On Fundamental Solutions of Generalized Schrödinger Operators

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Abstract. We consider the generalized Schrödinger operator $-\Delta + \mu$ where μ is a nonnegative Radon measure in $\mathbb{R}^n$, $n \geq 3$. Assuming that μ satisfies certain scale-invariant Kato condition and doubling condition, we establish the following bounds for the fundamental solution of $-\Delta + \mu$ in $\mathbb{R}^n$:

$$\frac{c\, e^{-\varepsilon_2 d(x,y,\mu)}}{|x-y|^{n-2}} \leq \Gamma_\mu(x,y) \leq \frac{C\, e^{-\varepsilon_1 d(x,y,\mu)}}{|x-y|^{n-2}}$$

where $d(x,y,\mu)$ is the distance function for the modified Agmon metric $m(x,\mu)dx^2$ associated with μ. We also study the boundedness of the corresponding Riesz transforms $\nabla(-\Delta+\mu)^{-1/2}$ on $L^p(\mathbb{R}^n, dx)$.

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Introduction

Consider the generalized Schrödinger operator

$$-\Delta + \mu \qquad \text{in } \mathbb{R}^n,\ n \geq 3 \tag{0.1}$$

where μ is a nonnegative Radon measure on $\mathbb{R}^n$. The main purpose of this paper is to establish optimal upper and lower bounds for the fundamental solution of $-\Delta+\mu$ under suitable conditions on the measure μ. We will also study the boundedness of the operators $(-\Delta+\mu)^{i\gamma}$ $(\gamma \in \mathbb{R})$, and $\nabla(-\Delta+\mu)^{-1/2}$ on $L^p(\mathbb{R}^n, dx)$.

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Throughout this paper we assume that μ satisfies the following conditions: there exist positive constants C_0, C_1 and δ such that

$$\mu(B(x,r)) \le C_0 \left(\frac{r}{R}\right)^{n-2+\delta} \mu(B(x,R)), \tag{0.2}$$

$$\mu(B(x,2r)) \le C_1 \left\{\mu(B(x,r)) + r^{n-2}\right\} \tag{0.3}$$

for all $x \in \mathbb{R}^n$ and $0 < r < R$, where $B(x,r)$ denotes the (open) ball centered at x with radius r. It can be shown that (0.2) is equivalent to the condition

$$\int_{B(x,R)} \frac{d\mu(y)}{|y-x|^{n-2}} \le C \cdot \frac{\mu(B(x,R))}{R^{n-2}} \tag{0.4}$$

(see Remark 0.13). Thus (0.2) may be regarded as a scale-invariant Kato condition. Note that (0.3) merely says that the measure μ is doubling on balls satisfying $\mu(B(x,r)) \ge c\, r^{n-2}$. We will also assume that $\mu \ne 0$.

To state the main results, we first introduce the auxiliary function $m(x,\mu)$ defined by

$$\frac{1}{m(x,\mu)} = \sup\left\{ r > 0 : \quad \frac{\mu(B(x,r))}{r^{n-2}} \le C_1 \right\} \tag{0.5}$$

where C_1 is the constant in (0.3) (see [11]). With the modified Agmon metric

$$ds^2 = m(x,\mu)\{dx_1^2 + \cdots + dx_n^2\}, \tag{0.6}$$

we define the distance function

$$d(x,y,\mu) = \inf_\gamma \int_0^1 m(\gamma(t),\mu)|\gamma'(t)|dt \tag{0.7}$$

where $\gamma : [0,1] \to \mathbb{R}^n$ is absolute continuous and $\gamma(0) = x$, $\gamma(1) = y$ (see [8, 12]).

The following is one of the main results of the paper.

Theorem 0.8. *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n \ge 3$. Assume that μ satisfies the conditions (0.2)-(0.3). Let $\Gamma_\mu(x,y)$ denote the fundamental solution of $-\Delta + \mu$ in $\mathbb{R}^n$. Then we have*

$$\frac{c\, e^{-\varepsilon_2 d(x,y,\mu)}}{|x-y|^{n-2}} \le \Gamma_\mu(x,y) \le \frac{C\, e^{-\varepsilon_1 d(x,y,\mu)}}{|x-y|^{n-2}} \tag{0.9}$$

where C, c, ε_1 and ε_2 are positive constants depending only on n and constants C_0, C_1 and δ in (0.2)-(0.3).

A few remarks are in order.

Remark 0.10. If $d\mu = V(x)dx$ and $V \geq 0$ is in the reverse Hölder class $(RH)_{n/2}$, i.e.,

$$\left\{ \frac{1}{|B(x,r)|} \int_{B(x,r)} V(y)^{n/2} dy \right\}^{2/n} \leq C \left\{ \frac{1}{|B(x,r)|} \int_{B(x,r)} V(y) dy \right\}, \tag{0.11}$$

then μ satisfies the conditions (0.2)-(0.3) for some $\delta > 0$ [11]. However, in general, measures which satisfy (0.2)-(0.3) need not to be absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. Indeed, if $d\mu = d\sigma(x_1, x_2)dx_3 \cdots dx_4$ where σ is a doubling measure on $\mathbb{R}^2$, then μ satisfies (0.2)-(0.3) for some $\delta > 0$. Also, if σ is the surface measure on a Lipschitz graph $S = \left\{ (x', \varphi(x')) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1} \right\}$ and $\mu(E) \equiv \sigma(E \cap S)$, then μ satisfies (0.2)-(0.3) with $\delta = 1$.

Remark 0.12. In the case $n = 2$, it is possible to establish an essentially optimal upper bound of $\Gamma_\mu(x, y)$ under the assumptions (0.2)-(0.3). This was pointed out by M. Christ in [4], who derived upper bounds for certain distribution kernels associated with the $\overline{\partial}$ equation in a weighted L^2 space. Our study of $-\Delta + \mu$ with conditions (0.2)-(0.3) is partially motivated by [4]. In the case $n \geq 3$ and $d\mu = V(x)dx$, certain power-decay estimate of $\Gamma_\mu(x, y)$ was obtained by the author in [11] under the condition $V \in (RH)_{n/2}$. It played an important role in the study of the L^p boundedness of various operators associated with $-\Delta + V(x)$. Also see [13] for the case of magnetic Schrödinger operator $H(\mathbf{a}, V) = -(\nabla - i\mathbf{a})^2 + V$, and [10] for some results on the heat kernel of $H(\mathbf{a}, V)$.

Remark 0.13. It is easy to see that (0.2) implies (0.4). To see they are in fact equivalent, let

$$\phi(r) = \frac{\mu(B(x,r))}{r^{n-2}}.$$

By Fubini's Theorem, (0.4) implies

$$\int_0^R \phi(r) \, \frac{dr}{r} \leq C \, \phi(R).$$

It follows that

$$\begin{aligned}\phi(R) &\geq c\int_0^R \phi(r)\,\frac{dr}{r} \geq c_1\int_0^R \left\{\int_0^r \phi(t)\frac{dt}{t}\right\}\frac{dr}{r}\\ &= c_1\int_0^R \phi(t)\ln\left(\frac{R}{t}\right)\frac{dt}{t}\\ &\geq c_1\int_{r/2}^r \phi(t)\ln\left(\frac{R}{t}\right)\frac{dt}{t}\\ &\geq c_2\,\phi(r/2)\ln\left(\frac{R}{r}\right)\end{aligned}$$

where $0 < r < R$. Hence, if C is large, $\phi(R/C) \leq (1/2)\phi(R)$. From this (0.2) follows easily.

Our main tool in the proof of Theorem 0.8 is a new Poincaré inequality. It extends a Poincaré inequality in [4, Proposition 4.1] for the case $n = 2$.

Lemma 0.14. *(Poincaré inequality) Let μ be a nonnegative Radon measure satisfying (0.2). Let $B = B(x_0, R)$. Then, for any $\psi \in C^1(B)$,*

$$\int_B\int_B |\psi(x)-\psi(y)|^2 d\mu(y)\,dx \leq CR^2\mu(3B)\int_B |\nabla\psi(x)|^2 dx \tag{0.15}$$

where $3B = B(x_0, 3R)$.

Lemma 0.14 replaces the Fefferman-Phong inequality used in [11]. It allows us to bound the operator $-\Delta+\mu$ from below by $c\,m(x,\mu)^2$:

$$c\int_{\mathbb{R}^n} m(x,\mu)^2|\psi(x)|^2 dx \leq \int_{\mathbb{R}^n}|\nabla\psi|^2 dx + \int_{\mathbb{R}^n}|\psi|^2 d\mu \tag{0.16}$$

(see Theorem 1.13). This, together with Agmon's argument for exponential decay [1], gives the upper bound of $\Gamma_\mu(x,y)$ in (0.9). The lower bound follows from a (uniform) Harnack inequality for the nonnegative weak solutions of $(-\Delta+\mu)u = 0$.

We remark that, if, in (0.2), $\delta > 1$, it is possible to bound the first derivatives of $\Gamma_\mu(x,y)$ pointwise.

Theorem 0.17. *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n \geq 3$. Assume that μ satisfies the conditions (0.2)-(0.3) for some $\delta > 1$. Then there exist constants $C > 0$, $\varepsilon > 0$ such that*

$$|\nabla_x\Gamma_\mu(x,y)| \leq \frac{C\,e^{-\varepsilon d(x,y,\mu)}}{|x-y|^{n-1}}. \tag{0.18}$$

With the estimates on the fundamental solution at our disposal, in the second part of this paper, we study the boundedness of the operator $(-\Delta+\mu)^{i\gamma}$ ($\gamma \in \mathbb{R}$) and the Riesz transforms $\nabla(-\Delta+\mu)^{-1/2}$ on $L^p(\mathbb{R}^n, dx)$. Our basic argument will be similar to that in [11], where we assumed that $d\mu = V(x)dx$ and $V \in (RH)_{n/2}$.

Theorem 0.19. *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n \geq 3$. Suppose that μ satisfies the conditions (0.2)-(0.3) for some $\delta > 0$. Then, for $\gamma \in \mathbb{R}$, $(-\Delta+\mu)^{i\gamma}$ is a Calderón-Zygmund operator.*

Theorem 0.20. *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n \geq 3$. Suppose that μ satisfies (0.2)-(0.3) for some $\delta \in (0,1)$. Then*

$$\|\nabla(-\Delta+\mu)^{-1/2} f\|_p \leq C\,\|f\|_p \tag{0.21}$$

for $1 < p < (2-\delta)/(1-\delta)$. Furthermore, if μ satisfies (0.2)-(0.3) for some $\delta > 1$, then $\nabla(-\Delta+\mu)^{-1/2}$ and $\nabla(-\Delta+\mu)^{-1}\nabla$ are Calderón-Zygmund operators.

In (0.21) and thereafter, $\|\cdot\|_p$ denotes the norm in $L^p(\mathbb{R}^n, dx)$.

Remark 0.22. We recall that an operator T taking $C_c^\infty(\mathbb{R}^n)$ into $L^1_{\text{loc}}(\mathbb{R}^n, dx)$ is called a Calderón-Zygmund operator if (a) T extends to a bounded operator on $L^2(\mathbb{R}^n, dx)$, (b) there exists a kernel K such that, for every $f \in L_c^\infty(\mathbb{R}^n, dx)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,dy \quad \text{a.e. on} \quad \{\text{supp} f\}^c, \tag{0.23}$$

and, (c) the kernel K satisfies

$$\begin{cases} |K(x,y)| \leq \dfrac{C}{|x-y|^n}, \\ |K(x+h,y) - K(x,y)| \leq \dfrac{C\,|h|^\delta}{|x-y|^{n+\delta}}, \\ |K(x,y+h) - K(x,y)| \leq \dfrac{C\,|h|^\delta}{|x-y|^{n+\delta}} \end{cases} \tag{0.24}$$

for x, y, $h \in \mathbb{R}^n$ with $|h| \leq |x-y|/2$, and some $\delta > 0$. It is well known that Calderón-Zygmund operators are bounded on $L^p(\mathbb{R}^n, dx)$ for $1 < p < \infty$, and of weak-type (1,1) [14].

The paper is organized as follows. In section 1 we prove the Poincaré inequality (0.15), and then use it to establish the estimate (0.16). In section 2 we study the properties of weak solutions of the equation $(-\Delta+\mu)u=0$, using analytic method. We show the existence of the fundamental solution. Sections 3 and 4 are devoted to the proof of Theorem 0.8. In section 5 we give the proof of Theorem 0.17. Finally Theorems 0.19 and 0.20 are proved in sections 6 and 7 respectively.

Throughout this paper we use C, c and ε to denote positive constants, which may be different at each occurrence and which depend at most on the dimension n and constants in the assumptions (0.2)-(0.3).

1. The Poincaré Inequality

We begin with the

Proof of Lemma 0.14 (Poincaré inequality)

We shall adapt an argument found in [4].

Let $B=B(x_0,R)$ and $\psi\in C^1(B)$. For x, $y\in B$, we write

$$\psi(x)-\psi(y)=\int_0^{|x-y|}\nabla\psi(y+t\frac{x-y}{|x-y|})\cdot\frac{x-y}{|x-y|}\,dt.$$

Thus, it suffices to show that

$$\int_B\int_B\Big|\int_0^{|x-y|}f(y+t\frac{x-y}{|x-y|})dt\Big|^2d\mu(y)dx\le CR^2\mu(3B)\int_B|f(x)|^2dx \tag{1.1}$$

for any $f\in C(B)$.

To this end, we let

$$T_jf(x,y)=\int_{\substack{0<t<|x-y|\\ 2^{-j}R\le t<2^{-j+1}R}}f(y+t\frac{x-y}{|x-y|})dt,\qquad j=0,1,2,\dots. \tag{1.2}$$

Clearly,

$$\|T_jf\|_{L^\infty(B\times B,\,d\mu(y)dx)}\le 2^{-j}R\,\|f\|_{L^\infty(B,\,dx)}. \tag{1.3}$$

Also, note that

$$
\begin{aligned}
&\int_B \int_B |T_j f(x,y)| d\mu(y) dx \\
&\qquad \le \int_B d\mu(y) \int_0^{2R} r^{n-1} dr \int_{S^{n-1}} d\sigma \int_{\substack{2^{-j}R \le t \le 2^{-j+1}R \\ y+t\sigma \in B}} |f(y+t\sigma)| dt \\
&\qquad \le CR^n \int_B d\mu(y) \int_{S^{n-1}} d\sigma \int_{\substack{2^{-j}R \le t \le 2^{-j+1}R \\ y+t\sigma \in B}} |f(y+t\sigma)| dt \\
&\qquad \le \frac{CR^n}{(2^{-j}R)^{n-1}} \int_B d\mu(y) \int_{B \cap B(y, 2^{-j+1}R)} |f(z)|\, dz \\
&\qquad \le \frac{CR^n}{(2^{-j}R)^{n-1}} \int_B |f(z)|\, \mu(B \cap B(z, 2^{-j+1}R))\, dz \\
&\qquad \le CR \cdot 2^{-j(\delta-1)} \cdot \mu(3B) \|f\|_{L^1(B,dx)}
\end{aligned}
$$

where we have used (0.2) in the last inequality. This, together with (1.3), yields that

$$\|T_j f\|_{L^2(B \times B, d\mu(y)dx)} \le C(\sqrt{2})^{-j\delta} R \cdot \mu(3B)^{1/2} \|f\|_{L^2(B,dx)}$$

by interpolation. By summation, we have

$$\sum_{j=0}^{\infty} \|T_j f\|_{L^2(B \times B, d\mu(y)dx)} \le CR \cdot \mu(3B)^{1/2} \|f\|_{L^2(B,dx)}.$$

(1.1) now follows by Minkowski's inequality. The proof is complete.

It follows from Poincaré inequality (0.15) that

$$\int_B |\psi(x)|^2 d\mu(x) \le C \left\{ \frac{\mu(3B)}{R^{n-2}} \int_B |\nabla \psi(x)|^2 dx + \frac{\mu(B)}{R^n} \int_B |\psi(x)|^2 dx \right\} \tag{1.4}$$

for any $\psi \in C^1(B)$. This implies that $W^{1,2}(B)$ is imbedded in $L^2(B, d\mu)$ (the imbedding is in fact compact, see Lemma 2.24). One may also deduce (1.4) from a general result on the trace inequality by Kerman and Sawyer [9].

To define the operator $-\Delta + \mu$, we consider the quadratic form

$$q[\phi, \psi] = \int_{\mathbb{R}^n} < \nabla\phi, \nabla\psi > dx + \int_{\mathbb{R}^n} < \phi, \psi > d\mu \tag{1.5}$$

with domain $W^{1,2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, d\mu)$. Clearly, $q[\,,\,]$ is a semibound, symmetric closed form. Hence there exists a unique self-adjoint operator, which we call $-\Delta + \mu$, such that

$$q[\phi, \psi] =< (-\Delta + \mu)\phi, \psi >_{L^2(\mathbb{R}^n, dx)}$$

for any $\phi \in \text{Domain}(-\Delta+\mu)$ and $\psi \in W^{1,2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, d\mu)$. Moreover,

$$\begin{aligned} &\text{Domain}(-\Delta+\mu) \\ &= \left\{\phi \in W^{1,2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, d\mu) : \quad (-\Delta+\mu)\phi \in L^2(\mathbb{R}^n, dx)\right\}. \end{aligned} \tag{1.6}$$

Proposition 1.7. *Suppose μ satisfies (0.2). Then $C_0^\infty(\mathbb{R}^n)$ is form core for $q[\,,\,]$, i.e., $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, d\mu)$.*

Proof. Given $f \in W^{1,2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, d\mu)$, we need to approximate it by functions in $C_0^\infty(\mathbb{R}^n)$. By multiplying a smooth cut-off function to f, we may assume that f has compact support. Now suppose that $\text{supp}(f) \subset B$. Then $f \in W_0^{1,2}(B)$ and there exists a sequence $\{f_j\} \subset C_0^\infty(B)$ such that $f_j \to f$ in $W_0^{1,2}(B)$. Since $W_0^{1,2}(B) \subset L^2(B, d\mu)$, we also have $f_j \to f$ in $L^2(B, d\mu)$. Thus $f_j \to f$ in $W^{1,2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, d\mu)$.

Finally in this section we give the proof of the inequality (0.16). We will need some properties of the auxiliary function $m(x, \mu)$ defined by (0.5).

Proposition 1.8. *Suppose μ satisfies (0.2)-(0.3). Then*

(a) $0 < m(x, \mu) < \infty$ *for every* $x \in \mathbb{R}^n$,

(b) *if* $r = \frac{1}{m(x,\mu)}$, *then* $r^{n-2} \le \mu(B(x, r)) \le C_1 r^{n-2}$,

(c) *if* $|x - y| \le \frac{C}{m(x,\mu)}$, *then* $m(x, \mu) \approx m(y, \mu)$,

(d) *there exists* $k_0 > 0$ *such that, for* x, $y \in \mathbb{R}^n$,

$$m(x, \mu) \le C m(y, \mu) \left\{1 + |x - y| m(y, \mu)\right\}^{k_0}.$$

Proof. Note that, by (0.2),

$$\lim_{r\to 0} \frac{\mu(B(x, r))}{r^{n-2}} = 0 \quad \text{and} \quad \lim_{r\to\infty} \frac{\mu(B(x, r))}{r^{n-2}} = \infty.$$

Part (a) follows by definition.

To see part (b), let $r = 1/m(x, \mu)$. By definition,

$$\mu(B(x, r)) = \lim_{t\to r^-} \mu(B(x, t)) \le C_1 r^{n-2}.$$

Also, by definition, $\mu(B(x,2r)) > C_1(2r)^{n-2}$. It follows from (0.3) that

$$C_1(2r)^{n-2} \le C_1 \left\{ \mu(B(x,r)) + r^{n-2} \right\}.$$

This gives $\mu(B(x,r)) \ge r^{n-2}$.

To prove part (c), we again let $r = 1/m(x,\mu)$. Suppose that $|x-y| < C\,r$. Then $B(y,r) \subset B(x,(C+1)r)$. Using (0.3) and part (b), we have $\mu(B(x,(C+1)r)) \le C\,r^{n-2}$. Hence, by (0.2),

$$\begin{aligned}\frac{\mu(B(y,tr))}{(tr)^{n-2}} &\le C\,t^\delta\,\frac{\mu(B(y,r))}{r^{n-2}} \\ &\le C\,t^\delta\,\frac{\mu(B(x,(C+1)r))}{r^{n-2}} \le Ct^\delta \le 1\end{aligned}$$

if t is small. Thus, by definition,

$$\frac{1}{m(y,\mu)} \ge t\,r.$$

We obtain $m(y,\mu) \le C\,m(x,\mu)$. Since this implies that

$$|x-y| < \frac{C'}{m(y,\mu)},$$

we also obtain $m(x,\mu) \le C\,m(y,\mu)$ by reversing the roles of x and y.

Finally we prove part (d). We may assume $|x-y|m(y,\mu) \ge 1$. For otherwise it follows from part (c).

Let $R = 1/m(y,\mu)$. Suppose that

$$2^{j-1}R \le |x-y| < 2^j R \qquad \text{for some } j \ge 1.$$

Note that $B(x,R) \subset B(y,(2^j+1)R)$. By (0.3), we have

$$\mu(B(x,R)) \le \left\{C_1 + 2^{n-2}\right\}^{j+2} R^{n-2}$$

where we also used $\mu(B(y,R)) \le C_1 R^{n-2}$.

Now, using (0,2), we get

$$\frac{\mu(x,tR)}{(tR)^{n-2}} \le C\,t^\delta\,\frac{\mu(B(x,R))}{R^{n-2}} \le C\,t^\delta\,C^j.$$

Choosing t such that $C\,t^{\delta}\,C^{j}=1/2$. Then

$$\frac{1}{m(x,\mu)} \geq t\,R.$$

Thus

$$m(x,\mu) \leq m(y,\mu)\cdot\frac{1}{t} \leq C\,m(y,\mu)\,\{1+|x-y|m(y,\mu)\}^{k_0}$$

for some $k_0>0$.

Remark 1.9. It follows easily from parts (c) and (d) of Proposition 1.8 that, for $x,\ y\in\mathbb{R}^n$,

$$1+|x-y|m(x,\mu) \leq C\,\{1+|x-y|m(y,\mu)\}^{k_0+1} \tag{1.10}$$

and

$$m(y,\mu) \geq \frac{c\,m(x,\mu)}{\{1+|x-y|m(x,\mu)\}^{\frac{k_0}{k_0+1}}}. \tag{1.11}$$

Also, by (0.3) and part (b) of Proposition 1.8, there exists $k_1>0$ such that

$$\frac{\mu(B(x,R))}{R^{n-2}} \leq C\,\{1+R\,m(x,\mu)\}^{k_1} \tag{1.12}$$

for all $x\in\mathbb{R}^n$ and $R>0$.

We are now ready to prove (0.16).

Theorem 1.13. *Suppose that μ satisfies (0.2)-(0.3). Assume that $u\in W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ and $\nabla u\in L^2(\mathbb{R}^n,dx)$.*

(a) If $u\in L^2(\mathbb{R}^n,d\mu)$, then $m(\cdot,\mu)u\in L^2(\mathbb{R}^n,dx)$ and

$$\int_{\mathbb{R}^n}|u(x)|^2 m(x,\mu)^2 dx \leq C\left\{\int_{\mathbb{R}^n}|\nabla u|^2 dx+\int_{\mathbb{R}^n}|u|^2 d\mu\right\}.$$

(b) If $m(\cdot,\mu)u\in L^2(\mathbb{R}^n,dx)$, then $u\in L^2(\mathbb{R}^n,d\mu)$ and

$$\int_{\mathbb{R}^n}|u|^2 d\mu \leq C\left\{\int_{\mathbb{R}^n}|\nabla u|^2 dx+\int_{\mathbb{R}^n}|u(x)|^2 m(x,\mu)^2 dx\right\}.$$

Proof. We will only prove part (a). Part (b) may be shown in the same manner.

Let $B = B(y, R)$ where $R = 1/m(y, \mu)$. It follows from Poincaré inequality (0.15) that

$$\mu(B) \int_B |u(x)|^2 dx \le CR^2 \mu(3B) \int_B |\nabla u(x)|^2 dx + 2|B| \int_B |u(x)|^2 d\mu(x). \tag{1.14}$$

Note that, by part (b) of Proposition 1.8,

$$\mu(B) \ge R^{n-2} \quad \text{and} \quad \mu(3B) \le C \left\{ \mu(B) + R^{n-2} \right\} \le CR^{n-2}.$$

Hence, (1.14) gives

$$\frac{1}{R^{n+2}} \int_B |u(x)|^2 dx \le \frac{C}{R^n} \int_B |\nabla u(x)|^2 dx + \frac{C}{R^n} \int_B |u(x)|^2 d\mu(x). \tag{1.15}$$

Recall that $B = B(y, R)$ and $R = 1/m(y, \mu)$. We integrate both sides of (1.15) with respect to y over $\mathbb{R}^n$. By Fubini's Theorem, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |u(x)|^2 dx \int_{|x-y|<\frac{1}{m(y,\mu)}} &\{m(y, \mu)\}^{n+2} dy \\ &\le C \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \int_{|x-y|<\frac{1}{m(y,\mu)}} \{m(y, \mu)\}^n dy \\ &\quad + C \int_{\mathbb{R}^n} |u(x)|^2 d\mu(x) \int_{|x-y|<\frac{1}{m(y,\mu)}} \{m(y, \mu)\}^n dy. \end{aligned}$$

The inequality in part (a) now follows, since part (c) of Proposition 1.8 yields

$$\int_{|x-y|<\frac{1}{m(y,\mu)}} \{m(y, \mu)\}^{n+2} dy \ge c\{m(x, \mu)\}^{n+2} \int_{|x-y|<\frac{c}{m(x,\mu)}} dy \ge c \{m(x, \mu)\}^2$$

and

$$\int_{|x-y|<\frac{1}{m(y,\mu)}} \{m(y, \mu)\}^n dy \le C\{m(x, \mu)\}^n \int_{|x-y|<\frac{C}{m(x,\mu)}} dy \le C.$$

Remark 1.16. Let

$$\mathcal{H} = \left\{ u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) : \ \nabla u \in L^2(\mathbb{R}^n, dx) \text{ and } \ m(\cdot, \mu)|u| \in L^2(\mathbb{R}^n, dx) \right\} \tag{1.17}$$

equipped with the norm

$$\|u\|_{\mathcal{H}} = \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx + \int_{\mathbb{R}^n} m(x, \mu)^2 |u(x)|^2 dx. \tag{1.18}$$

Then $\mathcal{H}$ is a Hilbert space. Using (1.11), one may show that $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{H}$. By Theorem 1.13, we have

$$\begin{aligned} \mathcal{H} &= \left\{ u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) : \ \nabla u \in L^2(\mathbb{R}^n, dx) \ \text{ and } \ u \in L^2(\mathbb{R}^n, d\mu) \right\} \\ &= \text{Domain}((-\Delta + \mu)^{1/2}). \end{aligned} \tag{1.19}$$

2. The Fundamental Solution of $-\Delta + \mu$

Definition 2.1. Let Ω be an open set in $\mathbb{R}^n$. Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $f \in L^1_{\mathrm{loc}}(\Omega, dx)$. u is called a weak solution of $(-\Delta + \mu)u = f$ in Ω if

$$\int_\Omega < \nabla u, \nabla \psi > dx + \int_\Omega < u, \psi > d\mu = \int_\Omega < f, \psi > dx \tag{2.2}$$

for any $\psi \in C^1_0(\Omega)$.

Using Remark 1.16 and the Lax-Milgram Theorem, we may deduce the following.

Proposition 2.3. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n, dx)$. Assume $m(\cdot, \mu)^{-1} f \in L^2(\mathbb{R}^n, dx)$. Then $(-\Delta + \mu)u = f$ in $\mathbb{R}^n$ has a unique weak solution u_f in $\mathcal{H}$.*

Next we will show that there exists a unique kernel function $\Gamma_\mu(x, y)$ such that the solution u_f in Proposition 2.3 is by

$$u_f(x) = \int_{\mathbb{R}^n} \Gamma_\mu(x, y) f(y) dy \tag{2.4}$$

for any $f \in L^2_c(\mathbb{R}^n, dx)$. We shall call $\Gamma_\mu(x, y)$ the fundamental solution of $-\Delta + \mu$ in $\mathbb{R}^n$.

The proof of (2.4) is fairly standard. We include it for the sake of completeness. We remark that the fundamental solution as well as the Harnack inequality for the operator $-\Delta + \mu$ with a positive measure was investigated in [5] by a probabilistic method.

Definition 2.5. Let Ω be an open set in $\mathbb{R}^n$ and $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. u is called a subsolution of $(-\Delta + \mu)u = 0$ in Ω if

$$\int_\Omega < \nabla u, \nabla \psi > dx + \int_\Omega < u, \psi > d\mu \leq 0 \tag{2.6}$$

for any nonnegative function ψ in $C^1_0(\Omega)$.

Lemma 2.7. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a subsolution of $(-\Delta + \mu)u = 0$ in Ω. Then $u^+ = \max(u, 0)$ is subharmonic in Ω.*

Proof. We need to show that

$$\int_\Omega < \nabla u^+, \nabla \phi > dx \leq 0 \quad \text{for any } \phi \in C^1_0(\Omega), \ \ \phi \geq 0. \tag{2.8}$$

Note that $\nabla u^+ = \nabla u$ on $\{x \in \Omega : \ u(x) > 0\}$, and $\nabla u^+ = 0$ on $\{x \in \Omega : \ u(x) \le 0\}$. (2.8) follows by taking

$$\psi = \psi_\varepsilon = \phi \cdot \frac{u^+}{u^+ + \varepsilon}$$

in (2.6) and letting $\varepsilon \to 0$. We omit details.

Lemma 2.9. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a weak solution of $(-\Delta + \mu)u = 0$ in Ω. Then u^+ and $|u|$ are subharmonic in Ω. Hence,*

$$|u(x)| \le \frac{1}{|B|} \int_B |u(y)|\, dy \qquad \text{if } \ B = B(x,R) \subset \Omega. \tag{2.10}$$

Proof. Since u and $-u$ are subsolutions, u^+ and $u^- = (-u)^+$ are subharmonic in Ω by Lemma 2.7. It follows that $|u| = u^+ + u^-$ is also subharmonic.

Lemma 2.11. *Let $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$. Suppose that $f \ge 0$ and u is a weak solution of $(-\Delta + \mu)u = f$ in $\mathbb{R}^n$. Also assume that*

$$\lim_{R\to\infty} \sup_{|x|=R} \frac{1}{R^n} \int_{B(x,R/2)} |u(y)|\, dy = 0. \tag{2.12}$$

Then $u \ge 0$ in $\mathbb{R}^n$.

Proof. Since $f \ge 0$, $-u$ is a subsolution in $\mathbb{R}^n$. By Lemma 2.7, $u^- = (-u)^+$ is subharmonic in $\mathbb{R}^n$. It follows from the maximal principle that

$$\sup_{\Omega_R} u^- \le \sup_{\partial\Omega_R} u^-$$

where $\Omega_R = B(0,R)$. Note that, for $x \in \partial\Omega_R$,

$$u^-(x) \le \frac{1}{|B(x,R/2)|} \int_{B(x,R/2)} u^-(y) dy \le \frac{1}{B(x,R/2)} \int_{B(x,R/2)} |u(y)| dy.$$

Hence, by (2.12), $\sup_{\partial\Omega_R} u^- \to 0$ as $R \to \infty$. This implies that $u^- \equiv 0$. So $u = u^+ \ge 0$.

Lemma 2.13. *Let $f \in L^2_c(\mathbb{R}^n, dx)$ and $f \ge 0$. Let $u = u_f$ be the solution of $(-\Delta + \mu)u = f$ in $\mathbb{R}^n$ given by Proposition 2.3. Then*

$$0 \le u(x) \le \int_{\mathbb{R}^n} \Gamma_0(x,y) f(y) dy \tag{2.14}$$

where $\Gamma_0(x,y)=c_n|x-y|^{2-n}$ *is the fundamental solution of* $-\Delta$ *in* $\mathbb{R}^n$.

Proof. First, note that, by (1.11),

$$\frac{1}{R^2}\int_{R/2\le|x|\le 2R}|u|^2dx \le C_\mu \int_{R/2\le|x|\le 2R} m(x,\mu)^2|u|^2dx \le C_\mu\,\|u\|^2_{\mathcal{H}}. \tag{2.15}$$

Hence u satisfies the condition (2.12). It follows that $u\ge 0$ in $\mathbb{R}^n$.

To prove the remaining inequality in (2.14), let

$$v(x)=\int_{\mathbb{R}^n}\Gamma_0(x,y)f(y)dy.$$

Then $v\in W^{2,2}_{\rm loc}(\mathbb{R}^n)$, $-\Delta v=f$ in $\mathbb{R}^n$ and $v\ge 0$. Furthermore, $u-v$ is a subsolution of $(-\Delta+\mu)u=0$ in $\mathbb{R}^n$. Hence, by Lemma 2.7, $(u-v)^+$ is subharmonic in $\mathbb{R}^n$. This implies that

$$\sup_{\Omega_R}(u-v)^+\le \sup_{\partial\Omega_R}(u-v)^+\le \frac{C}{R^n}\int_{R/2\le|x|\le 2R}(|u|+|v|)dx$$

where $\Omega_R=B(0,R)$. It follows from (2.15) and $v(x)=O(|x|^{2-n})$ as $|x|\to\infty$ that $(u-v)^+=0$ in $\mathbb{R}^n$. Hence $u\le v$ in $\mathbb{R}^n$.

The following theorem is an easy consequence of Lemma 2.13. We omit its proof.

Theorem 2.16. *For each* $x\in\mathbb{R}^n$, *there exists a unique* $\Gamma_\mu(x,\cdot)\in L^p_{\rm loc}(\mathbb{R}^n,dy)$ *(*$1<p<n/(n-2)$*) such that, for any* $f\in L^2_c(\mathbb{R}^n,dx)$, *the unique solution* $u_f\in\mathcal{H}$ *of* $(-\Delta+\mu)u=f$ *in Proposition 2.3 is given by*

$$u_f(x)=\int_{\mathbb{R}^n}\Gamma_\mu(x,y)f(y)dy.$$

Moreover, we have

$$0\le\Gamma_\mu(x,y)\le\frac{c_n}{|x-y|^{n-2}}\qquad \textit{for any}\ \ x,\,y\in\mathbb{R}^n. \tag{2.17}$$

The rest of this section is devoted to the proof of the following theorem.

Theorem 2.18. *Let* $\Gamma_\mu(x,y)$ *be the fundamental solution of* $-\Delta+\mu$ *in* $\mathbb{R}^n$ *given by Theorem 2.16. Then* $\Gamma_\mu(x,\cdot)$ *is a weak solution of* $(-\Delta+\mu)u=0$ *in* $\mathbb{R}^n\setminus\{x\}$ *and*

$$\Gamma_\mu(x,y)=\Gamma_\mu(y,x)\qquad \textit{for any}\ \ x\ne y. \tag{2.19}$$

We need two lemmas before we carry out the proof of Theorem 2.18.

The first lemma shows that the weak solution of $(-\Delta + \mu)u = 0$ is Hölder continuous (see Lemma 6.1 for a stronger version).

Lemma 2.20. *Assume that μ satisfies (0.2)-(0.3) for some $\delta \in (0,1)$. Let $B = B(x_0, R)$. Suppose that $u \in W^{1,2}(B)$ is a weak solution of $(-\Delta + \mu)u = 0$ in B. Then*

$$|u(x) - u(y)| \leq C \left(\frac{|x-y|}{R} \right)^{\delta} \left\{ 1 + \frac{\mu(B)}{R^{n-2}} \right\} \sup_B |u| \tag{2.21}$$

for any x, $y \in B(x_0, R/2)$.

Proof. We may assume that $\sup_B |u| < \infty$.

Let

$$v(x) = u(x) + \int_B \Gamma_0(x,z) u(z) d\mu(z). \tag{2.22}$$

We claim that v is harmonic in B.

Assume the claim for a moment. It follows that, for x, $y \in B(x_0, R/2)$,

$$\begin{aligned} |v(x) - v(y)| &\leq C \left(\frac{|x-y|}{R} \right) \sup_B |v| \\ &\leq C \left(\frac{|x-y|}{R} \right) \sup_B |u| \left\{ 1 + \sup_{\xi \in B} \int_B \frac{d\mu(z)}{|\xi - z|^{n-2}} \right\} \\ &\leq C \left(\frac{|x-y|}{R} \right) \sup_B |u| \left\{ 1 + \frac{\mu(B)}{R^{n-2}} \right\} \end{aligned}$$

where we used (0.4) and (0.3) in the last inequality. Thus it suffices to estimate $|W(x) - W(y)|$ where

$$W(x) = \int_B \Gamma_0(x,z) u(z) d\mu(z).$$

To this end, let $r = |x-y| < R$. Note that

$$\begin{aligned} |W(x) - W(y)| &\leq \sup_B |u| \cdot c_n \int_B \left| \frac{1}{|x-z|^{n-2}} - \frac{1}{|y-z|^{n-2}} \right| d\mu(z) \\ &= c_n \sup_B |u| \left\{ I_1 + I_2 + I_3 \right\} \end{aligned}$$

where I_1, I_2, I_3 represent the integrals over $B \cap B(x, 2r)$, $B \cap B(y, 2r)$ and

$$\{z \in B : \ |z-y| \geq 2r, \ |z-x| \geq 2r\}$$

respectively. Clearly, by (0.4) and (0.3),

$$\begin{aligned} I_1 &\leq C \int_{B(x,2r)} \frac{d\mu(z)}{|z-x|^{n-2}} + C \int_{B(y,3r)} \frac{d\mu(z)}{|z-y|^{n-2}} \\ &\leq C \left(\frac{r}{R}\right)^{\delta} \left\{1 + \frac{\mu(B)}{R^{n-2}}\right\}. \end{aligned}$$

I_2 can be handled in the same manner.

For I_3, we have

$$\begin{aligned} I_3 &\leq C\, r \int_{r \leq |z-x| < 2R} \frac{d\mu(z)}{|z-x|^{n-1}} \\ &\leq C\, r \int_r^{2R} \frac{\mu(B(x,t))}{t^{n-1}} \cdot \frac{dt}{t} \\ &\leq C\, r \int_r^{2R} \left(\frac{t}{R}\right)^{\delta} \frac{dt}{t^2} \cdot \frac{\mu(B(x,2R))}{R^{n-2}} \\ &\leq C \left(\frac{r}{R}\right)^{\delta} \left\{1 + \frac{\mu(B)}{R^{n-2}}\right\} \end{aligned}$$

where we have used (0.2) in the third inequality. The desired estimate for W now follows.

It remains to show that v is harmonic in B.

Let $g \in C(B)$. Note that

$$\begin{aligned} &\left| \int_B g(x)dx \int_B \frac{d\mu(y)}{|x-y|^{n-1}} \right| = \left| \int_B d\mu(y) \int_B \frac{g(x)dx}{|x-y|^{n-1}} \right| \\ &\leq \int_B |f(y)| d\mu(y) \leq \{\mu(B)\}^{1/2} \left\{ \int_B |f(y)|^2 d\mu(y) \right\}^{1/2} \end{aligned} \tag{2.23}$$

where

$$f(y) = \int_B \frac{g(x)dx}{|x-y|^{n-1}}.$$

By (1.4) and the well-known estimates for fractional integrals [14], we have

$$\int_B |f(y)|^2 d\mu(y) \leq C_B \int_B |g(x)|^2 dx.$$

This, together with (2.23), implies that

$$\int_B \frac{d\mu(y)}{|x-y|^{n-1}} \in L^2(B, dx).$$

It follows that $v \in W^{1,2}(B)$. Finally, observe that, for any $\psi \in C_0^1(B)$,

$$\int_B < \nabla v, \nabla \psi > dx = \int_B < \nabla u, \nabla \psi > dx + \int_B < \nabla W, \nabla \psi > dx$$
$$= \int_B < \nabla u, \nabla \psi > dx + \int_B u(y) d\mu(y) \int_B < \nabla_x \Gamma_0(x,y), \nabla_x \psi(x) > dx$$
$$= \int_B < \nabla u, \nabla \psi > dx + \int_b < u, \psi > d\mu(y) = 0$$

since u is a weak solution of $(-\Delta + \mu)u = 0$ in B. We conclude that v is harmonic in B. The proof is thus complete.

Lemma 2.24. *Let B be a ball in $\mathbb{R}^n$. Then $W^{1,2}(B)$ is compactly imbedded in $L^2(B, d\mu)$.*

Proof. Let Q be a closed cube containing B. It suffices to show that $W^{1,2}(Q)$ is compactly imbedded in $L^2(Q, d\mu)$.

Let R be the side length of Q. We divide Q into a finite number of closed subcubes $\{Q_j\}$ of equal size. Let r denote the side length of Q_j.

Note that the inequality (1.4) still holds if we replace the ball B by a closed cube (the proof is the same). We apply (1.4) on each subcube Q_j to obtain

$$\int_{Q_j} |\psi(x)|^2 d\mu(x) \le C \left\{ \frac{\mu(3Q_j)}{r^{n-2}} \int_{Q_j} |\nabla \psi(x)|^2 dx + \frac{\mu(Q_j)}{r^n} \int_{Q_j} |\psi(x)|^2 dx \right\}$$

where $\psi \in C^1(Q)$. Summing in j and using (0.2), we get

$$\int_Q |\psi|^2 d\mu \le C \frac{\mu(3Q)}{R^{n-2}} \left\{ \left(\frac{r}{R}\right)^\delta \int_Q |\nabla \psi|^2 dx + \frac{1}{r^2} \left(\frac{r}{R}\right)^\delta \int_Q |\psi|^2 dx \right\}. \tag{2.25}$$

By choosing r small, we see that, for any $\epsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\int_Q |u|^2 d\mu \le C \frac{\mu(3Q)}{R^{n-2}} \left\{ \varepsilon \int_Q |\nabla u|^2 dx + C_\varepsilon \int_Q |u|^2 dx \right\} \tag{2.26}$$

for all $u \in W^{1,2}(Q)$. The compactness of the imbedding follows easily from (2.26).

We are now in a position to give the

Proof of Theorem 2.18.

Fix x_0, $y_0 \in \mathbb{R}^n$ such that $x_0 \neq y_0$. Let $r = |x_0 - y_0|$ and

$$f = \chi_{B(t,\varepsilon_1)}, \qquad g = \chi_{B(s,\varepsilon_2)}$$

where $t \in B(x_0, r/8)$, $s \in B(y_0, r/8)$ and $0 < \varepsilon_1$, $\varepsilon_2 < r/16$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} < f, u_g > dx &= \int_{\mathbb{R}^n} < \nabla u_f, \nabla u_g > dx + \int_{\mathbb{R}^n} < u_f, u_g > d\mu \\ &= \int_{\mathbb{R}^n} < u_f, g > dx\end{aligned}$$

where u_f and u_g are weak solutions, given by Proposition 2.3, with right hand sides f and g respectively. It follows from Theorem 2.16 that

$$\int_{B(t,\varepsilon_1)} dx \int_{B(s,\varepsilon_2)} \Gamma_\mu(x,y)dy = \int_{B(s,\varepsilon_2)} dy \int_{B(t,\varepsilon_1)} \Gamma_\mu(y,x)dx. \tag{2.27}$$

Since $\int_{B(s,\varepsilon_2)} \Gamma_\mu(x,y)dy$ is a weak solution of $(-\Delta + \mu)u = 0$ in $B(x_0, r/2)$, it is continuous in $B(x_0, r/4)$ by Lemma 2.20. Thus, dividing both sides of (2.27) by $|B(t, \varepsilon_1)|$ and letting $\varepsilon_1 \to 0$, we obtain

$$\int_{B(s,\varepsilon_2)} \Gamma_\mu(t,y)dy = \lim_{\varepsilon_1 \to 0} \int_{B(s,\varepsilon_2)} dy \frac{1}{|B(t,\varepsilon_1)|} \int_{B(t,\varepsilon_1)} \Gamma_\mu(y,x)dx. \tag{2.28}$$

Now consider

$$u_{t,\varepsilon_1}(y) = \frac{1}{|B(t,\varepsilon_1)|} \int_{B(t,\varepsilon_1)} \Gamma_\mu(y,x)dx. \tag{2.29}$$

u_{t,ε_1} is a nonnegative weak solution of $(-\Delta + \mu)u = 0$ in $B(y_0, r/2)$. Since u_{t,ε_1} is subharmonic in $B(y_0, r/2)$ by Lemma 2.7, we have Caccioppoli's inequality

$$\int_{B(y_0,r/4)} |\nabla u_{t,\varepsilon_1}|^2 dy \le \frac{C}{r^2} \int_{B(y_0,r/2)} |u_{t,\varepsilon_1}|^2 dy.$$

This, together with the size estimate (2.17), implies that $\|u_{t,\varepsilon_1}\|_{W^{1,2}(B(y_0,r/2))} \le C_r$ where C_r does not depend on t and ε_1. Hence there exist a sequence $\{\varepsilon_{1,j}\}$ and $u_t \in W^{1,2}(B(y_0, r/4))$ such that, as $j \to \infty$, $\varepsilon_{1,j} \to 0$, $u_{t,\varepsilon_{1,j}} \to u_t$ weakly in $W^{1,2}(B(y_0, r/4))$ and $u_{t,\varepsilon_{1,j}} \to u_t$ strongly in $L^2(B(y_0, r/4), dy)$. Using Lemma

2.24, we may verify easily that u_t is also a weak solution of $(-\Delta+\mu)u=0$ in $B(y_0,r/4)$. By (2.28), we have

$$\int_{B(s,\varepsilon_2)} \Gamma_\mu(t,y)dy = \int_{B(s,\varepsilon_2)} u_t(y)dy.$$

Since $s\in B(y_0,r/4)$ and $0<\varepsilon_2<r/16$ are arbitrary, we conclude that $\Gamma_\mu(t,y)=u_t(y)$ for any $y\in B(y_0,r/8)$. This implies that $\Gamma_\mu(x_0,y)$ is a weak solution of $(-\Delta+\mu)u=0$ in $\mathbb{R}^n\setminus\{x_0\}$. In particular, it is continuous in $\mathbb{R}^n\setminus\{x_0\}$.

Finally, since $\Gamma_\mu(y_0,\cdot)$ is continuous in $\mathbb{R}^n\setminus\{y_0\}$, in view of (2.29), we have $u_{x_0,\varepsilon_1}(y_0)\to\Gamma_\mu(y_0,x_0)$ as $\varepsilon_1\to 0$. Thus $\Gamma_\mu(x_0,y_0)=u_{x_0}(y_0)=\Gamma_\mu(y_0,x_0)$.

3. The Upper Bound of $\Gamma_\mu(x,y)$

In this section we use the operator bound (0.14) and Agmon's argument to establish the upper bound of $\Gamma_\mu(x,y)$ in the estimate (0.9).

Lemma 3.1. *Let $u\in W^{1,2}_{\text{loc}}(\mathbb{R}^n\setminus B)$ be a weak solution of $(-\Delta+\mu)u=0$ in $\mathbb{R}^n\setminus B$, and $\phi\in C_0^\infty(\mathbb{R}^n)$ such that $\phi=0$ on $2B$. Let $g\in C^1(\mathbb{R}^n)$ be a nonnegative function satisfying $|\nabla g(x)|\le C_2 m(x,\mu)$ for every $x\in\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} m(x,\mu)^2|u\phi|^2 e^{2\varepsilon g}dx \le C\int_{\mathbb{R}^n}|u|^2|\nabla\phi|^2 e^{2\varepsilon g}dx \tag{3.2}$$

for any $\varepsilon\in(0,\varepsilon_0)$, where ε_0 depends only on C_0, C_1, C_2 and n.

Proof. We may assume that u and ϕ are real-valued functions.

Let $\psi=\psi_\varepsilon=u\phi e^{\varepsilon g}$. Since ψ is a function in $W^{1,2}(\mathbb{R}^n)$ with compact support, we may apply Theorem 1.13 to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} m(x,\mu)^2|u\phi|^2e^{2\varepsilon g}dx &= \int_{\mathbb{R}^n} m(x,\mu)^2|\psi|^2dx\\ &\le C\left\{\int_{\mathbb{R}^n}|\nabla\psi|^2dx+\int_{\mathbb{R}^n}|\psi|^2d\mu\right\}.\end{aligned}$$

Let $f=\phi e^{\varepsilon g}$. Note that $\psi=uf$ and

$$\begin{aligned}|\nabla\psi|^2 &=<\nabla\psi,\nabla\psi>\\ &=<\nabla u,\nabla u>|f|^2+2<\nabla u,\nabla f>uf+|u|^2|\nabla f|^2\\ &=<\nabla u,\nabla(u|f|^2)>+|u|^2|\nabla f|^2.\end{aligned}$$

Thus

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla \psi|^2 dx + \int_{\mathbb{R}^n} |\psi|^2 d\mu \\
&= \int_{\mathbb{R}^n} < \nabla u, \nabla (u|f|^2) > dx + \int_{\mathbb{R}^n} < u, u|f|^2 > d\mu + \int_{\mathbb{R}^n} |u|^2 |\nabla f|^2 dx \\
&= \int_{\mathbb{R}^n} |u|^2 |\nabla f|^2 dx
\end{aligned}
$$

where we have used the assumption that $(-\Delta + \mu)u = 0$ in $\mathbb{R}^n \setminus B$, and $u|f|^2 \in W_0^{1,2}(\mathbb{R}^n \setminus B)$. It follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^n} m(x,\mu)^2 |u\phi|^2 e^{2\varepsilon g} dx \le C \int_{\mathbb{R}^n} |u|^2 |\nabla f|^2 dx \\
&\le C \int_{\mathbb{R}^n} |u|^2 |\nabla \phi|^2 e^{2\varepsilon g} dx + \varepsilon^2 C \int_{\mathbb{R}^n} |u\phi|^2 e^{2\varepsilon g} |\nabla g|^2 dx \\
&\le C \int_{\mathbb{R}^n} |u|^2 |\nabla \phi|^2 e^{2\varepsilon g} dx + \varepsilon^2 C \int_{\mathbb{R}^n} m(x,\mu)^2 |u\phi|^2 e^{2\varepsilon g} dx.
\end{aligned}
$$

This implies that, if $\varepsilon^2 C \le 1/2$, then

$$
\int_{\mathbb{R}^n} m(x,\mu)^2 |u\phi|^2 e^{2\varepsilon g} dx \le 2C \int_{\mathbb{R}^n} |u|^2 |\nabla \phi|^2 e^{2\varepsilon g} dx.
$$

To use Lemma 3.1, we need to regularize the distance function $d(x,y,\mu)$ defined by (0.7).

Lemma 3.3. *For each $y \in \mathbb{R}^n$, there exists a nonnegative function $\varphi_\mu(\cdot, y) \in C^\infty(\mathbb{R}^n)$ such that, for every $x \in \mathbb{R}^n$,*

$$
|\varphi_\mu(x,y) - d(x,y,\mu)| \le C, \tag{3.4}
$$

and

$$
|\nabla_x \varphi_\mu(x,y)| \le C\, m(x,\mu). \tag{3.5}
$$

Proof. Since $m(x,\mu)$ is a slow-varying function (part (c) of Proposition 1.8), there exist a sequence $\{x_j\}$ in $\mathbb{R}^n$ and $\phi_j \in C_0^\infty(\mathbb{R}^n)$ such that

(a) $$\mathbb{R}^n = \cup_{j=1}^\infty B_j \quad \text{where} \quad B_j = B(x_j, \frac{1}{m(x_j,\mu)}),$$

(b) $$\phi_j \in C_0^\infty(B_j), \quad 0 \le \phi_j \le 1 \text{ and } \sum_j \phi_j \equiv 1,$$

(c) $$|\nabla \phi_j(x)| \le C\, m(x,\mu),$$

(d) $$\sum_{j=1}^\infty \chi_{B_j} \le C.$$

We define

$$\varphi_\mu(x,y) = \sum_j d(x_j, y, \mu)\phi_j(x). \tag{3.6}$$

We omit the proof of (3.4)-(3.5), which may be found in [12, p.4483].

For technique reasons, we have to approximate $\varphi_\mu(x,y)$ by a sequence of C^∞ bounded functions.

Lemma 3.7. *For each $y \in \mathbb{R}^n$, there exists a sequence of nonnegative C^∞ bounded functions $\{\varphi_{\mu,j}(\cdot, y)\}$ such that, for every $x \in \mathbb{R}^n$,*

$$\varphi_{\mu,j}(x,y) \le \varphi_\mu(x,y) \quad \textit{and} \quad \varphi_{\mu,j}(x,y) \to \varphi_\mu(x,y) \quad \textit{as } j \to \infty, \tag{3.8}$$

and

$$|\nabla_x \varphi_{\mu,j}(x,y)| \le C\, m(x,\mu). \tag{3.9}$$

Proof. Fix $F \in C^\infty((0,\infty))$ such that $F(t) = t$ if $t \in (0, 1/2)$, $F(t) = 0$ if $t \ge 2$, and $0 \le F(t) \le t$ for all $t \ge 0$.

Let

$$\varphi_{\mu,j}(x,y) = jF\left(\frac{\varphi_\mu(x,y)}{j}\right), \qquad j \ge 1. \tag{3.10}$$

It is easy to check that $\varphi_{\mu,j}$ satisfies (3.8)-(3.9).

We are now ready to prove the upper bound of $\Gamma_\mu(x,y)$.

Theorem 3.11. *Assume that μ satisfies (0.2)-(0.3). Then*

$$\Gamma_\mu(x,y) \le \frac{C\, e^{-\varepsilon_1 d(x,y,\mu)}}{|x-y|^{n-2}} \tag{3.12}$$

for some $C>0$, $\varepsilon_1>0$.

Proof. Fix x_0, $y_0 \in \mathbb{R}^n$ and $x_0 \neq y_0$. Without the loss of generality, we may assume that $y_0 = 0$. Since

$$d(x,y,\mu) \le C \quad \text{if } \ |x-y| < \frac{C}{m(x,\mu)}, \tag{3.13}$$

in view of (2.17), we may also assume that $|x_0| \ge C/m(0,\mu)$ and

$$B(x_0, \frac{2}{m(x_0,\mu)}) \cap B(0, \frac{2}{m(0,\mu)}) = \emptyset. \tag{3.14}$$

Let $r = 1/m(0,\mu)$. Since $\Gamma_\mu(x,0)$ is a weak solution of $(-\Delta+\mu)u = 0$ in $\mathbb{R}^n \setminus \{0\}$ by Theorem 2.18, we may apply Lemma 3.1 with $u(x) = \Gamma_\mu(x,0)$, $g(x) = \varphi_{\mu,j}(x,0)$, and a suitable function ϕ in $C_0^\infty(B(0,2M) \setminus B(0,r))$ where $M \ge 4r$. We obtain

$$\begin{aligned} &\int_{2r\le|x|\le M} m(x,\mu)^2 |\Gamma_\mu(x,0)|^2 e^{2\varepsilon\varphi_{\mu,j}(x,0)}dx \\ &\le \frac{C}{r^2}\int_{r\le|x|\le 2r} |\Gamma_\mu(x,0)|^2 e^{2\varepsilon\varphi_{\mu,j}(x,0)}dx \\ &\quad + \frac{C}{M^2}\int_{M\le|x|\le 2M} |\Gamma_\mu(x,0)|^2 e^{2\varepsilon\varphi_{\mu,j}(x,0)}dx. \end{aligned} \tag{3.15}$$

Since $\varphi_{\mu,j}(x,0)$ is bounded and $|\Gamma_\mu(x,0)| \le C/|x|^{n-2}$, the second term on the right hand side of (3.15) goes to zero as $M \to \infty$. This implies that

$$\begin{aligned} &\int_{|x|\ge 2r} m(x,\mu)^2 |\Gamma_\mu(x,0)|^2 e^{2\varepsilon\varphi_{\mu,j}(x,0)}dx \\ &\le \frac{C}{r^2}\int_{r\le|x|\le 2r} |\Gamma_\mu(x,0)|^2 e^{2\varepsilon\varphi_{\mu,j}(x,0)}dx \\ &\le \frac{C}{r^{n-2}}. \end{aligned}$$

Now let $j \to \infty$. By Fatou's Lemma, we have

$$\int_{|x|\ge 2r} m(x,\mu)^2 |\Gamma_\mu(x,0)|^2 e^{2\varepsilon\varphi_\mu(x,0)}dx \le \frac{C}{r^{n-2}}.$$

Using (3.14) and (3.4), we obtain

$$\int_{B(x_0,R)} m(x,\mu)^2 |\Gamma_\mu(x,0)|^2 e^{2\varepsilon d(x,0,\mu)} dx \leq \frac{C}{r^{n-2}} \tag{3.16}$$

where $R = 1/m(x_0,\mu)$. It follows that

$$\left\{ \frac{1}{R^n} \int_{B(x_0,R)} |\Gamma_\mu(x,0)|^2 dx \right\}^{1/2} \leq \frac{C\, e^{-\varepsilon d(x_0,0,\mu)}}{(Rr)^{\frac{n-2}{2}}}. \tag{3.17}$$

By Lemma 2.9, this gives

$$\begin{aligned} \Gamma_\mu(x_0,0) &\leq \frac{C\, e^{-\varepsilon d(x_0,0,\mu)}}{(Rr)^{\frac{n-2}{2}}} \\ &= C\left\{m(x_0,\mu)m(0,\mu)\right\}^{\frac{n-2}{2}} e^{-\varepsilon d(x_0,0,\mu)}. \end{aligned} \tag{3.18}$$

To finish the proof, we claim that, if $|x-y|m(x,\mu) \geq 2$, then

$$d(x,y,\mu) \geq c\, \{1 + |x-y|m(x,\mu)\}^{\frac{1}{k_0+1}} \tag{3.19}$$

for some $k_0 > 0$.

Assume (3.19) for a moment. We have

$$|x_0|m(0,\mu) + |x_0|m(x_0,\mu) \leq C_\varepsilon e^{\frac{\varepsilon}{2} d(x_0,0,\mu)}. \tag{3.20}$$

for any $\varepsilon > 0$. In view of (3.18), this gives

$$\Gamma_\mu(x_0,0) \leq \frac{C\, e^{-\frac{\varepsilon}{2} d(x_0,0,\mu)}}{|x_0|^{n-2}}.$$

It remains to prove (3.19).

We choose $\gamma:\ [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = x$, $\gamma(1) = y$ and

$$2d(x,y,\mu) \geq \int_0^1 m(\gamma(t),\mu)|\gamma'(t)|dx.$$

It follows from (1.11) that

$$\begin{aligned} 2d(x,y,\mu) &\geq c\int_0^1 \frac{m(x,\mu)|\gamma'(t)|dt}{\{1+|\gamma(t)-x|m(x,\mu)\}^{k_0/(k_0+1)}} \\ &\geq c\cdot\left\{ \text{the geodesic distance from } x \text{ to } y \text{ in the metric} \right. \\ &\qquad\qquad \left. \frac{m(x,\mu)dz^2}{\{1+|z-x|m(x,\mu)\}^{k_0/(k_0+1)}} \right\} \\ &= c\int_0^1 \frac{m(x,\mu)|y-x|dt}{\{1+t|y-x|m(x,\mu)\}^{k_0/(k_0+1)}} \\ &\geq c\,\{1+|y-x|m(x,\mu)\}^{\frac{1}{k_0+1}}. \end{aligned}$$

(3.19) is then proved.

Remark 3.21. It follows from (3.19) and (3.12) that

$$\Gamma_\mu(x,y) \le C\, e^{-\varepsilon\{1+|x-y|m(x,\mu)\}^{\frac{1}{k_0+1}}} \cdot \frac{1}{|x-y|^{n-2}}. \tag{3.22}$$

We also point out that, using arguments similar to that in the proof of (3.19), one may show that, for any $x,\, y \in \mathbb{R}^n$,

$$d(x,y,\mu) \le C\,\{1+|x-y|m(x,y)\}^{k_0+1}. \tag{3.22}$$

4. The Lower Bound of $\Gamma_\mu(x,y)$

In this section we establish the lower bound of $\Gamma_\mu(x,y)$ in the estimate (0.9). Our main tool is the following Harnack inequality.

Lemma 4.1. *(Harnack inequality) Let $B = B(x_0,r)$ and $r < 1/m(x_0,\mu)$. Let $u \in W^{1,2}(2B)$ be a nonnegative weak solution of $(-\Delta+\mu)u = 0$ in $2B$. Then*

$$\sup_{x\in B} u(x) \le C \inf_{x\in B} u(x) \tag{4.2}$$

Proof. If $d\mu = V dx$ and V satisfies

$$\sup_{x\in 2B} \int_{|y-x|<s} \frac{|V(y)|dy}{|y-x|^{n-2}} \le \xi(s) \quad \text{and} \quad \xi(s) \to 0 \quad \text{as } \xi \to 0. \tag{4.3}$$

Harnack inequality (4.2) was proved by Aizenman and Simon [2], using a probabilistic method. The result was extended to the case $-\partial_j a_{jk}\partial a_k + V$ by Chiarenza, Fabes and Garofalo [3]. The proof in [3] was based on a real variable approach. We will deduce Lemma 4.1 from [3] by an approximation argument.

To do this, we choose a radial function φ in $C_0^\infty(\mathbb{R}^n)$ such that $\varphi \ge 0$, $\operatorname{supp}\varphi \subset B(0,1)$ and $\int \varphi = 1$. Let

$$V_t(x) = \varphi_t * \mu(x) \quad \text{where} \quad \varphi_t(x) = \frac{1}{t^n}\varphi(\frac{x}{t})$$

and $t \in (0,r)$. Since $V_t \in C^\infty(\mathbb{R}^n)$ and $V_t \ge 0$, there exists $v_t \in W^{1,2}(B)$ such that

$$\begin{cases} -\Delta v_t + V_t v_t = 0 \quad \text{in} \quad B \\ v_t - u \in W_0^{1,2}(B). \end{cases} \tag{4.4}$$

Since $u \geq 0$, by the maximum principle, $v_t \geq 0$. Using the assumption $r < 1/m(x_0, \mu)$ and (0.2), we see that V_t satisfies (4.3) with a function ξ independent of t and B. It follows that

$$\sup_B v_t \leq C \inf_B v_t$$

with constant C depending only on C_0, C_1 and n. Thus, to prove (4.2), it suffices to show that there exists a sequence $\{t_j\}$ such that $v_{t_j} \to u$ a.e. in B.

To this end, we first note that, by the maximum principle,

$$\sup_B v_t \leq \sup_B u. \tag{4.5}$$

By multiplying $v_t - u$ to the equation in (4.4) and then integrating both sides over B, it is not hard to show that

$$\|v_t\|_{W^{1,2}(B)} \leq C.$$

This implies that there exist a sequence $\{t_j\}$ and $v \in W^{1,2}(B)$ such that $v_{t_j} \to v$ weakly in $W^{1,2}(B)$, $v_{t_j} \to v$ strongly in $L^2(B, dx)$, and $v_{t_j} \to v$ a.e. on B. By Lemma 2.24, we may also assume that $v_{t_j} \to v$ strongly in $L^2(B, d\mu)$.

Next, let $\psi \in C_0^1(B)$. Then

$$\int_B < \nabla v_t, \nabla \psi > dx + \int_B < V_t v_t, \psi > dx = 0. \tag{4.6}$$

Clearly, as $j \to \infty$,

$$\int_B < \nabla v_{t_j}, \nabla \psi > dx \to \int_B < \nabla v, \nabla \psi > dx.$$

We claim that, as $j \to \infty$,

$$\int_B < V_{t_j} v_{t_j}, \psi > dx \to \int_B < v, \psi > d\mu. \tag{4.7}$$

Assume (4.7) for a moment. It follows that $v - u \in W_0^{1,2}(B)$ is a weak solution of $(-\Delta + \mu)u = 0$ in B. Note that, in the definition of the weak solution, (2.2) in fact holds for any $\psi \in W_0^{1,2}(\Omega)$. It is easy to see that this implies $v - u \equiv 0$ in B. Hence, $v_{t_j} \to u$ a.e. on B.

It remains to prove (4.7). We may assume that ψ is a real function.

We write

$$\int_B V_{t_j} v_{t_j} \psi dx = \int_B V_{t_j} (v_{t_j} - v)\psi dx + \int_B V_{t_j} v \psi dx.$$

Note that, by Minkowski's inequality,

$$\begin{aligned}
&\left| \int_B V_{t_j} (v_{t_j} - v)\psi dx \right| = \left| \int_{2B} [(v_{t_j} - v)\psi] * \varphi_{t_j} d\mu \right| \\
&\le \mu(2B) \left\{ \int_{2B} |[(v_{t_j} - v)\psi] * \varphi_{t_j}|^2 d\mu \right\}^{1/2} \\
&\le C_B \|(v_{t_j} - v)\psi\|_{L^2(2B, d\mu)} \to 0
\end{aligned}$$

as $j \to \infty$.

Similarly, by (1.4),

$$\begin{aligned}
&\left| \int_B V_{t_j} v \psi dx - \int_B v \psi d\mu \right| \le \mu(B)^{1/2} \left\{ \int_{2B} |(v\psi) * \varphi_{t_j} - v\psi|^2 d\mu \right\}^{1/2} \\
&\le C_B \|(v\psi) * \varphi_{t_j} - v\psi\|_{W^{1,2}(2B)} \to 0
\end{aligned}$$

as $j \to \infty$. Claim (4.7) is then proved. The proof of Lemma 4.1 is now complete.

Our next lemma compares $\Gamma_\mu(x, y)$ with $\Gamma_0(x, y)$ when $|x - y|$ is small.

Lemma 4.8. *Let x, $y \in \mathbb{R}^n$. Suppose that $|x - y| < 1/m(y, \mu)$. Then*

$$|\Gamma_\mu(x, y) - \Gamma_0(x, y)| \le \frac{C \{|x - y| m(y, \mu)\}^\delta}{|x - y|^{n-2}}.$$

Proof. By the uniqueness of the fundamental solution of $-\Delta$ in $\mathbb{R}^n$, it is not hard to show that, for any x, $y \in \mathbb{R}^n$,

$$\Gamma_0(x, y) = \Gamma_\mu(x, y) + \int_{\mathbb{R}^n} \Gamma_0(x, z) \Gamma_\mu(z, y) d\mu(z). \tag{4.9}$$

It follows from the upper bound of $\Gamma_\mu(x, y)$ in Theorem 3.11 that

$$\begin{aligned}
|\Gamma_\mu(x, y) - \Gamma_0(x, y)| &\le C \int_{\mathbb{R}^n} \frac{e^{-\varepsilon d(z, y, \mu)} d\mu(z)}{|z - x|^{n-2} |z - y|^{n-2}} \\
&= C\{I_1 + I_2 + I_3\}
\end{aligned} \tag{4.10}$$

where I_1, I_2 and I_3 denote the integrals over $B(x, r/2)$, $B(y, r/2)$ and

$$\{z \in \mathbb{R}^n : \; |z-x| \geq r/2, \; |z-y| \geq r/2\}$$

respectively, and $r = |x-y|$.

Clearly, by (0.4) and (0.2),

$$\begin{aligned} I_1 &\leq \frac{C}{r^{n-2}} \int_{|z-x|<r/2} \frac{d\mu(z)}{|z-x|^{n-2}} \leq \frac{C}{r^{n-2}} \cdot \frac{\mu(B(x, r/2))}{r^{n-2}} \\ &\leq \frac{C}{r^{n-2}} \cdot \left(\frac{r}{R}\right)^{\delta} \frac{\mu(B(x,R))}{R^{n-2}} \\ &\leq C\left(\frac{r}{R}\right)^{\delta} \cdot \frac{1}{r^{n-2}} \end{aligned} \tag{4.11}$$

where $R = 1/m(y, \mu) \approx 1/m(x, \mu)$.

Similarly,

$$I_2 \leq C\left(\frac{r}{R}\right)^{\delta} \cdot \frac{1}{r^{n-2}}. \tag{4.12}$$

To estimate I_3, we note that

$$\begin{aligned} I_3 &\leq C \int_{|z-y|\geq r/2} \frac{e^{-\varepsilon d(z,y,\mu)} d\mu(z)}{|z-y|^{2n-4}} \\ &\leq C \int_{2R>|z-y|\geq r/2} \frac{d\mu(z)}{|z-y|^{2n-4}} + C \int_{|z-y|\geq R/2} \frac{e^{-\varepsilon d(z,y,\mu)} d\mu(z)}{|z-y|^{2n-4}} \\ &= C\{I_{31} + I_{32}\}. \end{aligned}$$

By (0.2), we get

$$\begin{aligned} I_{31} &\leq C \int_{r/2}^{2R} \frac{\mu(B(y,t))}{t^{n-2}} \cdot \frac{dt}{t^{n-1}} \\ &\leq C \int_{r/2}^{2R} \left(\frac{t}{R}\right)^{\delta} \cdot \frac{\mu(B(x,R))}{R^{n-2}} \frac{dt}{t^{n-1}} \\ &\leq C\left(\frac{r}{R}\right)^{\delta} \cdot \frac{1}{r^{n-2}}. \end{aligned} \tag{4.13}$$

Finally we use (3.19) to obtain

$$
\begin{aligned}
I_{32} &= \sum_{j=1}^{\infty} \int_{2^j R \le |z-y| < 2^{j+1} R} \frac{e^{-\varepsilon d(z,y,\mu)} d\mu(z)}{|z-y|^{2n-4}} \\
&\le C \sum_{j=1}^{\infty} \frac{e^{-\varepsilon c j}}{(2^j R)^{2n-4}} \cdot \mu(B(y, 2^{j+1}R)) \\
&\le C \sum_{j=1}^{\infty} \frac{e^{-\varepsilon c j}}{(2^j R)^{2n-4}} \cdot C^j \cdot R^{n-2} \\
&\le \frac{C}{R^{n-2}} \sum_{j=1}^{\infty} \frac{e^{-\varepsilon c j}}{(2^j)^{2n-4}} \cdot C^j \\
&\le \frac{C}{R^{n-2}}.
\end{aligned}
$$

This, together with (4.13), gives

$$
I_3 \le C \left(\frac{r}{R}\right)^{\delta} \cdot \frac{1}{r^{n-2}}. \tag{4.14}
$$

The desired estimate now follows from (4.10)-(4.12) and (4.14).

We are now in a position to prove the lower bound of $\Gamma_\mu(x,y)$.

Theorem 4.15. *Assume that μ satisfies the conditions (0.2)-(0.3). Then*

$$
\Gamma_\mu(x,y) \ge \frac{c\, e^{-\varepsilon_2 d(x,y,\mu)}}{|x-y|^{n-2}} \tag{4.16}
$$

for some $c > 0$, $\varepsilon_2 > 0$.

Proof. Given x, $y \in \mathbb{R}^n$. Since $\Gamma_0(x,y) = c_n/|x-y|^{n-2}$, by Lemma 4.8,

$$
\Gamma_\mu(x,y) \ge \frac{c_n}{2|x-y|^{n-2}}
$$

if $|x-y| m(x,\mu) \le c$ and c is small. By Harnack inequality (Lemma 4.1), this implies that

$$
\Gamma_\mu(x,y) \ge \frac{c_A}{|x-y|^{n-2}} \quad \text{if} \quad |x-y| m(x,\mu) \le A. \tag{4.17}
$$

We fix a large A so that

$$
z \notin B(w, \frac{2}{m(w,\mu)}) \quad \text{whenever} \quad |z-w| \ge \frac{A}{m(z,\mu)}. \tag{4.18}
$$

By (4.17), it suffices to show (4.16) for any x, y such that $|x-y|m(x,\mu)>A$.

To this end, we choose $\gamma:\ [0,1]\to\mathbb{R}^n$ such that $\gamma(0)=x$, $\gamma(1)=y$ and

$$\int_0^1 m(\gamma(t),\mu)|\gamma'(t)|dt\le 2d(x,y,\mu).$$

Let

$$t_0=\sup\left\{t\in[0,1]:\quad |x-\gamma(t)|\le\frac{A}{m(x,\mu)}\right\}.$$

If $|y-\gamma(t_0)|\le 1/m(\gamma(t_0),\mu)$, (4.16) follows from (4.17). Suppose $|y-\gamma(t_0)|>1/m(\gamma(t_0),\mu)$. We define

$$t_1=\inf\left\{t\in[t_0,1]:\quad |\gamma(t)-\gamma(t_0)|\ge\frac{1}{m(\gamma(t_0),\mu)}\right\}.$$

Since $m(\cdot,\mu)$ is locally bounded, we obtain $t_0<t_1<\cdots<t_m\le 1$ with the following properties

$$\gamma(t)\in B(\gamma(t_{j-1}),\frac{1}{m(\gamma(t_{j-1}),\mu)})\quad\text{for}\quad t\in[t_{j-1},t_j),\quad j=1,2,\ldots,m \tag{4.19}$$

and

$$\gamma(t)\in B(\gamma(t_m),\frac{1}{m(\gamma(t_m),\mu)})\quad\text{for}\quad t\in[t_m,1]. \tag{4.20}$$

It follows that

$$\begin{aligned}\int_0^1 m(\gamma(t),\mu)|\gamma'(t)|dt &\ge\sum_{j=1}^{m-1}\int_{t_{j-1}}^{t_j} m(\gamma(t),\mu)|\gamma'(t)|dt\\ &\ge c\sum_{j=1}^{m-1} m(\gamma(t_{j-1},\mu)|\gamma(t_j)-\gamma(t_{j-1})|\\ &=c\,(m-1).\end{aligned}$$

Hence,

$$m-1\le C\,d(x,y,\mu). \tag{4.21}$$

Finally, let $u(z)=\Gamma_\mu(z,x)$. By (4.18),

$$x\notin B(\gamma(t_j),\frac{2}{m(\gamma(t_j),\mu)})$$

for $j = 0, 1, \ldots, m$. It follows that u is a weak solution of $(-\Delta + \mu)u = 0$ in $B(\gamma(t_j), \frac{2}{m(\gamma(t_j),\mu)})$. Since

$$|\gamma(t_j) - \gamma(t_{j-1})| = \frac{1}{m(\gamma(t_{j-1}), \mu)}, \qquad j = 1, 2, \ldots, m,$$

by Harnack inequality,

$$u(\gamma(t_0)) \le C u(\gamma(t_1)) \le \cdots \le C^m u(\gamma(t_m)) \le C^{m+1} u(y).$$

This gives

$$\begin{aligned} \Gamma_\mu(y, x) &\ge C^{-m-1} \Gamma_\mu(\gamma(t_0), x) \ge C^{-m-2} |\gamma(t_0) - x|^{2-n} \\ &\ge C^{-m-3} \{m(x, \mu)\}^{n-2} \end{aligned}$$

where we have used (4.17) and $|\gamma(t_0) - x| = A/m(x, \mu)$. It follows from (4.21) that

$$\begin{aligned} \Gamma_\mu(x, y) &\ge C \{m(x, \mu)\}^{n-2} e^{-\varepsilon d(x,y,\mu)} \\ &\ge \frac{c\, e^{-\varepsilon d(x,y,\mu)}}{|x - y|^{n-2}}. \end{aligned}$$

5. The Proof of Theorem 0.17

In this section we give the

Proof of Theorem 0.17

Fix x_0, $y_0 \in \mathbb{R}^n$ and $x_0 \ne y_0$. We need to show

$$\text{(5.1)} \qquad |\nabla_x \Gamma_\mu(x_0, y_0)| \le \frac{C\, e^{-\varepsilon d(x_0, y_0, \mu)}}{|x_0 - y_0|^{n-1}}.$$

To this end, let $R = |x_0 - y_0|/2$ and $u(x) = \Gamma_\mu(x, y_0)$. Then u is a weak solution of $(-\Delta + \mu)u = 0$ in $B(x_0, R)$. Let $r \le R$ and define

$$v(x) = u(x) + \int_{B(x_0,r)} \Gamma_0(x, y) u(y) d\mu(y).$$

By the proof of Lemma 2.20, v is harmonic in $B(x_0, r)$. It follows that

$$\text{(5.2)} \qquad \begin{aligned} \sup_{B(x_0,r/2)} |\nabla u| &\le \sup_{B(x_0,r/2)} |\nabla v| + C \sup_{B(x_0,r)} |u| \cdot \sup_{x\in B(x_0,r)} \int_{B(x_0,r)} \frac{d\mu(y)}{|y - x|^{n-1}} \\ &\le \frac{C}{r} \sup_{B(x_0,r)} |v| + C \sup_{B(x_0,r)} |u| \cdot \left\{1 + \frac{\mu(B(x_0, r))}{r^{n-2}}\right\} \cdot \frac{1}{r} \\ &\le \frac{C}{r} \sup_{B(x_0,r)} |u| \cdot \left\{1 + \frac{\mu(B(x_0, r))}{r^{n-2}}\right\} \end{aligned}$$

where we have used the assumption (0.2) with $\delta > 1$ in the second inequality.

Now suppose that $|x_0 - y_0| \le A/m(x_0, \mu)$. We choose $r = R = |x_0 - y_0|/2$. Then

$$\sup_{B(x_0,r)} |u| \le \frac{C}{r^{n-2}} \quad \text{and} \quad \frac{\mu(B(x_0,r))}{r^{n-2}} \le C.$$

We obtain

$$|\nabla_x \Gamma_\mu(x_0, y_0)| \le \frac{C}{r^{n-1}} \le \frac{C}{|x_0 - y_0|^{n-1}} \le \frac{C_A\, e^{-\varepsilon d(x_0,y_0,\mu)}}{|x_0 - y_0|^{n-1}}. \tag{5.3}$$

Finally, assume that $|x_0 - y_0| > A/m(x_0, \mu)$. We may also assume that $y_0 = 0$ and

$$B(x_0, \frac{2}{m(x_0, \mu)}) \cap B(0, \frac{2}{m(0, \mu)}) = \emptyset \tag{5.4}$$

by choosing A large.

Let $r = \frac{1}{2\, m(x_0,\mu)} < R/2$. Then

$$\begin{aligned}
|\nabla_x \Gamma_\mu(x_0, 0)| &\le \frac{C}{r} \sup_{B(x_0,r)} |u| \\
&\le \frac{C}{r} \left\{ \frac{1}{r^n} \int_{B(x_0,2r)} |u|^2 dx \right\}^{1/2} \\
&\le C\, m(x_0, \mu) \{ m(x_0, \mu) m(0, \mu) \}^{\frac{n-2}{2}} e^{-\varepsilon d(x_0,0,\mu)}
\end{aligned}$$

where we have used (3.17) in the third inequality. Using (3.20), we then obtain

$$|\nabla_x \Gamma_\mu(x_0, 0)| \le \frac{C\, e^{-\frac{\varepsilon}{2} d(x_0,0,\mu)}}{|x_0|^{n-1}}.$$

The proof is complete.

6. The Proof of Theorem 0.19

This section is devoted to the proof of Theorem 0.19, which states that $(-\Delta+\mu)^{i\gamma}$ is a Calderón-Zygmund operator if μ satisfies (0.2)-(0.3) for some $\delta > 0$.

We begin with a uniform Hölder estimate for the weak solutions of $(-\Delta+\mu)u = 0$.

Lemma 6.1. *Let $B = B(x_0, R)$. Suppose that $u \in W^{1,2}(B)$ is a weak solution of $(-\Delta + \mu)u = 0$ in B. Then*

$$|u(x) - u(y)| \le C \left(\frac{|x-y|}{R} \right)^{\delta} \sup_B |u| \tag{6.2}$$

for any x, $y \in B(x_0, R/4)$.

Proof. By Lemma 2.20 and (1.12), it suffices to show that

$$\{1 + R\ m(x_0, \mu)\}^{k_1} \sup_{B(x_0,R/2)} |u| \le C \sup_{B(x_0,R)} |u|. \tag{6.3}$$

To this end, let $\varphi \in C_0^\infty(B(x_0, 3R/4))$ such that $0 \le \varphi \le 1$, $\varphi = 1$ on $B(x_0, 5R/8)$, and $|\nabla \varphi| \le C/R$, $|\nabla^2 \varphi| \le C/R^2$. Since

$$(-\Delta + \mu)(u\varphi) = -2\nabla u \cdot \nabla \varphi - u \Delta \varphi \quad \text{in } \mathbb{R}^n, \tag{6.4}$$

we have

$$u(x)\varphi(x) = \int_{\mathbb{R}^n} \Gamma_\mu(x, y) \{-2\nabla u \cdot \nabla \varphi - u\Delta\varphi\}\, dy.$$

It follows that, for any $x \in B(x_0, R/2)$,

$$\begin{aligned} |u(x)| &\le \frac{C}{R} \int_{5R/8 \le |y-x_0| \le 3R/4} \Gamma_\mu(x, y) |\nabla u(y)| dy \\ &\quad + \frac{C}{R^2} \int_{5R/8 \le |y-x_0| \le 3R/4} \Gamma_\mu(x, y) |u(y)| dy \\ &\le \frac{C}{R^2} \left\{ \int_{5R/8 \le |y-x_0| \le 3R/4} |\Gamma_\mu(x, y)|^2 dy \right\}^{1/2} \left\{ \int_{B(x_0,R)} |u(y)|^2 dy \right\}^{1/2} \end{aligned}$$

where we have used Hölder inequality and Caccioppoli's inequality.

Using the upper bound of $\Gamma_\mu(x, y)$ in Theorem 0.8, we obtain

$$|u(x)| \le \frac{C}{R^{n/2}} \sup_{B(x_0,R)} |u| \left\{ \int_{5R/8 \le |y-x_0| \le 3R/4} e^{-2\varepsilon d(x,y,\mu)} dy \right\}^{1/2} \tag{6.5}$$

for $x \in B(x_0, R/2)$. It follows from (3.19) and (1.10) that, for $x \in B(x_0, R/2)$

$$|u(x)| \le \frac{C}{\{1 + Rm(x_0, \mu)\}^k} \sup_{B(x_0,R)} |u|$$

for any $k > 0$. (6.3) is then proved.

We are now in a position to give the

Proof of Theorem 0.19

Since $-\Delta + \mu$ is a positive self-adjoint operator, by the spectral theorem, $(-\Delta + \mu)^{i\gamma}$ is bounded on $L^2(\mathbb{R}^n, dx)$. We need to show that the kernel associated with $(-\Delta + \mu)^{i\gamma}$ satisfies the estimate (0.24).

By functional calculus, we may write, at least formally

$$(-\Delta + \mu)^{i\gamma} = -\frac{\sin(\pi\gamma i)}{\pi} \int_0^\infty \lambda^{i\gamma} (-\Delta + \mu + \lambda)^{-1} d\lambda. \tag{6.6}$$

Hence,

$$(-\Delta + \mu)^{i\gamma} f(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy \tag{6.7}$$

where

$$K(x, y) = -\frac{\sin(\pi\gamma i)}{\pi} \int_0^\infty \lambda^{i\gamma} \Gamma_{\mu+\lambda}(x, y) d\lambda. \tag{6.8}$$

We remark that (6.7)-(6.8) may be justified by a limiting argument, using

$$A^{i\gamma} f \equiv \lim_{\varepsilon \to 0^+} A^{i\gamma - \varepsilon} f.$$

Also note that the measure $\mu + \lambda$ satisfies (0.2)-(0.3) with constants independent of $\lambda \geq 0$.

Since

$$\begin{aligned} & m(x, \mu + \lambda) \geq m(x, \lambda) \geq c\sqrt{\lambda}, \\ & d(x, y, \mu + \lambda) \geq d(x, y, \lambda) \geq c\sqrt{\lambda}\,|x - y| \end{aligned} \tag{6.9}$$

for $\lambda \geq 0$, by Theorem 0.8, we have

$$0 \leq \Gamma_{\mu+\lambda}(x, y) \leq \frac{C\, e^{-c\sqrt{\lambda}|x-y|}}{|x - y|^{n-2}}. \tag{6.10}$$

This, together with (6.8), gives

$$|K(x, y)| \leq \frac{C}{|x - y|^{n-2}}. \tag{6.11}$$

Next, consider $u(x) = \Gamma_{\mu+\lambda}(x, y_0)$. u is a weak solution of $(-\Delta + \mu + \lambda)u = 0$ in $B(x_0, R)$ where $R = |x_0 - y_0|/2$. It follows from Lemma 6.1 that, if $|h| < R/4 = |x_0 - y_0|/8$,

$$\begin{aligned}|\Gamma_{\mu+\lambda}(x_0 + h, y_0) - \Gamma_{\mu+\lambda}(x_0, y_0)| &= |u(x_0 + h) - u(x_0)| \\ &\le C\left(\frac{|h|}{R}\right)^{\delta} \sup_{B(x_0,R)} |u| \\ &\le C\left(\frac{|h|}{|x_0 - y_0|}\right)^{\delta} \frac{e^{-c\sqrt{\lambda}|x_0 - y_0|}}{|x_0 - y_0|^{n-2}}\end{aligned}$$

where we have used (6.10) in the last inequality. In view of (6.8), we have

$$|K(x_0 + h, y_0) - K(x_0, y_0)| \le C\left(\frac{|h|}{|x_0 - y_0|}\right)^{\delta} \frac{1}{|x - y|^n}$$

where $|h| < |x_0 - y_0|/8$. (0.24) now follows since $K(x, y) = K(y, x)$.

7. The Riesz Transforms

In this section we study the boundedness of the Riesz Transform $\nabla(-\Delta+\mu)^{-1/2}$ on $L^p(\mathbb{R}^n, dx)$.

Theorem 7.1. *Suppose that μ satisfies (0.2)-(0.3) for some $\delta \in (0,1)$. Then, for $1 < p < (2-\delta)/(1-\delta)$,*

$$\|\nabla(-\Delta + \mu)^{-1/2}\psi\|_p \le C\|\psi\|_p. \tag{7.2}$$

By functional calculus, we may write

$$(-\Delta + \mu)^{-1/2} = \frac{1}{\pi}\int_0^{\infty} \lambda^{-1/2}(-\Delta + \mu + \lambda)^{-1} d\lambda. \tag{7.3}$$

Thus,

$$\nabla(-\Delta + \mu)^{-1/2} f(x) = \int_{\mathbb{R}^n} K_1(x, y) f(y) dy \tag{7.4}$$

where

$$K_1(x, y) = \frac{1}{\pi}\int_0^{\infty} \lambda^{-1/2}\nabla_x \Gamma_{\mu+\lambda}(x, y) d\lambda. \tag{7.5}$$

As in [11], let

$$Tf(x) = \int_{\mathbb{R}^n} K_1(y,x) f(y). \tag{7.6}$$

By duality, (7.2) is equivalent to

$$\|Tf\|_p \le C\|f\|_p \quad \text{for} \quad p > 2-\delta. \tag{7.7}$$

To prove (7.7), we write

$$\begin{aligned} Tf(x) = &\int_{|y-x|>r} K_1(y,x) f(y) dy + \int_{|y-x|\le r} \{K_1(y,x) - K_1^0(y,x)\} f(y) dy \\ &+ \int_{|y-x|\le r} K_1^0(y,x) f(y) dy \end{aligned} \tag{7.8}$$

where $r = 1/m(x,\mu)$ and $K_1^0(x,y)$ is the kernel for the operator $\nabla(-\Delta)^{-1/2}$.

To estimate the first term in the right hand side of (7.8), we need the following lemma.

Lemma 7.9. *Let $B = B(x_0, r)$ and*

$$I(x) = \int_B \frac{d\mu(y)}{|y-x|^{n-1}}.$$

Suppose that μ satisfies (0.2) for some $\delta \in (0,1)$. Then, for $1 \le q < (2-\delta)/(1-\delta)$,

$$\|I\|_{l^q(B,dx)} \le C\,\frac{\mu(3B)}{r^{n(1-\frac{1}{q})-1}}.$$

Proof. Write $I = \sum_{j=0}^\infty I_j$ where

$$I_j(x) = \int_{\substack{y\in B \\ 2^{-j}r \le |y-x| < 2^{-j+1}r}} \frac{d\mu(y)}{|y-x|^{n-1}}.$$

Then

$$\begin{aligned} \|I_j\|_{L^\infty(B,dx)} &\le \frac{1}{(2^{-j}r)^{n-1}} \cdot \sup_{x\in B} \mu\{y \in B : |y-x| \le 2^{-j+1}r\} \\ &\le \frac{C}{2^{-j}r} \cdot \frac{\mu(3B)}{r^{n-2}} \cdot (2^{-j})^\delta \end{aligned}$$

where we have used (0.3). Also, note that,

$$\begin{aligned} \|I_j\|_{L^1(B,dx)} &\le \int_{x\in B} dx \int_{\substack{y\in B \\ 2^{-j}r \le |y-x| < 2^{-j+1}r}} \frac{d\mu(y)}{|y-x|^{n-1}} \\ &= \int_{y\in B} d\mu(y) \int_{\substack{x\in B \\ 2^{-j}r \le |y-x| < 2^{-j+1}r}} \frac{dx}{|y-x|^{n-1}} \\ &\le C\, 2^{-j} r\, \mu(B) \end{aligned}$$

It follows that, for $1 \le q \le \infty$,

$$\begin{aligned}\|I_j\|_{L^q(B,dx)} &\le \|I_j\|_{L^\infty(B)}^{1-\frac{1}{q}} \|I_j\|_{L^1(B)}^{\frac{1}{q}}\\ &\le C\,(2^{-j})^{\frac{2}{q}-1+\delta(1-\frac{1}{q})} \cdot \frac{\mu(3B)}{r^{n(1-\frac{1}{q})-1}}.\end{aligned}$$

Since $(2/q)-1+\delta(1-(1/q))>0$ if $q<(2-\delta)/(1-\delta)$, we have

$$\|I\|_{L^q(B,dx)} \le \sum_{j=0}^\infty \|I_j\|_{L^q(B,dx)} \le C\,\frac{\mu(3B)}{r^{n(1-\frac{1}{q})-1}}.$$

Lemma 7.10. *Suppose μ satisfies (0.2)-(0.3) for some $\delta \in (0,1)$. Then*

$$\left|\int_{|y-x|>R} K_1(y,x) f(y)\,dy\right| \le C_p \left\{M(|f|^p)(x)\right\}^{1/p}$$

for $p>2-\delta$, where $R=1/m(x,\mu)$ and $M(g)$ denotes the Hardy-Littlewood maximal function of g.

Proof. Fix $x_0,\ y_0 \in \mathbb{R}^n$ with $x_0 \ne y_0$. Let $u(x)=\Gamma_{\mu+\lambda}(x,x_0)$ where $\lambda \ge 0$. Then u is a weak solution of $(-\Delta+\mu+\lambda)u=0$ in $B(y_0,2r)$ where $r=|x_0-y_0|/4$. It follows from the proof of Theorem 0.17 that

$$|\nabla u(x)| \le \frac{C}{r}\sup_{B(y_0,r)}|u| + C\int_{B(y_0,r)} \frac{d\mu(y)}{|y-x|^{n-1}} \sup_{B(y_0,r)}|u| \tag{7.11}$$

for any $x \in B(y_0,r/2)$. Hence, by Lemma 7.9, if $1 \le q \le (2-\delta)/(1-\delta)$,

$$\begin{aligned}\left\{\int_{B(y_0,r/2)} |\nabla u(x)|^q dx\right\}^{1/q} &\le \frac{C}{r}\sup_{B(y_0,r)}|u|\cdot r^{n/q} + C\cdot\frac{\mu(3B)}{r^{n(1-\frac{1}{q})-1}}\sup_{B(y_0,r)}|u|\\ &= C\,r^{\frac{n}{q}-1}\sup_{B(y_0,r)}|u|\left\{1+\frac{\mu(3B)}{r^{n-2}}\right\}\\ &\le C\,r^{\frac{n}{q}-1}\sup_{B(y_0,2r)}|u|\end{aligned}$$

where the last inequality follows from (1.12) and (6.3). This gives

$$\begin{aligned}\left\{\int_{B(y_0,r/2)} |\nabla_x \Gamma_{\mu+\lambda}(y,x_0)|^q dy\right\}^{1/q} &\le C\,r^{\frac{n}{q}-1}\sup_{y\in B(y_0,2r)}|\Gamma_{\mu+\lambda}(y,x_0)|\\ &\le C\,r^{\frac{n}{q}-n+1}\sup_{y\in B(y_0,2r)} e^{-\varepsilon d(y,x_0,\mu+\lambda)}\\ &\le C\,r^{\frac{n}{q}-n+1}\cdot\frac{e^{-c\sqrt{\lambda}r}}{\{1+r\,m(x_0,\mu)\}}\end{aligned}$$

where we have used (3.19) in the last inequality. Using (7.5) and Minkowski's inequality, we now obtain

$$\left\{ \int_{B(y_0, r/2)} |K_1(y, x_0)|^q dy \right\}^{1/q} \le \frac{C\, r^{\frac{n}{q}-n}}{\{1 + r\, m(x_0, \mu)\}} \tag{7.12}$$

for $1 \le q \le (2-\delta)/(1-\delta)$.

Finally, for $p > 2 - \delta$, we have

$$\begin{aligned}
&\left| \int_{|y-x_0| \ge R} K_1(y, x_0) f(y) dy \right| \\
&\le \sum_{j=1}^{\infty} \int_{2^{j-1}R \le |y-x_0| < 2^j R} |K_1(y, x_0)|\, |f(y)| dy \\
&\le \sum_{j=1}^{\infty} \left\{ \int_{2^{j-1}R \le |y-x_0| < 2^j R} |K_1(y, x_0)|^q dy \right\}^{1/q} \left\{ \int_{|y-x_0| \le 2^j R} |f(y)|^p dy \right\}^{1/p} \\
&\le C \sum_{j=1}^{\infty} \frac{(2^j R)^{\frac{n}{q}-n}}{1 + 2^j} \left\{ \int_{|y-x_0| \le 2^j R} |f(y)|^p dy \right\}^{1/p} \\
&\le C \left\{ M(|f|^p)(x_0) \right\}^{1/p}
\end{aligned}$$

where $q = p'$, $R = 1/m(x_0, \mu)$, and we used (7.12) in the third inequality.

The next lemma handles the second term in (7.8).

Lemma 7.13. *Suppose that μ satisfies (0.2)-(0.3) for some $\delta \in (0,1)$. Then*

$$\left| \int_{B(x,R)} \left\{ K_1(y, x) - K_1^0(y, x) \right\} f(y) dy \right| \le C_p \left\{ M(|f|^p)(x) \right\}^{1/p}$$

for $p > 2 - \delta$ and $R = 1/m(x, \mu)$.

Proof. Since

$$\Gamma_\lambda(y, x) = \Gamma_{\mu+\lambda}(y, x) + \int_{\mathbb{R}^n} \Gamma_\lambda(y, z) \Gamma_{\mu+\lambda}(z, x) d\mu(z), \tag{7.14}$$

we have

$$\begin{aligned}
&\left| \nabla_y \left\{ \Gamma_{\mu+\lambda}(y, x) - \Gamma_\lambda(y, x) \right\} \right| \\
&\le \int_{\mathbb{R}^n} |\nabla_y \Gamma_\lambda(y, z)|\, \Gamma_{\mu+\lambda}(z, x) d\mu(z) \\
&\le C \int_{\mathbb{R}^n} \frac{e^{-c\sqrt{\lambda}|y-z|} e^{-c\sqrt{\lambda}|x-z|} e^{-\varepsilon d(z,x,\mu)}}{|y-z|^{n-1} |z-x|^{n-2}} d\mu(z).
\end{aligned}$$

Let $R = 1/m(x,\mu)$ and $r = |x-y|/2$. Using the same argument as in the proof of Lemma 4.8, we may show that, if $|x-y| \le 1/m(x,\mu)$,

$$
\begin{aligned}
&|\nabla_y \{\Gamma_{\mu+\lambda}(y,x) - |\Gamma_\lambda(y,x)\}| \\
&\le C\, e^{-c\sqrt{\lambda}\, r} \left\{ \frac{1}{r^{n-2}} \int_{B(y,r)} \frac{d\mu(z)}{|z-y|^{n-1}} + \frac{(r\, m(x,\mu))^\delta}{r^{n-1}} \right\}.
\end{aligned} \tag{7.15}
$$

By (7.5), this implies that

$$
|K_1(y,x) - K_1^0(y,x)| \le C \left\{ \frac{1}{r^{n-1}} \int_{B(y,r)} \frac{d\mu(z)}{|z-y|^n} + \frac{(r\, m(x,\mu))^\delta}{r^n} \right\}
$$

This, together with Lemma 7.9, gives

$$
\begin{aligned}
&\left\{ \int_{2^{-j}R<|y-x|\le 2^{-j+1}R} |K_1(y,x) - K_1^0(y,x)|^q dy \right\}^{1/q} \\
&\le C \left\{ \frac{1}{(2^{-j}R)^{n-1}} \cdot \frac{\mu(B(x,2^{-j+3}R))}{(2^{-j}R)^{n(1-\frac{1}{q})-1}} + (2^{-j}R)^{\frac{n}{q}-n} \cdot (2^{-j})^\delta \right\} \\
&\le C\, (2^{-j})^\delta \cdot (2^{-j}R)^{n(\frac{1}{q}-1)}
\end{aligned}
$$

where $1 \le q < (2-\delta(/(1-\delta)$. It follows that, for $p > 2-\delta$,

$$
\begin{aligned}
&\left| \int_{B(x,R)} \{K_1(y,x) - K_1^0(y,x)\} f(y) dy \right| \\
&\le \sum_{j=1}^\infty \left\{ \int_{2^{-j}R<|y-x|\le 2^{-j+1}R} |K_1(y,x) - K_1^0(y,x)|^{p'} dy \right\}^{1/p'} \left\{ \int_{B(x,2^{-j+1}R)} |f(y)|^p dy \right\}^{1/p} \\
&\le C\, \{M(|f|^p)(x)\}^{1/p} \sum_{j=1}^\infty (2^{-j})^\delta \\
&\le C\, \{M(|f|^p)(x)\}^{1/p}.
\end{aligned}
$$

We are now ready to give the

Proof of Theorem 7.1

In view of (7.8), by Lemmas 7.10 and 7.13, we have

$$
|Tf(x)| \le C\, \{M(|f|^p)(x)\}^{1/p} + 2 \sup_{\varepsilon>0} \left| \int_{|y-x|>\varepsilon} K_1^0(y,x) f(y) dy \right|
$$

for any $p > 2-\delta$. (7.7) follows from the well-known estimates for the Hardy-Littlewood maximal function and Riesz transforms $\nabla(-\Delta)^{1/2}$ [14].

Remark 7.16. It is interesting to note that, for any $1 \le p \le \infty$, we have

$$\|m(\cdot,\mu)(-\Delta+\mu)^{-1/2} f\|_p \le C\|f\|_p. \tag{7.17}$$

This follows easily from the fact that the integral kernel of $(-\Delta+\mu)^{-1/2}$ is bounded by

$$\frac{C\, e^{-\varepsilon d(x,y,\mu)}}{|x-y|^{n-1}}$$

and (3.22).

The rest of this section deals with the remaining part of Theorem 0.20.

Theorem 7.18. *Assume that μ satisfies (0.2)-(0.3) for some $\delta > 1$. Then $\nabla(-\Delta+\mu)^{-1/2}$ is a Calderón-Zygmund operator.*

Proof. By Theorem 7.1, $\nabla(-\Delta+\mu)^{-1/2}$ is bounded on $L^p(\mathbb{R}^n, dx)$ for any $1 < p < \infty$. We need to show that the kernel of $\nabla(-\Delta+\mu)^{-1/2}$ satisfies the estimate (0.24).

First, by Theorem 0.17,

$$|\nabla_x \Gamma_{\mu+\lambda}(x,y)| \le \frac{C\, e^{-c\sqrt{\lambda}|x-y|} e^{-c\, d(x,y,\mu)}}{|x-y|^{n-1}}. \tag{7.19}$$

It follows from (7.4)-(7.5) that

$$|K_1(x,y)| \le \frac{C\, e^{-c\, d(x,y,\mu)}}{|x-y|^n}. \tag{7.20}$$

Fix $x_0,\ y_0 \in \mathbb{R}^n$ such that $x_0 \neq y_0$. Let $r = |x_0 - y_0|/2$. To estimate $|K_1(x_0+h, y_0) - K_1(x_0,y_0)|$ for $|h| < |x_0-y_0|/4$, we use the identity

$$\Gamma_\lambda(x,y) = \Gamma_{\mu+\lambda}(x,y) + \int_{\mathbb{R}^n} \Gamma_\lambda(x,z)\Gamma_{\mu+\lambda}(z,y)\,d\mu(z) \tag{7.21}$$

to obtain

$$\begin{aligned}
&|\nabla_x \Gamma_{\mu+\lambda}(x_0+h,y_0) - \nabla_x \Gamma_{\mu+\lambda}(x_0,y_0)| \\
&\le |\nabla_x \Gamma_\lambda(x_0+h,y_0) - \nabla_x \Gamma_\lambda(x_0,y_0)| \\
&\quad + \int_{\mathbb{R}^n} |\nabla_x \Gamma_\lambda(x_0+h,z) - \nabla_x \Gamma_\lambda(x_0,z)|\Gamma_{\mu+\lambda}(z,y_0)\,d\mu(z) \\
&= I + J.
\end{aligned} \tag{7.22}$$

It is easy to show that

$$I \le \frac{C\,|h|e^{-c\sqrt{\lambda}|x_0-y_0|}}{|x_0-y_0|^n}. \tag{7.23}$$

To handle J, we write $J = J_1 + J_2$ where J_1 and J_2 are integrals over $B(x_0, r)$ and $\mathbb{R}^n \setminus B(x_0, r)$ respectively.

Note that, by Theorem 0.8 and (3.19),

$$\begin{aligned}
J_1 &\le \frac{C}{r^{n-2}} \int_{B(x_0,r)} |\nabla_x \Gamma_\lambda(x_0+h, z) - \nabla_x \Gamma_\lambda(x_0, z)| e^{-\varepsilon d(z, y_0, \mu+\lambda)} d\mu(z) \\
&\le \frac{C_k e^{-c\sqrt{\lambda}\, r}}{r^{n-2}\{1 + r\, m(y_0, \mu)\}^k} \int_{B(x_0,r)} |\nabla_x \Gamma_\lambda(x_0+h, z) - \nabla_x \Gamma_\lambda(x_0, z)| d\mu(z) \\
&= \frac{C_k e^{-c\sqrt{\lambda}\, r}}{r^{n-2}\{1 + r\, m(y_0, \mu)\}^k} \left\{ \int_{B(x_0, 2|h|)} + \int_{2|h| \le |z-x_0| < r} \right\} \\
&\le \frac{C_k e^{-c\sqrt{\lambda}\, r}}{r^{n-2}\{1 + r\, m(y_0, \mu)\}^k} \left\{ \frac{\mu(B(x_0, 3|h|))}{|h|^{n-1}} + |h| \int_{2|h| \le |z-x_0| < r} \frac{d\mu(z)}{|z-x_0|^n} \right\}
\end{aligned}$$

where we also used the assumption (0.2) with $\delta > 1$.

It follows from (0.2) that

$$\begin{aligned}
J_1 &\le \frac{C_k e^{-c\sqrt{\lambda}\, r}}{r^{n-2}\{1 + r\, m(y_0, \mu)\}^k} \left\{ \left(\frac{|h|}{r}\right)^{\delta-1} \cdot \frac{\mu(B(x_0, r))}{r^{n-1}} + |h| \int_{2|h|}^{r} \frac{\mu(B(x_0, t))}{t^{n-2}} \cdot \frac{dt}{t^3} \right\} \\
&\le \frac{C_k e^{-c\sqrt{\lambda}\, r}}{r^{n-2}\{1 + r\, m(y_0, \mu)\}^k} \left\{ \left(\frac{|h|}{r}\right)^{\delta-1} \cdot \frac{\mu(B(x_0, r))}{r^{n-1}} + |h| \int_{2|h|}^{r} \left(\frac{t}{r}\right)^{\delta} \cdot \frac{dt}{t^3} \frac{\mu(B(x_0, r))}{r^{n-2}} \right\} \\
&\le \frac{C_k e^{-c\sqrt{\lambda}\, r}}{r^{n-2}\{1 + r\, m(y_0, \mu)\}^k} \cdot \left(\frac{|h|}{r}\right)^{\delta-1} \cdot \frac{\mu(B(x_0, r))}{r^{n-1}}.
\end{aligned}$$

Since

$$\frac{\mu(B(x_0, r))}{r^{n-2}} \le C\, \frac{\mu(B(y_0, 4r))}{(4r)^{n-2}} \le C\, \{1 + r\, m(y_0, \mu)\}^{k_1},$$

we get

$$J_1 \le C \left(\frac{|h|}{r}\right)^{\delta-1} \frac{e^{-c\sqrt{\lambda}}}{r^{n-1}}. \tag{7.24}$$

Next, for J_2, we have

$$\begin{aligned}
J_2 &\le \frac{C\,|h|e^{-c\sqrt{\lambda}\, r}}{r^n} \int_{\mathbb{R}^n} \frac{e^{-\varepsilon d(z, y_0, \mu)}}{|z-y_0|^{n-2}} d\mu(z) \\
&\le \frac{C\,|h|e^{-c\sqrt{\lambda}\, r}}{r^n}
\end{aligned}$$

where the second inequality follows from (3.19) and (1.12). This, together with (7.23)-(7.24), yields that

$$
\begin{aligned}
&|\nabla_x \Gamma_{\mu+\lambda}(x_0+h, y_0) - \nabla_x \Gamma_{\mu+\lambda}(x_0, y_0)| \\
&\quad \le C \left(\frac{|h|}{|x_0 - y_0|} \right)^{\delta - 1} \cdot \frac{e^{-c\sqrt{\lambda}|x_0 - y_0|}}{|x_0 - y_0|^{n-1}}.
\end{aligned} \tag{7.25}
$$

In view of (7.4)-(7.5), we obtain

$$
|K_1(x_0+h, y_0) - K_1(x_0, y_0)| \le C \left(\frac{|h|}{|x_0 - y_0|} \right)^{\delta - 1} \cdot \frac{1}{|x_0 - y_0|^n}. \tag{7.26}
$$

Finally, we estimate $|K_1(x_0, y_0+h) - K_1(x_0, y_0)|$.

Using Theorem 0.17, Caccioppli's inequality and Lemma 2.24, one may show that $\nabla_x \Gamma_{\mu+\lambda}(x_0, y)$ is a weak solution of $(-\Delta + \mu + \lambda)u = 0$ in $\mathbb{R}^n \setminus \{x_0\}$. It follows from Lemma 6.1 and Theorem 0.17 that

$$
\begin{aligned}
&|\nabla_x \Gamma_{\mu+\lambda}(x_0, y_0+h) - \nabla_x \Gamma_{\mu+\lambda}(x_0, y_0)| \\
&\le C \left(\frac{|h|}{r} \right)^{\delta_1} \sup_{B(y_0, r)} |\nabla_x \Gamma_{\mu+\lambda}(x_0, y)| \\
&\le C \left(\frac{|h|}{r} \right)^{\delta_1} \frac{e^{-c\sqrt{\lambda}\, r}}{r^{n-1}}
\end{aligned}
$$

where $\delta_1 \in (0,1)$.

By (7.4)-(7.5), this implies that

$$
|K_1(x_0, y_0+h) - K_1(x_0, y_0)| \le C \left(\frac{|h|}{|x_0 - y_0|} \right)^{\delta_1} \frac{1}{|x_0 - y_0|^n}. \tag{7.27}
$$

The proof is complete.

Theorem 7.28. *Assume that μ satisfies (0.2) and (0.3) for some $\delta > 1$. Then $\nabla(-\Delta + \mu)^{-1}\nabla$ is a Calderón-Zygmund operator.*

Proof. By duality, Theorem 7.1 implies that $\nabla(-\Delta+\mu)^{-1} = \nabla(-\Delta+\mu)^{-1/2} \cdot (-\Delta+\mu)^{-1/2}\nabla$ is bounded on $L^p(\mathbb{R}^n, dx)$ for any $1 < p < \infty$. The desired estimates on the kernel may be proved by arguments similar to that in the proof of Theorem 7.18. We omit the details.

Remark 7.29. Putting Theorems 7.1, 7.18 and 7.28 together, we get Theorem 0.24.

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