Conditionally Invariant Probability Measures in Dynamical Systems.

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Abstract

Let T be a measurable map on a Polish space X, let Y be a non trivial subset of X. We give conditions ensuring existence of conditionally invariant probability measures (to non absorption in Y). We also supply sufficient conditions for these probability measures to be absolutely continuous with respect to some regular measure.

1 Introduction.

The notion of conditionally invariant probability measures *c.i.p.m.* was introduced for countable state Markov chains with absorbing state in [9]. It was proven in [6] that geometric absorption was a necessary and sufficient condition for the existence of c.i.p.m. for a wide class of Markov chains. In [7] and recently in [2, 3, 4] it was investigated the existence of such measures for expanding dynamical systems and topolgical Markov chains with holes. More recently, these questions were also studied for Anosov systems in [1].
In this article, we are concerned with general dynamical systems. In the first section, we give necessary and sufficient conditions of existence of a c.i.p.m. following closely the ideas introduced in [6]. In the second one, we study existence and properties of c.i.p.m. which are absolutely continuous with respect to some regular reference measure. The construction of such measures is inspired by [8].

Let us introduce the framework and main definitions. Let (X, d) be a Polish space, $\mathcal{B}$ be the Borel σ-field and $T^t : X \to X$ be a measurable dynamics which may be either continuous or discrete (i.e. t belongs either to $\mathbb{N}$ or $\mathbb{R}$). Let $Y \subset X$ a non trivial subset of X. Denote $X_0 = X \setminus Y$. Set $\tau(x) = \inf\{t \geq 0 \ / \ T^t x \in Y\}$ for $x \in X$ the time of absorption at the hole and $X_t = \{x \ / \ \tau(x) > t\}$. We have $X_t = \bigcap_{0 \leq s \leq t} T^{-s} X_0$ and observe that $T^{-t} X_s \cap X_t = X_{t+s}$.

Let $\mathcal{M}(X_0)$ be the set of probability measures on X concentrated on X_0. A probability measure $\nu \in \mathcal{M}(X_0)$ is said to be a c.i.p.m. if

$$\nu(T^{-t} A \cap X_t) = \nu(A)\nu(X_t) \ \forall A \in \mathcal{B}, \ \forall t,$$

i.e. when conditioned to be at X_t the measure remains invariant. By taking $A = X_t$, we find that a c.i.p.m. verifies $\nu(X_{t+s}) = \nu(X_t)\nu(X_s) \ \forall s, t$. Then $\nu(X_t) = \alpha^t$ for some α in $[0, 1]$.

2 Some properties of c.i.p.m. and existence condition.

For $\nu \in \mathcal{M}(X_0)$ consider the measures ν_t given by $\nu_t(B) = \nu(T^{-t} B \cap X_t)$. We have $\nu_0 = \nu$ and $\nu_t(X_0) = \nu(X_t)$ for every $t \geq 0$. Since $T^{-t} X_s \cap X_t = X_{s+t}$, we have $\nu_t(X_s) = \nu(X_{s+t}) = \nu_{s+t}(X_0)$. We denote by F^ν the distribution function of τ, $F^\nu(t) = \mathbb{P}_\nu\{\tau \leq t\} = 1 - \nu(X_t)$, then if ν is a c.i.p.m. the distribution F^ν is exponential (of parameter θ_0 for some θ_0) i.e. $F^\nu(t) = 1 - e^{\theta_0 t} \ \forall t \geq 0$. Also $\mathbb{E}_\nu(\tau) = \theta_0^{-1}$. For any probability distribution function F

on $\mathbb{R}_+$ we put $m_1(F) = \int_0^\infty t dF(t) = \int_0^\infty (1-F(t))dt$. Assume $m_1(F) < \infty$. We can define the probability distribution ψF given by $1-\psi F(t) = (m_1(F))^{-1} \int_t^\infty (1-F(s))ds$. Observe that $m_1(F^\nu) = \mathbb{E}_\nu(\tau)$, so $1-\psi F^\nu(t) = (\mathbb{E}_\nu(\tau))^{-1} \int_t^\infty (1-F^\nu(s))ds$. Under this assumption we can define the measure $\varphi\nu$ by

$$\varphi\nu(B) = (\mathbb{E}_\nu(\tau))^{-1} \int_0^\infty \nu_t(B)dt.$$

Since $\int_0^\infty \nu_t(X_0)dt = \int_0^\infty (1-F^\nu(t))dt = \mathbb{E}_\nu(\tau)$ we get that $\varphi\nu \in \mathcal{M}(X_0)$. Observe that

$$\begin{aligned} 1 - F^{\varphi\nu}(t) &= \varphi\nu(X_t) = (\mathbb{E}_\nu(\tau))^{-1} \int_0^\infty \nu_s(X_t)ds = (\mathbb{E}_\nu(\tau))^{-1} \int_0^\infty \nu_{s+t}(X_0)ds \\ &= (\mathbb{E}_\nu(\tau))^{-1} \int_t^\infty \nu_s(X_0)ds = (m_1(F^\nu))^{-1} \int_t^\infty (1-F^\nu(s))ds. \end{aligned}$$

Hence

$$\text{if } \mathbb{E}_\nu(T) < \infty \text{ then } F^{\varphi\nu} = \psi F^\nu. \tag{1}$$

Lemma 2.1 *$\nu \in \mathcal{M}(X_0)$ is a c.i.p.m. if and only if $\varphi\nu = \nu$.*

Proof: Assume ν is a c.i.p.m. then for any $B \subset X_0, B \in \mathcal{B}$, $\nu_t(B) = \nu(f^{-t}B \cap X_t) = \nu(B)\nu(X_t)$, so

$$\varphi\nu(B) = (\mathbb{E}_\nu(\tau))^{-1} \int_0^\infty \nu_t(B)dt = \nu(B)(\mathbb{E}_\nu(\tau))^{-1} \int_0^\infty \nu(X_t) = \nu(B).$$

Reciprocally assume that $\varphi\nu = \nu$. Fix $B \subset X_0, B \in \mathcal{B}$. Denote $\theta_0 = (\mathbb{E}_\nu(\tau))^{-1}$ and $h(t) = \nu_t(B)$. From $\nu_t(B) = \theta_0 \int_0^\infty \nu_{t+s}(B)ds$, we get $h(t) = \theta_0 \int_t^\infty h(s)ds$. We get $h'(t) = -\theta_0 h(t)$ for $t \geq 0$ and $h(0) = \nu(B)$. Then $h(t) = \nu(B)e^{-\theta_0 t}$ for $t \geq 0$. By applying it for $B = X_0$ we find $\nu(X_t) = e^{-\theta_0 t}$ so $\nu_t(B) = \nu(B)\nu(X_t)$ and the result holds. $\triangle$

Theorem 2.2 *Assume $\nu \in \mathcal{M}(X_0)$ is such that the trajectories are exponentially absorbed starting from ν i.e. there exist $C < \infty, \theta_0 > 0$ such that*

$$(1 - F^\nu(t)) \le Ce^{-\theta_0 t} \text{ for any } t \ge 0, \tag{2}$$

and for t big enough $X \setminus X_t$ is contained in a compact set. Then there exists a c.i.p.m.

Observe that the condition "$X \setminus X_t$ is contained in a compact set for t big enough" is obviously verified if X is compact. It is also fulfilled when the hole Y is compact, T^1 is continuous and time is dicrete.

For proving the theorem we shall use the following result shown in [6] (Proposition 3.3).

Lemma 2.3 *If for some $C < \infty, \theta_0 > 0$ a probability distribution function F verifies $1 - F(t) \le Ce^{-\theta_0 t}$ for all $t \ge 0$, then there exists $\theta \ge \theta_0$ and a sequence $\mathcal{N} = \{n_1 < n_2 < ..\}$ such that $\psi^n F$ converges to an exponential of parameter θ as $n \to \infty, n \in \mathcal{N}$.*

Proof of Theorem 2.2: It is entirely similar to the proof given in [6] for Markov chains. Denote $\nu^n = \varphi^n \nu$. Let $\mathcal{N}$ be given by Lemma 2.3, so it satisfies that $(F^{\nu^n})_{n\in\mathcal{N}}$ converges in distribution to an exponential of parameter θ with $\theta \ge \theta_0$. Let $\varepsilon > 0$ and take n big enough in order that:

$$\varepsilon \ge 1 - F^{\nu^n}(t(\varepsilon)) = \mathbb{P}_{\nu^n}\{\tau > t(\varepsilon)\} = \nu^n(X_{t(\varepsilon)}),$$

hence $\{\nu^n : n \in \mathcal{N}\}$ is tight. Let $\mathcal{N}' \subset \mathcal{N}$ be a subsequence such that $\nu^n \to \nu^\infty$ when $n \to \infty, n \in \mathcal{N}'$. We have

$$F^{\nu^\infty}(t) = \nu^\infty(X \setminus X_t) = \lim_{\substack{n\to\infty \\ n\in\mathcal{N}'}} \nu^n(X \setminus X_t) = \lim_{\substack{n\to\infty \\ n\in\mathcal{N}'}} F^{\nu^n}(t) = 1 - e^{-\theta t}$$

Hence the set $K_\theta = \{\nu \in \mathcal{M}(X_0) : F^\nu \sim \text{exponential of parameter } \theta\}$ is non empty. From (1) it follows

$$\mathbb{P}_{\varphi\nu}\{\tau > t\} = (\mathbb{E}_\nu(\tau))^{-1} \int_t^\infty (1 - F^\nu(s))ds = e^{-\theta t},$$

then $\varphi : K_\theta \to K_\theta$. On K_θ we put the weak topology. Let us show φ is continuous on K_θ. Let $\nu_k \to \nu$, with $\nu_k, \nu \in K_\theta$. We have $\mathbb{E}_{\nu_k}(\tau) = \mathbb{E}_\nu(\tau) = \theta$. Let $Z \subset X_0$ be a Borel set with $\nu(\partial Z) = 0$. We have

$$\liminf_{k\to\infty} \varphi\nu_k(Z) = \liminf_{k\to\infty} \frac{1}{\mathbb{E}_{\nu_k}(\tau)} \int_0^\infty \nu_k(T^{-t}Z \cap X_t)dt$$

$$= \frac{1}{\theta} \int_0^\infty \liminf_{t\to\infty} \nu_k(T^{-t}Z \cap X_t)dt \ge \frac{1}{\theta} \int_0^\infty \nu(T^{-t}Z \cap X_t)dt = \varphi\nu(Z).$$

By taking $X_0 \setminus Z$ we obtain $\liminf_{k\to\infty} \varphi\nu_k(X_0 \setminus Z) = 1 - \limsup_{k\to\infty} \varphi\nu_k(Z) \geq 1 - \varphi\nu(Z)$. Hence $\lim_{k\to\infty} \varphi\nu_k(Z) = \varphi\nu(Z)$ and the continuity follows. Now K_θ is convex, let us show it is also compact. Since the weak topology is metrizable we must show that K_θ is sequentially compact. But for the purpose it suffices to show that any sequence $(\nu_k) \subset K_\theta$ is tight. But this holds because $F^\nu(t) = \nu(X \setminus X_t) = 1 - e^{\theta t}$ for any $\nu \in K_\theta$. Hence $\varphi : K_\theta \to K_\theta$ has a fixed point ν, which is a c.i.p.m. $\triangle$

Now we give a condition which ensure that (2) is verified. This condition is related with mixing properties of the invariant probability measure for the original system (without the hole).

Proposition 2.4 *Consider $X_0 \in \mathcal{B}$ and $\nu \in \mathcal{M}(X)$ a T^t-invariant probability measure such that $\nu(X_0) < 1$. Assume that there exists $\varepsilon_t = \varepsilon_t(X_0)$ with $\varepsilon_t \to 0$ as $t \to \infty$, such that $\forall B \in \mathcal{B}$,*

$$|\nu(T^{-t}X_0 \cap B) - \nu(X_0)\nu(B)| \leq \nu(B)\varepsilon_t \quad \forall t \geq 0.$$

Then there exists $\theta_0 > 0, C < \infty$ such that

$$1 - F^\nu(t) \leq Ce^{-\theta_0 t} \quad \forall t \geq 0.$$

Proof: Fix $r > 0$ such that $\varepsilon' = \nu(X_0) + \varepsilon_r \in (0,1)$. For $t > 0$ take $q \in \mathbb{N}$ such that $t = qr + r_0$ with $r_0 < r$. We have

$$1 - F^\nu(t) = \nu(\bigcap_{0\leq s\leq t} T^{-s}X_0) \leq \nu(\bigcap_{\ell=0}^{q} T^{-\ell r}X_0).$$

We have from hypothesis:

$$\int \varphi \cdot \mathbf{1}_{X_0} \circ T^r d\nu \leq \varepsilon' \int \varphi d\nu \text{ for } \varphi \in L^1_+(d\nu).$$

We can define the Perron-Frobenius operator $P^t : L^1_+(d\nu) \to L^1_+(d\nu)$ as
$P^t\varphi = \frac{d\nu_\varphi \circ T^{-t}}{d\nu}$ where ν_φ is the measure $\nu_\varphi(B) = \int_B \varphi d\nu$ and $\nu_\varphi \circ T^{-t}(B) = \nu_\varphi(T^{-t}B)$ $\forall B \in \mathcal{B}$. We have $\int \mathbf{1}_B \cdot P^t\varphi d\nu = \nu_\varphi \circ T^{-t}(B) = \int \mathbf{1}_B \circ T^t \cdot \varphi d\nu$. Since ν if T^t-invariant $P^t\mathbf{1} = \mathbf{1}$. Now,

$$\begin{aligned}\nu(\bigcap_{\ell=0}^{q} T^{-\ell r}X_0) &= \nu(\bigcap_{\ell=0}^{q-1} T^{-\ell r}X_0 \cap T^{-qr}X_0) \\ &= \int P^{(q-1)r}\mathbf{1}_{\bigcap_{\ell=0}^{q-1} T^{-\ell r}X_0} \cdot \mathbf{1}_{X_0} \circ T^r d\nu\end{aligned}$$

$$\leq \varepsilon' \int P^{(q-1)r} \mathbf{1}_{\bigcap_{\ell=0}^{q-1} T^{-\ell r} X_0} d\nu = \varepsilon' \nu(\bigcap_{\ell=0}^{q-1} T^{-\ell r} X_0).$$

From recurrence $\nu(\bigcap_{\ell=0}^{q} T^{-\ell r} X_0) \leq {\varepsilon'}^q$ and we find: $1 - F^\nu(t) \leq {\varepsilon'}^q = {\varepsilon'}^{\frac{t-r_0}{q}}$ and the result follows. $\triangle$

Our purpose is now to give a sufficient condition to ensure existence of a c.i.p.m., absolutely continuous with respect to some reference probability λ (this will be denoted by a.c.c.i.p.m.).

3 Absolutely continuous c.i.p.m.

In what follows, we are concern with discrete dynamics, so T^n is the n^{th} iterate of $T : X \to X$. Let λ be a regular probability on X. We assume that T is *non singular with respect to* λ, that is $\lambda(A) = 0$ implies $\lambda(T^{-1}A) = 0$. The Perron-Frobenius operator $P : L^1(\lambda) \to L^1(\lambda)$ is defined by

$$\int_X (Pf) g d\lambda = \int_X fg \circ T d\lambda \;\forall f \in L^1(\lambda), g \in L^\infty(\lambda).$$

The operator P is a positive contraction of $L^1(\lambda)$. Moreover P^n verifies

$$\int_X (P^n f) g d\lambda = \int_X fg \circ T^n d\lambda \;\forall f \in L^1(\lambda), g \in L^\infty(\lambda) \;\forall n \in \mathbb{N}. \tag{3}$$

Let $P_0 : L^1(\lambda) \to L^1(\lambda)$ be defined by $P_0 f = P(f \mathbf{1}_{X_0})$, note that $P_0^n f = P^n(f \mathbf{1}_{X_{n-1}})$, thus using (3), we get

$$\int_{X_0} (P_0^n f) g d\lambda = \int_{X_n} fg \circ T^n d\lambda \;\forall f \in L^1(\lambda), g \in L^\infty(\lambda) \;\forall n \in \mathbb{N}. \tag{4}$$

Lemma 3.1 *There exists an a.c.c.i.p.m. if and only if there exists $h \neq 0$ in $L^1_+(\lambda)$ such that $\mathbf{1}_{X_0} P_0 h = \alpha h \mathbf{1}_{X_0}$ with $\alpha \in [0,1]$.*

Proof: We can assume that h satisfies $\int_{X_0} h d\lambda = 1$ and $\mathbf{1}_{X_0} P_0 h = \alpha h \mathbf{1}_{X_0}$ with $\alpha \in [0,1]$.

Then $\mathbf{1}_{X_0} P_0^n h = \alpha^n h \mathbf{1}_{X_0}$ $\forall n \geq 1$. Let ν be the probability defined by $\nu = (h \mathbf{1}_{X_0})\lambda$. Let us prove that ν is a c.i.p.m. We have for $B \in \mathcal{B}$,

$$\nu(T^{-n} B \cap X_n) = \int_{X_n} \mathbf{1}_B \circ T^n h d\lambda = \int_{X_0} P_0^n h \mathbf{1}_B d\lambda \text{ using (4)} = \alpha^n \int_{X_0} h \mathbf{1}_B d\lambda = \alpha^n \nu(B).$$

Taking $B = X_0$, we obtain $\nu(X_n) = \alpha^n$, thus $\nu(T^{-n}B \cap X_n) = \nu(B)\nu(X_n)$ which shows that ν is a c.i.p.m.

Reciprocally, assume that ν is an absolutely continuous c.i.p.m. and let $\frac{d\lambda}{d\nu} = \mathbf{1}_{X_0}h$ (remind that ν is concentrated on X_0, so we can assume $\frac{d\lambda}{d\nu}$ is zero on $Y = X \setminus X_0$). Using (4) with $g = \mathbf{1}_B$ and $f = h\mathbf{1}_{X_0}$, we have

$$\begin{aligned} \nu(T^{-n}B \cap X_n) &= \int\limits_{X_n} h\mathbf{1}_B \circ T^n d\lambda \\ &= \int\limits_{X_0} P_0^n h\mathbf{1}_B d\lambda = \nu(B)\nu(X_n) \text{ since } \nu \text{ is a c.i.p.m.} \end{aligned}$$

Hence, we obtain $\mathbf{1}_{X_0}h\nu(X_n) = P_0^n h\mathbf{1}_{X_0}$. Since ν is a c.i.p.m., $\nu(X_n) = \alpha^n$ for some $\alpha \in [0,1]$, so $\mathbf{1}_{X_0}P_0 h = \alpha\mathbf{1}_{X_0}h$. △

Definition 3.2 *Let* $(\mu_n)_{n\in\mathbb{N}}$ *be a sequence of probabilities, absolutely continuous with respect to* λ. *We say that* the sequence $(\mu_n)_{n\in\mathbb{N}}$ is uniformly absolutely continuous with respect to λ *if*

$$\forall\gamma > 0, \exists\delta > 0 \textit{ such that } \lambda(A) < \delta \textit{ implies } \mu_n(A) < \gamma \ \forall n \in \mathbb{N}. \tag{5}$$

For any $n \in \mathbb{N}$, we define μ_n by $\mu_n(A) = \dfrac{\lambda(T^{-n}A \cap X_n)}{\lambda(X_n)}$.

Theorem 3.3 *If the sequence* $(\mu_n)_{n\in\mathbb{N}}$ *is uniformly absolutely continuous with respect to* λ *and* X_0 *is compact then there exists an absolutely continuous c.i.p.m.*

Remarks. **1**. The asumption on uniformly absolue continuity of $(\mu_n)_{n\in\mathbb{N}}$ is related with asymptotic laws of entrance times.
2. Note that if ν is an a.c.c.i.p.m. with density h bounded away from zero and infinity then the sequence $(\mu_n)_{n\in\mathbb{N}}$ is uniformly absolutely continuous with respect to λ.
Indeed, let h be the density of an a.c.c.i.p.m. such that $M^{-1} < h < M$. Then

$$\begin{aligned} M^{-1}(P_0^n h)\mathbf{1}_{X_0} &\leq \mathbf{1}_{X_0}(P_0^n\mathbf{1}) \leq M(P_0^n h)\mathbf{1}_{X_0} \\ M^{-2}\alpha^n\mathbf{1}_{X_0} &\leq \mathbf{1}_{X_0}(P_0^n\mathbf{1}) \leq M^2\alpha^n\mathbf{1}_{X_0}, \end{aligned}$$

thus,

$$\frac{\lambda(T^n A \cap X_n)}{\lambda(X_n)} = \left(\int\limits_{X_0}(P_0^n\mathbf{1})d\lambda\right)^{-1}\left(\int\limits_{X_0}(P_0^n\mathbf{1})\mathbf{1}_A d\lambda\right) \leq M^4\lambda(A),$$

this implies uniform absolute continuity of $(\mu_n)_{n\in\mathbb{N}}$. △

3.1 Proof of theorem 3.3

Proof of Theorem 3.3: The operator P_0 acts on $L^1(\lambda)$, consider the canonic injection of L^1 into its bidual $(L^1)^{**} = (L^\infty)^*$. $(L^\infty)^*$ is identified with the space of finitely additive bounded set function on $\mathcal{B}$, which are absolutely continuous with respect to λ ([5] th IV 8.15-16, [10] 2.). The positive operator P_0 may be continued to a positive operator on $(L^\infty)^*$ by $P_0 z(A) = z(T^{-1}A \cap X_1)$. We will first obtain an eigenvalue for P_0 in $(L^\infty)^*$ and then use this eigenvalue to construct the a.c.c.i.p.m.

Consider the bounded sequence

$$a_n = 1/n \sum_{i=0}^{n-1} \frac{\lambda(X_{i+1})}{\lambda(X_i)} \tag{6}$$

and let α be any limit point of this sequence, $\alpha = \lim_k a_{n_k}$. In the following, α and the subsequence n_k are fixed. Now, let Q_k be the following sequence in L^1,

$$Q_k = \frac{1}{n_k} \sum_{i=0}^{n_k-1} \frac{1}{\lambda(X_i)} P_0^i \mathbf{1},$$

for any k, $\displaystyle\int_{X_0} Q_k = 1$. Thus, it admits limits points for the weak-$*$ topology of $(L^\infty)^*$. Let z be such a limit point, it verifies $P_0 z = \alpha z$. Indeed, let $A \in \mathcal{B}$ and $\varphi(k)$ a subsequence such that

$$\begin{aligned} z(A) &= \lim_{k\to\infty} \frac{1}{\varphi(k)} \sum_{i=0}^{\varphi(k)-1} \frac{\lambda(T^{-i}A \cap X_i)}{\lambda(X_i)} \text{ and} \\ z(T^{-1}A \cap X_1) &= \lim_{k\to\infty} \frac{1}{\varphi(k)} \sum_{i=0}^{\varphi(k)-1} \frac{\lambda\left(T^{-i}[T^{-1}A \cap X_1] \cap X_i\right)}{\lambda(X_i)}. \end{aligned}$$

Now,

$$\frac{1}{\varphi(k)} \sum_{i=0}^{\varphi(k)-1} \frac{\lambda\left(T^{-i}[T^{-1}A \cap X_1] \cap X_i\right)}{\lambda(X_i)} = \\ \frac{1}{\varphi(k)} \sum_{i=0}^{\varphi(k)-1} \frac{\lambda\left(T^{-i-1}A \cap X_{i+1}\right)}{\lambda(X_{i+1})} \frac{\lambda(X_{i+1})}{\lambda(X_i)}.$$

We have

$$\frac{1}{\varphi(k)} \sum_{i=0}^{\varphi(k)-1} \frac{\lambda(X_{i+1})}{\lambda(X_i)} \longrightarrow \alpha$$

and

$$\frac{1}{\varphi(k)}\sum_{i=0}^{\varphi(k)-1}\frac{\lambda\left(T^{-i-1}A\cap X_{i+1}\right)}{\lambda(X_{i+1})}\longrightarrow z(T^{-1}A\cap X_1).$$

This shows that $P_0 z = \alpha z$. Moreover z have its support in X_0, thus z also defines a linear form on $C(X_0)$. Therefore there exists a regular measure μ on X_0 such that

$$z(f)=\int_{X_0} f d\mu = \mu(f),\ f\in C(X_0). \tag{7}$$

By $z\wedge\mu$, we denote the infimum between z and μ in the lattice of bounded, finitely additive set function on $\mathcal{B}$, for the inclusion ordering ([10], [5] III.1.8 for the definition of $\mu^- = \mu\wedge 0$)

$$\text{for } A\in\mathcal{B},\ z\wedge\mu(A)=\inf_{B\subset A}\{\mu(B)+z(A\setminus B)\}.$$

Since $0\le z\wedge\mu\le\mu$ and $0\le z\wedge\mu\le z$, $z\wedge\mu$ is a positive regular measure, absolutely continuous with respect to λ. We claim that $P_0(z\wedge\mu)\le\alpha(z\wedge\mu)$. Indeed, it suffices to shows that $\forall A\in\mathcal{B}$,

$$\begin{aligned} P_0(z\wedge\mu)(A) &\le \alpha z(A) \text{ and}\\ P_0(z\wedge\mu)(A) &\le \alpha\mu(A). \end{aligned} \tag{8}$$

The first inequality is trivial: $P_0(z\wedge\mu)(A)\le P_0 z(A)$ (P_0 is a positive operator) and $P_0 z=\alpha z$. If $\alpha=0$, it implies $P_0(z\wedge\mu)=0$ and $z\wedge\mu$ is a a.c.c.i.p.m. provided it is not zero. The fact that $z\wedge\mu(X_0)\ne 0$ will be clear at the end of the proof.
Let us consider the case $\alpha\ne 0$. Remark that both sides of (8) define a regular Borel measure on X_0, so to establish (8), it suffices to prove it for compact sets. Let $K\subset X_0$ be a compact set of X_0, there exists a decreasing sequence $(g_n)_{n\in\mathbb{N}}$ of positive continuous functions on X_0 such that $\mu(K)=\inf_n\mu(g_n)$. We get

$$\begin{aligned} \mu(K) &= \inf_{n\in\mathbb{N}}\mu(g_n)=\inf_{n\in\mathbb{N}} z(g_n)\ge z(K)\\ &= \alpha^{-1}P_0 z(K)\ge\alpha^{-1}P_0(z\wedge\mu)(K). \end{aligned}$$

For $A\in\mathcal{B}$ consider the sequence

$$\nu_n(A)=\frac{P_0^n}{\alpha^n}(z\wedge\mu)(A),$$

using the preceding remark, we get that this sequence is decreasing. Thus there exists a regular absolutely continuous measure ν satisfying $\nu(A)=\lim_n\nu_n(A)$ for all $A\in\mathcal{B}$ and $P_0\nu=\alpha\nu$. To conclude the proof, it remains to show that $0<\nu(X_0)<\infty$. This will follows from the uniform absolute continuity of μ_n. We will only consider the case $\alpha\ne 0$,

if $\alpha = 0$, it may be proved that $\mu \wedge z(X_0) \neq 0$ in the same way.
We have $\nu(X_0) = \lim_n \alpha^{-n}(z \wedge \mu)(X_n)$, so $\nu(X_0) \leq 1$ (note that $P_0 z = \alpha z$ implies $z(X_n) = \alpha^n$). Let us assume that $\nu(X_0) = 0$. Let $\varepsilon < 1/2$ be fixed and n such that $z \wedge \mu(X_n) < \varepsilon\alpha^n$. There exists $A \subset X_n$ such that

$$\mu(A) < \varepsilon\alpha^n \text{ and } z(X_n \setminus A) < \varepsilon\alpha^n.$$

Let $\gamma = \alpha^n(1-2\varepsilon)$ and $\delta > 0$ associated to γ by the uniform absolute continuity of $(\mu_k)_{k\in\mathbb{N}}$. Since λ is regular, there exists a compact $K \subset A$ such that $\lambda(A \setminus K) < \delta$. Since K is compact, $z(K) \leq \mu(K) \leq \mu(A) \leq \varepsilon\alpha^n$, so,

$$z(A \setminus K) = z(X_n \setminus K) - z(X_n \setminus A) > \alpha^n(1-2\varepsilon) = \gamma.$$

Yet, it follows from the construction of z and the choices of γ and δ that $z(A \setminus K) < \gamma$, which is a contradiction. So that ν is non zero. $\triangle$

3.2 Uniqueness problem.

Proposition 3.4 *Assume that T satisfies the hypothesis of theorem 3.3 and the following "irreducibility" condition: for all A and B in $\mathcal{B}$ such that $\lambda(A) > 0$ and $\lambda(B) > 0$ there exists $k \in \mathbb{N}$ such that*

$$\lambda(B \cap T^{-k}A \cap X_k) > 0.$$

Then any limit point α of the sequence $(a_n)_{n\in\mathbb{N}}$ (defined in (6)) is a simple eigenvalue of P_0.

Proof: Let us fix α a limit point of $(a_n)_{n\in\mathbb{N}}$. From section 3.1, α is an eigenvalue of P_0. Let us prove that it is simple.
First, we will show that if k if an eigenfunction of P_0, so are k^+ and k^-, where $k^+ = \max(0,k)$ and $k^- = \max(0,-k)$. Indeed, let $k \in L^1(\lambda)$ be such that $P_0 k = \alpha k$. Since P_0 is a positive operator, $P_0 k^\varepsilon \leq \alpha k^\varepsilon$ almost everywhere, $\varepsilon = +$ or $\varepsilon = -$, so $\displaystyle\int P_0 k^\varepsilon \leq \alpha \int k^\varepsilon$.
Moreover,

$$\int_{X_0} P_0 k^+ - P_0 k^- = \int_{X_0} P_0 k = \alpha \int_{X_0} k = \alpha \int_{X_0} k^+ - \alpha \int_{X_0} k^- \tag{9}$$

and $P_0|f| \geq |P_0 f| = \alpha|f|$, so $\displaystyle\int_{X_0} P_0|f| \geq \alpha \int_{X_0} |f|$ which is

$$\int_{X_0} P_0 k^+ + \int_{X_0} P_0 k^- \geq \alpha \int_{X_0} k^+ + \alpha \int_{X_0} k^- \tag{10}$$

then, adding (9) and (10) gives $\int_{X_0} P_0 k^+ \geq \alpha' \int_{X_0} k^+$ we deduce $\int_{X_0} P_0 k^+ = \alpha \int_{X_0} k^+$ and thus $P_0 k^+ = k^+$, the same reasoning gives also $P_0 k^- = \alpha k^-$.
Now, let us prove that if k is an almost everywhere non negative eigenfunction of P_0 then $k > 0$ a.e. Let $A = \{x \in X_0 \ / \ k(x) = 0\}$. Since k is no zero, $\lambda(A^c) > 0$, let us assume that $\lambda(A) > 0$.

$$\int_A k d\lambda = 0 = \alpha^{-n} \int_{X_n} \mathbf{1}_{T^{-n}A} k d\lambda \ \forall n.$$

Since $\lambda(A^c)$ and $\lambda(A)$ are non zero, there exists $n \in \mathbb{N}$ such that

$$\lambda(A^c \cap T^{-n}A \cap X_n) > 0.$$

In particular, $\lambda(T^{-n}A \cap X_n) > 0$, since $\int \mathbf{1}_{T^{-n}A \cap X_n} h = 0$, we deduce that $T^{-n}A \cap X_n \subset A$, but this contradicts $\lambda(A^c \cap T^{-n}A \cap X_n) > 0$.
We may now prove that α is simple as an eigenfunction of P_0. Let $f \in L^1(\lambda)$ be an eigenvalue of P_0 such that $\int_{X_0} f d\lambda \neq 0$, we may assume that $\int_{X_0} f = 1$, let h be the density of the probability ν constructed in theorem 3.3. By considering separately real and imaginary part, we may assume that f is a real function. Define $k = h - f$, k is an eigenvalue of P_0, by the preceding remarks, so are k^+ and k^-. For $\varepsilon = +$ or $\varepsilon = -$, k^ε is either zero a.e. or strictly positive a.e. Assume for example that $k^+ > 0$ a.e., then $h > f$ a.e. but this is not possible since $\int_{X_0} h = \int_{X_0} f = 1$. The same reasoning proves that $k^- > 0$ a.e. is also impossible, we deduce that $h = f$ a.e. It remains to consider the case $\int f d\lambda = 0$. We have that f^+ and f^- are eigenvalues of P_0 and $\int_{X_0} f^+ = \int_{X_0} f^- \neq 0$. We may assume that $\int_{X_0} f^+ = 1$ and the same arguments as above prove that $f^+ = f^- = h$ a.e. This concludes the proof of the proposition. $\triangle$

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References

[1] N. Chernov, R. Markarian & S. Troubetzkoy. *Conditionally invariant measures for*

Anosov maps with small holes. Ergodic Theory Dynam. Systems**18**, 5, 1049-1073 (1998).

[2] P. Collet, S. Martínez & B. Schmitt. *The Pianigiani-Yorke measure for topological Markov chains.* Israel Journal of Math. **97**, 61-70, (1997).

[3] P. Collet, S. Martínez & B. Schmitt. *The Pianigiani-Yorke measure and the asymptotic law on the limit Cantor set of expanding systems.* Nonlinearity **7**, 1437-1443, (1994).

[4] P. Collet, S. Martínez & B. Schmitt. *Quasi-stationary distribution and Gibbs measure of expanding systems.* Instabilities and nonequilibrium structures 205-219, (1996)

[5] N. Dunford & J. T. Schwartz. *Linear operators Part I.* Interscience, New-York (1966).

[6] P.A. Ferrari, H. Kesten, S. Martínez & P. Picco. *Existence of quasi-stationary distributions. A renewal dynamical approach.* Annals of Probability **23**, 501-521, (1995).

[7] G. Pianigiani & J.A. Yorke. *Expanding maps on sets which are almost invariant: decay and chaos.* Trans. Amer. Math. Society **252**, 433-497, (1989).

[8] E. Straube. *On the existence of invariant, absolutely continuous measures.* Commun. Math. Phys. **81**, 27-30, (1981).

[9] D. Vere-Jones. *Geometric ergodicity in denumerable Markov chains.* Quart. J. Math. **13**, 2, 2, 7-28, (1962).

[10] K. Yosida & E. Hewitt. *Finitely additive measures.* Trans. Amer. Math. Soc. **72**, 46-66, (1952).