LINEARIZATION OF CONTRACTIONS NEAR A FIXED POINT SATISFYING A GAP CONDITION

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ABSTRACT. In this work we study the linearization problem for $C^{k,1}, k \geq 1$, maps of $\mathbb{R}^n$ near a fixed point which satisfies a gap condition of order k. We show that for sufficiently small $\beta > 0$, there is a $C^{k,\beta}$ linearization in a sufficiently small neighbourhood of the fixed point.

1. Introduction

One of the central questions in dynamical systems is whether it is possible to linearize a map near a hyperbolic fixed point and if so, how smoothly and under what conditions. In this work we concentrate on contractions.

1.1. Setup and hypothesis. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^{k,1}, k \geq 1$, map with a fixed point. Without loss of generality we may assume that the fixed point is $0 \in \mathbb{R}^n$.

H1: Assume that the map takes the following form on $\mathbb{R}^n = \mathbb{R}^{N_o} \times \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_J} = \mathbb{R}^{N_o} \times \mathbb{R}^N$, $N = N_1 + \cdots + N_J$:

$$
\begin{aligned}
x_o^* &= \Lambda_o \eta + F_o(x) \\
x_I^* &= \Lambda_I x_I + F_I(x), \qquad I = 1, 2, \cdots, J \\
F(0) &= 0, \qquad DF(0) = 0 \\
\sigma(\Lambda) &= \bigcup_{I=0}^{J} \sigma_I \\
\sigma_o &\subset \{\zeta \in \mathbb{C} \mid 0 \leq |\zeta| < b_o\} \\
\sigma_I &= \sigma(\Lambda_I) \subset \{\zeta \in \mathbb{C} \mid |\zeta| = \mu_I\}, \qquad I = 1, 2, \cdots, J \\
0 &< b_o < \mu_1 < \mu_2 < \cdots < \mu_J < \gamma < 1
\end{aligned}
\tag{1.1}
$$

H2: Let $\hat{\Lambda} = blockdiag(\Lambda_1, \Lambda_2, \cdots, \Lambda_J)$. Assume that $\sigma(\hat{\Lambda})$ satisfies the following gap condition of order k which we denote by $G^{(k)}(\hat{\Lambda})$:

$$\mu_I < (\mu_{I+1})^k, \qquad I = 1, 2, \cdots, J-1 \tag{$G^{(k)}(\hat{\Lambda})$}$$

Notice that the gap condition does not depend on Λ_o.

Before we can state our main theorems we need to define mixed differentiability.

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1.2. Definition. For a map $\psi(\eta, z)$ define the following:

$$H_\delta(\psi) = \sup\left\{ \frac{|\psi(\eta,z) - \psi(\eta',z')|}{|\eta-\eta'|^\delta + |z-z'|^\delta} \;\mid\; (\eta,z) \neq (\eta',z') \right\}$$

$$H_\delta^{(z)}(\psi) = \sup\left\{ \frac{|\psi(\eta,z) - \psi(\eta,z')|}{|z-z'|^\delta} \;\mid\; z \neq z' \right\}$$

$$Lip^{(\eta)}(\psi) = \sup\left\{ \frac{|\psi(\eta,z) - \psi(\eta',z)|}{|\eta-\eta'|} \;\mid\; \eta \neq \eta' \right\}$$

We can define $Lip(\psi), Lip^{(z)}(\psi)$ and $H_\delta^{(\eta)}(\psi)$ similarly.

In this work we prove the following two theorems.

Theorem 1. *Let $B_\alpha(0) \subset \mathbb{R}^n$ be the closed ball with radius α centered at the origin. Then the following are true for sufficiently small $\alpha > 0$ and sufficiently small $0 \lneqq \beta \lneqq 1$ which depends on $\mu_1, \cdots, \mu_{J-1}$ and μ_J:*

1. *There is a $C^{k,\beta}$ diffeomorphism $x = T(\eta,z), (\eta,z) \in B_\alpha(0) \times B_\alpha(0) \subset \mathbb{R}^{N_o} \times \mathbb{R}^N$, which sends (1.1) to the map $(\eta,z) \mapsto (\eta^*, z^*)$ where*

$$\eta^* = \Lambda_o \eta + G(\eta,z) \tag{1.2}$$
$$z^* = \hat{\Lambda} z$$

2. *$H_\beta^{(z)}(D^k G) < \infty$ and $H_\beta^{(z)}(D^k Q) < \infty$.*
3. *$Lip(D_\eta^k G) < \infty$ and $Lip(D_\eta^k Q) < \infty$.*
4. *$Lip^{(\eta)}(D^k G) < \infty$ and $Lip^{(\eta)}(D^k Q) < \infty$.*
5. *If in addition $b_o < \mu_1^k$, then $G(0,z) = 0$.*

Theorem 2. *Assume that Φ in Theorem 1 is a diffeomorphism and that $\sigma(\Lambda_o) = \emptyset$. Then the $C^{k,\beta}$ linearization of Theorem 1 sends (1.1) to*

$$z^* = \Lambda z$$

Theorem 2 is a special case of Theorem 1.

1.3. Resonant and nonresonant fixed points. The linear part $D\Phi(0)$ and the fixed point are said to satisfy a nonresonant condition of order k, which we denote by $NR(k)$, iff for all $2 \leq j \leq k$

$$\lambda \neq \lambda_1\lambda_2\cdots\lambda_j, \qquad \lambda \in \sigma(\Lambda), \lambda_i \in \sigma(\Lambda), \qquad 1 \leq i \leq j \tag{$NR(k)$}$$

In other words,

$$\sigma(\Lambda)^j \cap \sigma(\Lambda) = \emptyset, \qquad 2 \leq j \leq k$$

where

$$\sigma(\Lambda)^j = \{\lambda_1\lambda_2\cdots\lambda_j \mid \lambda_i \in \sigma(\Lambda), 1 \leq i \leq j\}$$

If they satisfy a nonresonant condition of order k for all $k \geq 2$, they are said to be nonresonant.

1.4. Classical results on contractions.

1. Poincaré showed that if the map is analytic with analytic inverse and the fixed point is nonresonant, then there is an analytic linearization. See [1].
2. Sternberg showed that if the map is C^∞ with C^∞ inverse and the fixed point is nonresonant, then there is a C^∞ linearization [13].
3. In [7], Hartman showed that any $C^{1,1}$ contracting diffeomorphism is $C^{1,\beta}$ linearizable for sufficiently small $0 < \beta < 1$. The gap condition of order $k = 1$ is trivially satisfied.
4. It is known that if a fixed point has a resonance of order 2, a C^2 linearization may not exist. In [13]Sternberg showed that the following contraction on $\mathbb{R}^2$ does not have a C^2 linearization for any $0 < a < 1$:
$$x^* = ax$$
$$y^* = a^2 y + x^2$$
Notice that $\lambda_2 = a^2 = \lambda_1^2$. Thus the term x^2 is resonant.
5. In the literature on linearization of contracting diffeomorphisms (in this case $\sigma_o = \emptyset$) one finds the following statements stated in terms of the natural log:
   (a) *Assume that the map is C^k and the fixed point satisfies NR(k). If*
$$(\mu_N)^k < \mu_1, \tag{St}$$
   then the contraction is C^k linearizable.
   This statement appears in [13] where Sternberg is dealing with a nonresonant C^∞ contraction and can afford to eliminate monomials up to arbitrary high degree. If we restrict the condition given in [11] to vector fields with an attracting fixed point we obtain the vector field version of (St).
   (b) The following statement, when stated for vectors fields, appears in [12]:
   Assume that the map is C^{3k} and the fixed point satisfies NR(k). If
$$(\mu_N)^k \leq (\mu_1)^j, \tag{Se}$$
   then the contraction is C^j linearizable.
   (c) In [2] the authors study diffeomorphisms that preserve volume, contact or symplectic structures. In the case of a contraction that does not preserve any structure they obtain a condition which is slightly weaker than (S). Namely:
   Assume that the map is C^k and the fixed point satisfies NR(k). If
$$(\mu_N)^{k+1} < \mu_1, \tag{BLW}$$
   then the contraction is C^k linearizable.
6. Now, if the fixed point satisfies $NR(k)$ but has a resonance of order $(k+1)$ then
$$(\mu_1)^j \leq \mu_1 \leq (\mu_N)^{k+1} < (\mu_N)^k$$
It follows that (St), (Se) and (BLW) can never be attained for a resonant fixed point.
7. In view of the above we can see that there is no theorem that provides a sufficient condition for $C^k, k \geq 2$, linearization near a resonant fixed point. Theorem 2 provides such a condition.

2. **Proof of Theorem 1**

2.1. Smallness assumptions on $F(x)$. The conclusions of Theorem 1 are true in a sufficiently small closed ball $B_\alpha(0)$ of radius $\alpha > 0$. We will replace the small $\alpha > 0$ requirement by a smallness condition on $F(x)$ as follows:

1. First we multiply $\Phi(x)$ by a smooth cut-off function which vanishes outside a closed ball $B_{\alpha'}(0), \alpha' < \alpha$. We will continue to denote the new nonlinear part of the map by $F(x)$.
2. For $0 < \alpha < 1$ define
$$\Phi^{(\alpha)}(x) = \frac{1}{\alpha}\Phi(\alpha x) = D\Phi(0)x + F^{(\alpha)}(x) = D\Phi(0)x + \frac{1}{\alpha}F(\alpha x), \qquad |x| \leq 1$$
Straightforward calculations show that a diffeomorphism of the form
$$y = x + w(x), \qquad |x| \leq \alpha$$
linearizes the map Φ iff the diffeomorphism
$$y = w^{(\alpha)}(x), \qquad |x| \leq 1$$
linearizes the map $\Phi^{(\alpha)}$ where
$$w^{(\alpha)}(x) = \frac{1}{\alpha}w(\alpha x), \qquad |x| \leq 1$$
3. Thus for all $\epsilon > 0$ we can take $\alpha > 0$ sufficiently small so that our new map satisfies the following:
$$\begin{aligned} \|D^j F^{(\alpha)}\| &< \epsilon, \qquad j = 0, 1, \cdots, k \\ Lip(D^k F^{(\alpha)}) &< \epsilon \end{aligned} \tag{2.1}$$
We will drop the superscript (α) in order to simplify the notation.

2.2. Induction hypothesis. We will proof Theorem 1 by reversed finite induction on $\mu_1 < \mu_2 \cdots < \mu_{J-1} < \mu_J$.

1. We may assume that the map has been partially linearized for we can add a dummy variable, say z, and let $z^* = \gamma z$ with $b_J < \gamma^k < \gamma < 1$.
2. Let $1 \leq I < J$. We may assume that the map takes the form
$$\begin{aligned} \eta^* &= M\eta + F(\eta, z) \\ z^* &= Cz \\ \eta &= (x_o, x_1, \cdots, x_I) \\ z &= (x_{I+1}, \cdots, x_J) \\ M &= blockdiag(\Lambda_o, \Lambda_1, \cdots, \Lambda_I) \\ C &= blockdiag(\Lambda_{I+1}, \cdots, \Lambda_J) \\ F &= (F_o, F_1, \cdots, F_{I-1}, F_I) \end{aligned} \tag{2.2}$$

where for some $0 < \delta \leq 1, F(\eta, z)$ satisfies the following:

$$\|D^i F\| < \epsilon, \qquad 1 \leq i \leq k \tag{2.3}$$

$$Lip(D^k_\eta F) < \epsilon \tag{2.4}$$

$$Lip^{(\eta)}(D^k F) < \epsilon \tag{2.5}$$

$$H_\delta(D^k F) < \epsilon \tag{2.6}$$

2.3. Preparing for the induction step. In order to go one step further down $\mu_1 < \mu_2 \cdots < \mu_{J-1} < \mu_J$ (that is to eliminate $F_I(\eta, z)$) we observe that $\mathbb{R}^n$ and the map have the following decomposition also:

$$\begin{aligned}
(\eta, z) &= (x, y, z)\\
x &= (x_o, x_1, \cdots, x_{I-1})\\
y &= x_I\\
z &= (x_{I+1}, \cdots, x_J)\\
F &= (R, S)\\
R &= (F_o, F_1, \cdots, F_{I-1})\\
S &= F_I\\
\Lambda &= blockdiag(M, C) = blockdiag(A, B, C)\\
A &= blockdiag(\Lambda_o, \Lambda_1, \cdots, \Lambda_{I-1})\\
B &= \Lambda_I\\
C &= blockdiag(\Lambda_{I+1}, \cdots, \Lambda_J)
\end{aligned}$$

For any $\chi < \mu_I < b$ close to μ_I, for any $\mu_{I-1} < a < \mu_I$ and any $\mu_J < \gamma < 1$ we can use adapted norms so that

$$\begin{aligned}
\|A\| &< a\\
\|B\| &< b\\
\|B^{-1}\| &< \chi^{-1}\\
\|C\| &< \gamma
\end{aligned} \tag{2.7}$$

The map takes the following form in the variables (x, y, z):

$$\begin{aligned}
x^* &= Ax + R(x, y, z)\\
y^* &= By + S(x, y, z)\\
x^* &= Cz
\end{aligned} \tag{2.8}$$

2.4. Eliminating nonresonant monomials. The next step in the proof is to eliminate certain nonresonant monomials of degree 2 to k from the map.

1. By the gap condition $\mu_{I-1} < \mu_I < (\mu_{I+1})^k$. Thus we can eliminate all monomials in z of degree 2 to k from $R(\eta, z)$.

2. By the same inequality and the fact that $\sigma(\Lambda)$ lies in the interior of the unit disc, we can eliminate all monomials of degree 2 to k in (x, y, z) from $S(x, y, z)$.
3. Since $F(x)$ is $C^{k,\delta}$, we may assume that

$$|R(\eta, z)| < \epsilon[|z|^{k+\delta} + |\eta|(|\eta| + |z|)] \tag{2.9}$$

$$|S(\eta, z)| < \epsilon[|z|^{k+\delta} + |\eta|^{k+\delta}] \tag{2.10}$$

2.5. The weak-stable manifold. At this point we need to invoke the weak-stable manifold theorem under the gap condition. See for example Theorem 5.1 in [8, page 53]. However, we need a weak-stable manifold for maps that satisfy the vanishing properties (2.9) and (2.10). These vanishing properties are essential for the linearization step. Such a theorem is given in [5] and we state it here in the present notation.

Theorem 3. *Consider a $C^{k,\delta}, 0 < \delta \leq 1, k = 1, 2, \cdots,$ map*

$$\begin{aligned} \eta^* &= M\eta + F(\eta, z) \\ z^* &= Cz + T(\eta, z) \end{aligned}$$

Assume that there is $b < c$ such that

1. $b < c^{k+\delta}$.
2. $|\mu| < b, \mu \in \sigma(M)$.
3. $|\gamma| > c, \gamma \in \sigma(C)$.
4. $|F(\eta, z)| < \epsilon(|z|^{k+\delta} + |\eta|)$.
5. $\|D^j F\| < \epsilon, \|D^j T\| \leq \epsilon, j = 0, 1, \cdots, k$.
6. $H_\delta(D^k F) < \epsilon, H_\delta(D^k T) < \epsilon$.

Then, for sufficiently small $\epsilon > 0$, the map has a $C^{k,\delta}$ weak stable manifold of the form $\eta = w(z)$ which is unique in the class of functions that satisfy:

$$\begin{aligned} w(z^*) &= Mw(z) + F(w(z), z) \\ z^* &= Cz + T(w(z), z) \\ \|D^j w(z)\| &\leq |z|^{k+\delta-j}, \qquad j = 0, 1, \cdots, k \end{aligned} \tag{2.11}$$

2.6. Straightening up the weak-stable manifold. It is obvious that we can apply Theorem 3 to the map (2.2) by taking $T \equiv 0$ and $c < \mu_{I+1}$ sufficiently close to μ_{I+1}. Next we make a change of variables that straightens up the weak-stable manifold. Let $\xi = \eta - w(z)$. Thus (2.8) takes the form

$$\begin{aligned} \xi^* &= M\xi + G(\xi, z) \\ z^* &= Cz \\ G(\xi, z) &= F(\xi + w(z), z) - F(w(z), z) \\ G(0, z) &= 0 \end{aligned} \tag{2.12}$$

It follows that $\{(\xi, z) \mid \xi = 0\}$ is invariant under the map (2.12). The map (2.12) takes the following form where $G =: (f, g)$:

$$\begin{aligned} x^* &= Ax + f(\xi, z) \\ y^* &= By + g(\xi, z) \\ z^* &= Cz \end{aligned} \tag{2.13}$$

where,

$$\begin{aligned} f(\xi, z) &= R(\xi + w(z), z) - R(w(z), z) \\ g(\xi, z) &= S(\xi + w(z), z) - S(w(z), z) \end{aligned} \tag{2.14}$$

2.7. Estimates resulting from the existence of a weak-stable manifold. Straightening up the weak stable manifold has a significant effect on the map (2.13) which we describe in the following theorem:

Theorem 4. *Let $\mathfrak{D}^j = D^j - D^j_z - D^j_\xi$. Then the map (2.13) satisfies the following for $|\xi| \leq 1$ and $|z| \leq 1$ with $r = k + \delta, 0 \leq s \leq 1$ and $G = (f, g)$:*

$$|g(\xi, z)| < \epsilon|\xi|[|\xi|^{r-1} + |z|^{r-1}] \tag{2.15}$$

$$\|D^j_z G(\xi, z)\| < \epsilon|\xi| \qquad 0 \leq j \leq k \tag{2.16}$$

$$\|D^j_z g(\xi, z)\| < \epsilon|\xi|[|\xi|^{r-j-1} + |z|^{r-j-1}] \qquad 0 \leq j \leq k-1 \tag{2.17}$$

$$\|\mathfrak{D}^j g(\xi, z)\| < \epsilon[|\xi|^{r-j} + |z|^{r-j}] \qquad 0 \leq j \leq k \tag{2.18}$$

$$\|D^j_\xi g(\xi, z)\| < \epsilon[|\xi|^{r-j} + |z|^{r-j}] \qquad 0 \leq j \leq k \tag{2.19}$$

$$Lip(D^k_\xi G) < \epsilon \tag{2.20}$$

$$H_\delta(D^k G) < \epsilon \tag{2.21}$$

$$\|D^k G(\xi, z) - D^k G(\xi', z)\| < \epsilon|\xi - \xi'| \tag{2.22}$$

$$\|D^j_z G(\xi, z) - D^j_z G(\xi, z')\| < \epsilon|\xi| \qquad 0 \leq j \leq k \tag{2.23}$$

$$\|D^j_z G(\xi, z) - D^j_z G(\xi, z')\| < \epsilon|\xi|^s|z - z'|^{\delta(1-s)}, \qquad 0 \leq j \leq k \tag{2.24}$$

The proof of Theorem 4 will be given in Section 4 in order not to obstruct the flow of the discussion.

Theorem 5. *Consider The map (2.13): By Theorem 4 the nonlinear part (f, g) satisfies (2.15)-(2.24). Recall (2.7). Let $0 < s < 1$ close enough to 1 so that :*

$$\gamma^{k+1} < (\mu_I)^{1-s} \tag{2.25}$$

If $\gamma^{k+1} < \mu_I$ take $s = 0$. Let $\beta = \delta(1 - s)$.

Then the following are true for sufficiently small $\epsilon > 0$:

1. *For any $0 < d \leq 1$ there is a function $u \in C^{k,\beta}(B_1(0), \mathbb{R}^{N_I})$ such that the change of variables*

$$\begin{aligned} &(x, y, z) \mapsto (X, Y, Z) \\ &(X, Y, Z) = Q(x, y, z) = (x, y + u(x, y, z), z) \end{aligned} \tag{2.26}$$

transforms the map (2.13) to the map

$$X^* = AX + H(X,Y,Z) \quad (2.27)$$
$$Y^* = BY$$
$$Z^* = CZ$$

Moreover

$$Lip(D_\xi^k u) \le d \quad (2.28)$$

$$Lip^{(\xi)}(D^k u) \le d \quad (2.29)$$

$$H_\delta^{(z)}(\mathfrak{D}^k u) \le d \quad (2.30)$$

$$\|D_z^k u(\xi,z) - D_z^k u(\xi,z')\| \le d|\xi|^s|z-z'|^\beta \quad (2.31)$$

2. *For sufficiently small $0 < d < 1$, $Q(x,y,z)$ is a $C^{k,\beta}$ diffeomorphism of a neighbourhood of the origin. (This assertion follows immediately from the Inverse Function Theorem).*
3. *Let the inverse of $Q(x,y,z)$ be given by $Q^{-1}(X,Y,Z) = (X, Y + v(X,Y,Z), Z)$. Then $v(X,Y,Z) = -u(x,y,z)$. Moreover,*

$$H(X,Y,Z) = f \circ Q^{-1}(X,Y,Z) = f(X, Y + v(X,Y,Z), Z)$$

satisfy the following with $\Xi = (X,Y)$:

$$\|D^j H\| < O(\epsilon), \qquad 0 \le j \le k \quad (2.32)$$

$$H_\beta(D^k H) < O(\epsilon) \quad (2.33)$$

$$Lip(D_\Xi^k H) < O(\epsilon) \quad (2.34)$$

$$Lip^{(\Xi)}(D^k H) < O(\epsilon) \quad (2.35)$$

Proof of Theorem 1 granting Theorem 5. The new map (2.27) satisfies the Induction Hypothesis 2.2 with I replaced by $I-1$. This finishes the proof of Theorem 1 since we are performing a finite induction and do not have to worry about neighbourhoods shrinking to the origin. □

3. Proof of Theorem 5

In this section we prove Theorem 5. A change of variables of the form (2.26) eliminates $g(x,y,z)$ from the map (2.13) iff $u(\xi,z)$ satisfies

$$u(\xi,z) = B^{-1}u(\xi^*,z^*) + B^{-1}g(\xi,z) \quad (3.1)$$

We should expect $u(\xi,z)$ to be a fixed point for a transformation on a closed subset of an appropriately constructed Banach space which we define presently.

3.1. The space. Let $r = k + \delta, 0 < \delta \le 1$. For $0 \le j \le k-1$ let $p_j = r - j$ and $q_j = r - j - 1 = p_j - 1$. Let $p_k = r - k = \delta$ and $q_k = 0$. Thus, $q_j = p_{j+1}, 0 \le j \le k-1$. For $u \in C^{k,\delta}(B_1(0), \mathbb{R}^{N_I})$ define

$$N_o(u) = \sup\left\{ \frac{|u(\xi,z)|}{|\xi|[|\xi|^{q_o} + |z|^{q_o}]} \;\Big|\; (\xi,z) \ne (0,0), \xi \ne 0 \right\}$$

For $j = 1, \cdots, k$, let

$$
\begin{aligned}
K_j(u) &= \sup\left\{\frac{\|D^j_\xi u(\xi,z)\|}{|\xi|^{p_j}+|z|^{p_j}} \;\middle|\; (\xi,z)\neq(0,0)\right\}\\
N_j(u) &= \sup\left\{\frac{\|D^j_z u(\xi,z)\|}{|\xi|[|\xi|^{q_j}+|z|^{q_j}]} \;\middle|\; \xi\neq 0\right\}\\
\mathfrak{D}^j &= D^j - D^j_z - D^j_\xi = \{D^l_\xi D^i_z \mid i+l=j, i\geq 1, l\geq 1\}\\
m_{l,i}(u) &= \sup\left\{\frac{\|D^l_\xi D^i_z u(\xi,z)\|}{|\xi|^{p_j}+|z|^{p_j}} \;\middle|\; (\xi,z)\neq(0,0)\right\}, \qquad i+l=j, i\geq 1, l\geq 1\\
M_j(u) &= \max\{m_{l,i}(u) \mid i+l=j, i\geq 1, l\geq 1\}\\
\|u\|_* &= N_o(u) + \sum_{j=1}^{k}[K_j(u)+N_j(u)+M_j(u)]\\
\beta &= \delta(1-s), \qquad 0<s<1\\
\kappa(D^k_z u) &= \sup\left\{\frac{\|D^k_z u(\xi,z) - D^k_z u(\xi,z')\|}{|\xi|^s|z-z'|^\beta} \;\middle|\; \xi\neq 0, z\neq z'\right\}\\
H^{(z)}_\delta(\mathfrak{D}^k u) &= \max\{H^{(z)}_\delta(D^l_\xi D^i_z u) \mid l+i=k, l\geq 1, i\geq 1\}\\
\sigma(u) &= \max\{Lip^{(\xi)}(D^k u), Lip(D^k_\xi u), \kappa(D^k_z u), H^{(z)}_\delta(\mathfrak{D}^k u)\}\\
\mathfrak{Z} &= \{u\in C^k(B_1(0),\mathbb{R}^{N_I}) \mid \|u\|_*<\infty\}\\
\mathfrak{X}_d &= \{u\in\mathfrak{Z} \mid \|u\|_*\leq d,\ \sigma(u)\leq d,\ D^j u(0)=0, j=0,1,\cdots,k\}
\end{aligned}
$$

Since convergence in the norm $\|\cdot\|_*$ implies uniform convergence on bounded sets, it follows that $(\mathfrak{Z}, \|\cdot\|_*)$ is a Banach space and that $\mathfrak{X}_d$ is a closed subset.

It is also obvious that $H_\beta(D^k u) \leq 2^{1-\beta} d$ for all $u \in \mathfrak{X}_d$. This is because for $0 < \beta < \alpha \leq 1$, $|X - X'|^\alpha \leq 2^{\alpha-\beta}|X - X'|^\beta$. Also we use the norm

$$\|D^k u\| = \max\{\|D^l_\xi D^i_z u\| \mid l+i=k\}$$

It follows that $\|D^j u\| \leq 2d, j = 0, 1, \cdots, k$, for all $u \in \mathfrak{X}_d$.

3.2. The transformation. Let Φ be the map (2.13). Define $\Omega : \mathfrak{Z} \to \mathfrak{Z}$ and $\Gamma : \mathfrak{Z} \to \mathfrak{Z}$ by

$$
\begin{aligned}
\Omega u &= \Gamma u + B^{-1} g \\
\Gamma u &= B^{-1} u\circ\Phi
\end{aligned}
\tag{3.2}
$$

In other words,

$$
\begin{aligned}
\Omega u(\xi,z) &= B^{-1}u(\xi^*,z^*) + B^{-1}g(\xi,z)\\
\Gamma u(\xi,z) &= B^{-1}u(\xi^*,z^*)
\end{aligned}
$$

It is obvious that $B^{-1}g$ does not depend on u, Γ is linear and Ω is affine. These properties will simplify the proof drastically.

3.3. The steps. We prove Theorem 5 in four steps:

Step 1: For $\epsilon < d$, Theorem 4 tells us that $\|g\|_* < O(\epsilon)$ and $\sigma(g) < O(\epsilon)$. Thus, for sufficiently small $\epsilon, g \in \mathfrak{X}_d$.

Step 2: We show that for sufficiently small $\epsilon > 0$ there is $\theta \in (0,1)$ such that for all $u \in \mathfrak{X}_d$

$$\|\Gamma u\|_* \leq \theta \|u\|_* \tag{3.3}$$

$$\sigma(\Gamma u) \leq \theta\sigma(u) + O(\epsilon) \tag{3.4}$$

Step 3: For sufficiently small $\epsilon > 0$, Ω maps $\mathfrak{X}_d$ to itself and is a contraction and hence has a unique attractive fixed point in $\mathfrak{X}_d$. This can be seen as follows: It is obvious that if $D^j u(0) = 0, j = 0, \cdots, k$, then then $D^j \Omega u(0) = 0, j = 0, \cdots, k$. By step 1 and step 2, for sufficiently small $\epsilon > 0$ for all $u \in \mathfrak{X}_d$

$$\|\Omega u\|_* \leq \theta\|u\|_* + \|g\|_* < \theta d + O(\epsilon) < d$$
$$\sigma(\Omega u) \leq \theta\sigma(u) + \sigma(g) < \theta + O(\epsilon) < d$$

Thus Ω maps $\mathfrak{X}_d$ to itself. Also by step 2, since Γ is linear and $g(\xi, z)$ is independent of $u \in \mathfrak{X}_d$ we have

$$\|\Omega u - \Omega v\|_* = \|\Gamma u - \Gamma v\|_* \leq \theta\|u - v\|_*, \qquad u, v \in \mathfrak{X}_d$$

Thus Ω is a contraction and hence has a unique attractive fixed point in $\mathfrak{X}_d$.

Step 4: $H(X, Y, Z)$ in the new map (2.27) has the stated properties.

To finish the proof of Theorem 5 we only need to prove steps 2 and 4.

3.4. Notation.

1. In what follows t will stand for a continuous function $t(\epsilon) \to 0$ as $\epsilon \to 0$ which is independent of $u \in \mathfrak{X}_d$ and depends only on the map.
2. All norms are assumed to be box norms.

3.5. Lemma. *Let $|\xi| \leq 1$ and $|z| \leq 1$.*

1. *By Theorem 4, the following are true for sufficiently small $\epsilon > 0$:*

$$|z^*| \leq \gamma|z| < 1 \tag{3.5}$$

$$|\xi^*| \leq b(1+t)|\xi| < 1 \tag{3.6}$$

$$|\xi^*|^{p_j} + |z^*|^{p_j} \leq \gamma^{p_j}(1+t)[|\xi|^{p_j} + |z|^{p_j}] \tag{3.7}$$

$$|\xi^*|[|\xi^*|^{q_j} + |z^*|^{q_j}] \leq b\gamma^{q_j}(1+t)|\xi|[|\xi|^{q_j} + |z|^{q_j}] \tag{3.8}$$

$$\begin{aligned} |\xi|[|\xi|^{q_j} + |z|^{q_j}] &\leq O(|\xi|^{q_j+1} + |z|^{q_j+1}) \\ &\leq O(|\xi|^{p_j} + |z|^{p_j}) \end{aligned} \tag{3.9}$$

2. *Since we are using the box norm, it follows from the first two estimates that $\Gamma u(\xi, z)$ is well defined on $B_1(0)$.*
3. *Let $\chi < \mu_I < b$ close enough to μ_I so that*

$$\begin{aligned} \theta_o &:= \chi^{-1} b\gamma^{k+\delta-1} < 1 \\ \theta_2 &:= \chi^{-1} b^k \gamma < 1 \\ \theta_3 &:= \chi^{-1} b\gamma^k < 1 \end{aligned} \tag{3.10}$$

In fact, since $b < \gamma$ and $0 < \delta \le 1$, we have

$$\theta_2 < \theta_3 < \theta_o$$

4. *By (2.25), we can choose $\chi < \mu_I < b$ close enough to μ_I so that*

$$\theta_1 := \chi^{-1} b^s \gamma^{k+\beta} < 1 \tag{3.11}$$

3.6. Proposition. *For sufficiently small $\epsilon > 0$, for all $u \in \mathfrak{X}_d$*

$$N_o(\varGamma u) \le \theta N_o(u)$$
$$\theta = b\chi^{-1}\gamma^{r-1}(1+t) = \theta_o(1+t) < 1$$

Proof. By (3.5) and (3.6) and Theorem 4

$$\begin{aligned} |\varGamma u(\xi, z)| &\le \chi^{-1} N_o(u) |\xi^*| [|\xi^*|^{q_o} + |z^*|^{q_o}] \\ &\le [b\chi^{-1}\gamma^{q_o}(1+t)] N_o(u) |\xi| [|\xi|^{q_o} + |z|^{q_o}] \end{aligned}$$

and $q_o = r - 1$ which proves the proposition. □

3.7. Proposition. $\mathfrak{D}^1 = 0$. *For sufficiently small $\epsilon > 0$, for all $u \in \mathfrak{X}_d$*

$$N_1(D_z \varGamma u) \le \theta N_1(u) + t K_1(u)$$
$$K_1(D_\xi \varGamma u) \le \theta K_1(u)$$
$$\theta = b\chi^{-1}\gamma^{r-1}(1+t) = \theta_o(1+t) < 1$$

Proof. Recall that $p_1 = q_1 + 1 = r - 1$, $|z^*| < 1$ and that $|\xi^*| < 1$. Thus, by (2.16) and Lemma 3.5 we have

$$\begin{aligned} D_z \varGamma u(\xi, z) &= B^{-1} D_\xi u(\xi^*, z^*) D_z G(\xi, z) + B^{-1} D_z u(\xi^*, z^*)[C] \\ \|D_z \varGamma u(\xi, z)\| &\le \chi^{-1} K_1(u) [|\xi^*|^{p_1} + |z^*|^{p_1}][\epsilon|\xi|] \\ &\quad + \gamma\chi^{-1} N_1(u) |\xi^*| [|\xi^*|^{q_1} + |z^*|^{q_1}] \\ &\le [tK_1(u) + b\chi^{-1}\gamma^{r-1}(1+t)N_1(u)] |\xi| [|\xi|^{q_1} + |z|^{q_1}] \end{aligned}$$

Since $z^* = Cz$ is independent of ξ, we have

$$\begin{aligned} D_\xi \varGamma u(\xi, z) &= B^{-1} D_\xi u(\xi^*, z^*)[M + D_\xi G(\xi, z)] \\ \|D_\xi \varGamma u(\xi, z)\| &\le \chi^{-1} K_1(u) [|\xi^*|^{p_1} + |z^*|^{p_1}][b + t] \\ &\le b\chi^{-1}\gamma^{r-1}(1+t) K_1(u) [|\xi|^{p_1} + |z|^{p_1}] \end{aligned}$$

which proves the lemma. □

3.8. Proposition. *For sufficiently small $\epsilon > 0$, for all $u \in \mathfrak{X}_d$ and $2 \le j \le k$*

$$N_j(\varGamma u) \le \theta N_j(u) + t \sum_{i=1}^{j} [K_i(u) + N_i(u) + M_i(u)]$$
$$\theta = b\chi^{-1}\gamma^{r-1}(1+t) = \theta_o(1+t) < 1$$

Proof. Notice that $D^n_\xi z^* = 0, n \geq 1$, and $D^n_z z^* = 0, n \geq 2$. Thus

$$
\begin{aligned}
D^j_z \Gamma u(\xi, z) &= B^{-1} D^j_z u(\xi^*, z^*)[C]^j + P_j(\xi, z) \\
P_j(\xi, z) &= \mathcal{P}_j(\tilde{\Delta}^{(j)} u(\xi^*, z^*), \Delta^{(j)}_z G(\xi, z)) \\
\mathcal{P}_j(0, \cdot) &\equiv 0 \\
\mathcal{P}_j(\cdot, 0) &\equiv 0 \\
\Delta^{(l)}\psi &= (D\psi, \cdots, D^l \psi) \\
\tilde{\Delta}^{(l)}\psi &= \Delta^{(l)}\psi - D^l_z \psi
\end{aligned}
\tag{3.12}
$$

Moreover, $\mathcal{P}_j(\Sigma, \Omega)$ is a polynomial in the components of Σ and Ω and linear in the components of Σ.

By (2.16) we have

$$\|\Delta^{(j)}_z G(\xi, z)\| \leq O(\epsilon)|\xi|$$

In view of this fact, the vanishing properties of $\mathcal{P}_j$ listed above and estimate (3.9) we obtain the following estimate:

$$
\begin{aligned}
\|P_j(\xi, z)\| &\leq t|\xi| \sum_{i=1}^{j} [K_i(u) + M_i(u) + N_i(u)][|\xi|^{p_i} + |z|^{p_i}] \\
&\leq t|\xi|[|\xi|^{q_j} + |z|^{q_j}] \sum_{i=1}^{j} K_i(u) + M_i(u) + N_i(u)
\end{aligned}
\tag{i}
$$

Moreover, by estimate (3.8) and since $q_j = r - j - 1$, we have

$$
\begin{aligned}
\|B^{-1} D^j_z u(\xi^*, z^*)[C]^j\| &\leq \chi^{-1}\gamma^j N_j(u)|\xi^*|[|\xi^*|^{q_j} + |z^*|^{q_j}] \\
&\leq b\chi^{-1}\gamma^{r-1}(1+t) N_j(u)|\xi|[|\xi|^{q_j} + |z|^{q_j}]
\end{aligned}
\tag{ii}
$$

Adding up estimates (i) and (ii) and taking the sup, we obtain the desired estimate. ∎

3.9. Proposition. *For sufficiently small $\epsilon > 0$, for all $u \in \mathfrak{X}_d$ and $2 \leq j \leq k$*

$$
\begin{aligned}
K_j(\Gamma u) &\leq \theta K_j(u) + t \sum_{i=1}^{j} K_i(u) \\
\theta &= b\chi^{-1}\gamma^{r-1}(1+t) = \theta_o(1+t) < 1
\end{aligned}
$$

Proof. Notice that $D^n_\xi z^* = 0, n \geq 1$. Thus

$$
\begin{aligned}
D^j_\xi \Gamma u(\xi, z) &= B^{-1} D^j_\xi u(\xi^*, z^*)[M + D_\xi G(\xi, z)]^j + Q_j(\xi, z) \\
Q_j(\xi, z) &= \mathcal{Q}_j(\Delta^{(j-1)}_\xi u(\xi^*, z^*), \Delta^{(j)}_\xi G(\xi, z)) \\
\mathcal{Q}_j(0, \cdot) &\equiv 0 \\
\mathcal{Q}_j(\cdot, 0) &\equiv 0
\end{aligned}
\tag{3.13}
$$

Moreover, $\mathcal{Q}_j(\Sigma, \Omega)$ is a polynomial in the components of Σ and Ω and linear in the components of Σ.

By Theorem 4

$$\|\Delta_\xi^{(j)} G(\xi, z)\| \le O(\epsilon)[|\xi|^{p_j} + |z|^{p_j}]$$

Thus in view of the vanishing properties of $\mathcal{Q}_j$ we have:

$$\text{(a)} \qquad \|\mathcal{Q}_j(\xi, z)\| \le t[|\xi|^{p_j} + |z|^{p_j}] \sum_{i=1}^{j-1} K_i(u)$$

Moreover, since $p_j = r - j$ and $b \le \gamma$,

$$\begin{aligned} \text{(b)} \qquad \|B^{-1} D_\xi^j u(\xi^*, z^*)[M + D_z G(\xi, z)]^j\| &\le \chi^{-1} K_j(u)[|\xi^*|^{p_j} + |z^*|^{p_j}][b^j(1+t)] \\ &\le b^j \chi^{-1} \gamma^{p_j}(1+t) K_j(u)[|\xi|^{p_j} + |z|^{p_j}] \\ &\le b \chi^{-1} \gamma^{r-1}(1+t) K_j(u)[|\xi|^{p_j} + |z|^{p_j}] \end{aligned}$$

Adding up estimates (a) and (b) and taking the sup, we obtain the desired estimate. □

3.10. Proposition. *For sufficiently small $\epsilon > 0$, for all $u \in \mathfrak{X}_d$ and $2 \le j \le k$*

$$M_j(u) \le \theta M_j(u) + t \sum_{i=1}^{j} [K_i(u) + N_i(u) + M_i(u)]$$
$$\theta = b\chi^{-1}\gamma^{r-1}(1+t) = \theta_o(1+t) < 1$$

Proof. Let (i, l) be fixed but arbitrary such that $i + l = j, i \ge 1$ and $l \ge 1$. Recall that $D_\xi^n z^* = 0, n \ge 1$. Thus

$$\begin{aligned} \text{(3.14)} \qquad D_\xi^l D_z^i \Gamma u(\xi, z) &= B^{-1} D_\xi^l D_z^i u(\xi^*, z^*)[C]^i [M + D_\xi G(\xi, z)]^l \\ &\quad + \mathcal{W}_j(\xi, z) + R_{j-1}(\xi, z) \end{aligned}$$

where $\mathcal{W}_j(\xi, z)$ is a finite sum of the form

$$\begin{aligned} \mathcal{W}_j(\xi, z) &= \sum_{n=1}^{i} c_n \theta_n \\ \theta_n &= D_\xi^{l+n} D_z^{i-n} u(\xi^*, z^*)[C]^{i-n}[D_z G(\xi, z)]^n [M + D_\xi G(\xi, z)]^l \\ R_{j-1}(\xi, z) &= \mathcal{R}_{j-1}(\Delta^{(j-1)} u(\xi^*, z^*), \Delta^{(j)} G(\xi, z)) \\ \mathcal{R}_{j-1}(0, \cdot) &\equiv 0 \\ \mathcal{R}_{j-1}(\cdot, 0) &\equiv 0 \end{aligned}$$

where $\mathcal{R}_{j-1}(\Sigma, \Omega)$ is a polynomial in the components of Σ and Ω and linear in the components of Σ. In order to estimate $\|\mathcal{W}_j(\xi, z)\|$ we notice that $n \ge 1$ in the sum defining it. Thus

$$\text{(a)} \qquad \|\mathcal{W}_j(\xi, z)\| \le t[|\xi|^{p_j} + |z|^{p_j}] \sum_{\nu=1}^{j} [M_\nu(u) + K_\nu(u)]$$

The vanishing properties of $\mathcal{R}_{j-1}$ and Theorem 4 imply that $\|\Delta^{(j-1)}G\| < O(\epsilon)$ and

$$\text{(b)} \qquad \|R_{j-1}(\xi,z)\| \le t[|\xi|^{p_j} + |z|^{p_j}] \sum_{\nu=1}^{j-1} [K_\nu(u) + M_\nu(u) + N_\nu(u)]$$

Finally, since $i + l - 1 + p_j = p_j + j - 1 = r - 1$,

$$\text{(c)} \qquad \begin{aligned} \|B^{-1} D_\xi^l D_z^i u(\xi^*, z^*)[C]^i [M + D_\xi G(\xi,z)]^l\| &\le \chi^{-1}\gamma^i b^l (1+t) m_{l,i}(u)[|\xi^*|^{p_j} + |z^*|^{p_j}] \\ &\le \chi^{-1}\gamma^{i+p_j} b^l (1+t) m_{l,i}(u)[|\xi|^{p_j} + |z|^{p_j}] \\ &\le \chi^{-1} b \gamma^{r-1}(1+t) m_{l,i}(u)[|\xi|^{p_j} + |z|^{p_j}] \end{aligned}$$

Adding up estimates (a)-(c) we can see that

$$m_{l,i}(\Gamma u) \le \chi^{-1} b \gamma^{r-1} m_{l,i}(u) + t \sum_{n=1}^{j} [K_n(u) + M_n(u) + N_n(u)]$$

Taking the maximum over all possible (l,i) such that $i + l = j, i \ge 1$ and $l \ge 1$ we obtain the desired estimate. $\square$

3.11. Corollary. *For sufficiently small $\epsilon > 0$*

$$\|\Gamma u\|_* \le \theta \|u\|_*, \qquad u \in \mathfrak{X}_d$$
$$\theta = \chi^{-1} b \gamma^{r-1} + t = \theta_o + t < 1$$

which proves (3.3) of step 2 in section 3.3.

Proof. By Propositions 3.6, 3.7, 3.8, 3.9 and 3.10

$$\|\Gamma u\|_* \le \theta \|u\|_*, \qquad u \in \mathfrak{X}_d$$
$$\theta = b\chi^{-1}\gamma^{r-1}(1+t) = \theta_o(1+t) < 1$$

By (3.10), $\theta_o < 1$. Thus $\theta < 1$ for sufficiently small $\epsilon > 0$. $\square$

Next we prove (3.4) of step 2 in section 3.3. This will be accomplished in four propositions. First we need a simple lemma.

3.12. Lemma. *Let $z \ne z', \xi \ne \xi'$,*

$$z^* = Cz, \qquad \hat{z} = Cz', \qquad \tilde{z} = Cz' = \hat{z}$$
$$\xi^* = M\xi + f(\xi, z), \qquad \hat{\xi} = M\xi + f(\xi, z'), \qquad \tilde{\xi} = M\xi' + f(\xi', z')$$

Then

$$\text{(3.15)} \qquad |\xi^* - \hat{\xi}| \le |f(\xi,z) - f(\xi,z')| \le \epsilon |z - z'|$$

$$\text{(3.16)} \qquad |\xi^* - \hat{\xi}| \le |f(\xi,z) - f(\xi,z')| \le \epsilon |\xi|^s |z - z'|^\beta$$

$$\text{(3.17)} \qquad |\xi^* - \tilde{\xi}| \le \gamma|\xi - \xi'| + |f(\xi,z) - f(\xi',z')| \le b(1+t)|\xi - \xi'| + \epsilon|z - z'|$$

Proof. Estimate (3.16) follows from (2.24). The others follow from the fact that $G(x,y)$ is uniformly Lipschitz continuous. The smallness assumptions lead to the factor ϵ in all three estimates. $\square$

3.13. Proposition. *For sufficiently small $\epsilon > 0$ and for $0 < s < 1$ sufficiently close to 1 the following hold for all $u \in \mathfrak{X}_d$:*

$$\kappa(D_z^k \Gamma u) \leq \theta_1 \kappa(D_z^k u) + t < 1$$
$$\theta_1 = \chi^{-1} b^s \gamma^{k+\beta} < 1$$

Proof. Recall that by (3.11) $\theta_1 < 1$. Recall (3.12) with $j = k$.

$$D_z^k \Gamma u(\xi, z) = B^{-1} D_z^k u(\xi^*, z^*)[C]^k + P_k(\xi, z)$$

Let $z \neq z'$ and let $\hat{\xi}$ and $\hat{z} = Cz'$ be as in Lemma 3.12. Then

$$\mathcal{A} := \|[B^{-1}[D_z^k u(\xi^*, z^*) - D_z^k u(\hat{\xi}, \hat{z})][C]^k\|$$
$$\mathcal{A} \leq \chi^{-1}\gamma^k [\|D_z^k u(\xi^*, z^*) - D_z^k u(\xi^*, \hat{z})\| + \|D_z^k u(\xi^*, \hat{z}) - D_z^k u(\hat{\xi}, \hat{z})\|]$$

Since $Lip^{(\xi)}(D_z^k u) \leq d$, by (3.16) we have

$$\begin{aligned} \mathcal{A} &\leq \chi^{-1}\gamma^k (1+t)\kappa(D_z^k u)|\xi^*|^s |C(z-z')|^\beta + td|\xi^* - \hat{\xi}| \\ &\leq [\chi^{-1} b^s \gamma^{k+\beta}(1+t)\kappa(D_z^k u) + t]|\xi|^s |z-z'|^\beta \\ &\leq [\theta_1 \kappa(D_z^k u) + t]|\xi|^s |z-z'|^\beta \end{aligned} \tag{1}$$

Let

$$\mathcal{B} := \|\mathcal{P}_k(\tilde{\Delta}^{(k)} u(\xi^*, z^*), \Delta_z^{(k)} G(\xi, z)) - \mathcal{P}_k(\tilde{\Delta}^{(k)} u(\hat{\xi}, \hat{z}), \Delta_z^{(k)} G(\xi, \hat{z}))\|$$

By (2.16)

$$\|\Delta_z^{(k)} G(\xi, z)\| \leq t|\xi|$$

By (2.24)

$$\|\Delta_z^{(k)} G(\xi, z) - \Delta_z^{(k)} G(\xi, \hat{z})\| \leq t|\xi|^s |z-z'|^\beta$$

Also notice that $H_\beta(\tilde{\Delta}^{(k)}(u)) \leq 2d$ which is independent of $u \in \mathfrak{X}_d$. By the vanishing properties of $\mathcal{P}_k$, given in (3.12), and by (3.16) we have

$$\mathcal{B} \leq t|\xi|^s |z-z'|^\beta \tag{2}$$

Adding (1) and (2) and taking the sup we obtain the desired estimate. □

3.14. Proposition. *For sufficiently small $\epsilon > 0$ for all $u \in \mathfrak{X}_d$:*

$$Lip(D_\xi^k \Gamma u) \leq \theta_2 Lip(D_\xi^k u) + t < 1$$
$$Lip^{(\xi)}(D_\xi^k \Gamma u) \leq \theta_2 Lip^{(\xi)}(D_\xi^k u) + t < 1$$
$$\theta_2 = \chi^{-1} b^k \gamma < \theta_3 = \chi^{-1} b \gamma^k < \chi^{-1} b \gamma^{r-1} = \theta_o < 1$$

Proof. Recall that $\theta_o < 1$ by (3.10).

Consider (3.13) with $j = k$:

$$D_\xi^k \Gamma u(\xi, z) = B^{-1} D_\xi^k u(\xi^*, z^*)[M + D_\xi G(\xi, z)]^k + Q_k(\xi, z) \tag{3.18}$$

Recall that $\Delta_\xi^{(k-1)}u(\xi^*,z^*)$ is $C^{1,\delta}$ in (ξ,z) and that $Lip(\Delta_\xi^{(k)}G) < O(\epsilon)$ by (2.20). It follows that

(a) $$Lip(Q_k) < t$$

To estimate the first term in (3.18) let

$$\mathcal{A}(\xi,z) = B^{-1}D_\xi^k u(\xi^*,z^*)[M + D_\xi G(\xi,z)]^k$$

Let $\xi \neq \xi'$ and $z \neq z'$. Let $\tilde{\xi}$ and $\tilde{z} = Cz'$ be as in Lemma 3.12. Then

$$\begin{aligned}\|M + D_\xi G(\xi,z)\|^k \|D_\xi^k u(\xi^*,z^*) - D_\xi^k u(\tilde{\xi},\tilde{z})\| &\le b^k(1+t)Lip(D_\xi^k u)[|\xi^* - \tilde{\xi}| + \gamma|z - z'|] \\ &\le b^k\gamma(1+t)Lip(D_\xi^k u)[|\xi - \xi'| + |z - z'|]\end{aligned}$$

Even if $k = 1, Lip(D_\xi G(\xi,z)) < \epsilon$. Thus

$$\|D_\xi^k u(\tilde{\xi},\tilde{z})\|\|[M + D_\xi G(\xi,z)]^k - [M + D_\xi G(\xi',z')]^k\| \le t(|\xi - \xi'| + |z - z'|)$$

Thus

(b) $$\|\mathcal{A}(\xi,z) - \mathcal{A}(\xi',z')\| \le [\theta_2 Lip(D_\xi^k u) + t][|\xi - \xi'| + |z - z'|], \qquad u \in \mathfrak{X}_d$$

The first estimate follows from estimates (a) and (b).

It is obvious that the second estimate follows from the first since $z^* = Cz$ is independent of ξ. □

3.15. Proposition. *The following hold for sufficiently small $\epsilon > 0$:*

$$Lip^{(\xi)}(D^k\Gamma u) \le \theta_3 Lip^{(\xi)}(D^k u) + t < 1, \qquad u \in \mathfrak{X}_d$$
$$\theta_3 = \chi^{-1}b\gamma^k < \chi^{-1}b\gamma^{r-1} = \theta_o < 1$$

Proof. Notice that $D^k u = (D_\xi^k u, D_z^k u, \mathfrak{D}^k u)$. So, we prove the estimate for the three parts separately.

We know from Proposition 3.14 that

(a) $$Lip^{(\xi)}(D_\xi^k\Gamma u) \le \theta_2 Lip^{(\xi)}(D_\xi^k u) + t < \theta_3 Lip^{(\xi)}(D_\xi^k u) + t$$

To consider $D_z^k u$ recall (3.12) with $j = k$:

$$D_z^k\Gamma u(\xi,z) = B^{-1}D_z^k u(\xi^*,z^*)[C]^k + P_k(\xi,z) \tag{3.19}$$

By Theorem 4 we conclude that

$$Lip^{(\xi)}(\Delta_z^{(k)}G) < O(\epsilon)$$

¿From the definition of $\mathfrak{X}_d$, it is obvious that $Lip^{(\xi)}(\tilde{\Delta}_z^{(k)}u)$ is uniformly bounded with a bound which is independent of $u \in \mathfrak{X}_d$. Thus, from the vanishing properties of $\mathcal{P}_j$, it follows that

(1) $$Lip^{(\xi)}(P_k) = O(\epsilon)$$

To consider the first term in (3.19) let $\bar{\xi} = M\xi' + f(\xi',z)$. Thus $|\xi^* - \bar{\xi}| \le b(1+t)|\xi - \xi'|$. It follows that

(2) $$\|B^{-1}[D_z^k u(\xi^*,z^*) - D_z^k u(\bar{\xi},z^*)][C]^k\| \le \chi^{-1}b\gamma^k(1+t)Lip^{(\xi)}(D_z^k u)|\xi - \xi'|$$

It follows from (1) and (2) that

$$Lip^{(\xi)}(D_z^k \Gamma u) \le \chi^{-1} b\gamma^k Lip^{(\xi)}(D_z^k u) + t = \theta_2 Lip^{(\xi)}(D_z^k u) + t \tag{b}$$

It remains to consider $Lip^{(\xi)}(\mathfrak{D}^k \Gamma u)$. Recall (3.14) with $i + l = j = k, i \ge 1$, and $l \ge 1$:

$$\begin{aligned} D_\xi^l D_z^i \Gamma u(\xi, z) &= B^{-1} D_\xi^l D_z^i u(\xi^*, z^*)[C]^i [M + D_\xi G(\xi, z)]^l \\ &\quad + \mathcal{W}_k(\xi, z) + R_{k-1}(\xi, z) \end{aligned} \tag{3.20}$$

It is obvious that $Lip^{(\xi)}(\mathcal{W}_k) = O(\epsilon), Lip^{(\xi)}(R_{k-1}) = O(\epsilon)$ and $Lip^{(\xi)}([M + D_\xi G]^n) = O(\epsilon)$ for $0 \le n \le k$. To consider the first term in (3.20) notice that

$$\chi^{-1} b^l \gamma^i (1 + t)\|D_\xi^l D_z^i u(\xi^*, z^*) - D_\xi^l D_z^i u(\bar{\xi}, z^*)\| \le \chi^{-1} b^{l+1} \gamma^i (1 + t) Lip^{(\xi)}(D_\xi^l D_z^i u)|\xi - \xi'|$$

Since $\chi^{-1} b^{l+1} \gamma^i < \chi^{-1} b\gamma^k$, we have

$$Lip^{(\xi)}(D_\xi^l D_z^i \Gamma u) \le \theta_3 Lip^{(\xi)}(D_\xi^l D_z^i u) + t$$

Taking the maximum over all $l + i = k, i \ge 1, l \ge 1$, we obtain

$$Lip^{(\xi)}(\mathfrak{D}^k \Gamma u) \le \theta_3 Lip^{(\xi)}(\mathfrak{D}^k u) + t \tag{c}$$

Estimates (a), (b) and (c) proves the proposition. ∎

3.16. Proposition. *The following hold for sufficiently small $\epsilon > 0$:*

$$\begin{gathered} H_\delta^{(z)}(\mathfrak{D}^k \Gamma u) \le \theta_o H_\delta^{(z)}(\mathfrak{D}^k u) + t < 1, \qquad u \in \mathfrak{X}_d \\ \theta_o = \chi^{-1} b\gamma^{r-1} < 1 \end{gathered}$$

Proof. Recall (3.14) with $j = k = l + i, l \ge 1, i \ge 1$:

$$\begin{aligned} D_\xi^l D_z^i \Gamma u(\xi, z) &= B^{-1} D_\xi^l D_z^i u(\xi^*, z^*)[C]^i [M + D_\xi G(\xi, z)]^l \\ &\quad + \mathcal{W}_k(\xi, z) + R_{k-1}(\xi, z) \end{aligned} \tag{3.21}$$

It is obvious that

$$\begin{aligned} H_\delta^{(z)}(\mathcal{W}_k) &= O(\epsilon) \\ H_\delta^{(z)}(R_{k-1}) &= O(\epsilon) \\ H_\delta^{(z)}([M + D_\xi G]^l) &= O(\epsilon) \end{aligned} \tag{i}$$

Let $\hat{\xi} = M\xi + f(\xi, z')$. Thus $|\hat{\xi} - \xi^*| \le \epsilon|z - z'|$. It follows that

$$\begin{aligned} \|D_\xi^l D_z^i u(\xi^*, Cz) - D_\xi^l D_z^i u(\hat{\xi}, Cz')\| &\le H_\delta^{(z)}(D_\xi^l D_z^i u)|Cz - Cz'|^\delta \\ &\quad + Lip^{(\xi)}(D_\xi^l D_z^i u)|\xi^* - \hat{\xi}| \\ &\le \gamma^\delta H_\delta^{(z)}(D_\xi^l D_z^i u)(1 + t)|z - z'| \end{aligned}$$

Thus

$$\begin{aligned} \chi^{-1} \gamma^i b^l (1 + t)\|D_\xi^l D_z^i u(\xi^*, Cz) - D_\xi^l D_z^i u(\hat{\xi}, Cz')\| &\le \chi^{-1} b^l \gamma^{i+\delta} (1 + t) H_\delta^{(z)}(D_\xi^l D_z^i u)|z - z'| \\ &\le \chi^{-1} b\gamma^{r-1} (1 + t) H_\delta^{(z)}(D_\xi^l D_z^i u)|z - z'| \end{aligned} \tag{ii}$$

Estimates (i) and (ii) imply that

$$H_\delta^{(z)}(D_\xi^l D_z^i \Gamma u) \le \theta_o H_\delta^{(z)}(D_\xi^l D_z^i u) + t$$

Taking the maximum over all $l + i = k, i \ge 1, l \ge 1$, we obtain the desired result. □

3.17. Corollary. *For sufficiently small $\epsilon > 0$, there is $\theta < 1$ such that*

$$\sigma(\Gamma u) \le \theta\sigma(u) + t, \qquad u \in \mathfrak{X}_d$$

Proof. The corollary follows from Propositions 3.13, 3.14, 3.15 and 3.16. □

To finish the proof of Theorem 5 it remains to prove assertions (2) and (3) which will be done in Proposition 3.18.

3.18. Proposition. *For sufficiently small $0 < d < 1, Q(x, y, z) = (x, u(x, y, z), z)$ is a $C^{k,\beta}$ diffeomorphism. The new map (2.27) satisfies estimates (2.32)-(2.35).*

Proof. Since $0 < d < 1$, $Q = id + (0, u, 0)$ has an inverse in a neighbourhood of $(0, 0, 0)$. By the inverse function Theorem, Q^{-1} is as smooth as Q and hence $v(X, Y, Z)$ is as smooth as $u(x, y, z)$.

Recall that $H = f \circ Q^{-1}$. The $O(\epsilon)$ factors in estimates (2.32)-(2.35) follow from the presence of f. The smoothness part in these estimates follow from the fact that Q^{-1} is as smooth as Q. □

4. Proof of Theorem 4

We have proved Theorem 5 granting Theorem 4. In this section we prove Theorem 4.

Proof of (2.15). Recall estimate (2.10) on S. Thus, for some $0 \le c \le 1$, since $|\xi| \le 1$ and $|z| \le 1$

$$|g(\xi, z)| \le \|D_\eta S(c\xi + w(z), z)\||\xi| \le O(\epsilon)|\xi|[|\xi|^{r-1} + |z|^{r-1}]$$

which proves (2.15). □

Proof of (2.16) . Let $\rho(\xi, z) = (\xi + w(z), z)$. It is obvious that $\rho(\xi, z)$ is $C^{k,\delta}$ and that $D^j\rho(\xi, z), j = 1, \cdots, k$, are independent of ξ. Thus, for $1 \le j \le k$

$$D_z^j G(\xi, z) = [D_z^j(F \circ \rho)](\xi, z) - [D_z^j(F \circ \rho)](0, z) \tag{4.1}$$

$$[D_z^j(F \circ \rho)](\xi, z) = D^j F(\xi + w(z), z)[D_z\rho(\xi, z)]^j + C_{j-1}(\xi, z) \tag{4.2}$$

$$C_{j-1}(\xi, z) = \mathcal{C}_{j-1}(\Delta^{(j-1)}F(\xi + w(z), z), \Delta_z^{(j)}\rho(\xi, z))$$

$$\mathcal{C}_{j-1}(0, \cdot) \equiv 0$$

$$\mathcal{C}_{j-1}(\cdot, 0) \equiv 0$$

where $\mathcal{C}_{j-1}(\Sigma, \Omega)$ is a polynomial in the components of Σ and Ω and linear in the components of Σ. Recall that $Lip_\eta(D_z^k G) < \epsilon$. Thus, letting $j = k$ in (4.1) we obtain

$$\|D_z^k G(\xi, z)\| \le \epsilon|\xi|$$

which proves (2.16). □

Proof of (2.17). Consider (4.1) and (4.2) for (g,S). Let $C_{j-1} = (C^f_{j-1}, C^g_{j-1})$. Let $1 \le j \le k-1$. Then $D^j_z(S\circ\rho)$ is $C^{1,\delta}$ and hence for some $0 \le c \le 1$

$$\|D^j_z g(\xi,z)\| \le \|[D_\xi D^j_z(S\circ\rho)](c\xi,z)\||\xi|$$

Notice that $\Delta^j\rho$ is independent of ξ. Thus by (2.6) and (2.10)

$$\|D_\xi C^{(g)}_{j-1}(c\xi,z)\| \le O(\epsilon)[|\xi|^{r-j} + |z|^{r-j}]$$
$$\|D_\xi D^j_z S(c\xi + w(z), z)[D_z\rho(\xi,z)]^j\| \le O(\epsilon)[|\xi|^{r-j-1} + |z|^{r-j-1}]$$

which proves (2.17). $\square$

Proof of (2.18) and (2.19). Let

$$D^j_* = D^j - D^j_z = \{D^l_\xi D^i_z \mid i+l=j, l\ge 1\}$$

Since $l \ge 1$ in each term of D^j_* and since $D^m\rho(\xi,z), m=1,\cdots,k$, are independent of ξ it follows from (4.2) that

$$\begin{aligned}\|D^l_\xi D^i_z g(\xi,z)\| &= \|[D^l_\xi D^i_z(S\circ\rho)](\xi,z)\|\\ &\le \|D^l_\xi D^i_z S(\xi+w(z),z)[D_z\rho(\xi,z)]^i\| + \|D^l_\xi C^{(g)}_{i-1}(\xi,z)\|\\ &\le O(\epsilon)[(|\xi|+|w(z)|)^{r-j} + |z|^{r-j}] + O(\epsilon)[|\xi|^{r-(l+i-1)} + |z|^{r-(l+i-1)}]\\ &\le O(\epsilon)[|\xi|^{r-j} + |z|^{r-j}]\end{aligned}$$

which proves (2.18) and (2.19). $\square$

Proof of (2.20). Notice that

$$D^k_\xi G(\xi,z) = [D^k_\xi(F\circ\rho)](\xi,z) = D^k_\eta F(\xi+w(z),z)I^k_1$$

where I_1 is the identity map $I_1(\xi)=\xi$. Now, (2.20) follows from (2.4). $\square$

Proof of (2.21). Estimate (2.21) follows from (2.6) and the facts that $G(\xi,z)$ is $C^{k,\delta}$ and that each term in $D^kG(\xi,z)$ has a factor of $\Delta^kF(\xi+w(z),z)$ or $\Delta^kF(w(z),z)$ which produces an $O(\epsilon)$ factor. $\square$

Proof of (2.22). In view of (2.20), $D^k_\xi G(\xi,z)$ satisfies (2.22). Thus to prove (2.22) we need to consider only $D^k_z G(\xi,z)$ and $\mathfrak{D}^k G(\xi,z)$. By (4.1), (4.2) and (2.4), it follows that $D^k_z G(\xi,z)$ satisfies (2.22).

To study $\mathfrak{D}^kG(\xi,z)$ let (i,l) be such that $i+l=k, i\ge 1$ and $l\ge 1$. Since $i \ge 1$ we have

$$\begin{aligned}D^l_z D^i_\xi G(\xi,z) &= D^l_z[D^i_\eta F(\xi+w(z),z)]I^i_1\\ &= D^l_z D^i_\eta F(\xi+w(z),z)I^i_1 I^l_2 + \Psi_k(\xi,z) + B_{l-1}(\xi,z)\\ \Psi_k(\xi,z) &= \sum_{s=1}^{l} c_s D^{l-s}_z D^{i+s}_\eta F(\xi+w(z),z)I^i_1[Dw(z)]^s I^{l-s}_2\\ B_{l-1}(\xi,z) &= \mathcal{B}_{l-1}([\Delta^{(l-1)}(D^i_\eta F)](\rho(\xi,z)), \Delta^{(l-1)}\rho(\xi,z))\\ \mathcal{B}_{l-1}(0,\cdot) &\equiv 0\end{aligned}$$

where $\mathcal{B}_{l-1}(\Sigma, \Omega)$ is a polynomial in the components of its arguments and linear in Σ. Thus, $B_{l-1}(\xi, z)$ is $C^{1,\delta}$. ¿From the vanishing property of $\mathcal{B}_{l-1}$ it follows that $Lip^{(\xi)}(B_{l-1}) = O(\epsilon)$. Estimate (2.4) implies that $Lip^{(\xi)}$ of the remaining terms is also $O(\epsilon)$. This proves (2.22). $\square$

Proof of (2.23). For $0 \leq j \leq k-1$, estimate (2.23) follows from (2.17) because for some $0 \leq c \leq 1$

$$\|D_z^j G(\xi, z) - D_z^j G(\xi, z')\| \leq \|D_z^{j+1} G(\xi, z + c(z - z'))\||z - z'|$$

When $j = k$ estimate (2.23) follows from (2.4). $\square$

Proof of (2.24). Let

$$\mathcal{J} = \|D_z^j G(\xi, z) - D_z^j G(\xi, z'\|$$

Notice that for $0 \leq j \leq k-1, D_z^j G(\xi, z)$ is $C^{1,\delta}$ and $Lip(D_z^j G) < O(\epsilon)$. Thus by (2.21) and (2.23) we have

$$\begin{aligned} \mathcal{J} &= \mathcal{J}^s \mathcal{J}^{1-s} \\ &\leq O(\epsilon)|\xi|^s |z - z'|^{\delta(1-s)} \end{aligned}$$

$\square$

This finishes the proof of Theorem 4.

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