On Absolute Continuity of the Periodic Schrödinger Operators

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Abstract. This paper concerns the Schrödinger operator $-\Delta + V(\mathbf{x})$ in $\mathbb{R}^d$, $d \geq 3$, with periodic potential V. Under the assumption $V \in L^{d/2}_{\text{loc}}(\mathbb{R}^d)$, it is shown that the spectrum of $-\Delta + V(\mathbf{x})$ is purely absolutely continuous. The condition on the potential V is optimal in the context of L^p spaces. The proof relies on certain uniform Sobolev inequalities on the d-torus. We also establish the absolute continuity of $-\Delta + V(\mathbf{x})$ with certain periodic potential V in the weak-$L^{d/2}$ space.

1991 Mathematics Subject Classification. Primary 35J10.

Key Words and Phrases. Schrödinger Operator, Periodic Potential, Absolute Continuous Spectrum, Uniform Sobolev Inequalities.

1. Introduction

Let V be a real periodic function on $\mathbb{R}^d$, i.e.,

$$V(\mathbf{x} + \mathbf{e}_j) = V(\mathbf{x}), \quad j = 1, 2, \ldots, d, \tag{1.1}$$

for some basis $\{\mathbf{e}_j\}_{j=1}^d$ of $\mathbb{R}^d$. We shall be interested in the spectral properties of the periodic Schrödinger operator $-\Delta + V(\mathbf{x})$ in $\mathbb{R}^d$.

When $d = 3$, L. Thomas [19] proved that the spectrum of $-\Delta + V$ is purely absolutely continuous (a.c.) if $V \in L^2_{\text{loc}}(\mathbb{R}^3)$. Thomas' approach plays an important role in the subsequent development. In the book [14, vol.IV] by M. Reed and B. Simon, it was used to show that $-\Delta + V$ is a.c. if $V \in L^r_{\text{loc}}(\mathbb{R}^d)$, where $r \geq d-1$

[1]Research supported in part by the Erwin Schrödinger Institute in Vienna, Austria, the AMS Centennial Research Fellowship, and the NSF grant DMS-9732894.

if $d \geq 4$, and $r = 2$ if $d = 2$ or 3. In [4] the basic idea of Thomas was applied to the Dirac operator with periodic potential. Recently, the absolute continuity of the magnetic Schrödinger operator $(-i\nabla - \mathbf{a}(\mathbf{x}))^2 + V(\mathbf{x})$, with periodic potentials $\mathbf{a}$ and V, was investigated by R. Hempel and I. Herbst [6,7], M. Birman and T. Suslina [1,2], A. Morame [13], and A. Sobolev [15]. In particular, the results in [2], pertaining to the case $\mathbf{a} = 0$, give the absolute continuity of $-\Delta + V$ when $d = 2$ and $V \in L^r_{\text{loc}}(\mathbb{R}^2)$ for some $r > 1$. Very recently, M. Birman and T. Suslina [3] established the absolute continuity of $-\Delta + V$ for $d = 3$, $V \in L^{3/2}_{\text{loc}}(\mathbb{R}^3)$, and for $d \geq 4$, $V \in L^{d-2}_{\text{loc}}(\mathbb{R}^d)$.

In this paper we prove that the spectrum of $-\Delta + V$ is purely absolutely continuous if $d \geq 3$ and $V \in L^{d/2}_{\text{loc}}(\mathbb{R}^d)$. This improves the results in [3] when $d \geq 5$. In the context of L^p spaces, it is best possible in the sense that, with periodic condition (1.1), $L^{d/2}_{\text{loc}}$ is the largest space for which the self-adjoint operator $-\Delta + V$ may be defined by a quadratic form.

Let $\mathbf{D} = -i\nabla$. For a $d \times d$ matrix $A = (a_{jk})$, we use $\mathbf{D}A\mathbf{D}^T$ to denote $\sum_{j,k} D_j a_{jk} D_k$. The following is the main result of the paper.

Theorem 0.1. *Let $A = (a_{jk})_{d\times d}$ be a symmetric, positive-definite matrix with real constant entries. Let V be a real periodic function on $\mathbb{R}^d$. Suppose $d \geq 3$ and $V \in L^{d/2}_{\text{loc}}(\mathbb{R}^d)$. Then the spectrum of $\mathbf{D}A\mathbf{D}^T + V$ is purely absolutely continuous.*

We now describe the main idea of the proof. First, note that, by a change of variables, one may assume that $\mathbf{e}_j = 2\pi(0, \ldots, 1, \ldots, 0)$ in (1.1), i.e., V is periodic with respect to the lattice $(2\pi\mathbb{Z})^d$. Let $\Omega = [0, 2\pi)^d$ be a cell of the lattice. Fix $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$. For $t \in \mathbb{R}$, we consider the self-adjoint operator

$$\mathbb{H}_V(t) = (\mathbf{D} + t\mathbf{a} + \mathbf{b})A(\mathbf{D} + t\mathbf{a} + \mathbf{b})^T + V \tag{1.2}$$

on $L^2(\Omega)$, with periodic boundary conditions. It is known that $\mathbb{H}_V(t)$ has a discrete spectrum.

Next, using the Floquet-Bloch decomposition, we reduce the proof of Theorem 0.1 to the problem of showing that none of the eigenvalues $E_j(t)$ of $\mathbb{H}_V(t)$ is constant as a function of t. See section 2 for details.

Now, following the Thomas approach, one argues by contradiction. Suppose that, for some j, $E_j(t) \equiv E$ is a constant. By the analytic perturbation theory, E is an eigenvalue of $\mathbb{H}_V(z)$ for all $z \in \mathbb{C}$. This will be impossible if one can prove

$$\|\mathbb{H}_V(z)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)} \to 0 \quad \text{as} \quad |\operatorname{Im} z| \to \infty \tag{1.3}$$

for some $p \leq 2$. We remark that in all previous work, only the case $p = 2$ was considered. Our main estimate in this paper states that, if $V \in L^{d/2}(\Omega)$ and $z = \delta + i\rho$ (δ depends on $\mathbf{a}$ and $\mathbf{b}$), then (1.3) holds for $p = 2d/(d+2)$.

To establish estimate (1.3), we write

$$\mathbb{H}_V(z)^{-1} = \mathbb{H}_0(z)^{-1}\left\{I + V\mathbb{H}_0(z)^{-1}\right\}^{-1}. \tag{1.4}$$

We will show that

$$\|\mathbb{H}_0(\delta + i\rho)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)} \to 0 \quad \text{as} \quad |\rho| \to \infty, \tag{1.5}$$

(see Theorem 2.1) and

$$\|\mathbb{H}_0(\delta + i\rho)^{-1}\|_{L^p(\Omega)\to L^q(\Omega)} \leq C \tag{1.6}$$

with a constant C independent of ρ when $|\rho| \geq 1$, where $p = 2d/(d+2)$, $q = p' = 2d/(d-2)$ (see Theorem 2.2). Estimate (1.3) follows, since $V \in L^{d/2}(\Omega)$, (1.5)-(1.6) and Hölder inequality yield

$$\|V\mathbb{H}_0(\delta + i\rho)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)} \leq 1/2, \qquad \text{when} \quad |\rho| \quad \text{is large.} \tag{1.7}$$

While the proof of (1.5) is relatively straight forward, the proof of (1.6) is much involved. Note that $(1/q) = (1/p) - (2/d)$, and (1.6) is equivalent to

$$\|\psi\|_{L^q(\Omega)} \leq C\|\mathbb{H}_0(\delta + i\rho)\psi\|_{L^p(\Omega)}, \tag{1.8}$$

for $\psi \in C^\infty(\Omega)$. (1.8) is in fact a uniform Sobolev inequality on the d-torus for the second order elliptic operator $\mathbb{H}_0(\delta + i\rho)$. In the setting of $\mathbb{R}^d$, similar inequalities were established by C. Kenig, A. Ruiz and C. Sogge [10]. Such estimates play a key role in the study of unique continuation properties of differential operators (see e.g. [9]). Our estimate (1.8) may be viewed as an analogue on the d-torus. It is well known that the unique continuation properties can be used to eliminate the possibility of certain eigenfunctions. In the study of the absolute continuity of periodic Schrödinger operators, it turns out that the main issue is also to show the absence of eigenvalues in the spectrum [7], although this is not obvious in the Thomas method. In this regard, the use of uniform Sobolev inequalities seems to be quite natural in our case.

To prove (1.8), we shall adapt the approach developed in [10] for $\mathbb{R}^d$. We remark that certain forms of uniform Sobolev inequalities for second-order elliptic operators on compact manifolds were obtained by C. Sogge [16]. However it is not clear how to apply directly the results in [16] to our case, although a general result on the spectral projection in [16] will be used in the localization process.

With some modifications, the method outlined above also gives the optimal condition for the absolute continuity of $-\Delta+V$ with periodic potentials in Lorentz spaces. Indeed, assuming $V \in L^{d/2,\infty}(\Omega)$ where Ω be a periodic cell for V, we have the following result.

Theorem 0.2. *Let $A=(a_{jk})_{d\times d}$ be a symmetric, positive-definite matrix with real constant entries. Let V be a real periodic function on $\mathbb{R}^d$, $d\geq 3$. Then there exists $\varepsilon_0>0$ depending on d, $|\Omega|$ and A, such that, if*

$$\limsup_{t\to\infty}\; t\Big|\,\{x\in\Omega : |V(x)|>t\}\,\Big|^{2/d} < \varepsilon_0, \tag{1.9}$$

then the spectrum of $\mathbf{D}A\mathbf{D}^T+V$ is purely absolutely continuous.

To prove Theorem 0.2, we will show that

$$\|\mathbb{H}_V(z)^{-1}\|_{L^{p,2}(\Omega)\to L^{p,2}(\Omega)} \to 0 \quad \text{as} \quad |\rho|\to\infty, \tag{1.10}$$

in the place of (1.3), where $p=2d/(d+2)$ and $z=\delta+i\rho$ as before (see section 6). We remark that in the case $d=3$ or 4, the absolute continuity of $-\Delta+V$ with condition (1.9) was obtained in [3].

The paper is organized as follows. In section 2, we provide the details of the Thomas approach which reduces Theorem 0.1 to estimates (1.5) and (1.6). Section

3 contains the proof of estimate (1.5). Sections 4 and 5 are devoted to the proof of the key estimate (1.6). In section 6 we give the proof of Theorem 0.2.

Throughout the rest of the paper, we will assume that $d \geq 3$, $\Omega = [0, 2\pi)^d$, $V \in L^{d/2}(\Omega)$ (except in section 6 where we assume $V \in L^{d/2,\infty}(\Omega)$) and is periodic with respect to the lattice $(2\pi\mathbb{Z})^d$. We shall use $\|\cdot\|_p$ to denote the norm in $L^p(\Omega)$. Finally we will use C and c to denote positive constants which may depend on d and the matrix A, which are not necessarily the same at each occurrence.

2. The Thomas Approach

The materials in the subsections 2.1-2.3 below are standard. We include them here for the reader's convenience.

2.1. The definition of $\mathbf{D}A\mathbf{D}^T + V$ on $\mathbb{R}^d$.

Let $\Omega = [0, 2\pi)^d$. Given $\varepsilon > 0$, since $V \in L^{d/2}(\Omega)$, we may write $V = V_1 + V_2$ such that

$$\|V_1\|_{d/2} \leq \varepsilon \quad \text{and} \quad \|V_2\|_\infty \leq C_\varepsilon. \tag{2.1}$$

Let $f \in H^1(\mathbb{R}^d)$, i.e., $f \in L^2(\mathbb{R}^d)$ and $|\nabla f| \in L^2(\mathbb{R}^d)$. By the Sobolev imbedding, we have

$$\begin{aligned}
\int_\Omega |V|\,|f|^2 d\mathbf{x} &\leq \int_\Omega |V_1|\,|f|^2 d\mathbf{x} + \int_\Omega |V_2|\,|f|^2 d\mathbf{x} \\
&\leq \left\{\int_\Omega |V_1|^{\frac{d}{2}} d\mathbf{x}\right\}^{\frac{d}{2}} \left\{\int_\Omega |f|^{\frac{2d}{d-2}} d\mathbf{x}\right\}^{\frac{d-2}{d}} + C_\varepsilon \int_\Omega |f|^2 d\mathbf{x} \\
&\leq \varepsilon C \left\{\int_\Omega |\nabla f|^2 d\mathbf{x} + \int_\Omega |f|^2 d\mathbf{x}\right\} + C_\varepsilon \int_\Omega |f|^2 d\mathbf{x} \\
&\leq \varepsilon C \int_\Omega |\nabla f|^2 d\mathbf{x} + (\varepsilon C + C_\varepsilon) \int_\Omega |f|^2 d\mathbf{x}.
\end{aligned} \tag{2.2}$$

Since V is periodic with respect to $(2\pi\mathbb{Z})^d$, it follows from (2.2), by summation, that

$$\int_{\mathbb{R}^d} |V|\,|f|^2 d\mathbf{x} \leq \varepsilon C \int_{\mathbb{R}^d} |\nabla f|^2 d\mathbf{x} + \widetilde{C}_\varepsilon \int_{\mathbb{R}^d} |f|^2 d\mathbf{x} \tag{2.3}$$

for any $\varepsilon > 0$ and $f \in H^1(\mathbb{R}^d)$.

Let $<\ ,\ >$ denote the usual inner product in $\mathbb{C}$ or in $\mathbb{C}^d$. We define the quadratic form

$$q[f,g] = \int_{\mathbb{R}^d} \{<(\mathbf{D}f)A, \mathbf{D}g> + <Vf,g>\}\, d\mathbf{x} \tag{2.4}$$

for $f,\ g \in H^1(\mathbb{R}^d)$. Clearly, by (2.3), the symmetric quadratic form q is semi-bounded and closed. Thus there exists a unique self-adjoint operator, which we denoted by $\mathbf{D}A\mathbf{D}^T + V$, such that

$$q[f,g] = \int_{\mathbb{R}^d} <(\mathbf{D}A\mathbf{D}^T + V)f, g>\, d\mathbf{x} \tag{2.5}$$

for $f \in \mathrm{Domain}(\mathbf{D}A\mathbf{D}^T + V)$ and $g \in H^1(\mathbb{R}^d)$. Also,

$$\mathrm{Domain}(\mathbf{D}A\mathbf{D}^T + V) = \left\{ f \in H^1(\mathbb{R}^d) :\ (\mathbf{D}A\mathbf{D}^T + V)f \in L^2(\mathbb{R}^d) \right\}. \tag{2.6}$$

2.2. Definition of $(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T + V$ on Ω.

Let

$$H^1_{\mathrm{per}}(\Omega) = \left\{ \psi \in L^2(\Omega) :\ \psi(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} a_{\mathbf{n}} e^{i\mathbf{n}\mathbf{x}} \ \text{ and } \sum_{\mathbf{n}\in\mathbb{Z}^d} |\mathbf{n}|^2 |a_{\mathbf{n}}|^2 < \infty \right\} \tag{2.7}$$

be the Sobolev space on Ω of index one with periodic boundary conditions. For $\mathbf{k} \in \mathbb{C}^d$, we introduce the quadratic form

$$q(\mathbf{k})[\phi,\psi] = \int_\Omega \left\{ <[(\mathbf{D}+\mathbf{k})\phi]A, (\mathbf{D}+\bar{\mathbf{k}})\psi> + <V\phi,\psi> \right\} d\mathbf{x} \tag{2.8}$$

for $\phi, \psi \in H^1_{\mathrm{per}}(\Omega)$, where $\bar{\mathbf{k}}$ denotes the conjugate of $\mathbf{k}$. Using (2.2), we see that the form $q(\mathbf{k})$ is strictly m-sectorial. It follows that there exists a unique closed operator, which we denote by $(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T + V$, such that

$$q(\mathbf{k})[\phi,\psi] = \int_\Omega < \left\{ (\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T + V \right\} \phi, \psi>\, d\mathbf{x} \tag{2.9}$$

for any $\phi \in \mathrm{Domain}\left((\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T + V\right)$ and $\psi \in H^1_{\mathrm{per}}(\Omega)$. Furthermore,

$$\begin{aligned} &\mathrm{Domain}\left((\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T + V\right) \\ &\quad = \left\{ \phi \in H^1_{\mathrm{per}}(\Omega) :\ \left\{ (\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T + V \right\} \phi \in L^2(\Omega) \right\} \\ &\quad = \left\{ \phi \in H^1_{\mathrm{per}}(\Omega) :\ (\mathbf{D}A\mathbf{D}^T + V)\phi \in L^2(\Omega) \right\}, \end{aligned} \tag{2.10}$$

and

$$\{(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T+V\}^* = (\mathbf{D}+\bar{\mathbf{k}})A(\mathbf{D}+\bar{\mathbf{k}})^T+V. \tag{2.11}$$

2.3. The Floquet-Bloch decomposition.

Let $Q=[0,1)^d$ and

$$\mathcal{H} = \int_Q^{\oplus} L^2(\Omega)d\mathbf{k} \equiv L^2(Q, L^2(\Omega)). \tag{2.12}$$

For $f \in C_0^\infty(\mathbb{R}^d)$, we define

$$Uf(\mathbf{k},\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{-i\mathbf{k}(\mathbf{x}+2\pi\mathbf{n})} f(\mathbf{x}+2\pi\mathbf{n}), \quad \mathbf{x}\in\Omega,\ \mathbf{k}\in Q. \tag{2.13}$$

It is easy to check that

$$\|Uf\|_{\mathcal{H}} = \|f\|_{L^2(\mathbb{R}^d)}. \tag{2.14}$$

Furthermore, one may extend U to a unitary operator from $L^2(\mathbb{R}^d)$ to $\mathcal{H}$, by continuity.

A direct computation shows, for $f,\ g\in C_0^\infty(\mathbb{R}^d)$,

$$q[f,g] = \int_\Omega q(\mathbf{k})[Uf(\mathbf{k},\cdot),Ug(\mathbf{k},\cdot)]d\mathbf{k}. \tag{2.15}$$

In particular,

$$\int_{\mathbb{R}^d} |\nabla f|^2 d\mathbf{x} = \int_Q\int_\Omega |(\mathbf{D}+\mathbf{k})Uf|^2 d\mathbf{x}d\mathbf{k}. \tag{2.16}$$

Proposition 2.1.

$$U:\ H^1(\mathbb{R}^d) \to \int_Q^{\oplus} H^1_{\mathrm{per}}(\Omega)d\mathbf{k} \equiv L^2(Q, H^1_{\mathrm{per}}(\Omega))$$

is unitary.

Proof. It follows easily from (2.14) and (2.16) by a standard approximation argument that

$$U(H^1(\mathbb{R}^d)) \subset \int_Q^{\oplus} H^1_{\mathrm{per}}(\Omega)d\mathbf{k}.$$

To show U is onto, suppose that there exists $\eta \in \int_Q^{\oplus} H^1_{\rm per}(\Omega) d\mathbf{k}$ such that

$$\int_Q \int_\Omega \{< \mathbf{D}Uf, \mathbf{D}\eta > + < Uf, \eta >\} \, d\mathbf{x} d\mathbf{k} = 0$$

for any $f \in C_0^\infty(\mathbb{R}^d)$. Let $g \in C_0^\infty(\Omega)$ and $f(\mathbf{x}) = g(\mathbf{x} - 2\pi \mathbf{n}_0)$ for some $\mathbf{n}_0 \in \mathbb{Z}^d$. Then

$$Uf(\mathbf{k}, \mathbf{x}) = e^{-i\mathbf{k}(\mathbf{x}+2\pi\mathbf{n}_0)} g(\mathbf{x})$$

and

$$\mathbf{D}Uf(\mathbf{k}, \mathbf{x}) = e^{-i\mathbf{k}(\mathbf{x}+2\pi\mathbf{n}_0)} (\mathbf{D} - \mathbf{k}) g(\mathbf{x}).$$

Hence,

$$\int_Q e^{-2\pi i \mathbf{n}_0 \mathbf{k}} \int_\Omega e^{-i\mathbf{k}\mathbf{x}} \{< (\mathbf{D} - \mathbf{k})g, \mathbf{D}\eta > + < g, \eta >\} \, d\mathbf{x} d\mathbf{k} = 0.$$

Since $\mathbf{n}_0 \in \mathbb{Z}^d$ is arbitrary, we have

$$\int_\Omega e^{-i\mathbf{k}\mathbf{x}} \{< (\mathbf{D} - \mathbf{k})g, \mathbf{D}\eta > + < g, \eta >\} \, d\mathbf{x} = 0, \quad \text{a.e.} \quad \mathbf{k} \in Q.$$

Finally, since $g \in C_0^\infty(\Omega)$ is arbitrary, we conclude that

$$e^{i\mathbf{k}\mathbf{x}} (-\Delta + 1)\eta = 0 \quad \text{in } \Omega$$

(in the sense of distributions). Thus $\eta = 0$. The proof is complete.

Remark 2.1. It follows from Proposition 2.1 that (2.15) holds for any f, $g \in H^1(\mathbb{R}^d)$.

Proposition 2.2. *We have the following Floquet-Bloch decomposition of* $\mathbf{D}A\mathbf{D}^T + V$ *in a direct integral:*

$$U(\mathbf{D}A\mathbf{D}^T + V)U^{-1} = \int_Q^{\oplus} \left\{ (\mathbf{D} + \mathbf{k}) A (\mathbf{D} + \mathbf{k})^T + V \right\} d\mathbf{k}.$$

That is,

$$\mathit{Domain}\left(U(\mathbf{D}A\mathbf{D}^T + V)U^{-1}\right) = \mathit{Domain}\left(\int_Q^{\oplus} \left\{ (\mathbf{D} + \mathbf{k}) A (\mathbf{D} + \mathbf{k})^T + V \right\} d\mathbf{k} \right)$$

$$\equiv \Big\{ \phi \in \mathcal{H} : \phi(\mathbf{k}, \cdot) \in \mathit{Domain}\big((\mathbf{D} + \mathbf{k}) A (\mathbf{D} + \mathbf{k})^T + V\big) \ \textit{for a.e. } k \in Q \textit{ and} \\ \int_Q \| \left\{ (\mathbf{D} + \mathbf{k}) A (\mathbf{D} + \mathbf{k})^T + V \right\} \phi \|_2^2 d\mathbf{k} < \infty \Big\},$$

and, for any $\phi \in$ *Domain* $\big(U(\mathbf{D}A\mathbf{D}^T + V)U^{-1}\big)$,

$$U(\mathbf{D}A\mathbf{D}^T + V)U^{-1}\phi(\mathbf{k},\mathbf{x}) = \{(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T + V\}\phi(\mathbf{k},\mathbf{x}).$$

Proof. First, note that, by (2.6),

$$\begin{aligned}
&\text{Domain}\,\big(U(\mathbf{D}A\mathbf{D}^T + V)U^{-1}\big)\\
&\quad = \big\{\phi \in \mathcal{H} :\ U^{-1}\phi \in \text{Domain}(\mathbf{D}A\mathbf{D}^T + V)\big\}\\
&\quad = \big\{\phi \in \mathcal{H} :\ U^{-1}\phi \in H^1(\mathbb{R}^d) \text{ and } (\mathbf{D}A\mathbf{D}^T + V)U^{-1}\phi \in L^2(\mathbb{R}^d)\big\}\\
&\quad = \left\{\phi \in \int_Q^{\oplus} H^1_{\text{per}}(\Omega)d\mathbf{k} : (\mathbf{D}A\mathbf{D}^T + V)U^{-1}\phi \in L^2(\mathbb{R}^d)\right\}
\end{aligned}$$

where we have used Proposition 2.1 in the last equality.

Now, let $\phi \in \text{Domain}(U(\mathbf{D}A\mathbf{D}^T + V)U^{-1})$. By Remark 2.1, for any $\psi \in \int_Q^{\oplus} H^1_{\text{per}}(\Omega)d\mathbf{k}$,

$$\begin{aligned}
&\int_Q\int_\Omega \big[< [(\mathbf{D}+\mathbf{k})\phi]A, (\mathbf{D}+\mathbf{k})\psi > + < V\phi, \psi > \big] d\mathbf{x}d\mathbf{k}\\
&\qquad\qquad = \int_Q q(\mathbf{k})[\phi(\cdot,\mathbf{k}), \psi(\cdot,\mathbf{k})]d\mathbf{k}\\
&\qquad\qquad = q[U^{-1}\phi, U^{-1}\psi]\\
&\qquad\qquad = \int_{\mathbb{R}^d} < (\mathbf{D}A\mathbf{D}^T + V)U^{-1}\phi, U^{-1}\psi > d\mathbf{x}\\
&\qquad\qquad = \int_Q\int_\Omega < U(\mathbf{D}A\mathbf{D}^T + V)U^{-1}\phi, \psi > d\mathbf{x}d\mathbf{k}.
\end{aligned}$$

Since $\int_Q^{\oplus} H^1_{\text{per}}(\Omega)d\mathbf{k}$ is dense in $\mathcal{H}$, we obtain

$$\big\{(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T + V\big\}\,\phi = U(\mathbf{D}A\mathbf{D}^T + V)U^{-1}\phi \in \mathcal{H}.$$

This shows that the domain of $U(\mathbf{D}A\mathbf{D}^T + V)U^{-1}$ is contained in the domain of $\int_Q^{\oplus}[(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T + V]d\mathbf{k}$. The other side of the inclusion can be shown in the same manner.

Fix $\mathbf{a} \in \mathbb{R}^d$ such that

$$|\mathbf{a}| = 1,\quad \mathbf{a}A = (s_0, 0, \ldots, 0) \text{ and } s_0 > 0. \tag{2.17}$$

Then any $\mathbf{x} \in \mathbb{R}^d$ can be written as $\mathbf{x} = t\mathbf{a} + \mathbf{b}$ with $< \mathbf{a}, \mathbf{b} >= 0$. Note that, if $\mathbf{x} \in Q = [0,1)^d$, $|\mathbf{x}|^2 = t^2 + |\mathbf{b}|^2 \le d$. Hence, $|t| \le \sqrt{d}$ and $|\mathbf{b}| \le \sqrt{d}$.

Next, let

$$L = \left\{ \mathbf{b} \in \mathbb{R}^d : |\mathbf{b}| \le \sqrt{d} \ \text{ and } \ < \mathbf{b}, \mathbf{a} >= 0 \right\}. \tag{2.18}$$

For each $\mathbf{b} \in L$, we define

$$J(\mathbf{b}) = \{ t \in \mathbb{R} : \ t\mathbf{a} + \mathbf{b} \in Q \}. \tag{2.19}$$

Since Q is convex, $J(\mathbf{b})$ must be an interval of $\mathbb{R}$, if not empty.

We may now write

$$\begin{aligned} &U(\mathbf{D}A\mathbf{D}^T + V)U^{-1} \\ &\qquad = \int_Q \{ (\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T + V \} \, d\mathbf{k} \\ &\qquad = \int_{\mathbf{b}\in L} \left\{ \int_{t \in J(\mathbf{b})} \left[(\mathbf{D} + t\mathbf{a} + \mathbf{b})A(\mathbf{D} + t\mathbf{a} + \mathbf{b})^T + V \right] dt \right\} d\mathbf{b}. \end{aligned}$$

By [14, vol. IV, p.284, Theorem XIII.85], we have

Proposition 2.3. *If, for each* $\mathbf{b} \in L$*,*

$$\int_{t \in J(\mathbf{b})} \left[(\mathbf{D} + t\mathbf{a} + \mathbf{b})A(\mathbf{D} + t\mathbf{a} + \mathbf{b})^T + V \right] dt$$

has absolutely continuous spectrum, so does $\mathbf{D}A\mathbf{D}^T + V$*.*

2.4. The Thomas Approach.

Fix $\mathbf{b} \in L$. We consider the family of operators

$$\mathbb{H}_V(z) = (\mathbf{D} + z\mathbf{a} + \mathbf{b})A(\mathbf{D} + z\mathbf{a} + \mathbf{b})^T + V, \quad z \in \mathbb{C}, \tag{2.20}$$

defined by the quadratic form (2.9). Note that, by (2.10), the domain of $\mathbb{H}_V(z)$ is independent of z. Thus it is easy to check that $\{\mathbb{H}_V(z)\}$ is an analytic family of type (A) (see [8, p.375] for definition). Moreover, by (2.11), the family is self-adjoint in the sense that $\mathbb{H}_V(z)^* = \mathbb{H}_V(\bar{z})$.

The resolvent of $\mathbb{H}_V(z)$ is not empty for any $z \in \mathbb{C}$. This follows easily from (2.2). Also, since Domain$(\mathbb{H}_V(z)) \subset H^1_{\text{per}}(\Omega)$, $\mathbb{H}_V(z)$ has compact resolvent. Now, consider the family of self-adjoint operators $\{\mathbb{H}_V(t),\, t \in \mathbb{R}\}$. By the analytic perturbation theory [8, p.392, Theorem 3.9], there exist real analytic functions $E_j(t)$ and corresponding $L^2(\Omega)$ -valued functions $\eta_j(t)$ so that, for each $t \in \mathbb{R}$, $\{\eta_j(t)\}_{j=1}^{\infty}$ is an orthonormal basis for $L^2(\Omega)$ and

$$\mathbb{H}_V(t)\eta_j(t) = E_j(t)\eta_j(t). \tag{2.21}$$

Proposition 2.4. *If, as a function of $t \in \mathbb{R}$, no $E_j(t)$ is constant, then $\mathbf{DAD}^T + V$ has absolutely continuous spectrum.*

Proposition 2.4 follows directly from Proposition 2.3 and [14, vol.IV, p.286, Theorem XIII.86].

Let $\mathbf{a} = (a_1, \ldots, a_d)$. For $\mathbf{b} = (b_1, \ldots, b_d)$, choose

$$\delta = \frac{1}{a_1}(\frac{1}{2} - b_1). \tag{2.22}$$

Note that, since $a_1 s_0 =< \mathbf{a}A, \mathbf{a} >$ and $|\mathbf{a}| = 1, a_1 \neq 0$.

We now state the main estimates of the paper

Theorem 2.1. *Let δ be given by (2.22). Then, for $\rho \in \mathbb{R}$, $|\rho| \geq 1$,*

(i) $||\mathbb{H}_0(\delta + i\rho)^{-1}||_{L^p\Omega) \to L^2(\Omega)} \to 0$ as $|\rho| \to \infty$ where $1 \leq p \leq 2$ if $d = 3$, $2(d-2)/d < p \leq 2$ if $d \geq 4$,

(ii) there exists $C > 0$ depending only on the matrix A and d, such that, for $|\rho| \geq 1$,

$$||\mathbb{H}_0(\delta + i\rho)^{-1}||_{L^p(\Omega) \to L^2(\Omega)} \leq C$$

where $1 \leq p \leq 2$ if $d = 3$, $1 < p \leq 2$ if $d = 4$, and $2(d-2)/d \leq p \leq 2$ if $d \geq 5$.

Theorem 2.2. *Let $p = 2d(d+2)$ and $q = 2d/(d-2)$. Then there exists $C > 0$ depending on A and d, such that, for $|\rho| \geq 1$,*

$$||\mathbb{H}_0(\delta + i\rho)^{-1}||_{L^p(\Omega) \to L^q(\Omega)} \leq C.$$

Finally in this section, assuming Theorems 2.1 and 2.2, we give the

Proof of the Main Theorem. In view of Proposition 2.4, it suffices to show that $E_j(t)$ is not a constant function. We shall argue by contradiction.

Suppose that $E_j(t) \equiv E$ is a constant for some j. By the analytic perturbation theory [8, p.371, Theorem 1.10], E is an eigenvalue of $\mathbb{H}_V(z)$ for any $z \in \mathbb{C}$. In particular, for $\rho \in \mathbb{R}$, there exists $\phi_\rho \in \mathrm{Domain}(\mathbb{H}_V(\delta + i\rho))$ such that $\|\phi_\rho\|_2 = 1$ and

$$\mathbb{H}_V(\delta + i\rho)\phi_\rho = E\phi_\rho. \tag{2.23}$$

Since $\phi_\rho \in H^1_{\mathrm{per}}(\Omega)$, by the Sobolev imbedding, $\phi_\rho \in L^q(\Omega)$ where q=2d/(d-2). Let p=2d/(d+2). Since (1/p)=(2/d)+(1/q), by Hölder's inequality, $V\phi_\rho \in L^p(\Omega)$. This implies that

$$\mathbb{H}_0(\delta + i\rho)\phi_\rho = \mathbb{H}_V(\delta + i\rho)\phi_\rho - V\phi_\rho \in L^p(\Omega). \tag{2.24}$$

We remark that $\mathbb{H}_0(\delta + i\rho)\phi_\rho$ may be defined initially as an element in the dual space of $H^1_{\mathrm{per}}(\Omega)$.

We claim that, for $|\rho|$ sufficiently large,

$$\|V\mathbb{H}_0(\delta + i\rho)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)} \le 1/2. \tag{2.25}$$

Assume (2.25) for a moment. It is well known that (2.25) implies that $I + V\mathbb{H}_0(\delta + i\rho)^{-1} : \ L^p(\Omega) \to L^p(\Omega)$ is invertible and

$$\|\{I + V\mathbb{H}_0(\delta + i\rho)^{-1}\}^{-1}\|_{L^p(\Omega)\to L^p(\Omega)} \le 2 \ \text{ for } \ |\rho| \ \text{ large }. \tag{2.26}$$

Since

$$\left(I + V\mathbb{H}_0(\delta + i\rho)^{-1}\right)\mathbb{H}_0(\delta + i\rho)\phi_\rho = (\mathbb{H}_0(\delta + i\rho) + V)\phi_\rho = E\phi_\rho$$

in view of (2.24), we have

$$\mathbb{H}_0(\delta + i\rho)\phi_\rho = E\left\{I + VH_0(\delta + i\rho)^{-1}\right\}^{-1}\phi_\rho.$$

It follows that, for $|\rho|$ sufficiently large,

$$\phi_\rho = E\,\mathbb{H}_0(\delta+i\rho)^{-1}\left\{I + VH_0(\delta+i\rho)^{-1}\right\}^{-1}\phi_\rho. \tag{2.27}$$

Thus, by (2.26),

$$\|\phi_\rho\|_p \le 2|E|\,\|\mathbb{H}_0(\delta+i\rho)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)}\|\phi_\rho\|_p$$

or

$$1 \le 2|E|\,\|\mathbb{H}_0(\delta+i\rho)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)}.$$

This is impossible since

$$\|\mathbb{H}_0(\delta+i\rho)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)} \le 2\pi\|\mathbb{H}_0(\delta+i\rho)^{-1}\|_{L^p(\Omega)\to L^2(\Omega)} \to 0$$

as $|\rho|\to\infty$, by Theorem 2.1.

It remains to prove the claim (2.25).

For any $\varepsilon>0$, we write $V = V_1 + V_2$ so that

$$\|V_1\|_{d/2} \le \varepsilon \quad \text{and} \quad \|V_2\|_\infty \le C_\varepsilon.$$

Then

$$\begin{aligned}
&\|V\mathbb{H}_0(\delta+i\rho)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)}\\
&\quad\le \|V_1\mathbb{H}_0(\delta+i\rho)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)} + \|V_2\mathbb{H}_0(\delta+i\rho)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)}\\
&\quad\le \varepsilon\|\mathbb{H}_0(\delta+i\rho)^{-1}\|_{L^p(\Omega)\to L^q(\Omega)} + C_\varepsilon\|\mathbb{H}_0(\delta+i\rho)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)}\\
&\quad\le \varepsilon C + C_\varepsilon\|\mathbb{H}(\delta+i\rho)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)}.
\end{aligned}$$

Choose ε small so that $\varepsilon C \le 1/4$. Then let $|\rho|$ be so large that

$$C_\varepsilon\|\mathbb{H}(\delta+i\rho)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)} \le 1/4.$$

The claim is proved.

The next three sections are devoted to the proof of Theorems 2.1 and 2.2. We point out that Theorem 2.1 alone would give the absolute continuity of $-\Delta+V$ when $d=4$, $V\in L^r(\Omega)$ for some $r>2$, and $d\ge 5$, $V\in L^{d-2}(\Omega)$. In the case $d\ge 5$, this is the result obtained recently by Birman and Suslina [3].

3. The Proof of Theorem 2.1

Let $\psi \in L^2(\Omega)$, then

$$\psi(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} \hat{\psi}(\mathbf{n}) e^{i\mathbf{n}\mathbf{x}} \tag{3.1}$$

where

$$\hat{\psi}(\mathbf{n}) = \frac{1}{(2\pi)^d} \int_\Omega e^{-i\mathbf{y}\mathbf{n}} \psi(\mathbf{y}) d\mathbf{y}. \tag{3.2}$$

Note that

$$\mathbb{H}_0(\mathbf{k})\psi(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} (\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T e^{i\mathbf{n}\mathbf{x}}$$

where $(\mathbf{n}+\mathbf{k})^T$ denote the transpose of the $1 \times d$ vector $\mathbf{n}+\mathbf{k}$. Hence, at least formally,

$$\mathbb{H}_0(\mathbf{k})^{-1}\psi(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{\hat{\psi}(\mathbf{n}) e^{i\mathbf{n}\mathbf{x}}}{(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T}. \tag{3.3}$$

Let

$$\mathbf{k} = (\delta + i\rho)\mathbf{a} + \mathbf{b} \tag{3.4}$$

where $\mathbf{a}$ is fixed in (2.17), $\mathbf{b} \in L$, and δ is given by (2.22). Then

$$\begin{aligned}
&(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T \\
&= (\mathbf{n}+\mathbf{b})A(\mathbf{n}+\mathbf{b})^T + 2(\delta+i\rho)\mathbf{a}A(\mathbf{n}+\mathbf{b})^T + (\delta+i\rho)^2\mathbf{a}A\mathbf{a}^T \\
&= (\mathbf{n}+\mathbf{b})A(\mathbf{n}+\mathbf{b})^T + 2\delta(\mathbf{n}+\mathbf{b})A\mathbf{a}^T + (\delta^2-\rho^2)\mathbf{a}A\mathbf{a}^T + 2i\rho(\mathbf{n}+\mathbf{b}+\delta\mathbf{a})A\mathbf{a}^T \\
&= (\mathbf{n}+\mathbf{b})A(\mathbf{n}+\mathbf{b})^T + 2\delta(n_1+b_1)s_0 + (\delta^2-\rho^2)a_1 s_0 + 2i\rho(n_1+b_1+\delta a_1)s_0
\end{aligned}$$

where we have used $\mathbf{a}A = (s_0, \ldots, 0)$(see (2.17)) in the last equality.

Note that, by (2.22), $b_1 + \delta a_1 = 1/2$. Thus

$$\begin{aligned}
&(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T \\
&= (\mathbf{n}+\mathbf{b})A(\mathbf{n}+\mathbf{b})^T + 2\delta(n_1+b_1)s_0 + (\delta^2-\rho^2)a_1 s_0 + 2i\rho(n_1+\frac{1}{2})s_0.
\end{aligned} \tag{3.5}$$

Since n_1 is an integer, $|n_1 + 1/2| \geq 1/2$. It follows that

$$|\mathrm{Im}\,(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T| \geq |\rho| s_0. \tag{3.6}$$

Also, it is not hard to see that, if $|\rho| \geq 1$, $\mathbf{n} \in \mathbb{Z}^d$,

$$|(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T| \approx |(\mathbf{n}+\mathbf{b})A(\mathbf{n}+\mathbf{b})^T - \rho^2 a_1 s_0| + |\rho(n_1 + \frac{1}{2})|. \tag{3.7}$$

Now, for $\xi \in \mathbb{C}$, we consider the family of operators,

$$T_\xi \psi(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{\hat{\psi}(\mathbf{n}) e^{i\mathbf{n}\mathbf{x}}}{[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]^\xi}. \tag{3.8}$$

Lemma 3.1. *Suppose $Re\xi \geq 0$ Then*

$$\|T_\xi \psi\|_2 \leq \frac{Ce^{\pi|\mathrm{Im}\xi|}}{|s_0\rho|^{\mathrm{Re}\xi}} \|\psi\|_2.$$

Proof. The lemma follows directly from the estimate

$$|[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]^\xi| \geq |s_0\rho|^{\mathrm{Re}\xi} e^{-\pi|\mathrm{Im}\xi|} \quad \text{for } \mathrm{Re}\,\xi \geq 0. \tag{3.9}$$

Lemma 3.2. *Suppose $Re\xi = \gamma$ where $\gamma = 1$ if $d = 3$, $\gamma > 1$ if d=4, and $\gamma \geq (d/2) - 1$ if $d \geq 5$. Then*

$$\|T_\xi \psi\|_2 \leq C(\rho, \gamma) e^{\pi|\mathrm{Im}\xi|} \|\psi\|_1$$

with $C(\rho, \gamma)$ bounded for $|\rho| \geq 1$. Furthermore, $C(\rho, \gamma) \to 0$ as $|\rho| \to \infty$ for $d = 3$, $\gamma = 1$, and for $d \geq 4$, $\gamma > d/2 - 1$.

Assuming Lemma 3.2, we now give the

Proof of Theorem 2.1. Note that $\mathbb{H}_0(\delta + i\rho)^{-1} = T_1$. In the case $d = 3$, Theorem 2.1 follows directly from Lemma 3.2.

For $d \geq 4$, we apply the Stein Interpolation Theorem [18, p.205, Theorem 4.1] with estimates in Lemmas 3.1 ($\mathrm{Re}\,\xi = 0$) and 3.2. We obtain

$$\|\mathbb{H}_0(\delta + i\rho)^{-1}\psi\|_2 \leq \widetilde{C}(\rho)\|\psi\|_p$$

where $\widetilde{C}(\rho) \leq C[C(\rho, \gamma)]^M$ for some $M > 0$ and

$$\frac{1}{p} = \frac{t}{2} + \frac{1-t}{1}, \quad 1 = t \cdot 0 + (1-t) \cdot \gamma. \tag{3.10}$$

From (3.10), we have $p = 2\gamma/(\gamma+1)$. Hence p=2(d-2)/d if $\gamma = (d/2) - 1$. An inspection of Lemma 3.2 now yields Theorem 2.1.

Finally in this section we give the proof of Lemma 3.2 which completes the proof of Theorem 2.1.

Proof of Lemma 3.2. Note that, for $\mathrm{Re}\,\xi = \gamma > 0$,

$$\begin{aligned}
\|T_\xi \psi\|_2^2 &= (2\pi)^d \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{|\hat{\psi}(\mathbf{n})|^2}{\left|[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]^\xi\right|^2} \\
&\le (2\pi)^d e^{\pi|\mathrm{Im}\xi|} \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{|\hat{\psi}(\mathbf{n})|^2}{\left|[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]\right|^{2\gamma}} \\
&\le (2\pi)^{-d} e^{\pi|\mathrm{Im}\xi|} \|\psi\|_1^2 \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{|\hat{\psi}(\mathbf{n})|^2}{\left|[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]\right|^{2\gamma}}
\end{aligned}$$

where we have used $|\hat{\psi}(\mathbf{n})| \le (2\pi)^{-d}\|\psi\|_1$. Thus, by (3.7), we need to estimate

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{1}{\left\{|(\mathbf{n}+\mathbf{b})A(\mathbf{n}+\mathbf{b})^T - \rho^2 a_1 s_0| + |\rho(n_1+\frac{1}{2})|\right\}^{2\gamma}}. \tag{3.11}$$

We may assume $\rho \ge 1$.

Let B be a $d\times d$ symmetric, positive-definite matrix such that $A = B^2$, i.e., $B = \sqrt{A}$. Then, for $\rho \ge 1$,

$$\begin{aligned}
&|(\mathbf{n}+\mathbf{b})A(\mathbf{n}+\mathbf{b})^T - \rho^2 a_1 s_0| = \left||(\mathbf{n}+\mathbf{b})B|^2 - \rho^2 a_1 s_0\right| \\
&= \left\{|(\mathbf{n}+\mathbf{b})B| + \rho\sqrt{a_1 s_0}\right\} \left||(\mathbf{n}+\mathbf{b})B| - \rho\sqrt{a_1 s_0}\right| \\
&\approx (\rho + |\mathbf{n}|)\left||(\mathbf{n}+\mathbf{b})B| - \rho\sqrt{a_1 s_0}\right|.
\end{aligned}$$

Thus, (3.11) is bounded by $C\{I_1 + I_2\}$ where

$$I_1 = \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ |\mathbf{n}|\ge C\rho}} \frac{1}{|\mathbf{n}|^{4\gamma}},$$

and

$$I_2 = \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ |\mathbf{n}|< C\rho}} \frac{1}{\rho^{2\gamma}\left\{\left||(\mathbf{n}+\mathbf{b})B| - \rho\sqrt{a_1 s_0}\right| + |n_1+\frac{1}{2}|\right\}^{2\gamma}}.$$

Clearly,

$$I_1 \le C\int_{C\rho}^\infty \frac{r^{d-1}dr}{r^{4\gamma}} \le \frac{C_\gamma}{\rho^{4\gamma-d}} \tag{3.12}$$

where we have assumed that $\gamma > d/4$.

To estimate I_2, note that, if $\mathbf{x} \in \mathbb{R}^d$ and $|\mathbf{x} - \mathbf{n}| \le 1/4$, then

$$\begin{aligned}&\big||\mathbf{x}B| - \rho\sqrt{a_1 s_0}\big| + |x_1| + 1\\ &\qquad \le C\left\{\big|\,|(\mathbf{n}+\mathbf{b})B| - \rho\sqrt{a_1 s_0}\big| + |n_1 + \frac{1}{2}|\right\}\end{aligned}$$

where we have used the fact that $|\mathbf{b}| \le \sqrt{d}$ and $|n_1 + 1/2| \ge 1/2$. It follows that

$$\begin{aligned}I_2 &\le \frac{C}{\rho^{2\gamma}} \int_{\substack{\mathbf{x}\in\mathbb{R}^d\\ |\mathbf{x}|\le C\rho}} \frac{d\mathbf{x}}{\left\{\big||\mathbf{x}B| - \rho\sqrt{a_1 s_0}\big| + |x_1| + 1\right\}^{2\gamma}}\\ &\le \frac{C}{\rho^{2\gamma}} \int_{\substack{\mathbf{y}\in\mathbb{R}^d\\ |\mathbf{y}|\le C\rho}} \frac{d\mathbf{y}}{\left\{\big||\mathbf{y}| - \rho\sqrt{a_1 s_0}\big| + |(B^{-1}\mathbf{y})_1| + 1\right\}^{2\gamma}}\\ &\le \frac{C}{\rho^{2\gamma}} \int_{\substack{\mathbf{x}\in\mathbb{R}^d\\ |\mathbf{x}|\le C\rho}} \frac{d\mathbf{x}}{\left\{\big||\mathbf{x}| - \rho\sqrt{a_1 s_0}\big| + |x_1| + 1\right\}^{2\gamma}}\end{aligned}$$

where the last inequality follows by a rotation. A dilation now gives

$$I_2 \le \frac{C}{\rho^{4\gamma-d}} \int_{\substack{\mathbf{x}\in\mathbb{R}^d\\ |\mathbf{x}|\le C}} \frac{d\mathbf{x}}{\left\{\big||\mathbf{x}| - \sqrt{a_1 s_0}\big| + |x_1| + \frac{1}{\rho}\right\}^{2\gamma}}.$$

To take the advantage of the term $|x_1|$ in the integrand, we use polar coordinates in $\mathbb{R}^d$ with $x_1 = r\cos\theta$. We obtain

$$\begin{aligned}I_2 &\le \frac{C}{\rho^{4\gamma-d}} \int_0^C r^{d-1}dr \int_0^{\frac{\pi}{2}} \frac{(\sin\theta)^{d-2}d\theta}{\left\{|r - \sqrt{a_1 s_0}| + r\cos\theta + \frac{1}{\rho}\right\}^{2\gamma}}\\ &\le \frac{C_\gamma}{\rho^{4\gamma-d}} \int_0^C \frac{r^{d-2}dr}{\left\{|r - \sqrt{a_1 s_0}| + \frac{1}{\rho}\right\}^{2\gamma-1}}\\ &\le \frac{C_\gamma}{\rho^{4\gamma-d}} \int_0^C \frac{dr}{\left\{r + \frac{1}{\rho}\right\}^{2\gamma-1}}\\ &\le \frac{C_\gamma}{\rho^{2\gamma-d+2}} + \frac{C_\gamma}{\rho^{4\gamma-d}} \int_{\frac{1}{\rho}}^C \frac{dr}{r^{2\gamma-1}}.\end{aligned}$$

This, together with (3.12), shows that, if $\gamma > d/4$, the sum (3.11) is bounded by

$$\frac{C_\gamma}{\rho^{2\gamma-d+2}} + \frac{C_\gamma}{\rho^{4\gamma-d}} \int_{\frac{1}{\rho}}^C \frac{dr}{r^{2\gamma-1}}.$$

Thus, if $d = 3$ and $\gamma = 1$,

$$\|T_\xi\psi\|_2 \le Ce^{\pi|\mathrm{Im}\xi|} \frac{1}{|\rho|^{1/2}} \left\{1 + \log|\rho|\right\}^{1/2} \|\psi\|_1,$$

and

$$\|T_\xi\psi\|_2 \le C_\gamma e^{\pi|\mathrm{Im}\xi|}\frac{1}{|\rho|^{\gamma-1}}\,\|\psi\|_1 \text{ if } d=4, \mathrm{Re}\,\xi=\gamma>1,$$

$$\|T_\xi\psi\|_2 \le C_\gamma e^{\pi|\mathrm{Im}\xi|}\frac{1}{|\rho|^{\gamma-\frac{d}{2}+1}}\,\|\psi\|_1 \text{ if } d\ge 5, \mathrm{Re}\,\xi=\gamma\ge \frac{d}{2}-1.$$

The lemma is proved.

4. The Uniform Sobolev Inequalities

The goal of this section is to establish the following uniform Sobolev inequalities.

Theorem 4.1. *Let $p=2d/(d+2)$, $q=2d/(d-2)$, and $\mathbf{b}\in\mathbb{R}^d$. Let $z\in\mathbb{C}$ such that $Im\,z\neq 0$ and $Re\sqrt{z}\ge c_0>0$, then*

$$\|\big((\mathbf{D}+\mathbf{b})A(\mathbf{D}+\mathbf{b})^T+z\big)^{-1}\psi\|_q \le C\|\psi\|_p$$

with a constant C independent of $\mathbf{b}$ and z.

In the next section we will deduce Theorem 2.2 from Theorem 4.1 by a localization argument. We remark that our proof of Theorems 2.2 and 4.1 is based on an approach developed in [10] for the similar estimates in $\mathbb{R}^d$.

For $\xi\in\mathbb{C}$, we consider the family of operators

$$S_\xi\psi(\mathbf{x}) = (\xi-\frac{d}{2})\sum_{\mathbf{n}\in\mathbb{Z}^d}\frac{\hat{\psi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}}{[|(\mathbf{n}+\mathbf{b})B|^2+z]^\xi} \tag{4.1}$$

where $B=\sqrt{A}$, $\mathbf{b}\in\mathbb{R}^d$, and $z\in\mathbb{C}$ satisfies the conditions in Theorem 4.1. Clearly,

$$S_1 = (1-\frac{d}{2})\big((\mathbf{D}+\mathbf{b})A(\mathbf{D}+\mathbf{b})^T+z\big)^{-1}. \tag{4.2}$$

We begin with a L^2-L^2 estimate

Lemma 4.1. *Suppose $Re\xi\ge 0$ Then*

$$\|S_\xi\psi\|_{L^2(\Omega)} \le \frac{C(|\xi|+1)e^{\pi|\mathrm{Im}\xi|}}{|Im\,z|^{\mathrm{Re}\xi}}\|\psi\|_{L^2(\Omega)}.$$

As for Lemma 3.1, the proof of Lemma 4.1 is easy.

To apply the Stein interpolation theorem, we need to establish the $L^1 - L^\infty$ estimate for $\operatorname{Re}\xi = d/2$. To this end, we write

$$S_\xi\psi(\mathbf{x}) = \int_\Omega G_\xi(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})d\mathbf{y} \tag{4.3}$$

where

$$G_\xi(\mathbf{x}) = \frac{(\xi - \frac{d}{2})}{(2\pi)^d} \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{e^{i\mathbf{n}\mathbf{x}}}{[|(\mathbf{n}+\mathbf{b})B| + z]^\xi}. \tag{4.4}$$

Let $F_\xi(\cdot)$ denote the Fourier transform of $[|\cdot|^2 + z]^{-\xi}$ in $\mathbb{R}^d$.

Lemma 4.2. *The formula*

$$G_\xi(\mathbf{x}) = \frac{\xi - \frac{d}{2}}{\det(B)} \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{-i\mathbf{b}(\mathbf{x}+2\pi\mathbf{n})} F_\xi((\mathbf{x}+2\pi\mathbf{n})B^{-1})$$

holds for any ξ with which the right hand side converges absolutely.

Proof. We may assume $\operatorname{Re}\xi$ is large in order to justify the following computation. The formula can be extended then to any ξ for which the right hand side converges absolutely, by analytic continuation.

Note that

$$F_\xi(\mathbf{y}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{e^{-i\mathbf{x}\mathbf{y}}\,d\mathbf{x}}{[|\mathbf{x}|^2 + z]^\xi}. \tag{4.5}$$

By a change of variables,

$$F_\xi(\mathbf{y}B^{-1}) = \frac{\det(B)}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{e^{-i\mathbf{x}\mathbf{y}}\,d\mathbf{x}}{[|\mathbf{x}B|^2 + z]^\xi}.$$

The inverse Fourier transform then gives

$$\begin{aligned}
\frac{1}{[|\mathbf{x}B|^2 + z]^\xi} &= \frac{1}{\det(B)}\int_{\mathbb{R}^d} e^{i\mathbf{x}\mathbf{y}} F_\xi(\mathbf{y}B^{-1})d\mathbf{y} \\
&= \frac{1}{\det(B)} \sum_{\mathbf{m}\in\mathbb{Z}^d}\int_\Omega e^{i\mathbf{x}(\mathbf{y}+2\pi\mathbf{m})} F_\xi((\mathbf{y}+2\pi\mathbf{m})B^{-1})d\mathbf{y} \\
&= \frac{1}{\det(B)}\int_\Omega e^{i\mathbf{x}\mathbf{y}} \sum_{\mathbf{m}\in\mathbb{Z}^d} e^{2\pi i\mathbf{x}\mathbf{m}} F_\xi((\mathbf{y}+2\pi\mathbf{m})B^{-1})d\mathbf{y}.
\end{aligned}$$

In particular, if we let $\mathbf{x} = -(\mathbf{n}+\mathbf{b})$,

$$\frac{1}{[|(\mathbf{n}+\mathbf{b})B|^2+z]^\xi} = \frac{1}{\det(B)}\int_\Omega e^{-i\mathbf{n}\mathbf{y}}e^{-i\mathbf{b}\mathbf{y}}\sum_{\mathbf{m}\in\mathbb{Z}^d} e^{-2\pi i\mathbf{m}\mathbf{b}}F_\xi((\mathbf{y}+2\pi\mathbf{m})B^{-1})d\mathbf{y}.$$

This implies that

$$\sum_{\mathbf{n}\in\mathbb{Z}^d}\frac{e^{i\mathbf{n}\mathbf{x}}}{[|(\mathbf{n}+\mathbf{b})B|^2+z]^\xi} = \frac{(2\pi)^d}{\det(B)}\sum_{\mathbf{n}\in\mathbb{Z}^d} e^{-i(\mathbf{x}+2\pi\mathbf{n})\mathbf{b}}F_\xi((\mathbf{x}+2\pi\mathbf{n})B^{-1}).$$

The lemma follows.

By Lemma 4.2, we have

$$|G_\xi(\mathbf{x}) \le C|\xi-\frac{d}{2}|\sum_{\mathbf{n}\in\mathbb{Z}^d}|F_\xi((\mathbf{x}+2\pi\mathbf{n})B^{-1})|. \tag{4.6}$$

To estimate the sum, we use the following formula [5, pp. 288-289]:

$$F_\xi(\mathbf{x}) = \frac{2^{1-\xi}}{(2\pi)^{d/2}\Gamma(\xi)}\left[\frac{z}{|\mathbf{x}|^2}\right]^{\frac{1}{2}(\frac{d}{2}-\xi)} K_{\frac{d}{2}-\xi}(\sqrt{z|\mathbf{x}|^2}) \tag{4.7}$$

where $\Gamma(\xi)$ is the Gamma function and K_η is the modified Bessel function of the third kind [12, p.108].

We need two estimates on K_η for $\operatorname{Re}\eta = 0$:

$$|K_\eta(\omega)| \le Ce^{C|\mathrm{Im}\eta|}\frac{e^{-\operatorname{Re}\omega}}{|\omega|^{1/2}} \qquad \text{if } |\omega|\ge 1,\ \operatorname{Re}\omega>0, \tag{4.8}$$

$$|\sin(\pi\eta)K_\eta(\omega)| \le Ce^{C|\mathrm{Im}\eta|} \qquad \text{if } |\omega|\le 1,\ \operatorname{Re}\omega>0. \tag{4.9}$$

We remark that (4.8) follows from the formula

$$K_\eta(\omega) = \left(\frac{\pi}{2\omega}\right)^{1/2}\frac{e^{-\omega}}{\Gamma(\eta+\frac{1}{2})}\int_0^\infty e^{-s}s^{\eta-\frac{1}{2}}\left(1+\frac{s}{2\omega}\right)^{\eta-\frac{1}{2}}ds$$

which is valid for $\operatorname{Re}\eta > -1/2$ and $|\arg\omega| < \pi$ [12, p.140], while (4.9) is a consequence of

$$\begin{aligned}2\sin(\pi\eta)K_\eta(\omega) = \pi\Bigg\{ &\left(\frac{\omega}{2}\right)^\eta\sum_{j=0}^\infty\frac{(\frac{\omega}{2})^{2j}}{\Gamma(j+1)\Gamma(j+\eta+1)} \\ &- \left(\frac{\omega}{2}\right)^{-\eta}\sum_{j=0}^\infty\frac{(\frac{\omega}{2})^{2j}}{\Gamma(j+1)\Gamma(j-\eta+1)}\Bigg\}\end{aligned}$$

[12, p.108].

Lemma 4.3. *Suppose $Re\xi = d/2$, then*

$$\|S_\xi \psi\|_\infty \le Ce^{C|\mathrm{Im}\xi|}\|\psi\|_1.$$

Proof. In view of (4.3) and (4.6), it suffices to show that, for $\mathrm{Re}\,\xi = d/2$,

$$|\mathrm{Im}\,\xi| \sum_{\mathbf{n}\in\mathbb{Z}^d} \left|F_\xi((\mathbf{x}+2\pi\mathbf{n})B^{-1})\right| \le Ce^{C|\mathrm{Im}\xi|}, \quad \mathbf{x}\in\Omega. \tag{4.10}$$

To this end, we first note that, by (4.7), the left hand side of (4.10) is bounded by

$$C|\mathrm{Im}\,\xi|e^{C|\mathrm{Im}\xi|} \sum_{\mathbf{n}\in\mathbb{Z}^d} \left|K_{-i\mathrm{Im}\xi}(\sqrt{z}|(\mathbf{x}+2\pi\mathbf{n})B^{-1}|)\right|. \tag{4.11}$$

Next, we write the sum in (4.11) as $I_1 + I_2$, where I_1 is the sum over all $\mathbf{n}$ such that $\mathrm{Re}\sqrt{z}|(\mathbf{x}+2\pi\mathbf{n})B^{-1}| \le C$. It is easy to see that, by (4.8)-(4.9) and the estimate $|\mathrm{Im}\,\xi|\,|\sin(\pi i\mathrm{Im}\,\xi)| \le Ce^{C|\mathrm{Im}\xi|}$,

$$I_1 \le Ce^{C|\mathrm{Im}\xi|} \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d\\ |\mathbf{x}+2\pi\mathbf{n}|\le\widetilde{C}}} 1 \le Ce^{C|\mathrm{Im}\xi|}. \tag{4.12}$$

To estimate I_2, we use (4.9) to obtain

$$I_2 \le Ce^{C|\mathrm{Im}\xi|} \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d\\ \mathrm{Re}\sqrt{z}|\mathbf{x}+2\pi\mathbf{n}|\ge C}} e^{-c\mathrm{Re}\sqrt{z}|\mathbf{x}+2\pi\mathbf{n}|}\cdot\frac{1}{|z|^{1/4}|\mathbf{x}+2\pi\mathbf{n}|^{1/2}}.$$

Note that, if $|\mathbf{y}-\mathbf{x}| \le c_0/\mathrm{Re}\sqrt{z} \le 1$, then

$$e^{-c\mathrm{Re}\sqrt{z}|\mathbf{x}+2\pi\mathbf{n}|}\cdot\frac{1}{|z|^{1/4}|\mathbf{x}+2\pi\mathbf{n}|^{1/2}} \le Ce^{-c\mathrm{Re}\sqrt{z}|\mathbf{y}+2\pi\mathbf{n}|}\cdot\frac{1}{[\mathrm{Re}\sqrt{z}|\mathbf{y}+2\pi\mathbf{n}|]^{1/2}}.$$

Thus,

$$\begin{aligned} &e^{-c\mathrm{Re}\sqrt{z}|\mathbf{x}+2\pi\mathbf{n}|}\cdot\frac{1}{|z|^{1/4}|\mathbf{x}+2\pi\mathbf{n}|^{1/2}} \\ &\qquad \le C\left(\frac{\mathrm{Re}\sqrt{z}}{c_0}\right)^d \int_{|\mathbf{y}-\mathbf{x}|\le\frac{c_0}{\mathrm{Re}\sqrt{z}}} \frac{e^{-c\mathrm{Re}\sqrt{z}|\mathbf{y}+2\pi\mathbf{n}|}d\mathbf{y}}{[\mathrm{Re}\sqrt{z}|\mathbf{y}+2\pi\mathbf{n}|]^{1/2}}. \end{aligned}$$

This yields, by summation, that

$$\begin{aligned} I_2 &\le Ce^{C|\mathrm{Im}\xi|}\left(\frac{\mathrm{Re}\sqrt{z}}{c_0}\right)^d \int_{|\mathbf{y}|\ge\frac{c}{\mathrm{Re}\sqrt{z}}} \frac{e^{-c\mathrm{Re}\sqrt{z}|\mathbf{y}|}d\mathbf{y}}{[\mathrm{Re}\sqrt{z}|\mathbf{y}|]^{1/2}} \\ &\le Ce^{C|\mathrm{Im}\xi|}\int_0^\infty e^{-cr}\frac{dr}{\sqrt{r}} \\ &\le Ce^{C|\mathrm{Im}\xi|} \end{aligned}$$

which, together with (4.12), gives (4.10). The proof is finished.

We are now in a position to give the

Proof of Theorem 4.1. With estimates in Lemmas 4.1 ($\mathrm{Re}\,\xi = 0$) and 4.3, we apply the Stein interpolation theorem [18, p.205, Theorem 4.1] to obtain

$$\| \left\{(\mathbf{D}+\mathbf{b})A(\mathbf{D}+\mathbf{b})^T + z\right\}^{-1} \psi\|_q \le C\|\psi\|_p$$

where $q = p'$ and

$$\frac{1}{p} = \frac{t}{2} + \frac{1-t}{1}, \quad 1 = t\cdot 0 + (1-t)\cdot \frac{d}{2}. \tag{4.13}$$

The theorem follows since, by (4.13),

$$t = 1 - \frac{2}{d} \quad \text{and} \quad p = \frac{2}{2-t} = \frac{2d}{d+2}.$$

5. The Proof of Theorem 2.2

In this section we give the proof of Theorem 2.2 and thus complete the final step in the proof of Theorem 0.1.

Recall that

$$\mathbb{H}_0(\delta+i\rho)^{-1}\psi = \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{\hat{\psi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}}{(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T} \tag{5.1}$$

where $\mathbf{k} = (\delta+i\rho)\mathbf{a}+\mathbf{b}$ (see (3.3) - (3.5)) and we need to show

$$\|\mathbb{H}_0(\delta+i\rho)^{-1}\psi\|_q \le C\|\psi\|_p \tag{5.2}$$

for $p = 2d/(d+2)$, $q = p'$ with a constant C independent of ρ when $|\rho| \ge 1$. We may assume $\psi \in C_0^\infty(\Omega)$.

First, we write $\psi = \sum_{j=-\infty}^{\infty} \psi_j$ where

$$\begin{aligned} \psi_j &= \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ n_1\in[2^{j-1},2^j-1]}} \hat{\psi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}} \qquad \text{for } j\ge 1, \\ \psi_j &= \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ n_1\in[-2^{-j}+1,-2^{-j-1}]}} \hat{\psi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}} \qquad \text{for } j\le -1, \end{aligned} \tag{5.3}$$

and $\psi_0 = \hat{\psi}(0)$.

Lemma 5.1. *If estimate (5.2) holds for ψ_j with a constant C independent of j, then it holds for ψ.*

Proof. Since $\mathbb{H}_0(\delta+i\rho)^{-1}$ is a multiplier operator, by the one-dimensional Littlewood-Paley theory on $[0,2\pi]$ [20, Chapter XV], we have

$$\begin{aligned}
\|\mathbb{H}_0(\delta+i\rho)^{-1}\psi\|_q &\le C\|\left(\sum_{j=-\infty}^{\infty}|\mathbb{H}_0(\delta+i\rho)^{-1}\psi_j|^2\right)^{1/2}\|_q\\
&\le C\left(\sum_{j=-\infty}^{\infty}\|\mathbb{H}_0(\delta+i\rho)^{-1}\psi_j\|_q^2\right)^{1/2}\\
&\le C\left(\sum_{j=-\infty}^{\infty}\|\psi_j\|_p^2\right)^{1/2}\\
&\le C\|\left(\sum_{j=-\infty}^{\infty}|\psi_j|^2\right)^{1/2}\|_p\\
&\le C\|\psi\|_p
\end{aligned}$$

where we used the Minkowski's inequality [17, p.271] and $q>2>p$ in the second and fourth inequalities.

We will give the estimate of $\mathbb{H}_0(\delta+i\rho)^{-1}\psi_j$ for $j\ge 1$ in details. The case $j\le 0$ may be handled in the same manner.

Fix $j\ge 1$. Note that, $n_1\approx 2^j$ if $\hat{\psi}_j(\mathbf{n})\neq 0$. In view of (3.5), we let

$$z_j=-\rho^2a_1s_0+2i\rho\cdot 2^j\cdot s_0 \tag{5.4}$$

and consider the operator

$$\big((\mathbf{D}+\mathbf{b})A(\mathbf{D}+\mathbf{b})^T+z_j\big)^{-1}\psi=\sum_{n\in\mathbb{Z}^d}\frac{\hat{\psi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}}{|(\mathbf{n}+\mathbf{b})B|^2+z_j} \tag{5.5}$$

(see (4.1)-(4.2)).

Lemma 5.2. *Let $j\ge 1$ and $\rho\in\mathbb{R}$, $|\rho|\ge 1$. Then there exists a constant C independent of j and ρ, such that*

$$\|\mathbb{H}_0(\delta+i\rho)^{-1}\psi_j-\big((\mathbf{D}+\mathbf{b})A(\mathbf{D}+\mathbf{b})^T+z_j\big)^{-1}\psi_j\|_q\le C\|\psi_j\|_p.$$

Proof. Note that, by (5.1) and (5.5),

$$
\begin{aligned}
&\mathbb{H}_0(\delta+i\rho)^{-1}\psi_j-\left((\mathbf{D}+\mathbf{b})A(\mathbf{D}+\mathbf{b})^T+z_j\right)^{-1}\psi_j\\
&=\sum_{\mathbf{n}\in\mathbb{Z}^d}\frac{\hat{\psi}_j(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}\left\{|(\mathbf{n}+\mathbf{b})B|^2+z_j-(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T\right\}}{[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]\cdot[|(\mathbf{n}+\mathbf{b})B|^2+z_j]}\\
&=\sum_{M=1}^{\infty}\sum_{\substack{\mathbf{n}\in\mathbb{Z}^d\\ M-1\le|\mathbf{n}B|<M}}\frac{\hat{\psi}_j(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}\left\{|(\mathbf{n}+\mathbf{b})B|^2+z_j-(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T\right\}}{[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]\cdot[|(\mathbf{n}+\mathbf{b})B|^2+z_j]}.
\end{aligned}
\tag{5.6}
$$

Now we consider the second order elliptic operator $P=\mathbf{D}A\mathbf{D}^T$ on the d-torus $[0,2\pi)^d\approx\mathbb{R}^d/(2\pi\mathbb{Z})^d$. P has a complete set of eigenfunctions $\{e^{i\mathbf{n}\mathbf{x}},\mathbf{n}\in\mathbb{Z}^d\}$ with the corresponding eigenvalues $\{\mathbf{n}A\mathbf{n}^T,\mathbf{n}\in\mathbb{Z}^d\}$. Thus

$$
\chi_M\phi=\sum_{\substack{\mathbf{n}\in\mathbb{Z}^d\\ \mathbf{n}A\mathbf{n}^T\in[(M-1)^2,M^2)}}\hat{\phi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}=\sum_{\substack{\mathbf{n}\in\mathbb{Z}^d\\ |\mathbf{n}B|\in[M-1,M)}}\hat{\phi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}
\tag{5.7}
$$

is the projection of ϕ to the subspace of $L^2(\Omega)$, spanned by eigenfunctions with eigenvalues in $[(M-1)^2,M^2)$. It then follows from a general result in [16, p.127, Theorem 2.2, part (i)] that, for any $\phi\in L^2(\Omega)$,

$$
\|\chi_M\phi\|_2\le CM^{1/2}\|\phi\|_p.
\tag{5.8}
$$

since $p=2d/(d+2)<p_d=2(d+1)/(d+3)$ and $\sigma(p,d)=d[\frac{1}{p}-\frac{1}{p'}]-1=1$. By duality, (5.8) implies that, if $q=p'$,

$$
\|\chi_M\phi\|_q\le CM^{1/2}\|\phi\|_2.
\tag{5.9}
$$

Hence, by (5.6), the Minkowski's inequality and (5.8)-(5.9), we have

$$
\begin{aligned}
&\|\mathbb{H}_0(\delta+i\rho)^{-1}\psi_j - \left((\mathbf{D}+\mathbf{b})A(\mathbf{D}+\mathbf{b})^T + z_j\right)^{-1}\psi_j\|_q \\
&\le \sum_{M=1}^{\infty} \| \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ M-1\le|\mathbf{n}B|<M}} \frac{\hat{\psi}_j(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}\left\{|(\mathbf{n}+\mathbf{b})B|^2 + z_j - (\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T\right\}}{[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]\cdot[|(\mathbf{n}+\mathbf{b})B|^2+z_j]}\|_q \\
&\le C\sum_{M=1}^{\infty} M^{1/2}\| \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ M-1\le|\mathbf{n}B|<M}} \frac{\hat{\psi}_j(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}\left\{|(\mathbf{n}+\mathbf{b})B|^2 + z_j - (\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T\right\}}{[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]\cdot[|(\mathbf{n}+\mathbf{b})B|^2+z_j]}\|_2 \\
&\le C\sum_{M=1}^{\infty} M^{1/2} \sup_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ n_1\in[2^{j-1},2^j-1] \\ |\mathbf{n}B|\in[M-1,M)}} \left|\frac{\left\{|(\mathbf{n}+\mathbf{b})B|^2 + z_j - (\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T\right\}}{[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]\cdot[|(\mathbf{n}+\mathbf{b})B|^2+z_j]}\right| \\
&\qquad \cdot \| \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ |\mathbf{n}B|\in[M-1,M)}} \hat{\psi}_j(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}\|_2 \\
&\le C\|\psi_j\|_p \sum_{M=1}^{\infty} M \sup_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ n_1\in[2^{j-1},2^j-1] \\ |\mathbf{n}B|\in[M-1,M)}} \left|\frac{\left\{|(\mathbf{n}+\mathbf{b})B|^2 + z_j - (\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T\right\}}{[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]\cdot[|(\mathbf{n}+\mathbf{b})B|^2+z_j]}\right| \\
&\le C\|\psi_j\|_p \sum_{M=1}^{\infty} M \sup_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ |\mathbf{n}B|\in[M-1,M)}} \frac{|\rho|2^j}{\left\{\left||(\mathbf{n}+\mathbf{b})B|^2-\rho^2a_1s_0\right| + 2^j|\rho|\right\}^2}
\end{aligned}
$$

where we have used (3.5), (3.7) and (5.4) in the last inequality.

It remains to show that, for $\rho\in\mathbb{R}$ and $|\rho|\ge 1$,

$$
\sum_{M=1}^{\infty} M \sup_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ |\mathbf{n}B|\in[M-1,M)}} \frac{|\rho|2^j}{\left\{\left||(\mathbf{n}+\mathbf{b})B|^2-\rho^2a_1s_0\right| + 2^j|\rho|\right\}^2} \le C. \tag{5.10}
$$

We may assume $\rho\ge 1$.

To prove (5.10), note that, if $|\mathbf{n}B|\in[M-1,M)$ and $M\ge C\rho$,

$$
\left||(\mathbf{n}+\mathbf{b})B|^2-\rho^2a_1s_0\right| + 2^j\rho \ge c(M^2+2^j\rho).
$$

On the other hand, if $|\mathbf{n}B| \in [M-1, M)$ and $M \le C\rho$,

$$
\begin{aligned}
&\big||(\mathbf{n}+\mathbf{b})B|^2 - \rho^2 a_1 s_0\big| + 2^j\rho \\
&= \left(|(\mathbf{n}+\mathbf{b})B| + \rho\sqrt{a_1 s_0}\right)\big||(\mathbf{n}+\mathbf{b})B| - \rho\sqrt{a_1 s_0}\big| + 2^j\rho \\
&\ge \varepsilon\rho\big||(\mathbf{n}+\mathbf{b})B| - \rho\sqrt{a_1 s_0}\big| + 2^j\rho \\
&\ge \varepsilon\rho\big||\mathbf{n}B| - \rho\sqrt{a_1 s_0}\big| + (2^j - \varepsilon|\mathbf{b}B|)\rho \\
&\ge \varepsilon\rho\big|M - \rho\sqrt{a_1 s_0}\big| + (2^j - \varepsilon|\mathbf{b}B| - \varepsilon)\rho \\
&\ge \varepsilon\rho\left\{|M - \rho\sqrt{a_1 s_0}| + 2^j\right\}
\end{aligned}
$$

if ε is small. Thus, the left hand side of (5.10) is bounded by

$$
\begin{aligned}
&C\sum_{1\le M\le C\rho} \frac{M2^j}{\rho\left\{|M-\rho\sqrt{a_1 s_0}| + 2^j\right\}^2} + C\sum_{M\ge C\rho}\frac{M\rho 2^j}{(M^2+2^j\rho)^2} \\
&= I_1 + I_2.
\end{aligned}
$$

It is not hard to see that, if $|r-M| \le 1/2$ and $M \ge 1$,

$$
\frac{M2^j}{\rho\left\{|M-\rho\sqrt{a_1 s_0}| + 2^j\right\}^2} \le \frac{Cr2^j}{\rho\left\{|r-\rho\sqrt{a_1 s_0}| + 2^j\right\}^2}.
$$

It follows that

$$
\begin{aligned}
I_1 &\le C\int_0^{C\rho} \frac{r2^j\,dr}{\rho\left\{|r-\rho\sqrt{a_1 s_0}| + 2^j\right\}^2} \\
&\le C\int_0^{C} \frac{2^j\rho^{-1}\,dr}{\left\{|r-\sqrt{a_1 s_0}| + 2^j\rho^{-1}\right\}^2} \\
&\le C\int_{|r-\sqrt{a_1 s_0}|\le 2^j\rho^{-1}} \frac{dr}{2^j\rho^{-1}} + C\int_{|r-\sqrt{a_1 s_0}|\ge 2^j\rho^{-1}} \frac{2^j\rho^{-1}\,dr}{|r-\sqrt{a_1 s_0}|^2} \\
&\le C.
\end{aligned}
$$

Similarly,

$$
I_2 \le C\int_{C\rho}^{\infty} \frac{\rho 2^j r\,dr}{(r^2+2^j\rho)^2} \le C\int_0^{\infty}\frac{r\,dr}{(r^2+1)^2} \le C.
$$

(5.10) is now proved and thus the proof of Lemma 5.2 is complete.

Finally we are ready to give the

Proof of Theorem 2.2. In view of Lemmas 5.1 and 5.2, it suffices to show

$$
\|\left((\mathbf{D}+\mathbf{b})A(\mathbf{D}+\mathbf{b})^T + z_j\right)^{-1}\psi_j\|_q \le C\|\psi_j\|_p.
$$

But this follows immediately from Theorem 4.1, since for $z_j = -\rho^2 a_1 s_0 + 2i\rho\cdot 2^j \cdot s_0$, we have $\operatorname{Im} z_j = 2\rho \cdot 2^j \cdot s_0 \neq 0$ and

$$\begin{aligned}\operatorname{Re}\sqrt{z_j} = & |z_j|^{1/2}\cos\left(\frac{1}{2}\arg(z_j)\right) \geq \frac{1}{2}\cdot\frac{|\operatorname{Im} z_j|}{|z_j|^{1/2}} \\ & \geq \frac{c|\rho|2^j}{|\rho| + \sqrt{|\rho|2^j}} \geq c\min(2^j, \sqrt{|\rho|2^j}) \geq c_0 > 0.\end{aligned}$$

6. Potentials in weak-$L^{d/2}(\Omega)$ space

In this section we give the proof of Theorem 0.2. We will use $\|\cdot\|_{p,q}$ to denote the norm in the Lorentz space $L^{p,q}(\Omega)$ where $1 < p \leq \infty$, $1 \leq q \leq \infty$ (see [18, Chapter V]). We will need the following inequalities:

$$\|f\|_{q_0,2} \leq C\{\|\nabla f\|_2 + \|f\|_2\} \tag{6.1}$$

and

$$\|fg\|_{p_0,2} \leq C\|g\|_{d/2,\infty}\|f\|_{q_0,2} \tag{6.2}$$

where $p_0 = 2d/(d+2)$ and $q_0 = 2d/(d-2)$. (6.2) and (6.3), which may be proved by the real interpolation, are substitutes, in the case of Lorentz spaces, for the usual Sobolev inequality and Hölder inequality.

Let $V \in L^{d/2,\infty}(\Omega)$ be a real function satisfying the condition (1.9). We may write $V = V_1 + V_2$ such that

$$\|V_1\|_{d/2,\infty} \leq \varepsilon_0 \quad \text{and} \quad \|V_2\|_\infty \leq C. \tag{6.3}$$

It follows from (6.1)-(6.2) that

$$\int_\Omega |V|\,|f|^2 d\mathbf{x} \leq C\varepsilon_0 \int_\Omega |\nabla f|^2 d\mathbf{x} + C\int_\Omega |f|^2 d\mathbf{x}. \tag{6.4}$$

Using the same arguments as in section 2, we see that, if ε_0 is sufficiently small, we may define the operators $\mathbf{D}A\mathbf{D}^T + V$ on $\mathbb{R}^d$ and $(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T + V$ on Ω through the corresponding quadratic forms.

Next, note that, by (6.1)-(6.2), $V\phi \in L^{p_0,2}(\Omega)$ if $V \in L^{d/2,\infty}(\Omega)$ and $\phi \in H^1_{\text{per}}(\Omega)$. With this observation, a careful inspection of the arguments in the proof of Theorem 0.1 in section 2 shows that Theorem 0.2 follows from the following estimates:

$$(6.5)\qquad \|\mathbb{H}_0(\delta+i\rho)^{-1}\|_{L^{p_0,2}(\Omega)\to L^{p_0,2}(\Omega)}\to 0 \quad \text{as } |\rho|\to\infty,$$

$$(6.6)\qquad \|\mathbb{H}_0(\delta+i\rho)^{-1}\|_{L^{p_0,2}(\Omega)\to L^{q_0,2}(\Omega)}\le C \quad \text{for } |\rho|\ge 1.$$

Also it is not hard to see that (6.5) is a direct consequence of Theorem 2.1 and the real interpolation (see [18], p.197, Theorem 3.15). We omit the details.

To show (6.6), we need the following theorem.

Theorem 6.1. *Let $p_1\le p\le p_2$ and $(1/q)=(1/p)-(2/d)$ where $p_1=2d(d+1)/(d^2+3d+4)$ and $p_2=2(d+1)/(d+3)$. There there exists $C>0$ depending on A and d, such that, for $|\rho|\ge 1$,*

$$(6.7)\qquad \|\mathbb{H}_0(\delta+i\rho)^{-1}\|_{L^p(\Omega)\to L^q(\Omega)}\le C.$$

Note that $p_0=2d/(d+2)\in(p_1,p_2)$. Estimate (6.6) follows from Theorem 6.1, again by the real interpolation. Finally we give the

Proof of Theorem 6.1. By the same localization arguments as in the proofs of Lemmas 5.1 and 5.2, we may reduce the estimate (6.6) to

$$(6.8)\qquad \|\left((\mathbf{D}+\mathbf{b})A(\mathbf{D}+\mathbf{b})^T+z\right)^{-1}\psi\|_q\le C\|\psi\|_p$$

where $\mathbf{b}\in\mathbb{R}^d$, $z\in\mathbb{C}$, $\mathrm{Im}z\ne 0$ and $\mathrm{Re}\sqrt{z}\ge c_0>0$. We should point out that in the place of (5.8)-(5.9) in the proof of Lemma 5.2, one needs to use the assumption $p_1\le p\le p_2$. This gives $q'\le p_d=2(d+1)/(d+3)$, $p\le p_d$ and $\sigma(q',d)=d[\frac{1}{q'}-\frac{1}{q}]-1$, $\sigma(p,d)=d[\frac{1}{p}-\frac{1}{p'}]-1$. The result in [16, p.127, Theorem 2.2] then implies that

$$(6.9)\qquad \begin{aligned}\|\chi_M\|_{L^2\to L^q}\|\chi_M\|_{L^p\to L^2}&=\|\chi_M\|_{L^{q'}\to L^2}\|\chi_M\|_{L^p\to L^2}\\ &\le CM^{\sigma(q',d)/2}M^{\sigma(p,d)/2}=CM\end{aligned}$$

where we also used $(1/q)=(1/p)-(2/d)$ in the last equality.

To prove (6.8), we define $S_\xi\psi$ as in (4.1). We claim that, if $\mathrm{Re}\,\xi=(d-1)/2$, then

$$(6.10)\qquad |G_\xi(\mathbf{x})|\le Ce^{C|\mathrm{Im}\xi|}\left\{1+\sum_{|\mathbf{x}+2\pi\mathbf{n}|\le C}\frac{1}{|\mathbf{x}+2\pi\mathbf{n}|}\right\}$$

where $G_\xi(\cdot)$ is the integral kernel of the operator S_ξ (see (4.4)). Indeed, by (4.6)-(4.7), we have

$$(6.11)\qquad |G_\xi(\mathbf{x})| \le Ce^{C|\mathrm{Im}\xi|} \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{|z|^{1/4}}{|\mathbf{x}+2\pi\mathbf{n}|^{1/2}} \cdot \left|K_{\frac{1}{2}-i\mathrm{Im}\xi}(\sqrt{z}|(\mathbf{x}+2\pi\mathbf{n})B^{-1}|)\right|.$$

We need the following estimates on the Bessel function $K_\eta(\omega)$ for $\mathrm{Re}\,\eta = 1/2$:

$$(6.12)\qquad |K_\eta(\omega)| \le \frac{Ce^{C|\mathrm{Im}\eta|}}{|\omega|^{1/2}} \quad \text{if } |\omega| \le 1,$$

$$(6.13)\qquad |K_\eta(\omega)| \le \frac{Ce^{C|\mathrm{Im}\eta|}e^{-\mathrm{Re}\omega}}{|\omega|^{1/2}} \quad \text{if } |\omega| \ge 1 \text{ and } \mathrm{Re}\,w > 0.$$

Using (6.12)-(6.13), we obtain from (6.11) that

$$\begin{aligned}
|G_\xi(\mathbf{x})| &\le Ce^{C|\mathrm{Im}\xi|} \sum_{|z|^{1/2}|(\mathbf{x}+2\pi\mathbf{n})B^{-1}|\le 1} \frac{|z|^{1/4}}{|\mathbf{x}+2\pi\mathbf{n}|^{1/2}} \cdot \frac{1}{|z|^{1/4}|\mathbf{x}+2\pi\mathbf{n}|^{1/2}} \\
&\quad + Ce^{C|\mathrm{Im}\xi|} \sum_{|z|^{1/2}|(\mathbf{x}+2\pi\mathbf{n})B^{-1}|> 1} \frac{|z|^{1/4}}{|\mathbf{x}+2\pi\mathbf{n}|^{1/2}} \cdot \frac{e^{-\mathrm{Re}\sqrt{z}|(\mathbf{x}+2\pi\mathbf{n})B^{-1}|}}{|z|^{1/4}|\mathbf{x}+2\pi\mathbf{n}|^{1/2}} \\
&\le Ce^{C|\mathrm{Im}\xi|} \sum_{|\mathbf{x}+2\pi\mathbf{n}|\le C} \frac{1}{|\mathbf{x}+2\pi\mathbf{n}|} \\
&\quad + Ce^{C|\mathrm{Im}\xi|} \sum_{|\mathbf{x}+2\pi\mathbf{n}|\ge C} \frac{e^{-c|\mathbf{x}+2\pi\mathbf{n}|}}{|\mathbf{x}+2\pi\mathbf{n}|} \\
&\le Ce^{C|\mathrm{Im}\xi|} \left\{ 1 + \sum_{|\mathbf{x}+2\pi\mathbf{n}|\le C} \frac{1}{|\mathbf{x}+2\pi\mathbf{n}|} \right\}
\end{aligned}$$

where we also used the assumption $\mathrm{Re}\sqrt{z} \ge c_0$. (6.10) is then proved.

It follows from (6.10) and the well-known fractional integral estimates that, for $\mathrm{Re}\,\xi = (d-1)/2$,

$$(6.14)\qquad \|S_\xi\psi\|_{\tilde{q}} \le C\|\psi\|_{\tilde{p}},$$

where

$$(6.15)\qquad \frac{1}{\tilde{q}} = \frac{1}{\tilde{p}} - \frac{d-1}{d} \quad \text{and} \quad 1 < \tilde{p} < \frac{d}{d-1}.$$

With (6.14) and the L^2-estimate in Lemma 4.1 for $\mathrm{Re}\,\xi = 0$ at our disposal, we now apply the Stein complex interpolation theorem. We conclude that

$$(6.16)\qquad \|S_1\psi\|_q \le C\|\psi\|_p$$

where

$$
(6.17) \qquad \begin{aligned} &1 = t \cdot 0 + (1-t) \cdot \frac{d-1}{2}, \\ &\frac{1}{q} = \frac{t}{2} + \frac{1-t}{\tilde{q}}, \qquad \frac{1}{p} = \frac{t}{2} + \frac{1-t}{\tilde{p}}. \end{aligned}
$$

An inspection of (6.15) and (6.17) shows that (6.16), hence(6.8) holds for

$$
\frac{1}{q} = \frac{1}{p} - \frac{2}{d}, \quad \text{and} \quad \frac{1}{2} + \frac{2}{d} - \frac{1}{d-1} < \frac{1}{p} < \frac{1}{2} + \frac{1}{d-1}.
$$

In particular, (6.8) is true for any $p \in [p_1, p_2]$. The proof is then complete.

Acknowledgments. The work presented in this paper was done partly during the workshop “Schrödinger Operators with Magnetic Fields” at the Erwin Schrödinger Institute in Vienna, Austria. The author would like to thank T. Hoffmann-Ostenhof for his invitation and hospitality, and to the Institute for financial support. The author also wish to express thanks to I. Herbst for helpful conversations and suggestions, and to M. Birman and T. Suslina for making the English translation of their papers [1,2] available.

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