31) holds for n we prove it for $n+1$.

$$K_{n+1}^{(l)}(x,y) = -\int_0^y dz\ \left(V(x+y-z) K_n(x+y-z, z) \right)^{(l-1)}. \tag{2.32}$$

Then

$$\left| K_{n+1}^{(l)} \right| \leq 2^{k-1} \int_0^y dz\, g_k(x+y-z)\, 2^{n(k-1)}\, h_{1,k}(x+y) \frac{1}{n!} \left(\int_0^z dv\, g_k(x+y-v) \right)^n =$$

$$2^{(n+1)(k-1)} h_{1,k}(x+y) \frac{1}{(n+1)!} \left(\int_0^y g_k(x+y-z) dz \right)^{n+1}. \tag{2.33}$$

This proves (2.31). Equation (2.28) follows from (2.29) and (2.31).

■

We separate the proof of Theorem 1.1 into a high–energy estimate and a low–energy estimate. For this purpose let Φ be a function in C^∞ such that $\Phi(k) = 0$ for $|k| \leq \sqrt{k_1}$ and $\Phi(k) = 1$ for $|k| \geq \sqrt{k_2}$, for some $0 < k_1 < k_2$.

LEMMA 2.3. *(The High–Energy Estimate) Suppose that* $V \in L^1_\gamma$ *for some* $\gamma > 1$, *and that for some* $k \geq 1$, $V^{(l)} \in L^1$, $0 \leq l \leq k-1$. *Then* $W_\pm \Phi(H_0)$ *and* $\overline{\Phi(H_0)} W_\pm^*$ *originally defined on* $W_{k,p} \cap L^2$ *have extensions to bounded operators on* $W_{k,p}, 1 \leq p \leq \infty$, *and for some constants* C_p*:*

$$\|W_\pm \Phi(H_0) f\|_{W_{k,p}} \leq C_p \|f\|_{W_{k,p}}; \; \left\|\overline{\Phi(H_0)} W_\pm^* f\right\|_{W_{k,p}} \leq C_p \|f\|_{W_{k,p}}, \; f \in W_{k,p} \cap L^2. \tag{2.34}$$

Proof: We give the proof for W_+ and for W_+^*. The cases of W_- and of W_-^* follow similarly. Let $\chi \in C^\infty$ satisfy: $\chi = 0$, for $x \leq 0$ and $\chi = 1$ for $x \geq 1$. We denote

$$\Phi_+(k^2) := \begin{cases} \Phi(k^2), & k \geq 0, \\ \\ 0, & k < 0. \end{cases} \tag{2.35}$$

It follows from (2.8) and (2.20) that for any f in the space of Schwartz, $\mathcal{S}$,

$$\chi(x) W_\pm \Phi(H_0) f = \sum_{j=1}^{5} A_j f, \tag{2.36}$$

where

$$A_1 := \chi(x) C_1; \; C_1 := F^{-1} \Phi_+(k^2)(T(k)-1)F; \; A_2 := C_2\, C_1, \tag{2.37}$$

$$C_2 f := \chi(x) \int \frac{e^{ikx}}{\sqrt{2\pi}} (m_1(x,k)-1)(Ff)(k)\, dk; \; A_3 := C_2(I + C_3); \; C_3 := F^{-1}(\Phi(k^2)-1)F, \tag{2.38}$$

$$A_4 := \chi(x)\,(I + C_3); \; A_5 := C_2 C_4 P; \; C_4 := F^{-1} \Phi_+(k^2) R_1(k) F; \; (Pf)(x) := f(-x); \; A_6 := \chi(x)\, C_4 P. \tag{2.39}$$

By (2.9) and (2.10) we have that $\Phi_+(k^2)(T(k)-1) \in W_{1,2}$. Hence, $F^{-1}\left(\Phi_+(k^2)(T(k)-1)\right) \in L^1$ and it follows that $C_1 \in \mathcal{B}\left(W_{l,p}\right), l = 0, 1, \cdots, 1 \leq p \leq \infty$. We prove in a similar way that that $C_3, C_4 \in \mathcal{B}\left(W_{l,p}\right), l = 0, 1, \cdots, 1 \leq p \leq \infty$. Equation (2.22) implies that

$$C_2 f = \frac{\chi(x)}{2} \int_{-\infty}^{0} B_1(x, -y/2) f(x-y)\, dy. \tag{2.40}$$

Note that $\frac{d^l}{dx^l} C_2 f$ is a finite sum of terms of the type

$$\frac{\chi^{(l_1)}(x)}{2} \int_{-\infty}^{0} B_1^{(l_2)}(x, -y/2)\, f^{(l_3)}(x-y)\, dy = \int_x^\infty c(x,y)\, f^{(l_3)}(y)\, dy, \tag{2.41}$$

where $l_1 + l_2 + l_3 = l$, and

$$c(x,y) := \frac{\chi^{(l_1)}(x)}{2}\, B^{(l_2)}(x, (y-x)/2). \tag{2.42}$$

It is a consequence of (2.28) that c satisfies the standard criterion for L^p boundedness:

$$\sup_{y \in \mathbf{R}} \int |c(x,y)|\, dx < \infty; \quad \sup_{x \in \mathbf{R}} \int |c(x,y)|\, dy < \infty. \tag{2.43}$$

Then, C_2 is bounded on $W_{l,p}$ for $0 \leq l \leq k, 1 \leq p \leq \infty$. By (2.36) $\chi(x) W_+ \Phi(H_0)$ is bounded on $W_{l,p}, 0 \leq l \leq k, 1 \leq p \leq \infty$. We prove in a similar way that $(1-\chi(x)) W_+ \Phi(H_0)$ is bounded on $W_{l,p}, 0 \leq l \leq k, 1 \leq p \leq \infty$. This proves the Lemma for $W_+ \Phi(H_0)$. We prove that $\overline{\Phi(H_0)} W_+^* \chi(x)$ is bounded on $W_{l,p}, 0 \leq l \leq k,\ 1 \leq p \leq \infty$ observing that for $f \in \mathcal{S}$ (see (2.36))

$$\overline{\Phi(H_0)} W_+^* \chi(x) f = \sum_{j=1}^{5} A_j^* f, \tag{2.44}$$

and arguing as above. We establish in an analogous way that $\overline{\Phi(H_0)} W_+^* (1-\chi(x))$ is bounded on $W_{l,p}, 0 \leq l \leq k, 1 \leq p \leq \infty$.

■

Let Ψ be any function on C_0^∞ such that $\Psi(k) = 1, |k| \leq \sqrt{k_1}$, for some $k_1 > 0$.

LEMMA 2.4. *(Low–Energy Estimate) Suppose that $V \in L^1_\gamma$, where in the* generic case $\gamma > 3/2$ *and in the* exceptional case $\gamma > 5/2$, *and that for some $k = 1, 2, \cdots, V^{(l)} \in L^1, l = 0, 1, \cdots, k-1$. Then $W_\pm\, \Psi(H_0)$ and $\overline{\Psi(H_0)}\, W_\pm^*$ originally defined on $W_{k,p} \cap L^2, 1 \leq p \leq \infty$, have extensions to bounded operators on $W_{k,p}, 1 < p < \infty$. Moreover, for some constants $C_p, 1 < p < \infty$,*

$$\| W_\pm\, \Psi(H_0) f \|_{k,p} \leq C_p \, \|f\|_{k,p}; \ \left\| \overline{\Psi(H_0)}\, W_\pm^* f \right\| \leq C_p \, \|f\|_{k,p}, f \in W_{k,p} \cap L^2, 1 < p < \infty. \tag{2.45}$$

If furthermore, V is exceptional *and $a := \lim_{x \to -\infty} f_1(x,0) = 1$, then $W_\pm\, \Psi(H_0)$ and $\overline{\Psi(H_0)}\, W_\pm^*$ have extensions to bounded operators on $W_{k,1}$ and on $W_{k,\infty}$, and there are constants C_1, C_∞ such that (2.45) holds for $p = 1$ and $p = \infty$.*

Proof: We give the proof in the case of W_+. The case of W_- follows in a similar way. Let χ be as in the proof of Lemma 2.3. For $f \in \mathcal{S}$ we decompose $\chi W_+ \Psi(H_0)$ as follows (see (2.8) and (2.20)):

$$\chi(x) W_+ \Psi(H_0) f = W_{+,1} f + \frac{i\chi(x)}{2} \mathcal{H}\, C_3\, [T(0) - 1 + R_1(0) P]\, f, \tag{2.46}$$

where,

$$W_{+,1} f = \sum_{j=1}^{7} A_j f, \tag{2.47}$$

$$A_1 := \chi(x) C_1;\ C_1 := F^{-1} T_+(k) F;\ T_+(k) := \begin{cases} \Psi(k^2)(T(k) - T(0)), & k \geq 0, \\ 0, & k < 0, \end{cases} \tag{2.48}$$

$$A_2 := \chi(x)\frac{T(0)+1}{2}C_3;\ C_3 := F^{-1}\Psi(k^2)F;\ A_3 := \chi(x)\ C_4\,P;\, C_4 := F^{-1}R_+(k)F; \tag{2.49}$$

$$R_+(k) := \begin{cases} (R_1(k)-R_1(0))\Psi(k^2), & k \geq 0, \\ \\ 0, & k<0 \end{cases};\ A_4 := \chi(x)\frac{R_1(0)}{2}\,C_3\,P, \tag{2.50}$$

$$A_5 f(x) := \chi(x)\int dk\, \frac{e^{ikx}}{\sqrt{2\pi}}\frac{(1+\ \mathrm{sign}(k))}{2}T(k)\Psi(k^2)(m_1(x,k)-1)\hat{f}(k), \tag{2.51}$$

$$A_6 f := \chi(x)\int dk\, \frac{e^{ikx}}{\sqrt{2\pi}}\frac{(1+\ \mathrm{sign}(k))}{2}R_1(k)\Psi(k^2)(m_1(x,k)-1)(F(Pf))(k), \tag{2.52}$$

$$A_7 f(x) := \chi(x)\int dk\, \frac{e^{ikx}}{\sqrt{2\pi}}\frac{(1-\mathrm{sign}(k))}{2}\Psi(k^2)(m_1(x,k)-1)\hat{f}(k), \tag{2.53}$$

and where $(Pf)(x) := f(-x)$. It follows from Theorem 2.1 and from Proposition 4 in page 139 of [21] that $T_+ \in \mathcal{L}^2_{\alpha}$, for some $\alpha > 1/2$. Then, $F^{-1}T_+ \in L^1$, and we have that $C_1 \in \mathcal{B}(W_{k,p}), k = 0,1,\cdots,1 \leq p \leq \infty$. We prove in the same way that C_4 is bounded in $W_{k,p}, k = 0,1,\cdots,1 \leq p \leq \infty$. Note that,

$$A_5 = C_2\frac{(1+i\mathcal{H})}{2}C_5;\ C_5 := F^{-1}T(k)\Psi(k^2)F, \tag{2.54}$$

with C_2 as in (2.40). It follows from Theorem 2.1, as above, that $C_5 \in \mathcal{B}(W_{k,p}).k = 0,1,\cdots,1 \leq p \leq \infty$. Then, $A_5 \in \mathcal{B}(W_{k,p}), 1 < p < \infty$. We prove in a similar way that $A_6,\, A_7 \in \mathcal{B}(W_{k,p}), 1 < p < \infty$. We now complete the proof of the Lemma as in the proof of Lemma 2.3 and using the properties of the Hilbert transform quoted at the beginning of this section. Observe that in the exceptional case with $a = 1$, $T(0) = 1$, and $R_j(0) = 0$. Then, in this case there are no singular terms that contain the Hilbert transform in the second term in the right–hand side of (2.46). Moreover, in this case

$$A_5 + A_7 = C_2\,C_1 + C_2\,C_3 \in \mathcal{B}(W_{k,p});\ A_6 = C_2\,C_4\,P \in \mathcal{B}(W_{k,p}), 1 \leq p \leq \infty. \tag{2.55}$$

PROPOSITION 2.5. *Suppose that $V \in L^1_{\gamma}, \gamma > 5/2$, and that for some $k = 0,1,\cdots, V^{(l)} \in L^1_1, l = 0,1,\cdots,k-1$. Then, given $\chi \in C^{\infty}$ with $\chi(x) = 0, x \leq 0$, and $\chi(x) = 1, x \geq 1$, and given $\Psi \in C_0^{\infty}$ with $\Psi(k) = 1, |k| \leq \sqrt{k_1}$, for some $|k_1| > 0$, we can decompose $W_{\pm}$ as follows:*

$$W_{\pm} = W_{\pm,r} \pm i\chi(x)f_1(x,0)\frac{\mathcal{H}C_3}{2}\left[T(0)-1+R_1(0)P\right] \pm i(1-\chi(x))f_2(x,0)\frac{\mathcal{H}C_3}{2}\left[1-T(0)-R_2(0)P\right], \tag{2.56}$$

where $W_{\pm,r}, W^*_{\pm,r}$ are bounded operators on $W_{k,p}\,, 1 \leq p \leq \infty$.

Proof: We give the proof in the case of W_+. The case of W_- follows in a similar way. For $f \in \mathcal{S}$ we decompose $\chi W_+ \Psi(H_0)$ as follows (see (2.8) and (2.20)):

$$\chi(x)W_+\Psi(H_0)f = W_{+,1}f + \frac{i\chi(x)\,m_1(x,0)}{2}\mathcal{H}\,C_3\,\left[T(0)-1+R_1(0)P\right]f, \tag{2.57}$$

where,

$$W_{+,1}f = \sum_{j=1}^{7} A_j f, \tag{2.58}$$

$$A_1 := \chi(x) m_1(x,0) C_1;\;\; A_2 := \chi(x) m_1(x,0) \frac{T(0)+1}{2} C_3, \tag{2.59}$$

$$A_3 := \chi(x) m_1(x,0)\;\; C_4\, P;\; A_4 := \chi(x) m_1(x,0) \frac{R_1(0)}{2} C_3 P, \tag{2.60}$$

where C_1, C_3, C_4, C_5, are as in Lemma 2.4, and

$$A_5 f(x) := \chi(x) \int dk\, \frac{e^{ikx}}{\sqrt{2\pi}} \frac{(1+\ \mathrm{sign}(k))}{2} T(k) \Psi(k^2)(m_1(x,k) - m_1(x,0)) \hat{f}(k), \tag{2.61}$$

$$A_6 f := \chi(x) \int dk\, \frac{e^{ikx}}{\sqrt{2\pi}} \frac{(1+\ \mathrm{sign}(k))}{2} R_1(k) \Psi(k^2)(m_1(x,k) - m_1(x,0))(F(Pf))(k), \tag{2.62}$$

$$A_7 f(x) := \chi(x) \int dk\, \frac{e^{ikx}}{\sqrt{2\pi}} \frac{(1-\mathrm{sign}(k))}{2} \Psi(k^2)(m_1(x,k) - m_1(x,0)) \hat{f}(k). \tag{2.63}$$

We prove as in Lemma 2.4 that $A_j \in \mathcal{B}(W_{k,p}), 1 \leq p \leq \infty, 1 \leq j \leq 4$. For this purpose note that by (2.22) and (2.28), $m_1^{(l)}(x,0) \in L^\infty(\mathbf{R}^+), 0 \leq l \leq k$. It follows from (2.22) that

$$A_5 f(x) = C_6\, C_5\, f(x); C_5 := F^{-1} T(k)\, \Xi(k^2)\, F;\; C_6\, f(x) := \int c(x,y) f(y)\, dy, \tag{2.64}$$

with $\Xi \in C_0^\infty$, $\Xi(k^2) = 1, k \in$ support $\Psi(k^2)$, and

$$c(x,y) := \chi(x) \int_0^\infty B_1(x,z) g_z(y)\, dz; g_z(y) := \frac{1}{2\pi} \int dk\, \Psi(k^2) \frac{(1+\mathrm{sign}(k))}{2} [e^{i2kz} - 1] e^{-iky}. \tag{2.65}$$

But,

$$\|g_z(\cdot)\|_{L^1} \leq C \left\| \Psi(k^2) \frac{(1+\mathrm{sign}(k))}{2} [e^{2ikz} - 1] \right\|_{1,2} \leq C\, |z|. \tag{2.66}$$

Then, by (2.23) $c(x,y)$ satisfies condition (2.43) and it follows that $A_5 \in \mathcal{B}(L^p), 1 \leq p \leq \infty$. We prove in a similar way that $A_5 \in \mathcal{B}(W_{k,p}), 1 \leq p \leq \infty$. By an analogous argument we establish that $A_6, A_7 \in \mathcal{B}(W_{k,p}), 1 \leq p \leq \infty$. This proves that $W_{+,1} \in \mathcal{B}(W_{k,p}), 1 \leq p \leq \infty$. The proof of the Proposition is now completed decomposing $(1 - \chi(x)) W_+ \Psi(H_0)$ in a similar way and arguing as in the proof of Lemma 2.3. Furthermore,by Lemma 2.3 $(I - \Psi(H_0)) W_\pm$ contributes to $W_{\pm,r}$.

Proof of Theorem 1.1: The theorem follows from lemmas 2.3 and 2.4.

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