RENORMALIZATION–GROUP TRANSFORMATIONS UNDER STRONG MIXING CONDITIONS: GIBBSIANNESS AND CONVERGENCE OF RENORMALIZED INTERACTIONS.

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Abstract

In this paper we study a renormalization-group map: the block averaging transformation applied to Gibbs measures relative to a class of finite range lattice gases, when suitable strong mixing conditions are satisfied. Using block decimation procedure, cluster expansion (like in [HK]) and detailed comparison between statistical ensembles, we are able to prove Gibbsianess and convergence to a trivial (i.e. Gaussian and product) fixed point. Our results apply to 2D standard Ising model *at any* temperature above the critical one and arbitrary magnetic field.

Keywords: Renormalization–group; Gibbsianess; Finite–size conditions; Complete analyticity; Strong mixing; Equivalence of ensembles; Ising model.

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1. Introduction

This paper concerns the rigorous analysis of some real–space renormalization group transformations (RGT) (see, for instance, ref. [NL] for a general introduction to this subject). In the recent years many works have been devoted to the question of well–definedness of RGT. We refer to ref. [EFS] for a clear and complete discussion of the problematic as well as for an exhaustive description of the general setup of renormalization maps from the point of view of rigorous statistical mechanics. Already in the seventies (see [GaK], [CG], [GP], [I]) the question was raised of whether or not some typical RGT give rise to a well defined renormalized interaction. In other words, calling $\mu^{(\ell)}$ the renormalized measure arising from the application of a RGT on scale ℓ to the Gibbs measure μ, we pose the question of Gibbsianess of $\mu^{(\ell)}$, namely we ask ourselves whether $\mu^{(\ell)}$ is a Gibbs measure corresponding to a finite–norm translationally invariant potential so that the "renormalized Hamiltonian" is well defined.

More explicitly: let us assume that our RGT can be expressed as

$$\mu^{(\ell)}(\sigma') = \sum_{\sigma} T^{(\ell)}(\sigma', \sigma)\mu(\sigma) \tag{1.1}$$

where $T^{(\ell)}(\sigma', \sigma)$ is a normalized non–negative kernel. The system described in terms of the σ variables by the original measure μ is called *object system.* The σ''s are the *renormalized variables* of the *image system* described by the renormalized measure $\mu^{(\ell)}$. We can think of the transformation T_ℓ as directly acting at infinite volume or we can consider a finite volume version and subsequently try to perform the thermodynamic limit (see [EFS]).

The above mentioned pathological behavior (non–Gibbsianess of $\mu^{(\ell)}$) can be a consequence of the violation of a necessary condition for Gibbsianess called *quasi–locality* (see [Ko], [EFS]). The latter is a continuity property of the finite volume conditional probabilities of $\mu^{(\ell)}$ which, roughly speaking, says that they are almost independent of very far away conditioning spins. In many interesting examples (see [E1], [E2], [EFS], [EFK]) violation of quasi–locality and consequently non–Gibbsianess of the renormalized measure $\mu^{(\ell)}$ is a direct consequence of the appearance of a first order phase transition for the original (object) system *conditioned* to some particular configuration of the image system. More precisely, given a configuration σ', let us consider the probability measure on the original spin variables given by

$$\mu_{\sigma'}(\sigma) \;=\; \frac{T^{(\ell)}(\sigma', \sigma)\mu(\sigma)}{\sum_\eta T^{(\ell)}(\sigma', \eta)\mu(\eta)}$$

It defines the *constrained* model corresponding to σ' (which here plays the role of an external parameter). For some particular σ' it may happen that the corresponding measure $\mu_{\sigma'}(\sigma)$ exhibits long range order inducing violation of quasi–locality and then non–Gibbsianess for the image system. See [EFS] and also [GP], [I] where this mechanism was first pointed out.

This pathological behavior is often induced by configurations σ' highly non–typical with respect to the measure $\mu^{(\ell)}$. This suggests the introduction of a weaker notion of Gibbsianess requiring well–definedness of renormalized interactions not *for all* renormalized configurations σ'

but, rather, *for* $\mu^{(\ell)}$*–almost all* σ' (see [D2], [BKL], [LM]). However also this point of view poses various other problems (see [ES] and references therein).

It is also natural to ask ourselves about *robustness* of this pathology. Sometimes it can be shown that non–Gibbsianess is an artifact due to a wrong choice of the scale ℓ of the transformation in terms of the thermodynamic parameters of the object system. For instance in [MO4] has been considered the case of the measure $\mu^{(\ell)} = T_d^{(\ell)} \mu_{\beta,h}$, where $T_d^{(\ell)}$ is the so–called decimation transformation on scale ℓ (see [EFS]) and $\mu_{\beta,h}$ is the Gibbs measure for the standard 2D Ising model, h and β being, respectively, the external field and the inverse temperature. In [EFS] the authors show non–Gibbsianess for some choices of h, β, ℓ. On the contrary, in [MO4] it is shown that, for the same values of h, β for which, for suitable ℓ, in [EFS] the authors got non–Gibbsianess, by changing ℓ into a sufficiently large ℓ', depending on β, h, one gets again Gibbsianess. We also mention it is possible to show that, by iterating the transformation, one has convergence to a (trivial) fixed point, see [MO4] and also [Ka] for the high temperature case. The above behavior is related to the fact that, given suitable values of the parameters β, h (close to the coexistence line $h = 0, \beta > \beta_c$), on a suitable scale ℓ some constrained models can undergo a phase transition (somehow related to the phase transition of the object system); whereas, given the same h, β, for sufficiently large scale ℓ any constrained model is in the weak coupling region. Another notion of robustness of the pathology is related to the application of decimation transformations, see [LV], [MO5].

Let us stress that the fact that the object system is very well behaved in the sense that it is in the unique phase region (in the strongest possible sense) does not preclude the possibility that some constrained model undergoes a dangerous phase transition inducing the pathology.

On the positive side, since the pioneering paper [CG], there are many indications that if the constrained models are in the weak coupling regime, then Gibbsianess of the renormalized measure follows. Recently Haller and Kennedy gave very interesting new rigorous results in this direction. They proved, under very general hypotheses, that if *all* constrained models satisfy a uniform (in the constraint) version of the Dobrushin–Shlosman complete analyticity condition (see [DS2], [DS3]) then the renormalized measure is Gibbsian with a finite norm potential which can be computed via a convergent cluster expansion.

Another interesting question, which, in a sense, is the main object of the present paper, is the convergence of the iterates of RGT or, in other words, the behavior of the transformation $T^{(\ell)}$ for large ℓ. This problem has not been, up to now, studied very much from a point of view of rigorous statistical mechanics. Here we present results referring to non–critical systems and so we have convergence to a trivial fixed point, i.e. Gaussian *and* product (which correspond to infinite temperature). Indeed most of the recent papers concerning rigorous results on RGT refer to the non–critical region with some exceptions, see [BMO], [CiO], [HK], where the authors consider 2D critical Ising system but only for one step of RGT.

Let us now introduce, for the standard 2D Ising model, the Block–Averaging Transformation (BAT). It is convenient to use the lattice gas variables. For a standard Ising system enclosed in a finite volume $\Lambda \subset \mathbb{Z}^2$ the configuration space is therefore $\{0,1\}^\Lambda$; given a configuration $\eta \in \{0,1\}^\Lambda$ and a site $x \in \Lambda$, $\eta_x \in \{0,1\}$ represents the occupation number at x. For free or periodic boundary conditions the energy associated to a configuration $\eta \in \{0,1\}^\Lambda$ is:

$$E_\Lambda(\eta) := -\beta \sum_{\langle x,y\rangle \subset \Lambda} \eta_x \eta_y - \lambda \sum_{x\in\Lambda} \eta_x \tag{1.2}$$

where $\langle x,y\rangle$ is a pair of nearest neighbor sites, β is the inverse temperature and λ is β times the chemical potential so that the Boltzmann factor is $\sim \exp\{-E_\Lambda(\eta)\}$. Given β let $\lambda^* = \lambda^*(\beta)$ be the value of λ corresponding to the value zero of the magnetic field h appearing in the expression of E in terms of spin variables $\sigma_x = 2\eta_x - 1$. For β, λ in the uniqueness region: $(\beta,\lambda) \in \{\beta < \beta_c\} \cup \{\beta \geq \beta_c, \lambda \neq \lambda^*\}$ (β_c is the inverse critical temperature), let $\mu_{\beta,\lambda}$ be the unique infinite volume Gibbs measure. We partition $\mathbb{Z}^2$ into square blocks $Q_\ell(i)$ of side ℓ and centers at the points i belonging to the rescaled lattice $(\ell\mathbb{Z})^2$. Let $N_i = N_i(\eta) := \sum_{x\in Q_\ell(i)} \eta_x$ be the number of particles in the block $Q_\ell(i)$ in the η configuration, $\rho = \rho(\beta,\lambda) = \mu_{\beta,\lambda}(\eta_0)$ be the equilibrium density, $\chi = \chi(\beta,\lambda) := \sum_{x\in\mathbb{Z}^2}[\mu_{\beta,\lambda}(\eta_0\eta_x) - \mu_{\beta,\lambda}(\eta_0)\mu_{\beta,\lambda}(\eta_x)]$ be the susceptibility; we then set:

$$M_i := \frac{N_i - \rho|Q_\ell|}{\sqrt{|Q_\ell|\chi}} \tag{1.3}$$

the random variables M_i are centered and normalized; they take values in

$$\bar{\Omega}_i^{(\ell)} := \left\{ \frac{-\rho|Q_\ell|}{\sqrt{|Q_\ell|\chi}}, \frac{1-\rho|Q_\ell|}{\sqrt{|Q_\ell|\chi}}, \dots, \frac{|Q_\ell|(1-\rho)}{\sqrt{|Q_\ell|\chi}} \right\} \tag{1.4}$$

We expect M_i, $i \in (\ell\mathbb{Z})^2$ to have a product (Gaussian) limiting distribution as $\ell \to \infty$.

The renormalized measure $\mu^{(\ell)} = \mu^{(\ell)}_{\beta,\lambda}$ (arising from the application of the BAT transformation on scale ℓ to $\mu_{\beta,\lambda}$) is the joint distribution of the random variables M_i's under $\mu_{\beta,\lambda}$; i.e. it is obtained by assigning to each block $Q_\ell(i)$ a value $m_i \in \bar{\Omega}_i^{(\ell)}$ and by computing the probability, w.r.t. the original Gibbs measure $\mu_{\beta,\lambda}$ of the event: $M_i(\eta) = m_i$. In other words, in the notation of (1.1) in the case of BAT we have: $\sigma = \{\eta_x\}, \sigma' = \{m_i\}$ and

$$T^{(\ell)}_{BAT}(m,\eta) = \begin{cases} 1 & \text{if } M_i(\eta) = m_i \ \forall i \\ 0 & \text{otherwise} \end{cases} .$$

In this case a constrained model is a *multi–canonical* Ising model; namely an Ising model subject to the constraint of having a fixed number of particles in each block $Q_\ell(i)$.

Theorem 1.1. *Consider a 2D Ising system with $\beta < \beta_c$ and $\lambda \in \mathbb{R}$ given. Then there exists $\ell_0 \in \mathbb{N}$ such that $\forall\, \ell > \ell_0$ $\mu^{(\ell)}_{\beta,\lambda}$ is Gibbsian with a finite norm translationally invariant potential $\Phi^{(\ell)} = \{\Phi^{(\ell)}_X,\ X \subset (\ell\mathbb{Z})^d\}$.*

Furthermore it is possible to decompose the potential into a short and a long range part, $\Phi^{(\ell)} = \Phi^{(\ell),sr} + \Phi^{(\ell),lr}$, where $\exists \kappa \in \mathbb{N}$: $\Phi^{(\ell),sr}_X \equiv 0$ if $\mathrm{diam}(X) \geq \kappa$ and we have the following:

(i) there is $\alpha > 0$ such that

$$\lim_{\ell\to\infty} \sum_{X\ni 0} e^{\alpha|X|} \sup_{m_i \in \bar{\Omega}_i^{(\ell)}} \left| \Phi^{(\ell),lr}_X(m_i,\, i\in X) \right| = 0$$

(ii) there exist $a > 0$ such that

$$\lim_{\ell\to\infty} \sup_{\substack{m_i\in\bar\Omega_i^{(\ell)}\\ |m_i|\le \ell^a}} \left|\Phi_X^{(\ell),sr}(m_i, i\in X)\right| = 0 \qquad \text{for} \;\; |X|\ge 2$$

$$\lim_{\ell\to\infty} \sup_{\substack{m_i\in\bar\Omega_i^{(\ell)}\\ |m_i|\le \ell^a}} \left|\Phi_{\{i\}}^{(\ell),sr}(m_i) + \frac{1}{2}m_i^2\right| = 0 \qquad \text{for} \;\; i\in(\ell\mathbb{Z})^d$$

We want to stress that the results hold for ℓ sufficiently large. Certainly, in particular, we cannot exclude that, very near to T_c, for some, not sufficiently large ℓ, the renormalized measure is not Gibbsian. On the other side, it is easily seen that taking the limit $\ell\to\infty$ is equivalent to iterate BAT transformation on a given scale ℓ_0; to show this it is sufficient to take $\ell = \ell_0^n$ with $n\in\mathbb{N}$. Theorem 1.1 above says that not only the renormalized measure $\mu_{\beta,\lambda}^{(\ell)}$, for any sufficiently large ℓ is Gibbsian but the corresponding renormalized potential $\Phi^{(\ell)}$ actually converges, as $\ell\to\infty$, to the one of a system of non-interacting harmonic oscillators.

We notice that the limiting image system as $\ell\to\infty$ becomes an unbounded spin system and the usual setup of Gibbsianess does not apply to it (see [EFS]). It is therefore clear that we have to introduce a large field cutoff. Indeed our result is almost optimal as we introduce this cutoff only for the short range part of the interaction and, moreover, the cutoff diverges as a power law in ℓ. On the other side it is not difficult to convince ourselves that the convergence result, at least in the form given above, cannot hold without any restriction on the large fields.

This paper contains also other, much weaker, results that apply to Ising model below T_c at $\lambda\neq\lambda^*$, see Theorem 2.3 below. In that case we are forced to restrict the possible values of m_i also in the computation of long–range part of the renormalized potential; indeed we have of course to forbid that m_i lies in the phase coexistence interval.

Results in the same direction as Theorem 1.1 were obtained by Cammarota [C]; the main differences w.r.t. the present paper are that Cammarota considers a high temperature (much higher than $T_c = \beta_c^{-1}$) situation and that he introduces a finite (not growing to infinity as $\ell\to\infty$) field cutoff. The approach of [C] is substantially different w.r.t. ours; [C] uses a high temperature expansion: the small parameter is β and the system is supposed to be weakly coupled on scale one; whereas since we want to treat a system with $T=\beta^{-1}$ higher but arbitrarily close to T_c, we have to use an approach supposing weak coupling only on a sufficiently large scale depending on the temperature $T>T_c$ that we have chosen; indeed we are forced to act in the scenario of the so called restricted complete analyticity. Let us try to clarify this point. Exactly in the spirit of renormalization group theory we can say that a system above its critical point is very weakly coupled on a scale large compared to the correlation length; as we want to consider *any* $T>T_c$ we have to take into account the divergence of the correlation length when approaching T_c (from above). The above statement: "the system is weakly coupled on a scale larger than the correlation length" seems a tautology; in fact it is not since we need a suitable mathematical setup in order to be able to implement the above simple observation. The basic idea is to obtain a perturbative expansion on the basis of very strong mixing conditions satisfied by the Gibbs measures; the small parameter ceases to be the inverse temperature but it will, rather, be related to the ratio between the correlation length and the scale on which we are analyzing our system. The geometrical objects (polymers) in terms of which we perform our perturbative expansions will not live any more on scale one (like in [C]) but on a scale sufficiently larger than the correlation length.

A possible notion of strong mixing is the exponential decay of truncated correlations for any finite volume Gibbs measure with decay constants uniform in the volume and in the boundary conditions. This is a stronger notion w.r.t uniqueness of the Gibbs state or exponential clustering of *infinite volume* truncated correlations. In [DS2], [DS3] Dobrushin and Shlosman introduced and studied the so called completely analytical interactions showing, in those cases, the above strong mixing behavior *for any* finite or infinite domain of arbitrary shape. This complete analyticity turns out to be a too strong notion in the context of renormalization group theory. Indeed Dobrushin–Shlosman's complete analyticity implies that exponential clustering takes place even inside volumes with very anomalous shapes (for instance with anomalous ratio between boundary and bulk) so that one is forced to take into account the influence of boundary conditions up to a scale of order one. There are cases of systems perfectly well behaved on regular domains, say cubes, which, however, do not satisfy D-S complete analyticity because of their behavior for anomalous shape domains (see example in [MO2]). Another point of view, introduced in [O], [OP], [MO1], [MO2], [MO3] leading to what can be called restricted complete analyticity, takes into account only regular domains. In this approach there is a minimal basic length L and one never goes below L in the sense that one only considers domains obtained as disjoint unions of cubes of side L (for instance cubes of side nL).

The algorithm used in the present paper to compute the renormalized potential is the following. We start, as basic hypothesis, from restricted complete analyticity for the constrained, multi–canonical systems, with a minimal length L proportional to the scale ℓ of our BAT transformation and with decay constants uniform in the constraint. We then construct a convergent cluster expansion which allows us to compute the renormalized potential. Since studying directly the mixing properties of a canonical or multi–canonical measure is a very difficult task we instead deduce it by using a sharp form of equivalence, or better comparison, between canonical and grancanonical ensembles. Indeed, the main key novel technical point of this paper is to get a very precise notion of equivalence of ensembles, implying the validity of a finite size condition, which, in turn, will imply a strong mixing condition for the constrained multi–canonical systems. See also [DT], [CM], [Y] for a further discussion on the equivalence of ensembles.

Certainly assuming strong mixing for the object system with a given value λ of the chemical potential is not sufficient to imply the strong mixing property of the constrained models even at the level of regular domains. It is, rather, necessary to assume for the object grancanonical system a strong mixing condition *uniform* in λ. Quite surprisingly, this condition is not sufficient in general. Indeed it turns out that what we really need is a strong mixing condition for a *multi–grancanonical* object system; by multi–grancanonical we mean a grancanonical measure which is not translationally invariant because in each cube $Q_\ell(i)$ we put a different chemical potential λ_i whereas we leave the original, translationally invariant, mutual interaction. It happens, as it is shown by an example in Appendix A.2 that uniform (in λ) strong mixing for a grancanonical measure *does not imply* uniform in $\underline{\lambda} = \{\lambda_i\}$ strong mixing for the multi–grancanonical measure; this pathology is due to the possibility of a sort of layering phase transition, with long range order, taking place along the interface between two contiguous large cubes with different chemical potentials λ_1, λ_2 even though, introducing the same chemical potential $\lambda_1 = \lambda_2 = \lambda$ in both cubes the resulting system, is, $\forall\, \lambda$, very well behaved. On the other side we show that since the interface between two regular two–dimensional domains is one–dimensional, this layering phase transition cannot occur when the object system lives in two–dimensions. Then the result of Theorem 1.1 ultimately follows from strong mixing, uniform in λ, exploiting the two–dimensionality of the Ising system. The latter follows from the general result of [MOS] saying that in two dimensions the so–called weak mixing implies strong mixing, provided one is able to prove weak mixing for our particular model. This is a weaker notion of mixing of a finite volume Gibbs measure saying, roughly speaking, that the influence of a change in a conditioning spin on a site x outside a domain Λ decays, inside Λ, with the distance from the boundary $\partial\Lambda$ and not, like it would be the case assuming strong mixing, with the distance from x. Weak mixing, uniform in the chemical potential λ, for Ising model above T_c has been proved by Higuchi in [H] exploiting general results by Aizenman *et al.* [ABF] about boundedness of susceptibility above T_c. We thus use the two–dimensionality in two crucial points: i) in deducing uniform strong mixing for multi–grancanonical measure from the same property for simple grancanonical measure and ii) in deducing strong mixing from weak mixing. On the other side, given the strong mixing condition for the multi–grancanonical measure, the results on the RGT of this paper apply to any dimension.

The general results about Gibbsianess of renormalized measures that have been obtained by Haller and Kennedy in [HK], use a strategy very similar to ours. Indeed their computations, also based on the methods developed in [O], [OP] are much simpler and more transparent then ours but apply only to the case when the image system is Ising-like; namely the σ' variables are dichotomic as in Majority rule or Kadanoff (see [EFS]) transformations. Haller and Kennedy for a given ℓ use the hypothesis of D–S complete analyticity of the constrained models to deduce Gibbsianess of the measure resulting from the application of one transformation.

We conclude by a brief outline of the various steps needed in the proof of Theorem 1.1. Higuchi [H] proves weak mixing, uniform in λ, for Ising model above T_c. [MOS] proves, in general, that in two dimensions, for regular domains, weak mixing implies strong mixing. In Appendix A.1 we prove that in two dimension strong mixing uniform in λ for grancanonical measure implies strong mixing uniform in $\underline{\lambda} = \{\lambda_i\}$ for the corresponding multi–grancanonical measure. In Section 4 we prove results about comparison, in a finite volume Λ, between multi–grancanonical and multi–canonical measures with precise estimates of the behavior in Λ. From this and previous points we deduce that, on a sufficiently large scale, an *effective* (propagating to arbitrarily large, regular domains) finite size condition is satisfied for multi–canonical constrained systems. Then, from this finite size conditions, using the theory developed in [O], [OP] we are able to perturbatively compute the renormalized Hamiltonian; and to extract the potentials. The long range terms of the interaction potential are computed starting from a cluster expansion whose convergence is directly related to the validity of the above finite size condition. Finally the short range terms are handled via a local central limit theorem for the multi–grancanonical measure.

2. Notation and results

We introduce the general setup: the one of the finite state space, lattice spin systems. Contrary to the usual treatments we drop the hypothesis of translation invariance; indeed it will be replaced by spatial uniformity of some basic estimates. We start by giving a list of basic definitions.

2.1. The lattice.

For $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ we let $|x| := \sup_{k=1,\cdots,d} |x_k|$. The spatial structure is modeled by the d–dimensional lattice $\mathcal{L} := \mathbb{Z}^d$ in which we let $e_i, i = 1, \ldots, d$ be the coordinate unit vectors. We shall denote by $x, y, \cdots$ the sites in $\mathcal{L}$ and by $\Lambda, \Delta, \cdots$ subsets of $\mathcal{L}$. We consider $\mathcal{L}$ endowed with the distance $d(x, y) = |x - y|$. We use $\Lambda^c := \mathcal{L} \setminus \Lambda$ to denote the complement of Λ. For Λ a finite subset of $\mathcal{L}$ (we use $\Lambda \subset\subset \mathcal{L}$ to indicate that Λ is finite), $|\Lambda|$ denotes the cardinality of Λ. For $x \in \mathcal{L}$ and ℓ an odd integer we let $Q_\ell(x) := \{y \in \mathcal{L} : d(y, x) \leq (\ell - 1)/2\}$ be the cube of side ℓ centered at x; for ℓ an even integer we let instead $Q_\ell(x) := \{y \in \mathcal{L} : |y - (x + \hat{e})| \leq \ell/2\}$, where $\hat{e} := (1/2, \cdots, 1/2)$, be the cube of side ℓ centered in $x + \hat{e}$ (which belongs to the dual lattice). We shall denote $Q_\ell(0)$ simply by Q_ℓ. Given $r > 0$ and $\Lambda \subset \mathcal{L}$ we introduce the outer boundary of Λ by $\partial_r \Lambda := \{x \notin \Lambda : d(x, \Lambda) \leq r\}$. We let also $\overline{\Lambda}^r := \Lambda \cup \partial_r \Lambda$.

Given an integer ℓ, we also introduce the rescaled lattice $\mathcal{L}_\ell := (\ell\mathbb{Z})^d$ which is naturally embedded in $\mathcal{L}$; we shall therefore regard points in $\mathcal{L}_\ell$ also as points in $\mathcal{L}$ without further mention, more precisely we will make the following identification: $\mathcal{L}_\ell \ni (i_1, \ldots, i_d) \equiv (\ell i_1, \ldots, \ell i_d) \in \mathcal{L}$. We use $i, j, \cdots$ to denote points in $\mathcal{L}_\ell$ and $I, \cdots$ to denote subsets of $\mathcal{L}_\ell$. Analogously the distance in $\mathcal{L}_\ell$ is denoted by $d_\ell(i, j)$, therefore for $i, j \in \mathcal{L}_\ell$ we have $d(i, j) = \ell d_\ell(i, j)$.

2.2. The configuration space.

We suppose given a positive integer $\mathcal{N} \in \mathbb{N}^+$ and, for every $x \in \mathcal{L}$, a positive integer $\mathcal{N}_x \leq \mathcal{N}$. We then introduce the following:

- *Configuration space of a single spin.* For any $x \in \mathcal{L}$ we have a finite set Ω_x, $|\Omega_x| = \mathcal{N}_x + 1$. We identify Ω_x with $\{0, 1, \ldots, \mathcal{N}_x\}$ which we consider endowed with the discrete topology;
- *Configuration space in a subset* $\Lambda \subset \mathcal{L}$. We set $\Omega_\Lambda^{(\mathcal{N})} := \otimes_{x \in \Lambda} \Omega_x$;
- *Configuration space in the whole* $\mathcal{L}$. We set $\Omega^{(\mathcal{N})} := \otimes_{x \in \mathcal{L}} \Omega_x$ and equip it with the product topology.

We can therefore look at a configuration $\sigma \in \Omega^{(\mathcal{N})}$ as a function $\sigma : \mathcal{L} \mapsto \{0, 1, \ldots, \mathcal{N}\}$. The integer $\sigma_x \equiv \sigma(x)$ is called value of the spin at the site $x \in \Lambda$ in the configuration σ. For $\Lambda \subset \mathcal{L}$, we use $\sigma_\Lambda := \{\sigma_x \in \Omega_x,\ x \in \Lambda\}$ to denote the collection of spins in Λ. For $x \in \mathcal{L}$ we define the shift ϑ_x (acting on $\Omega^{(\mathcal{N})}$) by $(\vartheta_x \sigma)_y := \sigma_{y-x}$.

We also introduce $C\left(\Omega^{(\mathcal{N})}\right)$ the space of continuous functions on Ω which becomes a Banach space under the norm $\|f\| := \sup_\sigma |f(\sigma)|$ and note that the local functions (i.e. the functions depending only on a finite number of spins) are dense in $C\left(\Omega^{(\mathcal{N})}\right)$. For f a local function depending on the spins in $\Lambda \subset\subset \mathcal{L}$, i.e. $f(\sigma) = f(\sigma_\Lambda)$, we let $S(f) \equiv \mathrm{supp}(f) := \Lambda$ be the support of f.

In the case $\mathcal{N} = 1$ the spin σ_x takes values in $\{0, 1\}$, i.e. we have a lattice gas. In such a case we use the notation $\Omega := \{0, 1\}^{\mathcal{L}}$ and denote by $\eta, \zeta, \cdots$ typical elements of Ω; the value $\eta_x \in \{0, 1\}$ is interpreted as the occupation number in x. Given $\eta \in \Omega$ we define a new configuration η^x which is obtained from η by flipping the occupation number in x, i.e.

$$(\eta^x)_y := \begin{cases} \eta_y & \text{if } y \neq x \\ 1-\eta_x & \text{if } y = x \end{cases}$$

2.3. The potential.

A potential $\Phi = \{\Phi_\Lambda, \Lambda \subset\subset \mathcal{L}\}$ is a family of functions labeled by finite subsets of $\mathcal{L}$ and $\Phi_\Lambda : \Omega_\Lambda^{(\mathcal{N})} \mapsto \mathbb{R}$. We introduce the following possible conditions on Φ:

- *Finite range.* There exists $r > 0$ such that $\Phi_\Lambda = 0$ if $\operatorname{diam}(\Lambda) > r$;
- *Translation invariance.* For each $x \in \mathcal{L}$, $\Phi_\Lambda(\sigma) = \Phi_{\Lambda+x}(\vartheta_x \sigma)$.

We note that the potentials (which do not need to satisfy the conditions above) form a linear space. Given $\alpha \geq 0$, we introduce in it the norm $\|\cdot\|_\alpha$ defined by

$$\|\Phi\|_\alpha := \sup_{x\in\mathcal{L}} \sum_{\Lambda \ni x} e^{\alpha|\Lambda|} \sup_{\sigma_\Lambda \in \Omega_\Lambda^{(\mathcal{N})}} |\Phi_\Lambda(\sigma_\Lambda)|$$

We also note that in the translation invariant case we can omit the first supremum above.

Given $\Lambda \subset\subset \mathcal{L}$ and a potential Φ with bounded $\|\cdot\|_0$ norm, the *energy* associated to a configuration σ when the boundary condition outside Λ is (the restriction to Λ^c of) $\tau \in \Omega$, is given by:

$$E_\Lambda(\sigma|\tau) := \sum_{\Gamma\cap\Lambda\neq\emptyset} \Phi_\Gamma(\sigma \circ_\Lambda \tau) \tag{2.1}$$

where

$$(\sigma \circ_\Lambda \tau)_x := \begin{cases} \sigma_x & \text{if } x \in \Lambda \\ \tau_x & \text{if } x \notin \Lambda \end{cases}$$

Note that the sum on the r.h.s. of (2.1) is absolutely convergent (uniformly in σ and τ) by the boundedness of $\|\Phi\|_0$. We also remark that for a finite range potential the map $\tau \mapsto E_\Lambda(\sigma|\tau)$ depends only on $\tau_{\partial_r\Lambda}$.

2.4. The Gibbs measures.

Given a potential Φ of bounded $\|\cdot\|_0$ norm, for each $\Lambda \subset\subset \mathcal{L}$ we define the (finite volume) Gibbs measure in Λ with boundary condition τ as

$$\mu_\Lambda^\tau(\sigma) := \frac{1}{Z_\Lambda^\tau} \exp\{-E_\Lambda(\sigma|\tau)\}$$

where Z_Λ^τ, called partition function, is the normalization constant, i.e.

$$Z_\Lambda^\tau = Z_\Lambda^\tau(\Phi) := \sum_{\sigma\in\Omega_\Lambda^{(\mathcal{N})}} e^{-E_\Lambda(\sigma|\tau)}$$

Note that we have included the inverse temperature in the definition of energy; in fact it will be kept fixed in our analysis.

We regard μ_Λ^τ also as a measure on the whole $\Omega^{(\mathcal{N})}$ by giving zero mass to the configurations σ which do not agree with τ on Λ^c. The (infinite volume) Gibbs states associated to the potential Φ are then the measures μ on $\Omega^{(\mathcal{N})}$ which satisfy the DLR equations

$$\int \mu(d\tau)\ \mu_\Lambda^\tau(f) = \mu(f), \quad \text{for any} \ \ \Lambda \subset\subset \mathcal{L},\ f \in C\left(\Omega^{(\mathcal{N})}\right)$$

For a translationally invariant lattice gas we observe that we have $\Phi_{\{x\}}(\eta) = -\lambda\eta_x + a$ for some constants $\lambda, a \in \mathbb{R}$. We neglect the constant a (which do not affect the definition of the Gibbs measure) and note that λ is interpreted as the *chemical potential*. We also introduce the *activity* $z \in \mathbb{R}^+$ by $z := e^\lambda$ which we use to parametrize lattice gases with different chemical potentials. In such a case we write $\Phi = (z, U)$ where $U = \{\Phi_\Lambda, \Lambda \subset\subset \mathcal{L}, |\Lambda| > 1\}$ and call U the interaction. We shall also write (sometimes) $\mu_{\Lambda,z}^\tau$ (resp. $Z_\Lambda^\tau(z)$) in order to indicate explicitly the dependence on the activity z.

2.5. Strong Mixing Conditions.

In what follows we recall notions concerning some mixing properties of Gibbs measures. Most of the theory has been, up to now, developed in the finite range, translationally invariant case. Extension to not translationally invariant cases, when suitable uniform conditions hold, is, in most of the cases, straightforward. In particular we will be concerned with the so–called strong mixing condition which can be formulated in terms of exponential clustering of truncated expectation with respect to the Gibbs measures in certain domains Λ with τ boundary conditions when this exponential clustering takes place uniformly in Λ and τ. This strong mixing condition implies uniqueness of infinite volume Gibbs measure and its exponential clustering. It can be shown that finite volume strong mixing condition, with constants uniform in the volume and in the boundary conditions, is strictly stronger than the equivalent infinite volume notion (see [Sh], [Ba], [DM], [CM]). As it has been shown by Dobrushin and Shlosman (see [DS2], [DS3]) this strong mixing condition, supposed to hold *for any* (finite or infinite) volume Λ, is equivalent to many other conditions like analyticity properties of thermodynamic and correlation functions or tree–decay of semi–invariants. Interactions giving rise to this kind of nice behavior have been called by Dobrushin and Shlosman *completely analytical.* Among their equivalent complete analyticity conditions, Dobrushin and Shlosman have introduced suitable finite size conditions that they call "constructive conditions". They show that, supposing that there exists a finite domain Λ such that strong mixing condition is satisfied with suitable (depending on Λ) decay constants *for all* subsets of Λ, then a strong mixing condition holds *for all* (finite or infinite) volumes.

We refer to [MO2] for a discussion on the applicability of this point of view. Indeed often the request of exponential clustering for *arbitrary* shape does not fit with many reasonable applications. There are examples (see [MO1]) where nice exponential mixing properties hold for regular domains (like, for instance, cubes) and in infinite volume, whereas they fail to hold for domains with anomalous ratio between boundary and bulk, implying violation of Dobrushin–Shlosman complete analyticity. In [O], [OP], [MO1], [MO2], [MO3] another scenario has been introduced, more suited to the renormalization group problematic. It can be called "restricted complete analyticity" or "complete analyticity for regular domains" This point of view refers to exponential mixing in finite volumes *multiples* of a given cube of size ℓ_0. In the framework of this theory one can develop a constructive condition of the following kind: if $\exists\, \ell_0$ such that a suitable (depending on ℓ_0) mixing condition holds in the cube Q_{ℓ_0}, then the same condition (possibly with worse constants) holds for any *multiple* of Q_{ℓ_0}. This possibility of propagation from finite to arbitrarily large (and even infinite) volumes is called "effectiveness" in [MO2]. Subsequently many results have been obtained in the framework of restricted complete analyticity that could have been problematic and even false in the context of Dobrushin–Shlosman complete analyticity (see, for instance [MO2], [MO3], [MOS], [SS]).

Given a measure μ and two square integrable random variables f, g we denote by $\mu(f;g) := \mu(fg) - \mu(f)\mu(g)$ the covariance between f and g. For $\Delta \subset\subset \mathcal{L}$ we introduce $\mu^\tau_{\Lambda;\Delta}$ as the relativization (projection) of the Gibbs measure μ^τ_Λ to $\Omega^{(\mathcal{N})}_\Delta$, i.e.

$$\mu^\tau_{\Lambda;\Delta}(\sigma_\Delta) := \int \mu^\tau_\Lambda(d\zeta)\ \mathbb{I}_{\zeta_\Delta = \sigma_\Delta}$$

We finally recall that the total variation distance between two measures μ, ν on a *finite* set S is given by

$$\mathrm{Var}(\mu,\nu) := \frac{1}{2}\sum_{\omega\in S} |\mu(\omega) - \nu(\omega)| \equiv \sup_{X\subset S} |\mu(X) - \nu(X)|$$

If $a, b \in \mathbb{R}$, we let $a \wedge b := \min\{a, b\}$. For a finite range potential we introduce the following strong mixing condition.

Condition SM(ℓ_0) (*Strong Mixing*).
Given an integer ℓ_0 we say that the potential Φ satisfies SM(ℓ_0) if there exist two constants $A, \gamma > 0$ such that for any volume

$$\Lambda = \bigcup_{i\in I} Q_{\ell_0}(i), \quad I \subset\subset \mathcal{L}_{\ell_0} \tag{2.2}$$

the following bound holds. For any $x \in \partial_r\Lambda$ and any $\Delta \subset \Lambda$ we have

$$\sup_{\tau\in\Omega^{(\mathcal{N})}} \sup_{a\in\Omega_x} \mathrm{Var}\left(\mu^{\tau\circ_x a}_{\Lambda;\Delta}, \mu^\tau_{\Lambda;\Delta}\right) \le A e^{-\gamma d(x,\Delta)} \tag{2.3}$$

We next discuss *finite size* conditions which imply SM(ℓ_0). Let m be an integer, $m > r$, and consider the cube $Q_{3m}(j)$, $j \in \mathcal{L}_m$. Given a particular lattice direction e_i we can partition $Q_{3m}(j)$ into three parallelepipeds having $d-1$ sides equal to $3m$ and the last one equal to m (slices) with the minimal side parallel to the e_i direction (slice orthogonal to e_i). We write

$$Q_{3m}(j) = Q^{(i,-)}_{3m}(j) \cup Q^{(i,0)}_{3m}(j) \cup Q^{(i,+)}_{3m}(j) \tag{2.4}$$

here $Q^{(i,0)}_{3m}(j)$ denotes the central slice.

Let $P^{(i)}_m(j)$ be the set of all subsets of $Q^{(i,0)}_{3m}(j)$ which are unions of cubes $Q_m(j)$. For $V \in P^{(i)}_m(j)$ let $\partial^{(i,+)}V$, $\partial^{(i,-)}V$ denote the part of $\partial_r V$ in the direction of $e_i, -e_i$ respectively (opposite

r–faces of V). Given $\sigma, \zeta, \tau \in \Omega$, $y \in \partial^{(i,+)}V$, $y' \in \partial^{(i,-)}V$, we denote by $\sigma^{(i,+)}, \zeta^{(i,-)}, \tau$ the configuration obtained from τ by substituting in $\partial^{(i,+)}V$,$\partial^{(i,-)}V$ the restrictions of σ, ζ, respectively:

$$\left(\sigma^{(i,+)}, \zeta^{(i,-)}, \tau\right)_x := \begin{cases} \sigma_x & \text{if } x \in \partial^{(i,+)}V \\ \zeta_x & \text{if } x \in \partial^{(i,-)}V \\ \tau_x & \text{otherwise} \end{cases}$$

analogously we denote by $\sigma_y, \zeta^{(i,-)}, \tau$ the configuration obtained from τ by substituting to τ in y, $\partial^{(i,-)}V$ the restrictions of σ, ζ, respectively:

$$\left(\sigma_y, \zeta^{(i,-)}, \tau\right)_x := \begin{cases} \sigma_x & \text{if } x = y \\ \zeta_x & \text{if } x \in \partial^{(i,-)}V \\ \tau_x & \text{otherwise} \end{cases}$$

finally we denote by $\sigma_y, \zeta_{y'}, \tau$ the configuration obtained by substituting to τ in y, y' the restrictions of σ, ζ:

$$(\sigma_y, \zeta_{y'}, \tau)_x := \begin{cases} \sigma_x & \text{if } x = y \\ \zeta_x & \text{if } x = y' \\ \tau_x & \text{otherwise} \end{cases}$$

of course

$$\tau^{(i,+)}, \tau^{(i,-)}, \tau \equiv \tau_y, \tau^{(i,-)}, \tau \equiv \tau_y, \tau_{y'}, \tau \equiv \tau.$$

We introduce the notation: $Z_V(\tau) := Z_V^\tau$.

Condition C1(m, ε_1) *(see [OP], Eq (1.8)]).*

$$\sup_{j \in \mathcal{L}_m} \sup_{i \in \{1,\cdots,d\}} \sup_{V \in P_m^{(i)}(j)} \sup_{\sigma,\tau \in \Omega^{(\mathcal{N})}} \left| \frac{Z_V\left(\sigma^{(i,+)}, \sigma^{(i,-)}, \tau\right) Z_V\left(\tau^{(i,+)}, \tau^{(i,-)}, \tau\right)}{Z_V\left(\sigma^{(i,+)}, \tau^{(i,-)}, \tau\right) Z_V\left(\tau^{(i,+)}, \sigma^{(i,-)}, \tau\right)} - 1 \right| < \varepsilon_1 \tag{2.5}$$

Condition C2(m, ε_2)

$$\sup_{j \in \mathcal{L}_m} \sup_{i \in \{1,\cdots,d\}} \sup_{V \in P_m^{(i)}(j)} \sup_{y \in \partial^{(i,+)}V} \sup_{\sigma,\tau \in \Omega^{(\mathcal{N})}} \left| \frac{Z_V(\sigma_y, \sigma^{(i,-)}, \tau) Z_V(\tau_y, \tau^{(i,-)}, \tau)}{Z_V(\sigma_y, \tau^{(i,-)}, \tau) Z_V(\tau_y, \sigma^{(i,-)}, \tau)} - 1 \right| < \frac{\varepsilon_2}{m^{d-1}} \tag{2.6}$$

Condition C3(m, ε_3) *(see [O]).*

$$\sup_{j \in \mathcal{L}_m} \sup_{i \in \{1,\cdots,d\}} \sup_{V \in P_m^{(i)}(j)} \sup_{\substack{y \in \partial^{(i,+)}V \\ y' \in \partial^{(i,-)}V}} \sup_{\sigma,\tau \in \Omega^{(\mathcal{N})}} \left| \frac{Z_V(\sigma_y, \sigma_{y'}, \tau) Z_V(\tau_y, \tau_{y'}, \tau)}{Z_V(\sigma_y, \tau_{y'}, \tau) Z_V(\tau_y, \sigma_{y'}, \tau)} - 1 \right| < \frac{\varepsilon_3}{m^{2(d-1)}} \tag{2.7}$$

It is easy to show, using a telescopic argument, that there exists a constants κ such that C2(m, ε_2) implies C1($m, \kappa\varepsilon_2$) and C3(m, ε_3) implies C2($m, \kappa\varepsilon_3$) (see [O]). It is also immediate to see that the results proven for the translationally invariant case in [O], [OP], [MO2] extend to the general case when the space uniform condition holds. We have indeed the following result. Let

$$\varepsilon(d) := [3(2^{d+1} + 1)]^{-d}\, 2^{-2d}\, e^{-4}, \tag{2.8}$$

then condition C1$(m, \varepsilon(d))$ implies the existence of a convergent cluster expansion which, in turn, implies SM(m).

We remark that once we have proven the crucial point which is the *effectiveness*, namely that C1$(m, \varepsilon(d))$ implies SM(m), then, considering the rescaled system whose new single spin variables, labeled by $j \in \mathcal{L}_m$, are the old spin configurations in the blocks $Q_m(j)$, we can apply Dobrushin–Shlosman's results [DS1], [DS2], [DS3] to get all their equivalent mixing and analyticity properties of the Gibbs state for every "multiple" of the Q_m's namely for all volumes Λ of the form (2.2). This is the *restricted complete analyticity* namely the validity of the D–S equivalent properties (see [DS2], [DS3]) for every volume of the form (2.2). In particular SM(ℓ_0) is equivalent to:

Condition SM2(ℓ_0)

Given an integer ℓ_0 we say that the potential Φ satisfies SM2(ℓ_0) if there exist two constants $A, \gamma > 0$ such that for every pair of local functions f, g and every volume of the form (2.2)

$$\sup_{\tau \in \Omega^{(\mathcal{N})}} |\mu_\Lambda^\tau(f; g)| \leq A \, (|S_f| \wedge |S_g|) \, \|f\| \;\; \|g\| \; e^{-\gamma d(S_f, S_g)} \tag{2.9}$$

where we recall S_f, S_g are the supports of f, g.

Indeed the implication C1$(\ell_0, \varepsilon(d)) \Rightarrow$ SM2(ℓ_0) can be obtained directly via cluster expansion by using the methods of references [O], [OP]. We do not reproduce here the results of [O], [OP] but, looking at the application to the renormalization group problem that will be developed in next section, the reader could easily understand these results.

2.6. Lattice gases: Uniform Strong Mixing Conditions.

We here consider just a finite range lattice gas (i.e. $\Omega = \{0,1\}^{\mathcal{L}}$) and introduce some uniform strong mixing conditions which are needed to study the RG map. These conditions say – rougly speaking – that SM(ℓ_0) holds *uniformly* in the external field (one body interaction). Unfortunately, as discussed in the Introduction, we need such a condition also for some non homogeneous external field. Such a condition plays also a crucial role in the ergodic properties of the Kawasaki (conservative) dynamics, [Y].

Given a finite range lattice gas with translationally invariant interaction we introduce the following Condition. We recall $z = \exp\{\lambda\}$ is the activity.

Condition USM($\mathcal{A}$) (*Uniform Strong Mixing*).

Given an open set $\mathcal{A} \subseteq [0, \infty)$, we say that the interaction U satisfies USM($\mathcal{A}$) if for each $z \in \mathcal{A}$ there exists $\ell_0 = \ell_0(z)$ such that condition SM(ℓ_0) holds for (z, U). Furthermore the following is to be satisfied:

(i) for any closed set $\mathcal{C} \subseteq \mathcal{A}$ we can take the constants ℓ_0, A, γ uniform for $z \in \mathcal{C}$;

(ii) we can take $A = A_0 z \wedge z^{-1}$ for some other constant A_0 independent of z.

Remark. We note that for $\mathcal{A} = [0, \varepsilon] \cup [\varepsilon^{-1}, \infty)$ with ε small enough (depending on d, r and $\|\Phi\|_0$) the above conditions holds. Indeed for $z \wedge z^{-1}$ small, SM(1) follows from standard perturbative theory (for instance by using Dobrushin single site criterion [D1]). We can therefore safely replace the set $[0, \infty)$ in Condition USM($\mathcal{A}$) by the compact set $[\varepsilon, \varepsilon^{-1}]$. To avoid delicate continuity questions we introduced (i) above as an independent hypothesis. The same argument shows (ii) is automatically satisfied; we have included it only for convenience.

Condition GUSM(*Global Uniform Strong Mixing*)
We say that Condition GUSM is satisfied if Condition USM($\mathcal{A}$) holds for $\mathcal{A} = [0, \infty)$.

2.7. The multi–grancanonical state.
Let ℓ be a positive integer and $\Lambda \subset \mathcal{L}$ a disjoint union of cubes of side ℓ, i.e.

$$\Lambda = \bigcup_{i \in I} Q_\ell(i), \quad I \subset \mathcal{L}_\ell \tag{2.10}$$

Given a lattice gas with a finite range translationally invariant interaction U, we next define a Gibbs measure in Ω_Λ which has a fixed chemical potential in each cube Q_ℓ. We call such a measure a *multi–grancanonical* state. Let $\underline{z} := \{z_i \in [0, \infty),\ i \in I\}$ the measure $\mu^\tau_{\Lambda,\underline{z}}$ is then defined as a Gibbs measure in Ω_Λ whose potential $\Phi^{\underline{z}} = (\underline{z}, U)$ is given by

$$\Phi^{\underline{z}}_\Gamma(\eta) := \begin{cases} -\eta_x \log z_i & \text{if } \Gamma = \{x\} \text{ and } x \in Q_\ell(i) \\ \Phi_\Gamma(\eta) & \text{if } |\Gamma| > 1 \end{cases}$$

If $I \subset\subset \mathcal{L}_\ell$ the finite volume multi–grancanonical measure is thus defined by

$$\mu^\tau_{\Lambda,\underline{z}}(\eta) = \frac{1}{Z^\tau_\Lambda(\underline{z})} \prod_{i \in I} z_i^{N_i} \cdot \exp\left\{ \sum_{\substack{\Gamma \cap \Lambda \neq \emptyset \\ |\Gamma| > 1}} \Phi_\Gamma(\eta \circ_\Lambda \tau) \right\} \tag{2.11}$$

where $Z^\tau_\Lambda(\underline{z})$ is the normalization constant and

$$N_i := \sum_{x \in Q_\ell(i)} \eta_x \tag{2.12}$$

is the total number of particles in $Q_\ell(i)$. We stress that the multi–grancanonical state $\mu^\tau_{\Lambda,\underline{z}}$ does depend on ℓ.

We shall need a stronger version of Condition USM which is formulated in terms of the multi–grancanonical state.

Condition MUSM($\mathcal{A}$) (*Uniform Strong Mixing for Multigrancanonical States*).
Given an open set $\mathcal{A} \subseteq [0, \infty)$, we say that the interaction U satisfies MUSM($\mathcal{A}$) if the following condition holds. For each closed set $\mathcal{C} \subseteq \mathcal{A}$ there are constants $\ell_0 \in \mathbb{N}, A, \gamma > 0$ such that for any ℓ integer multiple of ℓ_0, any $I \subset\subset \mathcal{L}_\ell$ and any $\underline{z} \in \mathcal{C}^I$ we have that for any Λ of the form given in (2.10) the multi–grancanonical measure $\mu^\tau_{\Lambda,\underline{z}}$ satisfies the bound (2.3).

Condition GMUSM (*Global Uniform Strong Mixing for Multigrancanonical States*).
If Condition MUSM($\mathcal{A}$) holds for $\mathcal{A} = [0, \infty)$ we finally say that Condition GMUSM is satisfied.

We also give an effective finite size condition of type C1 which implies MUSM($\mathcal{A}$). We note that if $V \in P^{(i)}_m(j)$ we have $V = \cup_{k \in \hat{V}} Q_m(k)$ for some $\hat{V} \subset \mathcal{L}_m$ uniquely determined by V. We denote below by $Z_{V,\underline{z}}(\tau)$ the multi–grancanonical partition function as defined in (2.11) with $\ell = m$.

Condition MUC1($\mathcal{A}$)
Given an open set $\mathcal{A} \subseteq [0,\infty]$ we say that MUC1($\mathcal{A}$) holds for the interaction U if for each closed set $\mathcal{C} \subseteq \mathcal{A}$ there exists an integer m such that

$$\sup_{i\in\{1,\cdots,d\}} \sup_{V\in P_m^{(i)}(0)} \sup_{\underline{z}\in\mathcal{C}^{\hat{V}}} \sup_{\sigma,\tau\in\Omega} \left| \frac{Z_{V,\underline{z}}\left(\sigma^{(i,+)},\sigma^{(i,-)},\tau\right) Z_{V,\underline{z}}\left(\tau^{(i,+)},\tau^{(i,-)},\tau\right)}{Z_{V,\underline{z}}\left(\sigma^{(i,+)},\tau^{(i,-)},\tau\right) Z_{V,\underline{z}}\left(\tau^{(i,+)},\sigma^{(i,-)},\tau\right)} - 1 \right| \le \varepsilon(d) \tag{2.13}$$

Indeed, by exploiting the translationally invariance of U and following the same argument as the one used in [O], [OP] it is easy to verify that if MUC1 holds we have that also MUSM($\mathcal{A}$) holds.

Remark 1. In the high temperature regime, $\|U\|_0 \le \varepsilon$ with ε small enough, it is not difficult to show (by using, for instance Dobrushin's single site condition [D1]) that Condition MUC1($[0,\infty)$) holds.

Remark 2. We recall that By [DS2], [DS3] if SM(ℓ_0) holds for the potential (z, U) we can find a neighborhood $\mathcal{O}_\varepsilon(z)$ of z such that MUSM($\mathcal{O}_\varepsilon(z)$) holds for the interaction U.

We stress that the above Remark 2 gives only a local implication. On the global side the relationship between MUSM($\mathcal{A}$) and USM($\mathcal{A}$) is not trivial. It is in fact possible to have a sort of layering phase transition which prevents MUSM($\mathcal{A}$) to hold even though USM($\mathcal{A}$) does hold. On the positive side, following an argument in the same spirit as [MOS] (i.e. that no phase transition may happen in a one–dimensional boundary of a regular two–dimensional domain), we rule out such a possibility in $d = 2$. We have in fact the following Theorem.

Theorem 2.1. *Let $d = 2$. Then USM($\mathcal{A}$) $\Longrightarrow$ MUSM($\mathcal{A}$).*

On the negative side we show that the aforementioned pathology may indeed happen. In Appendix A.2 we give in fact an *ad hoc* example of an interaction U (in $d = 3$) such that:

– GUSM holds;
– there exist $\underline{z}$ and Λ of the form (2.10) such that the multi–grancanonical measure associated to $(\underline{z}, U)$ exhibits long range order. In particular there exist τ, τ' such that

$$\liminf_{\ell\to\infty} \mathrm{Var}\left(\mu^{\tau'}_{\Lambda,\underline{z};\{0\}}, \mu^{\tau}_{\Lambda,\underline{z};\{0\}}\right) > 0$$

We finally mention that, in the context of the two–dimensional Ising model, the above Theorem implies the following. Consider a standard Ising model with a non–homogeneous external field which is however constant in cubes of side ℓ; then for each $\beta < \beta_c$, there exist ℓ and $L \gg \ell$ such that SM(L) holds uniformly in the external field.

2.8. Block Averaging Transformation (BAT)

Let μ_z be the (unique) infinite volume Gibbs measure of a finite range translationally invariant lattice gas satisfying Condition SM(ℓ_0) and ℓ an integer. In this case we can define the block averaging transformation directly in infinite volume. We partition the lattice $\mathcal{L} \equiv \mathbb{Z}^d$ into cubes of side ℓ, i.e. $\mathcal{L} = \bigcup_{i\in\mathcal{L}_\ell} Q_\ell(i)$. We recall that the random variable N_i has been defined in (2.12); it takes values in the set

$$\Omega_i^{(\ell)} := \{0, 1, \ldots, \ell^d\} \tag{2.14}$$

We then define the centered and renormalized random variables M_i as in (1.3); it takes values in the set $\bar\Omega_i^{(\ell)}$ defined in (1.4). We finally let $\underline{M} := \{M_i,\ i \in \mathcal{L}_\ell\}$. The BAT renormalized measure, that we denote by $\mu_z^{(\ell)}$, is then the (joint) probability distribution of $\underline{M}$ under μ_z. Denoting the renormalized configuration by $\underline{m} = \{m_i \in \bar\Omega_i^{(\ell)}\ , i \in \mathcal{L}_\ell\}$, the measure $\mu_z^{(\ell)}$ is formally given by $\mu_z^{(\ell)}(\underline{m}) = \mu_z(\underline{M} = \underline{m})$. We avoid the troublesome issue of describing Gibbs measures on non compact single spin space (see [EFS] for a discussion) and consider $\mu_z^{(\ell)}$ only for finite ℓ. Therefore the setup previously described applies to the finite single spin space $\bar\Omega_i^{(\ell)}$.

It is also possible to use a finite volume setup. Given the integer p we will denote by $\Lambda_p \subset\subset \mathcal{L}$ a cube with side $2d\ell p$. We have $\Lambda_p = \bigcup_{i \in I_p} Q_\ell(i)$ where $I_p \subset\subset \mathcal{L}_\ell$ is a cube of side $2dp$. Let $\mu^\tau_{\Lambda_p,z}$ be the finite volume Gibbs measure for our lattice gas with activity z enclosed in Λ_p with τ boundary condition. We denote by $\mu^{(\ell,\tau)}_{I_p,z}$ the finite volume renormalized measure arising from the application to $\mu_{z,p}$ of the Block Averaging Transformation on scale ℓ; it is defined as:

$$\mu^{(\ell,\tau)}_{I_p,z}\left(\{m_i,\ i \in I_p\}\right) := \mu^\tau_{\Lambda_p,z}\left(\{M_i = m_i,\ i \in I_p\}\right), \qquad m_i \in \bar\Omega_i^{(\ell)} \tag{2.15}$$

2.9. Main results.

We first discuss the case when the global Condition GMUSM holds. The most relevant example is the standard two–dimensional Ising model for $T > T_c$. In such a case we are able to prove that the renormalized measure $\mu_z^{(\ell)}$ is, for each (finite) ℓ large enough, Gibbsian w.r.t. a potential $\Phi^{(\ell)}$ of bounded $\|\cdot\|_\alpha$ norm (for suitable $\alpha > 0$). We can furthermore control the ℓ dependence of the norm $\|\Phi^{(\ell)}\|_\alpha$. We note (see [IS], [N]) that $\mu_z^{(\ell)}$ converges weakly to $\otimes_{i\in\mathcal{L}_\ell}\varphi_i$ where φ_i denotes a standard Gaussian measure. Accordingly, $\Phi^{(\ell)}$ should converge to the interaction of independent harmonic oscillators. Unfortunately, as the limiting interaction has not finite norm (since the limiting single spin space is unbounded), this convergence cannot be described in the $\|\cdot\|_\alpha$ norm. However this lack of convergence affects only the (somehow trivial) short range part of the interaction; we will decompose the potential into a short and a long range part $\Phi^{(\ell)} = \Phi^{(\ell),sr} + \Phi^{(\ell),lr}$ ($\exists \kappa \in \mathbb{N}$: $\Phi_X^{(\ell),sr} \equiv 0$ if $\mathrm{diam}(X) \geq \kappa$). We then introduce a large field cutoff (diverging as $\ell \to \infty$) to control the short range part: it will converge to the potential of independent harmonic oscillator for values of the image variables within the cutoff. We note that this result would be false for large image variables. The precise statement is given in the following Theorem.

Theorem 2.2. *Let U satisfy GMUSM. Then there exists $\alpha > 0$ such that for any $z \in (0,\infty)$ and ℓ large enough $\mu_z^{(\ell)}$ is a translationally invariant Gibbs measure w.r.t. a potential $\Phi^{(\ell)}$ for which*

$$\left\|\Phi^{(\ell)}\right\|_\alpha \leq K(\ell)$$

for some constant $K(\ell) < \infty$.
Furthermore it is possible to decompose the potential into a short and a long range part, $\Phi^{(\ell)} = \Phi^{(\ell),sr} + \Phi^{(\ell),lr}$, such that $\exists \kappa \in \mathbb{N} : \Phi_I^{(\ell),sr} \equiv 0$ if $\mathrm{diam}(I) \geq \kappa$ and the following holds:

(i) for the same α as before

$$\lim_{\ell\to\infty} \left\|\Phi^{(\ell),lr}\right\|_\alpha = 0$$

(ii) there exists a constant $a>0$ such that

$$\lim_{\ell\to\infty} \sup_{\substack{m_I\in\bar\Omega_I^{(\ell)}\\ |m_I|\le \ell^a}} \left|\Phi_I^{(\ell),sr}(m_I)\right| = 0, \qquad \text{for any } \ I\subset\subset \mathcal{L}_\ell,\ |I|\ge 2$$

$$\lim_{\ell\to\infty} \sup_{\substack{m_i\in\bar\Omega_i^{(\ell)}\\ |m_i|\le \ell^a}} \left|\Phi_{\{i\}}^{(\ell),sr}(m_i)+\frac{1}{2}m_i^2\right| = 0 \qquad \text{for any } \ i\in\mathcal{L}_\ell$$

When we assume only the local Condition MUSM($\mathcal{A}$) our results are much weaker. Before discussing them, let us first note that for the standard two–dimensional Ising model this Condition holds for $T\le T_c$ away from the phase coexistence line $z=z^*$ (z^* corresponds to zero magnetic field in the spin variables), i.e. for each $T\le T_c$, MUSM($\mathcal{A}$) holds for $\mathcal{A}=[0,\infty)\setminus\{z^*\}$. We are not able to deal directly with the BAT defined in infinite volume, but we have to start from the finite volume transformation and take the thermodynamic limit. Moreover, we need to introduce a large field cutoff also in the long range part of the interaction. These difficulties are of course related to the limiting single spin space for which the usual (i.e. uniform in all possible b.c.) Gibbsian formalism do not apply. We refer to [EFS] for a discussion on the problems connected with the introduction of a norm for interactions defined on a non compact spaces. Our results are formulated as follows.

Theorem 2.3. *Let U satisfies Condition MUSM($\mathcal{A}$) and $z>0$, $z\in\mathcal{A}$. Let also $\Phi^{(\ell,\tau)}$ be the (finite volume) potential associated to the (finite volume) renormalized measure $\mu_{I_p,z}^{(\ell,\tau)}$. Then it is possible to decompose the potential into a short and a long range part, $\Phi^{(\ell,\tau)}=\Phi^{(\ell,\tau),sr}+\Phi^{(\ell,\tau),lr}$, such that $\exists\kappa\in\mathbb{N}:\Phi_I^{(\ell,\tau),sr}\equiv 0$ if $\operatorname{diam}(I)\ge\kappa$ and the following holds. There is a constant $\varepsilon=\varepsilon(z)>0$ such that for any $I\subset\subset\mathcal{L}_\ell$ and any ℓ large enough*

$$\begin{aligned}
&\exists \lim_{p\to\infty}\Phi_I^{(\ell,\tau),lr}(m_I) =: \Phi_I^{(\ell),lr}(m_I), \qquad \text{uniformly for } \ m_I\in\bar\Omega_I^{(\ell)},\ |m_i|\le\varepsilon\sqrt{\chi|Q_\ell|},\ \tau\in\Omega\\
&\exists \lim_{p\to\infty}\Phi_I^{(\ell,\tau),sr}(m_I) =: \Phi_I^{(\ell),sr}(m_I) \quad , \quad \text{uniformly for } \ m_I\in\bar\Omega_I^{(\ell)},\ \tau\in\Omega
\end{aligned}\tag{2.16}$$

Furthermore, there are $\alpha=\alpha(z)>0, a=a(z)>0$ such that for the same $\varepsilon=\varepsilon(z)$ as before

$$\lim_{\ell\to\infty}\sum_{I\ni 0} e^{\alpha|I|} \sup_{\substack{m_I\in\bar\Omega_I^{(\ell)}\\ |m_I|\le\varepsilon\sqrt{\chi|Q_\ell|}}} \left|\Phi_I^{(\ell),lr}(m_I)\right| = 0$$

$$\lim_{\ell\to\infty} \sup_{\substack{m_I\in\bar\Omega_I^{(\ell)}\\ |m_I|\le \ell^a}} \left|\Phi_I^{(\ell),sr}(m_I)\right| = 0 \qquad \text{for any } \ I\subset\subset \mathcal{L}_\ell,\ |I|\ge 2$$

$$\lim_{\ell\to\infty} \sup_{\substack{m_i\in\bar\Omega_i^{(\ell)}\\ |m_i|\le \ell^a}} \left|\Phi_{\{i\}}^{(\ell),sr}(m_i)+\frac{1}{2}m_i^2\right| = 0 \qquad \text{for any } \ i\in\mathcal{L}_\ell$$

Warnings.

– Taking advantage of the symmetry of our Conditions w.r.t. the map $z \mapsto z^{-1}$, we shall assume, without loss of generality, that all the activities are bounded by 1. This will be used extensively without further mention.
– We denote by C a generic positive constant whose numerical value can change from line to line. From the statements it will appear clear from which parameters it depends on.

3. Computing the renormalized potential via cluster expansion

In this section we discuss the BAT transformation in finite volume. We will compute the renormalized interaction via a cluster expansion: the convergence of the expansion will be ensured by the validity of condition $\mathrm{C1}(m,\varepsilon(d))$ for the constrained (multi–canonical) systems. This condition C1, in turn, will be deduced from the MUSM property of the original system in Section 5.

To simplify notation we write the Boltzmann's factor (with τ boundary condition) for a configuration η in the volume Λ, $\eta \in \{0,1\}^\Lambda$, as $\exp\left(+H_\Lambda(\eta|\tau)\right)$ where

$$H_\Lambda(\eta|\tau) := -E_\Lambda(\eta|\tau) \tag{3.1}$$

Let us set $L := d\ell$; given the odd integer p, let Λ_p be the cube with side $2d\ell p$ given by

$$\Lambda_p := \begin{cases} \{x=(x_1,\dots,x_d)\in\mathcal{L} : -d\ell\left(p+\frac{1}{2}\right)+d\ell+1 \le x_j \le +d\ell\left(p+\frac{1}{2}\right), j=1,\dots,d\} & d\ell \text{ even} \\ \{x=(x_1,\dots,x_d)\in\mathcal{L} : -\left(d\ell p+\frac{d\ell-1}{2}\right)+d\ell \le x_j \le d\ell p+\frac{d\ell-1}{2}, j=1,\dots,d\} & d\ell \text{ odd} \end{cases}$$

We can write $\Lambda_p = \cup_{i\in I_p} Q_\ell(i)$ where $I_p \subset\subset \mathcal{L}_\ell$ is the cube of side $2dp$ given by

$$I_p = \begin{cases} \{i\in\mathcal{L}_\ell : -d\left(p+\frac{1}{2}\right)+d+1 \le x_j \le +d\left(p+\frac{1}{2}\right), j=1,\dots,d\} & \text{if } d \text{ is even} \\ \{i\in\mathcal{L}_\ell : -\left(dp+\frac{d-1}{2}\right)+d \le x_j \le dp+\frac{d-1}{2}, j=1,\dots,d\} & \text{if } d \text{ is odd} \end{cases}$$

Let us introduce the quantity:

$$Z^{(\ell,\tau)}_{\Lambda_p,\underline{n}} := \sum_{\eta\in\Omega^{(\underline{n})}_{\Lambda_p}} e^{H_{\Lambda_p}(\eta|\tau)} \tag{3.2}$$

where $\underline{n} = \{n_i,\ i\in I_p\} \in \Omega^{(\ell)}_{I_p} := \otimes_{i\in I_p}\Omega^{(\ell)}_i \equiv \{0,1,\dots,\ell^d\}^{I_p}$ and

$$\Omega^{(n_i)}_i := \left\{\eta\in\{0,1\}^{Q_\ell(i)} \ : \ N_i(\eta)=n_i\right\}, \qquad \Omega^{(\underline{n})}_{\Lambda_p} := \bigotimes_{i\in I_p}\Omega^{(n_i)}_i \tag{3.3}$$

It is convenient to look at the renormalized measure $\mu^{(\ell,\tau)}_{I_p,z}$ in (2.15) in terms of the variables $\underline{n}$; such measure is Gibbs w.r.t. to the renormalized Hamiltonian given by

$$H^{(\ell,\tau)}_{\Lambda_p}(\underline{n}) = \log Z^{(\ell,\tau)}_{\Lambda_p,\underline{n}} \tag{3.4}$$

Given $\underline{n}\in\Omega^{(\ell)}_{I_p}$; we can look at the quantity $Z^{(\ell,\tau)}_{\Lambda_p,\underline{n}}$ defined in (3.2) as the partition function of a (generally not translationally invariant) system which is the original lattice gas *constrained* to have fixed values of the total number of particles in each block $Q_\ell(i)$, $i \in I_p$. Its elementary configurational variables are the original spin configurations in each block $Q_\ell(i)$ compatible with the assigned value n_i of N_i namely the set $\Omega_{\Lambda_p}^{(\underline{n})}$ defined in (3.3). The elements of $\Omega_i^{(n_i)}$ will be called *block variables* not to be confused with the renormalized variables n_i. We also call *multi–canonical* these constrained systems.

We will consider blocks of these block variables of size d; these corresponds to the blocks $Q_L(i)$ with $L = d\ell$ in the original variables. The reason for this choice will appear clear in the following sections: it corresponds to the minimal size for which we are able to prove, for the constrained model, the validity of our Condition C1$(m, \varepsilon(d))$. In other words, to meet Condition C1$(m, \varepsilon(d))$ we have to choose $m = d$ and ℓ sufficiently large. With respect to the general setting of Section 2 we have $\Omega_i = \Omega_i^{(n_i)}$ whereas the potential is the one inherited by the original model. In particular, if we choose ℓ larger that the range r of the original interaction, then only contiguous blocks will interact. We repeat that the size of the blocks that in Section 2 was generically called m now equals d. The main result of this section is stated as follows, where, for $V \in P_L^{(k)}(i)$, we let $\hat{V} \subset\subset \mathcal{L}_\ell$ be such that $V = \cup_{j\in\hat{V}} Q_\ell(j)$.

Theorem 3.1. *Consider a d–dimensional lattice gas with finite range, translationally invariant interaction. Let $\ell \in \mathbb{N}$ and suppose there exists a closed $\mathcal{D} \subseteq [0,1]$ such that*

$$\sup_{i\in I_p} \sup_{k=1,\cdots,d} \sup_{V\in P_L^{(k)}(i)} \sup_{\underline{n}\in\mathcal{D}_{\hat{V}}^{(\ell)}} \sup_{\sigma,\zeta,\tau} \left| \frac{Z_{V,\underline{n}}\left(\sigma^{(k,+)},\sigma^{(k,-)},\tau\right) Z_{V,\underline{n}}\left(\zeta^{(k,+)},\zeta^{(k,-)},\tau\right)}{Z_{V,\underline{n}}\left(\sigma^{(k,+)},\zeta^{(k,-)},\tau\right) Z_{V,\underline{n}}\left(\zeta^{(k,+)},\sigma^{(k,-)},\tau\right)} - 1 \right| \le \delta(\ell) \tag{3.5}$$

where $\mathcal{D}_{\hat{V}}^{(\ell)} := (|Q_\ell|\mathcal{D})^{\hat{V}} \cap \Omega_{\hat{V}}^{(\ell)}$ and $\delta(\ell) \to 0$ as $\ell \to \infty$.

Then, the measure $\mu_{I_p,z}^{(\ell,\tau)}$ defined in (2.15) is Gibbsian w.r.t. a potential $\Phi^{(\ell,\tau)} = \{\Phi_X^{(\ell,\tau)},\ X \subset I_p\}$. Let

$$\mathcal{M}_X^{(\ell)} := \left(\frac{|Q_\ell|\mathcal{D} - \rho(z)}{\sqrt{\chi|Q_\ell|}}\right)^X \bigcap \bar{\Omega}_X^{(\ell)}$$

We have the following:

(i) For each $X \subset I_p$ with $d_\ell(X, I_p^c) > 3d$ and $m_X \in \mathcal{M}_X^{(\ell)}$, $\Phi_X^{(\ell,\tau)}$ does not depend on τ (and I_p). In particular for each $X \subset\subset \mathcal{L}_\ell$

$$\exists \lim_{p\to\infty} \Phi_X^{(\ell,\tau)}(m_X) =: \Phi_X^{(\ell)}(m_X), \qquad \textit{uniformly for } m_X \in \mathcal{M}_X^{(\ell)}, \tau \in \Omega \tag{3.6}$$

(ii) Let $\Phi^{(\ell)} = \{\Phi_X^{(\ell)},\ X \subset\subset \mathcal{L}_\ell\}$, we have a decomposition $\Phi^{(\ell)} = \Phi^{(\ell),sr} + \Phi^{(\ell),lr}$ where $\Phi_X^{(\ell),sr} \equiv 0$ if $\mathrm{diam}_\ell(X) > 3d$ and there are constants $\alpha > 0$, C such that:

$$\sum_{X\ni 0} e^{\alpha|X|} \sup_{m_X\in\mathcal{M}_X^{(\ell)}} \left|\Phi_X^{(\ell),lr}(m_X)\right| \le C\delta(\ell) \tag{3.7}$$

Remark. The potential $\Phi_X^{(\ell,\tau)}$ will be explicitly constructed (see (3.70) and (3.69)) below. In Section 5 we will show that we can take $\Phi_{\{i\}}^{(\ell),sr}(m_i) = -m_i^2/2$ and there exists a constant $a > 0$ such that for each X, $|X| \geq 2$

$$\sup_{\substack{m_X \in \mathcal{M}_X^{(\ell)} \\ |m_X| < \ell^a}} \left| \Phi_X^{(\ell),sr}(m_X) \right| < \gamma(\ell)$$

with $\gamma(\ell) \to 0$ if $\ell \to \infty$.

Similarly to what has been done in [HK], in order to compute the renormalized potential and prove Theorem 3.1, we are going to apply to the constrained systems the method developed in [O], [OP]. To simplify the exposition we will treat in detail only the two–dimensional case. An analogous treatment can be developed for the d–dimensional case along the lines of [OP]. For the same reason, we discuss only the case of periodic boundary condition in Λ_p; the case of general boundary condition can be treated along the same lines with minor changes giving rise to estimates uniform in τ.

In the rest of this section we will express the coordinates of points and components of vectors in $\mathcal{L}_L$ in L units. Let us denote by e_1, e_2, respectively, the coordinate unit vectors in $\mathcal{L}_L$: $e_1 = (1,0)$ horizontal, $e_2 = (0,1)$ vertical. Recall that since now $d = 2$, we have $L = 2\ell$. We further partition $\mathcal{L}_L$ into four sub–lattices of spacing $2L$:

$$\mathcal{L}_L = \mathcal{L}_{2L}^A \cup \mathcal{L}_{2L}^B \cup \mathcal{L}_{2L}^C \cup \mathcal{L}_{2L}^D$$

where:

$$\begin{aligned}
\mathcal{L}_{2L}^A &:= \{i = (i_1, i_2) \in \mathcal{L}_L : i_1 = 2j_1, i_2 = 2j_2, \text{for some integers } \ j_1, j_2\} \\
\mathcal{L}_{2L}^B &:= \mathcal{L}_{2L}^A + e_2 \\
\mathcal{L}_{2L}^C &:= \mathcal{L}_{2L}^A + e_1 + e_2 = \mathcal{L}_{2L}^B + e_1 \\
\mathcal{L}_{2L}^D &:= \mathcal{L}_{2L}^A + e_1 = \mathcal{L}_{2L}^C + e_2 = \mathcal{L}_{2L}^B + e_1 + e_2
\end{aligned} \tag{3.8}$$

We also set, for $i \in \mathcal{L}_L$:

$$A_i := Q_L(2i), \quad B_i := Q_L(2i + e_2) \quad C_i := Q_L(2i + e_1 + e_2) \quad D_i := Q_L(2i + e_1). \tag{3.9}$$

(See Fig. 1).

Then we can partition the torus Λ_p into the union of the L–blocks of the four types: A, B, C, D:

$$\Lambda_p = \mathcal{A}_p \cup \mathcal{B}_p \cup \mathcal{C}_p \cup \mathcal{D}_p$$

where

$$\mathcal{A}_p := \{A_i : \ |i_j| \leq \frac{p-1}{2}, \ j = 1, 2\}$$

and similarly for $\mathcal{B}_p, \mathcal{C}_p, \mathcal{D}_p$.

Given a renormalized configuration of our multi–canonical model and a block A_i we denote by α_i a generic original lattice gas configuration compatible with the four renormalized configurations $\{n_j, \ j \in \mathcal{L}_\ell : \ Q_\ell(j) \subset A_i\}$; in other words: $\alpha_i \in \otimes_{j: Q_\ell(j) \subset A_i} \Omega_j^{(n_j)}$ (recall that $\ell = \frac{L}{2}$). Similarly for $\beta_i, \gamma_i, \delta_i$. We simply denote by $\alpha, \beta, \gamma, \delta$ the configurations in $\mathcal{A}_p, \mathcal{B}_p, \mathcal{C}_p, \mathcal{D}_p$, respectively.

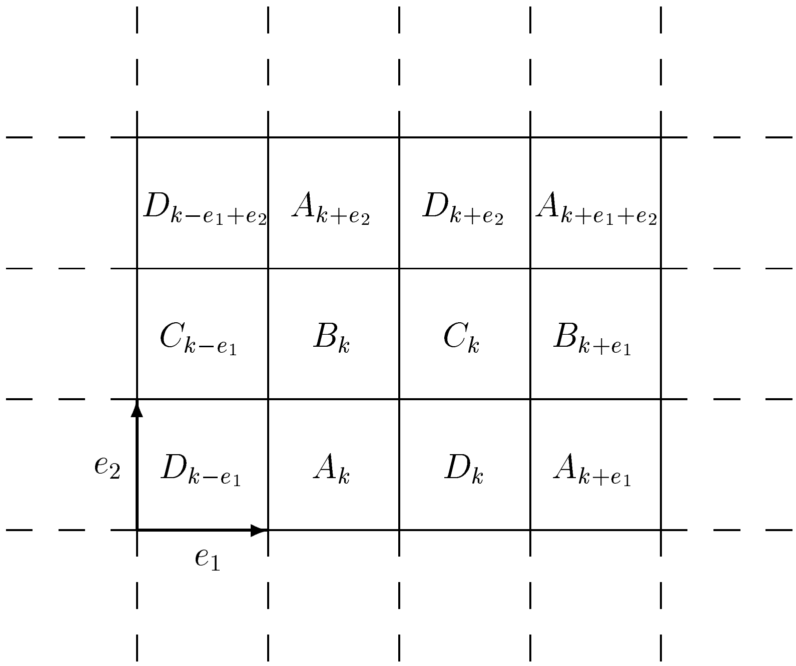

Fig. 1

Let us now quickly describe our strategy. We want to transform the constrained system into a polymer system (see, for instance, [GrK], [KP], [D3]) which, by condition (3.5) will turn out to be in the small activity region. To be more precise we shall prove the following formula:

$$Z^{(\ell)}_{\Lambda_p,\underline{n}} = \bar{Z}^{(\ell)}_{\Lambda_p,\underline{n}} \Xi^{(\ell)}_{\Lambda_p,\underline{n}} \tag{3.10}$$

where $\bar{Z}^{(\ell)}_{\Lambda_p,\underline{n}}$ is factorized in the sense that it has the form of a product of partition functions in suitable volumes not depending on p; the dependence on $\underline{n}$ of the single factors is local. The partition function $\bar{Z}^{(\ell)}_{\Lambda_p,\underline{n}}$ describes the *reference system* around which we perform a perturbative expansion. On the other hand, $\Xi^{(\ell)}_{\Lambda_p,\underline{n}}$ is the partition function of a gas of polymers; it has the form

$$\Xi^{(\ell)}_{\Lambda_p,\underline{n}} = 1 + \sum_{k\geq 1} \sum_{R_1,\cdots,R_k} \prod_{j=1}^{k} \zeta_{R_j}(\underline{n}) \tag{3.11}$$

where the polymers R_j, that will be defined below, are geometrical local objects living on scale $L = 2\ell$; the sum in (3.11) is restricted to "non–intersecting" polymers so that the unique interaction between polymers is a pairwise hard core exclusion. Finally the activity $\zeta_{R_j}(\underline{n})$ depends only on the n_i's with i localized on the polymer. It is already clear from this preliminary discussion that expression (3.11) is well suited to compute renormalized potential: in order to get good estimates of the norm of the renormalized potential we shall need that the polymer system described by $\Xi^{(\ell)}_{\Lambda_p,\underline{n}}$ is in the small activity region.

To get expression (3.10) we will perform a sequence of block decimations like in [O], [OP]. We start by integrating over the δ–variables, then the γ–variables, the β–variables and, finally, the α–variables. Using Condition (3.5) we will be able to prove that at each step of decimation the system described by the surviving variables is weakly coupled.

We use the following notation for the interaction (which is defined independently of the multi–canonical constraints) between two sets Λ_1 and Λ_2:

$$W_{\Lambda_1,\Lambda_2}(\eta_{\Lambda_1}|\eta_{\Lambda_2}) := W(\eta_{\Lambda_1}|\eta_{\Lambda_2}) = H_{\Lambda_1\cup\Lambda_2}(\eta_{\Lambda_1},\eta_{\Lambda_2}) \;-\; H_{\Lambda_1}(\eta_{\Lambda_1}) \;-\; H_{\Lambda_2}(\eta_{\Lambda_2}) \tag{3.12}$$

where $\eta_{\Lambda_1},\eta_{\Lambda_2} \in \{0,1\}^{\Lambda_1},\{0,1\}^{\Lambda_2}$, respectively. Recalling that L is larger than the range of the interaction, we can write:

$$\begin{aligned} H_{\Lambda_p}(\sigma_{\Lambda_p}) = &\sum_{k_1:A_{k_1}\in\mathcal{A}_p} H_{A_{k_1}}(\alpha_{k_1}) + \sum_{k_2:B_{k_2}\in\mathcal{B}_p} H_{B_{k_2}}(\beta_{k_2}) + W_{B_{k_2},\mathcal{A}_p}(\beta_{k_2}|\alpha) \\ &+ \sum_{k_3:C_{k_3}\in\mathcal{C}_p} H_{C_{k_3}}(\gamma_{k_3}) + W_{C_{k_3},\mathcal{A}_p\cup\mathcal{B}_p}(\gamma_{k_3}|\beta,\alpha) \\ &+ \sum_{k_4:D_{k_4}\in\mathcal{D}_p} H_{D_{k_4}}(\delta_{k_4}) + W_{D_{k_4},\mathcal{A}_p\cup\mathcal{B}_p\cup\mathcal{C}_p}(\delta_{k_4}|\gamma,\beta,\alpha) \end{aligned} \tag{3.13}$$

Again the above decomposition of H holds independently of the constraints on the number of particles in the blocks; in (3.13) we have only used that $L > r$ so that there is no direct interaction between blocks belonging to the same sub–lattice.

To simplify notation we shall often omit from H, W the subscripts referring to the various domains; the symbols used for the arguments of the functions H, W should be sufficiently clarifying; moreover we will also omit the explicit extensions of the sums (or products) over k_1, k_2, k_3, k_4 as well as the one over $\alpha \in \otimes_{i:Q_\ell(i)\subset\mathcal{A}_p}\Omega^{(n_i)}$, and similarly for β,γ,δ. We have:

$$\begin{aligned} Z^{(\ell)}_{\Lambda_p,\underline{n}} = &\sum_\alpha\prod_{k_1}\exp\left(H(\alpha_{k_1})\right)\sum_\beta\prod_{k_2}\exp\left(H(\beta_{k_2})+W(\beta_{k_2}|\alpha)\right) \\ &\times\sum_\gamma\prod_{k_3}\exp\left(H(\gamma_{k_3})+W(\gamma_{k_3}|\beta,\alpha)\right)\sum_\delta\prod_{k_4}\exp\left(H(\delta_{k_4})+W(\delta_{k_4}|\gamma,\beta,\alpha)\right) \end{aligned} \tag{3.14}$$

We first perform the sum over δ variables; using that the sums over different δ_{k_4} are decoupled since the size L of the blocks is larger than the range of the interaction, we get:

$$Z^{(\ell)}_{\Lambda_p} = \sum_\alpha\cdots\sum_\beta\cdots\sum_\gamma\cdots\prod_{k_4} Z_{D_{k_4}}\left((\beta,\gamma)^u,(\beta,\gamma)^d,\alpha)\right) \tag{3.15}$$

where by $Z_{D_{k_4}}\left((\beta,\gamma)^u,(\beta,\gamma)^d,\alpha)\right)$ we denote the partition function in D_{k_4} with boundary conditions $(\beta,\gamma)^u$ on the top (up) and $(\beta,\gamma)^d$ on the bottom (down) of D_{k_4} (see Fig. 2). More explicitly $(\beta,\gamma)^u$ is given by the restriction of β,γ to (simply called configuration in): $Q_L(2k_4+e_2)\cup Q_L(2k_4+e_2+e_1)\cup Q_L(2k_4+e_2+2e_1)\equiv B_{k_4}\cup C_{k_4}\cup B_{k_4+e_1}$ whereas $(\beta,\gamma)^d$ is the configuration in $Q_L(2k_4-e_2)\cup Q_L(2k_4-e_2+e_1)\cup Q_L(2k_4-e_2+2e_1)\equiv B_{k_4-e_2}\cup C_{k_4-e_2}\cup B_{k_4+e_1-e_2}$. Finally α in (3.15) denotes the configuration in $Q_L(2k_4)\cup Q_L(2k_4+2e_1)\equiv A_{k_4}\cup A_{k_4+e_1}$.

Notice that we are also presently omitting the explicit dependence on $\underline{n}$ and L. Let 0 denote a given reference configuration in $\Omega^{(\underline{n})}_{\Lambda_p}$. We write

$$\begin{aligned} Z_{D_{k_4}}\left((\beta,\gamma)^u,(\beta,\gamma)^d,\alpha)\right) = &\left(\frac{Z_{D_{k_4}}\left((\beta,\gamma)^u,(\beta,\gamma)^d,\alpha)\right) Z_{D_{k_4}}\left((0)^u,(0)^d,\alpha)\right)}{Z_{D_{k_4}}\left((\beta,\gamma)^u,(0)^d,\alpha)\right) Z_{D_{k_4}}\left((0)^u,(\beta,\gamma)^d,\alpha)\right)} - 1 + 1\right) \\ &\times\frac{Z_{D_{k_4}}\left((\beta,\gamma)^u,(0)^d,\alpha)\right) Z_{D_{k_4}}\left((0)^u,(\beta,\gamma)^d,\alpha)\right)}{Z_{D_{k_4}}\left((0)^u,(0)^d,\alpha)\right)} \end{aligned} \tag{3.16}$$

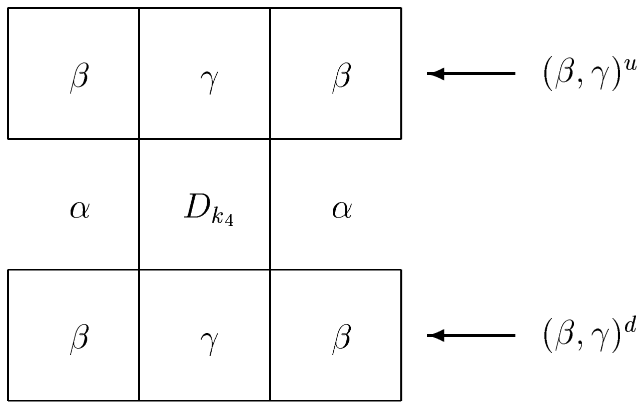

Fig. 2

Where by $(0)^u,(0)^d,\alpha$ we mean the boundary condition on D_{k_4} obtained from $(\beta,\gamma)^u,(\beta,\gamma)^d,\alpha$ by substituting (β,γ) with (0) both in the "up" and "down" blocks; similarly $(\beta,\gamma)^u,(0)^d,\alpha$; $(0)^u,(\beta,\gamma)^d,\alpha$ denote the boundary conditions on D_{k_4} obtained from $(\beta,\gamma)^u,(\beta,\gamma)^d,\alpha$ by substituting (β,γ) with (0) only in the "down", "up" blocks, respectively. We call the above operation "splitting" of the partition function $Z_{D_{k_4}}\big((\beta,\gamma)^u,(\beta,\gamma)^d,\alpha)\big)$ in the vertical e_2 direction. We set

$$\Phi_D^{(4)}(\alpha,\beta,\gamma):=\frac{Z_{D_{k_4}}\big((\beta,\gamma)^u,(\beta,\gamma)^d,\alpha)\big)\,Z_{D_{k_4}}\big((0)^u,(0)^d,\alpha)\big)}{Z_{D_{k_4}}\left((\beta,\gamma)^u,(0)^d,\alpha)\right)Z_{D_{k_4}}\left((0)^u,(\beta,\gamma)^d,\alpha)\right)}-1 \tag{3.17}$$

The quantity $\Phi_D^{(4)}(\alpha,\beta,\gamma)$ can be considered as an effective interaction potential between α,β,γ variables coming from decimation of the δ variables. In what follows we will exploit condition (3.5) above to deduce that $\Phi_D^{(4)}(\alpha,\beta,\gamma)$ and other similar quantities are uniformly small.

We can write:

$$\begin{aligned} Z_{\Lambda_p}^{(\ell)}=&\sum_\alpha\dots\sum_\beta\dots\sum_\gamma\prod_{k_3}\exp\left(H(\gamma_{k_3})+W(\gamma_{k_3}|\beta,\alpha)\right)\\ &\times Z_{D_{k_3+e_2}}\big((0)^u,(\beta,\gamma_{k_3})^d,\alpha\big)\,Z_{D_{k_3}}\big((\beta,\gamma_{k_3})^u,(0)^d,\alpha\big)\\ &\times\prod_{k_4}\left[Z_{D_{k_4}}\left((0)^u,(0)^d,\alpha)\right)\right]^{-1}\prod_{k_4}\Big(1+\Phi_{D_{k_4}}^{(4)}(\alpha,\beta,\gamma)\Big)\end{aligned} \tag{3.18}$$

In (3.18) above we associated to every C_{k_3} block in $\mathcal{C}_p$ the two terms $Z_{D_{k_3+e_2}}\big((0)^u,(\beta,\gamma_{k_3})^d,\alpha\big)$, $Z_{D_{k_3}}\big((\beta,\gamma_{k_3})^u,(0)^d,\alpha\big)$ coming from the splitting of the original partition functions over the volumes $D_{k_3+e_2},D_{k_3}$, respectively. Notice that

$$\begin{aligned}&\sum_{\gamma_{k_3}}\exp\left(H(\gamma_{k_3})+W(\gamma_{k_3}|\beta,\alpha)\right)Z_{D_{k_3+e_2}}\big((0)^u,(\beta,\gamma_{k_3})^d,,\alpha\big)\,Z_{D_{k_3}}\big((\beta,\gamma_{k_3})^u,(0)^d,\alpha\big)\\ &\quad=Z_{\widetilde{C}_{k_3}}\big((0)^u,(0)^d,\alpha,\beta\big)\equiv Z_{\widetilde{C}_{k_3}}\big((0)^u,(0)^d,(\alpha,\beta)^l,(\alpha,\beta)^r\big)\end{aligned} \tag{3.19}$$

Where:

$$\widetilde{C}_{k_3}:=C_{k_3}\cup D_{k_3}\cup D_{k_3+e_2} \tag{3.20}$$

is a $3L \times L$ rectangle DCD centered at C_{k_3} (see Fig 3) and $Z_{\widetilde{C}_{k_3}}\left((0)^u, (0)^d, (\alpha,\beta)^l, (\alpha,\beta)^r\right)$ is the partition function in $\widetilde{C}_{k_3}$ with (0) boundary condition on the top and on the bottom; $(\alpha,\beta)^l$ on the left and $(\alpha,\beta)^r$ on the right of C_{k_3}. Here by "on the left" of $\widetilde{C}_{k_3}$ we mean "in $A_{k_3+e_2} \cup B_{k_3} \cup A_{k_3}$" and by "on the right of $\widetilde{C}_{k_3}$ we mean "in $A_{k_3+e_2+e_1} \cup B_{k_3+e_1} \cup A_{k_3+e_1}$"; see Fig. 3. In what follows we will continue to use "on the top", "on the bottom", "on the left" and "on the right" for the boundary conditions to a volume in a similar sense.

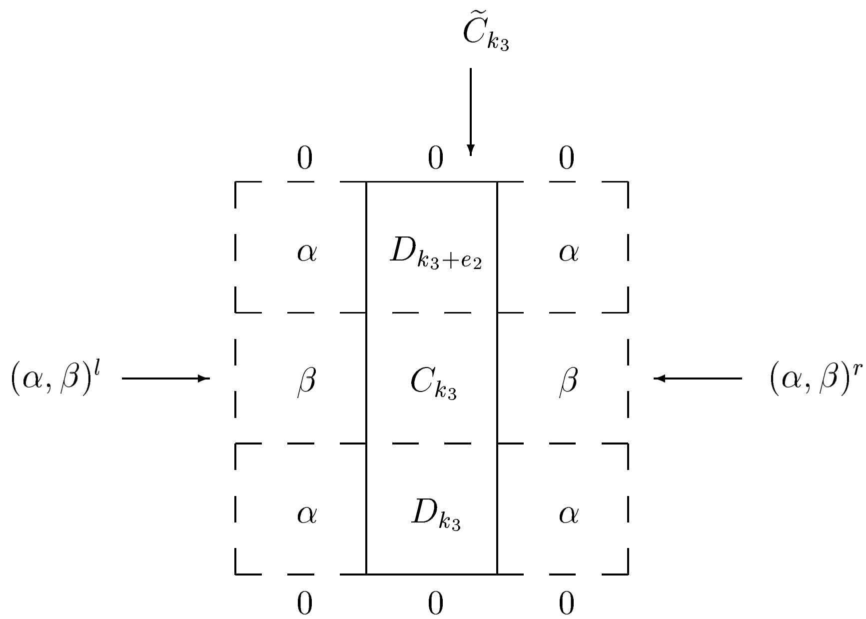

Fig. 3

The operation described by equation (3.19) above is called "gluing" of the partition functions $Z_{D_{k_3+e_2}}\left((0)^u, (\beta, \gamma_{k_3})^d, \alpha\right)$, $Z_{D_{k_3}}\left((\beta, \gamma_{k_3})^u, (0)^d, \alpha\right)$ on C_{k_3} in the vertical e_2 direction.

Now if in (3.18) we multiply and divide by

$$\prod_{k_3} Z_{\widetilde{C}_{k_3}}\left((0)^u, (0)^d, (\alpha,\beta)^l, (\alpha,\beta)^r\right),$$

we get:

$$\begin{aligned} Z^{(\ell)}_{\Lambda_p,\underline{n}} = &\sum_{\alpha} \prod_{k1} \exp\left(H(\alpha_{k_1})\right) \sum_{\beta} \prod_{k2} \exp\left(H(\beta_{k_2}) + W(\beta_{k_2}|\alpha)\right) \prod_{k_3} Z_{\widetilde{C}_{k_3}}\left((0)^u, (0)^d, (\alpha,\beta)^l, (\alpha,\beta)^r\right) \\ &\sum_{\gamma} \mu_3^{\alpha,\beta}(\gamma) \prod_{k_4} \left(1 + \Phi^{(4)}_{D_{k_4}}(\alpha,\beta,\gamma)\right) \prod_{k4} \left[Z_{D_{k_4}}\left((0)^u, (0)^d, (\alpha)\right)\right]^{-1} \end{aligned} \tag{3.21}$$

where $\mu_3^{\alpha,\beta}(\gamma)$ is the product (Bernoulli) probability measure on γ parametrically depending on α, β given by:

$$\mu_3^{\alpha,\beta}(\gamma) := \prod_{k_3} \mu^{\alpha,\beta}_{C_{k_3}}(\gamma_{k_3}) \tag{3.22}$$

where

$$\mu^{\alpha,\beta}_{C_{k_3}}(\gamma_{k_3}) := \frac{1}{Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(\alpha,\beta)^l,(\alpha,\beta)^r\right)} \exp\left(H(\gamma_{k_3}) + W(\gamma_{k_3}|\beta,\alpha)\right)$$
$$\times Z_{D_{k_3+e_2}}\left((0)^u,(\beta,\gamma_{k_3})^d,\alpha\right) Z_{D_{k_3}}\left((\beta,\gamma_{k_3})^u,(0)^d,\alpha\right) \tag{3.23}$$

At this moment we operate again a "splitting" but now we act on the partition function $Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(\alpha,\beta)^l,(\alpha,\beta)^r\right)$ in the horizontal e_1 direction; namely we write:

$$Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(\alpha,\beta)^l,(\alpha,\beta)^r\right)$$
$$= \left(\frac{Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(\alpha,\beta)^l,(\alpha,\beta)^r\right) Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(0)^l,(0)^r\right)}{Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(\alpha,\beta)^l,(0)^r\right) Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(0)^l,(\alpha,\beta)^r\right)} - 1 + 1\right) \tag{3.24}$$
$$\times \frac{Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(\alpha,\beta)^l,(0)^r\right) Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(0)^l,(\alpha,\beta)^r\right)}{Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(0)^l,(0)^r\right)}$$

We set

$$\left(\frac{Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(\alpha,\beta)^l,(\alpha,\beta)^r\right) Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(0)^l,(0)^r\right)}{Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(\alpha,\beta)^l,(0)^r\right) Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(0)^l,(\alpha,\beta)^r\right)} - 1\right) =: \Phi^{(3)}_{C_{k_3}}(\alpha,\beta) \tag{3.25}$$

We remark that:

$$\sum_{\beta_{k_2}} \exp\left(H(\beta_{k_2}) + W(\beta_{k_2}|\alpha)\right) \times Z_{\widetilde{C}_{k_2-e_1}}\left((0)^u,(0)^d,(0)^l,(\alpha,\beta_{k_2})^r\right)$$
$$Z_{\widetilde{C}_{k_2}}\left((0)^u,(0)^d,(\alpha,\beta_{k_2})^l,(0)^r\right) = Z_{\widetilde{B}_{k_2}}\left((0),\alpha\right) \tag{3.26}$$

where $\widetilde{B}_{k_2}$ is the set, centered at B_{k_3}, having the shape of a capital H given by:

$$\widetilde{B}_{k_2} := B_{k_2} \cup C_{k_2} \cup C_{k_2-e_1} \cup D_{k_2} \cup D_{k_2-e_1} \cup D_{k_2-e_1+e_2} \cup D_{k_2+e_2} \tag{3.27}$$

see Fig 4. The above operation, described in (3.26) above, is a "gluing" of the partition functions $Z_{\widetilde{C}_{k_2-e_1}}\left((0)^u,(0)^d,(0)^l,(\alpha,\beta_{k_2})^r\right)$, $Z_{\widetilde{C}_{k_2+e_1}}\left((0)^u,(0)^d,(\alpha,\beta_{k_2})^l,(0)^r\right)$ on B_{k_2} in the e_1 direction.

The boundary condition on $\widetilde{B}_{k_2}$ in the partition function $Z_{\widetilde{B}_{k_3}}\left((0),\alpha\right)$ are 0 everywhere except for the A–blocks $A_{k_2+e_2}$, A_{k_2} touching on the top and on the bottom, respectively, the block B_{k_3}. We write:

$$Z_{\widetilde{B}_{k_2}}\left((0),\alpha\right) =: Z_{\widetilde{B}_{k_2}}\left((0),(\alpha)^u,(\alpha)^d\right) \tag{3.28}$$

with $(\alpha)^u$, $(\alpha)^d$, given, respectively, by the restriction of α to $A_{k_2+e_2}$, A_{k_2}.

Similarly for the term $Z_{D_{k_4}}\left((0)^u,(0)^d,\alpha\right)$ appearing (at the power -1) in (3.18) we can write

$$Z_{D_{k_4}}\left((0)^u,(0)^d,\alpha\right) = Z_{D_{k_4}}\left((0)^u,(0)^d,(\alpha)^l,(\alpha)^r\right) \equiv Z_{D_{k_4}}\left((0),(\alpha)^l,(\alpha)^r\right) \tag{3.29}$$

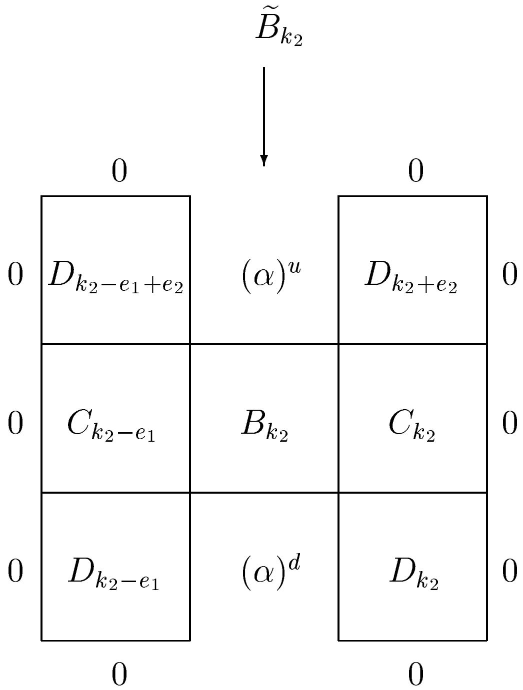

Fig. 4

where by $(0), (\alpha)^l, (\alpha)^r$ we mean the boundary conditions, outside D_{k_4} given by 0 everywhere except for the two blocks A_{k_4}, $A_{k_4+e_1}$, contiguous to D_{k_4}; $(\alpha)^l, (\alpha)^r$ are the restrictions of α to A_{k_4}, $A_{k_4+e_1}$, respectively.

Now we perform a "splitting" in the e_1 direction of the quantity $\left[Z_{D_{k_4}}\left((0), (\alpha)^l, (\alpha)^r\right)\right]^{-1}$; namely we write:

$$\begin{aligned} \left[Z_{D_{k_4}}\left((0), (\alpha)^l, (\alpha)^r\right)\right]^{-1} = &\left(\frac{Z_{D_{k_4}}\left((0), (\alpha)^l, (0)^r\right) Z_{D_{k_4}}\left((0), (0)^l, (\alpha)^r\right)}{Z_{D_{k_4}}\left((0), (\alpha)^l, (\alpha)^r\right) Z_{D_{k_4}}\left((0), (0)^l, (0)^r\right)} - 1 + 1 \right) \\ &\times \frac{Z_{D_{k_4}}\left((0), (0)^l, (0)^r\right)}{Z_{D_{k_4}}\left((0), (\alpha)^l, (0)^r\right) Z_{D_{k_4}}\left((0), (0)^l, (\alpha)^r\right)} \end{aligned} \tag{3.30}$$

We set:

$$\left(\frac{Z_{D_{k_4}}\left((0), (\alpha)^l, (0)^r\right) Z_{D_{k_4}}\left((0), (0)^l, (\alpha)^r\right)}{Z_{D_{k_4}}\left((0), (\alpha)^l, (\alpha)^r\right) Z_{D_{k_4}}\left((0), (0)^l, (0)^r\right)} - 1 \right) =: \Psi^{(4)}_{D_{k_4}}(\alpha) \tag{3.31}$$

We introduce the product probability measure $\mu_2^\alpha(\beta)$ on β, parametrically dependent on α, given by:

$$\mu_2^\alpha(\beta) := \prod_{k_2} \mu^\alpha_{B_{k_2}}(\beta_{k_2}) \tag{3.32}$$

where

$$\begin{aligned} \mu^\alpha_{B_{k_2}}(\beta_{k_2}) := &\frac{1}{Z_{\widetilde{B}_{k_2}}\left((0), (\alpha)^u, (\alpha)^d\right)} \exp\left(H(\beta_{k_2}) + W(\beta_{k_2}|\alpha)\right) \\ &\times Z_{\widetilde{C}_{k_2-e_1}}\left((0)^u, (0)^d, (0)^l (\alpha, \beta_{k_2})^r, \right) Z_{\widetilde{C}_{k_2}}\left((0)^u, (0)^d, (\alpha, \beta_{k_2})^l, (0)^r\right) \end{aligned} \tag{3.33}$$

Now we proceed similarly to the step leading to (3.21). We multiply and divide the expression on the r.h.s. of (3.21) by

$$\prod_{k_2} Z_{\widetilde{B}_{k_2}}\left((0),(\alpha)^u,(\alpha)^d\right);$$

by inserting in the r.h.s. of (3.21) the expression given by (3.33) and after operating the splitting described in (3.24), the gluing described in (3.26) and the splitting described in (3.30), we get:

$$\begin{aligned} Z^{(\ell)}_{\Lambda_p,\underline{n}} =& \sum_\alpha \prod_{k_1} \exp\left(H(\alpha_{k_1})\right)\left[Z_{D_{k_1+e_1}}\left((0),(\alpha_{k_1})^l,(0)^r\right) Z_{D_{k_1}}\left((0),(0)^l,(\alpha_{k_1})^r\right)\right]^{-1} \\ &\times \prod_{k_2} Z_{\widetilde{B}_{k_2}}\left((0),(\alpha)^u,(\alpha)^d\right)\sum_\beta \mu_2^\alpha(\beta)\prod_{k_3}\left[Z_{\widetilde{C}_{k_3}}((0))\right]^{-1}\prod_{k_3}\left(1+\Phi^{(3)}_{C_{k_3}}\right)\sum_\gamma \mu_3^{\alpha,\beta}(\gamma) \\ &\times \prod_{k_4}\left(1+\Phi^{(4)}_{D_{k_4}}(\alpha,\beta,\gamma)\right)\prod_{k_4}\left(1+\Psi^{(4)}_{D_{k_4}}(\alpha)\right)\prod_{k_4}\left[Z_{D_{k_4}}((0))\right]^{-1} \end{aligned} \tag{3.34}$$

where we used the shorthand notation $Z_{\widetilde{C}_{k_3}}((0))$ for $Z_{\widetilde{C}_{k_3}}\left((0)^u,(0)^d,(0)^l,(0)^r\right)$ and $Z_{D_{k_4}}((0)))$ for $Z_{D_{k_4}}\left((0)^u,(0)^d,(0)^l,(0)^r\right)$.

Now we perform, on the partition function $Z_{\widetilde{B}_{k_2}}\left((0),(\alpha)^u,(\alpha)^d\right)$, a splitting a bit different with respect to the previous ones. Let F_{k_2} be the horizontal $L\times 3L$ rectangle CBC contained in $\widetilde{B}_{k_2}$:

$$F_{k_2} = B_{k_2}\cup C_{k_2}\cup C_{k_2-e_1} \tag{3.35}$$

We can write:

$$\begin{aligned} Z_{\widetilde{B}_{k_2}}\left((0),(\alpha)^u,(\alpha)^d\right) =& \sum_{(\delta)^u,(\delta)^d}\exp\left[H((\delta)^u)+(H((\delta)^d)+W((\alpha)^u|(\delta)^u))+W((\alpha)^d|(\delta)^d))\right] \\ &\times\left(\frac{Z_{F_{k_2}}\left((0),(\alpha,\delta)^u,(\alpha,\delta)^d\right)Z_{F_{k_2}}\left((0),(0)^u,(0)^d\right)}{Z_{F_{k_2}}\left((0),(\alpha,\delta)^u,(0)^d\right)Z_{F_{k_2}}\left((0),(0)^u,(\alpha,\delta)^d\right)}-1+1\right) \\ &\times\frac{Z_{F_{k_2}}\left((0),(\alpha,\delta)^u,(0)^d\right)Z_{F_{k_2}}\left((0),(0)^u,(\alpha,\delta)^d\right)}{Z_{F_{k_2}}\left((0),(0)^u,(0)^d\right)} \end{aligned} \tag{3.36}$$

Where, for a generic $\delta\in\otimes_{j:Q_\ell(j)\subset\mathcal{D}_p}\Omega^{(n_j)}$ we denote by $(\delta)^u$ the restriction of δ to $D_{k_2-e_1+e_2}\cup D_{k_2+e_2}$ whereas we denote by $(\delta)^d$ the restriction of δ to $D_{k_2-e_1}\cup D_{k_2}$; by $(0),(\alpha,\delta)^u,(\alpha,\delta)^d$ we mean boundary conditions on F_{k_2} given by $(\alpha,\delta)^u$ on the top, $(\alpha,\delta)^d$ on the bottom and 0 elsewhere (see Fig. 5).

Let $F^{(u)}_{k_2}$, $F^{(d)}_{k_2}$ be the "horseshoe" shaped domains given by:

$$\begin{aligned} F^{(u)}_{k_2} &:= B_{k_2}\cup C_{k_2}\cup C_{k_2-e_1}\cup D_{k_2-e_1+e_2}\cup D_{k_2+e_2}, \\ F^{(d)}_{k_2} &:= B_{k_2}\cup C_{k_2}\cup C_{k_2-e_1}\cup D_{k_2}\cup D_{k_2-e_1} \end{aligned} \tag{3.37}$$

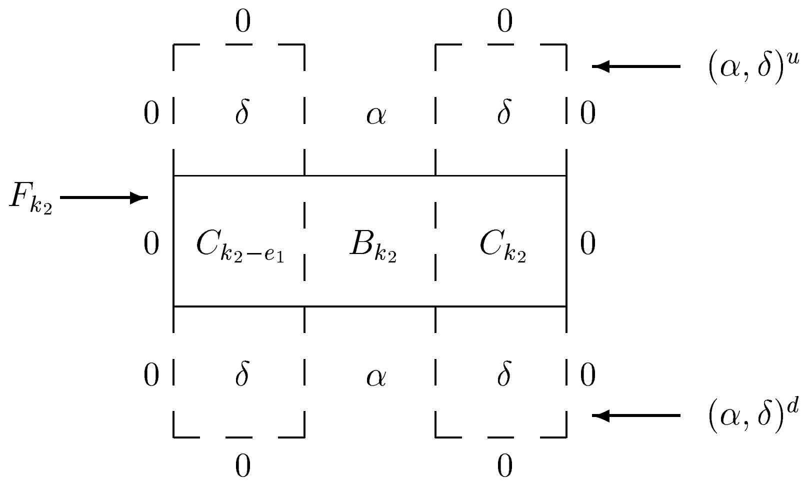

Fig. 5

(see Fig (6)). From (3.36) we easily get:

$$Z_{\widetilde{B}_{k_2}}\left((0),(\alpha)^u,(\alpha)^d\right)=\frac{Z_{F^{(u)}_{k_2}}\left((0),(\alpha)^u\right)Z_{F^{(d)}_{k_2}}\left((0),(\alpha)^d\right)}{Z_{F_{k_2}}\left((0),(0)^u,(0)^d\right)}\left(1+\Phi^{(2)}_{B_{k_2}}(\alpha)\right) \tag{3.38}$$

where $Z_{F^{(u)}_{k_2}}((0),(\alpha)^u)$ is the partition function on the domain $F^{(u)}_{k_2}$ with boundary conditions 0 everywhere except for $A_{k_2+e_2}$ where they take the value $(\alpha)^u$($\equiv$ the restriction of α to $A_{k_2+e_2}$); similarly $Z_{F^{(d)}_{k_2}}\left((0),(\alpha)^d\right)$ is the partition function on the domain $F^{(d)}_{k_2}$ with boundary conditions 0 everywhere except for A_{k_2} where they take the value $(\alpha)^d$($\equiv$ the restriction of α to A_{k_2}); finally $\Phi^{(2)}_{B_{k_2}}(\alpha)$ is defined as:

$$\Phi^{(2)}_{B_{k_2}}(\alpha):=\sum_{(\delta)^u,(\delta)^d}\widetilde{\mu}^{\alpha}_{k_2}((\delta)^u,(\delta)^d)\left(\frac{Z_{F_{k_2}}\left((0),(\alpha,\delta)^u,(\alpha,\delta)^d\right)Z_{F_{k_2}}\left((0),(0)^u,(0)^d\right)}{Z_{F_{k_2}}\left((0),(\alpha,\delta)^u,(0)^d\right)Z_{F_{k_2}}\left((0),(0)^u,(\alpha,\delta)^d\right)}-1\right) \tag{3.39}$$

where $\widetilde{\mu}^{\alpha}_{k_2}((\delta)^u,(\delta)^d)$ is a probability measure on $\otimes_{i:Q_l(i)\subset\mathcal{D}_p\cap\widetilde{B}_{k_2}}\Omega^{(n_i)}$ parametrically dependent on $\alpha_{k_2+e_2},\alpha_{k_2}$ given by:

$$\begin{aligned}\widetilde{\mu}^{\alpha}_{k_2}((\delta)^u,(\delta)^d)=&\exp\left(H((\delta)^u)+(H((\delta)^d)+W((\alpha)^u|(\delta)^u))+W((\alpha)^d|(\delta)^d))\right)\\&\times\frac{Z_{F_{k_2}}\left((0),(\alpha,\delta)^u,(0)^d\right)Z_{F_{k_2}}\left((0),(0)^u,(\alpha,\delta)^d\right)}{Z_{F^{(u)}_{k_2}}((0),(\alpha)^u)\,Z_{F^{(d)}_{k_2}}((0),(\alpha)^d)}\end{aligned} \tag{3.40}$$

Indeed $\widetilde{\mu}^{\alpha}_{k_2}((\delta)^u,(\delta)^d)$ has the form of a product measure over the "up" and "down" variables but in (3.39) we are averaging, with respect to $\widetilde{\mu}^{\alpha}_{k_2}$, a function which couples these variables so that the result is a $\Phi^{(2)}_{B_{k_2}}(\alpha)$ which is a non–factorized function of $(\alpha)^u,(\alpha)^d$.

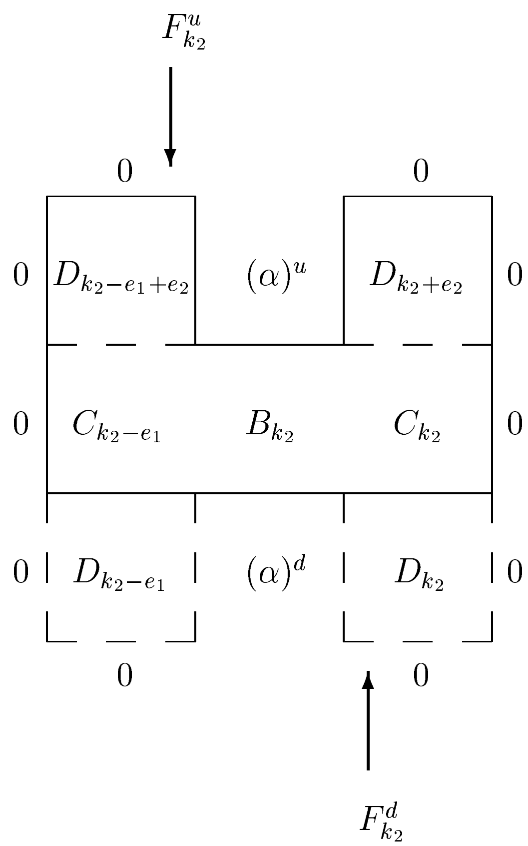

Fig. 6

By inserting (3.38) into (3.34) we get

$$\begin{aligned} Z^{(\ell)}_{\Lambda_p,\underline{n}} = &\sum_{\alpha}\prod_{k_1}\exp\left(H(\alpha_{k_1})\right)\left[Z_{D_{k_1}}\left((0),(\alpha_{k_1})^l,(0)^r\right)Z_{D_{k_1}-e_1}\left((0),(0)^l,(\alpha_{k_1})^r\right)\right]^{-1} \\ &\times Z_{F^{(u)}_{k_1-e_1}}\left((0),(\alpha_{k_1})^u\right)Z_{F^{(d)}_{k_1}}\left((0),(\alpha_{k_1})^d\right)\prod_{k_2}\left(1+\Phi^{(2)}_{B_{k_2}}(\alpha)\right)\left[Z_{F_{k_2}}(0)\right]^{-1} \\ &\times\sum_{\beta}\mu^{\alpha}_2(\beta)\prod_{k_3}\left[Z_{\widetilde{C}_{k_3}}((0))\right]^{-1}\prod_{k_3}\left(1+\Phi^{(3)}_{C_{k_3}}(\alpha,\beta)\right) \\ &\times\sum_{\gamma}\mu^{\alpha,\beta}_3(\gamma)\prod_{k_4}\left(1+\Phi^{(4)}_{D_{k_4}}(\alpha,\beta,\gamma)\right)\prod_{k_4}\left(1+\Psi^{(4)}_{D_{k_4}}(\alpha)\right)\prod_{k_4}Z_{D_{k_4}}((0)) \end{aligned} \tag{3.41}$$

where we have used the shorthand forms $Z_{D_{k_1}}\left((0),(\alpha_{k_1})^l\right)$, respectively $Z_{D_{k_1}-e_1}\left((0),(\alpha_{k_1})^r\right)$, for $Z_{D_{k_1}}\left((0),(\alpha_{k_1})^l,(0)^r\right)$, $Z_{D_{k_1}-e_1}\left((0),(0)^l,(\alpha_{k_1})^r\right)$ and $Z_{F_{k_2}}(0)$ for $Z_{F_{k_2}}\left((0),(0)^u,(0)^d\right)$.

We notice that if in (3.41) above we neglect all the "small quantities" Φ and Ψ and we use that $\mu^{\alpha}_2(\beta)$ and $\mu^{\alpha,\beta}_3(\gamma)$ are normalized measures, then, by performing the sum over the γ,β variables, we get a factorized partition function describing a system of independent α variables. So we substantially have already reached our goal; we want now to manipulate a little bit these factorized terms (the product over k_1) in order to get a simpler expression with a more transparent physical meaning.

We use the notation $\widetilde{A}_{k_1}$ to denote the $3L \times 3L$ cube centered at the block A_{k_1}:

$$\widetilde{A}_{k_1} := Q_{3L}(2k_1), \qquad k_1 \in \mathcal{L}_L \tag{3.42}$$

Let G_{k_1} denote the annulus obtained from $\widetilde{A}_{k_1}$ by removing the block A_{k_1} itself:

$$G_{k_1} := \widetilde{A}_{k_1} \setminus A_{k_1} \equiv B_{k_1} \cup B_{k_1-e_2} \cup C_{k_1} \cup C_{k_1-e_1} \cup C_{k_1-e_1-e_2} \cup C_{k_1-e_2} \cup D_{k_1} \cup D_{k_1-e_1} \tag{3.43}$$

We denote by $Z_{G_{k_1}}((0), \alpha_{k_1})$ the partition function on G_{k_1} with boundary conditions α_{k_1} on the "hole" A_{k_1} and 0 elsewhere. Moreover let $Z_{D_{k_1-e_1} \cup D_{k_1}}\left((0), \alpha_{k_1}, (\beta\gamma)^u, (\beta\gamma)^d\right)$ denote the partition function on the (non–connected) set $D_{k_1-e_1} \cup D_{k_1}$ with boundary conditions α_{k_1} on A_{k_1}, $(\beta\gamma)^u$ on the up part of $G_{k_1} \setminus (D_{k_1-e_1} \cup D_{k_1})$ (namely in $C_{k_1-e_1} \cup B_{k_1} \cup C_{k_1}$), $(\beta\gamma)^d$ in the down part $C_{k_1-e_1-e_2} \cup B_{k_1-e_2} \cup C_{k_1-e_2}$ and 0 elsewhere. (see Fig (7)).

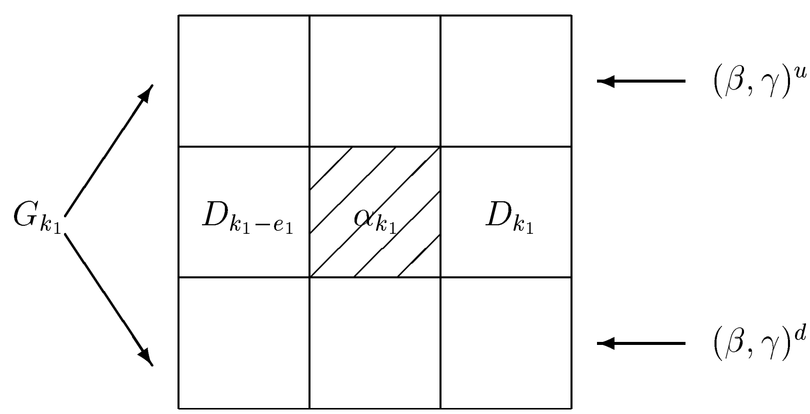

Fig. 7

Indeed we have the following factorization:

$$Z_{D_{k_1-e_1} \cup D_{k_1}}\left((0), \alpha_{k_1}, (\beta\gamma)^u, (\beta\gamma)^d\right) = Z_{D_{k_1-e_1}}\left((0), \alpha_{k_1}, (\beta\gamma)^u, (\beta\gamma)^d\right) Z_{D_{k_1}}\left((0), \alpha_{k_1}, (\beta\gamma)^u, (\beta\gamma)^d\right) \tag{3.44}$$

We have:

$$Z_{G_{k_1}}((0), \alpha_{k_1}) = \sum_{(\beta\gamma)^u, (\beta\gamma)^d} \exp\left(H((\beta\gamma)^u) + H((\beta\gamma)^d)\right) Z_{D_{k_1-e_1} \cup D_{k_1}}\left((0), \alpha_{k_1}, (\beta\gamma)^u, (\beta\gamma)^d\right) \tag{3.45}$$

We can write:

$$
\begin{aligned}
Z_{G_{k1}}((0),\alpha_{k_1}) &= \sum_{(\beta\gamma)^u,(\beta\gamma)^d} \exp\left(H((\beta\gamma)^u)+H((\beta\gamma)^d)\right) \\
&\times \left[\frac{Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(\beta\gamma)^u,(\beta\gamma)^d\right) Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(0)^u,(0)^d\right)}{Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(\beta\gamma)^u,(0)^d\right) Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(0)^u,(\beta\gamma)^d\right)} - 1 + 1\right] \\
&\times \frac{Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(\beta\gamma)^u,(0)^d\right) Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(0)^u,\beta\gamma)^d\right)}{Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(o)^u,(0)^d\right)} \\
&= \frac{Z_{F^{(u)}_{k_2}}\left((0),(\alpha_{k_1})^u\right) Z_{F^{(d)}_{k_2}}\left((0),(\alpha_{k_1})^d\right)}{Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(0)^u,(0)^d\right)} \left(1+\Phi^{(1)}_{A_{k_1}}(\alpha_{k_1})\right)
\end{aligned}
\tag{3.46}
$$

where

$$
\begin{aligned}
\Phi^{(1)}_{A_{k_1}}(\alpha_{k_1}) &:= \sum_{(\beta\gamma)^u,(\beta\gamma)^d} \widetilde{\mu}^{(\alpha_{k_1})}_{k_1}((\beta\gamma)^u,(\beta\gamma)^d) \\
&\times \left[\frac{Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(\beta\gamma)^u,(\beta\gamma)^d\right) Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(0)^u,(0)^d\right)}{Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(\beta\gamma)^u,(0)^d\right) Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(0)^u,(\beta\gamma)^d\right)} - 1\right]
\end{aligned}
\tag{3.47}
$$

and

$$
\begin{aligned}
&\widetilde{\mu}^{(\alpha_{k_1})}_{k_1}((\beta\gamma)^u,(\beta\gamma)^d) := \exp\left(H((\beta\gamma)^u)+H((\beta\gamma)^d)\right) \\
&\quad\times \frac{Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(\beta\gamma)^u,(0)^d\right) Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(0)^u,(\beta\gamma)^d\right)}{Z_{F^{(u)}_{k_2}}\left((0),(\alpha_{k_1})^u\right) Z_{F^{(d)}_{k_2}}\left((0),(\alpha_{k_1})^d\right)}
\end{aligned}
\tag{3.48}
$$

We write:

$$
\Psi^{(1)}_{A_{k_1}}(\alpha_{k_1}) := (1+\Phi^{(1)}_{A_{k_1}}(\alpha_{k_1}))^{-1} - 1 \tag{3.49}
$$

From (3.46),(3.47),(3.48),(3.49) we get

$$
\frac{Z_{F^{(u)}_{k_2}}\left((0),(\alpha_{k_1})^u\right) Z_{F^{(d)}_{k_2}}\left((0),(\alpha_{k_1})^d\right)}{Z_{D_{k_1-e_1}\cup D_{k_1}}\left((0),\alpha_{k_1},(0)^u,(0)^d\right)} = Z_{G_{k_1}}((0),\alpha_{k_1})\left(1+\Psi^{(1)}_{A_{k_1}}(\alpha_{k_1})\right) \tag{3.50}
$$

We define the Bernoulli probability measure $\mu_1(\alpha)$ as

$$
\mu_1(\alpha) := \prod_{k_1} \mu_{A_{k_1}}(\alpha_{k_1}) \tag{3.51}
$$

where

$$
\mu_{A_{k_1}}(\alpha_{k_1}) := \frac{1}{Z_{\widetilde{A}_{k_1}}((0))} \exp(H_{A_{k_1}}(\alpha_{k_1}) Z_{G_{k_1}}((0),\alpha_{k_1}) \tag{3.52}
$$

in which by $Z_{\widetilde{A}_{k_1}}((0))$ we denote the partition function in $\widetilde{A}_{k_1}$ with 0 boundary conditions.

In conclusion, from (3.41), (3.42), (3.50), (3.51) and (3.52) we get :

$$
\begin{aligned}
Z^{(\ell)}_{\Lambda_p,\underline{n}} = & \prod_{k_1} Z_{\widetilde{A}_{k_1}}((0)) \prod_{k_2} \left[Z_{F_{k_2}}(0)\right]^{-1} \prod_{k_3} \left[Z_{\widetilde{C}_{k_3}}((0))\right]^{-1} \prod_{k_4} Z_{D_{k_4}}((0)) \\
& \times \sum_{\alpha} \mu_1(\alpha) \prod_{k_1} \left(1+\Psi^{(1)}_{A_{k_1}}(\alpha_{k_1})\right) \prod_{k_2} \left(1+\Phi^{(2)}_{B_{k_2}}(\alpha)\right) \prod_{k_4} \left(1+\Psi^{(4)}_{D_{k_4}}(\alpha)\right) \\
& \times \sum_{\beta} \mu^{\alpha}_{B_{k_2}}(\beta_{k_2}) \prod_{k_3} \left(1+\Phi^{(3)}_{C_{k_3}}(\alpha,\beta)\right) \sum_{\gamma} \mu^{\alpha,\beta}_3(\gamma) \prod_{k_4} \left(1+\Phi^{(4)}_{D_{k_4}}(\alpha,\beta,\gamma)\right)
\end{aligned}
\tag{3.53}
$$

We write

$$
Z^{(\ell)}_{\Lambda_p,\underline{n}} = \bar{Z}^{(\ell)}_{\Lambda_p,\underline{n}} \Xi^{(\ell)}_{\Lambda_p,\underline{n}} \tag{3.54}
$$

with

$$
\bar{Z}^{(\ell)}_{\Lambda_p,\underline{n}} := \prod_{k_1} Z_{\widetilde{A}_{k_1}}((0)) \prod_{k_2} \left[Z_{F_{k_2}}(0)\right]^{-1} \prod_{k_3} \left[Z_{\widetilde{C}_{k_3}}((0))\right]^{-1} \prod_{k_4} Z_{D_{k_4}}((0)) \tag{3.55}
$$

and

$$
\begin{aligned}
\Xi^{(\ell)}_{\Lambda_p,\underline{n}} = & \sum_{\alpha} \mu_1(\alpha) \prod_{k_1} \left(1+\Psi^{(1)}_{A_{k_1}}(\alpha_{k_1})\right) \prod_{k_2} \left(1+\Phi^{(2)}_{B_{k_2}}(\alpha)\right) \prod_{k_4} \left(1+\Psi^{(4)}_{D_{k_4}}(\alpha)\right) \\
& \times \sum_{\beta} \mu^{\alpha}_{B_{k_2}}(\beta_{k_2}) \prod_{k_3} \left(1+\Phi^{(3)}_{C_{k_3}}(\alpha,\beta)\right) \sum_{\gamma} \mu^{\alpha,\beta}_3(\gamma) \prod_{k_4} \left(1+\Phi^{(4)}_{D_{k_4}}(\alpha,\beta,\gamma)\right)
\end{aligned}
\tag{3.56}
$$

We are now ready to express $\Xi^{(\ell)}_{\Lambda_p,\underline{n}}$ as the partition function of a gas of polymers whose only interaction is a hard core exclusion.

We have to analyze the various interaction terms (the Φ's and Ψ's) appearing in (3.56). We see from (3.17) that the term $\Phi^{(4)}_{D_{k_4}}(\alpha,\beta,\gamma)$, involving the α,β,γ variables in the annulus $Q_{3L}(2k_4+e_1)\setminus D_{k_4}$, corresponds to an "eight body" interaction among the A,B,C blocks adjacent to D_{k_4}; we see from (3.25) that $\Phi^{(3)}_{C_{k_3}}(\alpha,\beta)$ is a six body interaction involving the A and B blocks adjacent to C_{k_3}; $\Phi^{(2)}_{B_{k_2}}(\alpha)$, $\Psi^{(4)}_{D_{k_4}}(\alpha)$ are two body terms involving the pair of A blocks contiguous to B_{k_2} , D_{k_4}, respectively. Finally $\Psi^{(1)}_{A_{k_1}}(\alpha_{k_1})$ is just a one body term.

Looking at (3.22), (3.23) we can say that $\Phi^{(4)}_{D_{k_4}}(\alpha,\beta,\gamma)$ extends its action to all A and B blocks adjacent to the C blocks in $Q_{3L}(2k_4+e_1)$ (see Fig. 8), becoming a "twelve body" interaction. Indeed we have to average $\Phi^{(4)}_{D_{k_4}}(\alpha,\beta,\gamma)$ with respect to the product of the measures $\mu^{\alpha,\beta}_{C_{k_3}}(\gamma_{k_3})$ which are parametrically dependent on the α,β variables adjacent to C_{k_3} . On the other hand, looking at (3.33), it is easily seen that we do not have to extend any more the region of influence of $\Phi^{(4)}_{D_{k_4}}(\alpha,\beta,\gamma)$ because of the parametric dependence on α of $\mu^{\alpha}_{B_{k_2}}(\beta_{k_2})$. Moreover, still looking at (3.33), we easily see that also the term $\Phi^{(3)}_{C_{k_3}}(\alpha,\beta)$ does not extend at all its influence. Of course $\Phi^{(2)}_{B_{k_2}}(\alpha)$, $\Psi^{(4)}_{D_{k_4}}(\alpha)$ $\Psi^{(1)}_{A_{k_1}}(\alpha_{k_1})$ do not extend, as well, their action.

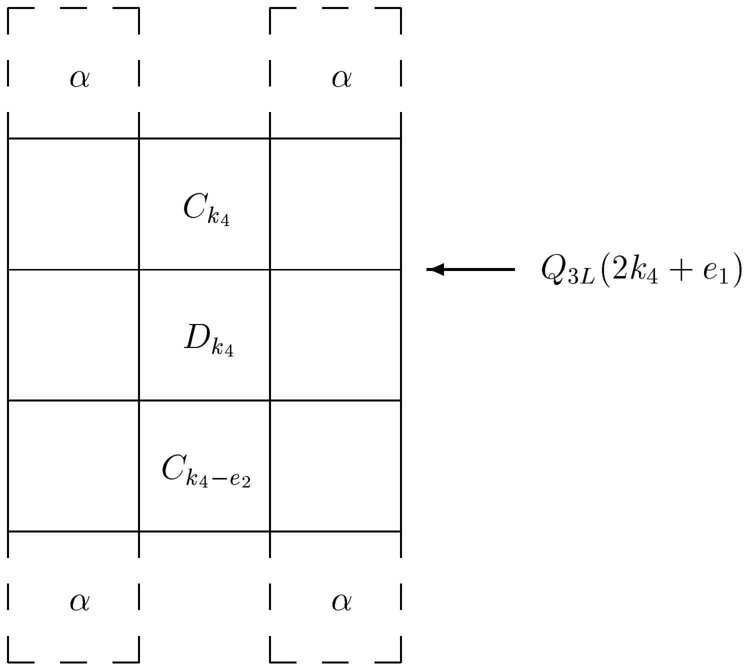

Fig. 8

So it is natural to define different kind of (many body) bonds corresponding to the above interaction terms. As a consequence of the above discussion we have the following kind of bonds; the bond $D_{k_4}^{(\Phi)}$, to which corresponds the weight $\Phi^{(4)}_{D_{k_4}}(\alpha,\beta,\gamma)$ which is given by the set of A, B and C blocks contiguous to D_{k_4} united with the other A blocks adjacent from the exterior of $Q_{3L}(2k_4+e_1)$ to the already considered B blocks. So a $D_{k_4}^{(\Phi)}$–bond contains twelve blocks. We similarly define (now without any extension) the bond $C_{k_3}^{(\Phi)}$ with weight $\Phi^{(3)}_{C_{k_3}}(\beta,\alpha)$; the bond $B_{k_2}^{(\Phi)}$ with weight $\Phi^{(2)}_{B_{k_2}}(\alpha)$; the bond $D_{k_4}^{(\Psi)}$ with weight $\Psi^{(4)}_{D_{k_4}}(\alpha)$ and the bond $A_{k_1}^{(\Psi)}$ with weight $\Psi^{(1)}_{A_{k_1}}(\alpha_{k_1})$.

Given a bond b of one of the above kinds we define its support $\widetilde{b}$ as the subset of $\mathcal{L}$ obtained as the union of the Q_L blocks making part of b. For any bond b we denote by ξ_b the corresponding weight. Notice that ξ_b will be, in general, a function of the α,β,γ variables associated to the blocks in $\widetilde{b}$. For instance a bond $b=D_{k_4}^{(\Phi)}$ can be seen as an element of $(\mathcal{L}_L)^{12}$ whereas $\widetilde{b}$ is a subset of the original lattice $\mathcal{L}$ given by the union of the twelve interacting blocks.

We say that two bonds b_1,b_2 are *connected* if $\widetilde{b}_1\cap\widetilde{b}_2\neq\emptyset$. A *polymer* R is a set of bonds $b_1,\dots,b_k$ which is connected in the sense that $\forall\ i,\ j$: $1\le i<j\le k$ there exists a chain of connected bonds in R joining b_i to b_j namely $\exists\ b_{i_1},\dots b_{i_h}; b_{i_m}\ \in R,\ m=1,\dots h,\ b_{i_1}=b_i, b_{i_h}=b_j$: $\widetilde{b}_{i_m}\cap\widetilde{b}_{i_{m+1}}\neq\emptyset,\ m=1,\dots h-1$.

The support $\widetilde{R}$ of a polymers $R=b_1,\dots,b_k$ is simply $\widetilde{R}=\cup_{i=1}^k\widetilde{b}_i$. We call $\mathcal{R}_{\Lambda_p}$ the set of all possible polymer with support in Λ_p and $\mathcal{R}$ the set of all possible polymers with arbitrary support in $\mathcal{L}$. Two polymers R_1,R_2 are said to be compatible if $\widetilde{R}_1\cap\widetilde{R}_2=\emptyset$; otherwise they are called incompatible.

Given a polymer $R = b_1, \dots, b_k$ we define its *activity* ζ_R as:

$$\zeta_R \;:=\; \sum_\alpha \mu_1(\alpha) \sum_\beta \mu_2^\alpha(\beta) \sum_\gamma \mu_3^{\alpha,\beta}(\gamma) \prod_{i=1}^k \xi_{b_i}(\alpha,\beta,\gamma) \tag{3.57}$$

Notice that, due to the Bernoulli character of the above probability measures, we can, as well, write:

$$\sum_{\alpha_{\widetilde{R}}} \mu_{1,\widetilde{R}}(\alpha_{\widetilde{R}}) \sum_{\beta_{\widetilde{R}}} \mu_{2,\widetilde{R}}^{\alpha_{\widetilde{R}}}(\beta_{\widetilde{R}}) \sum_{\gamma_{\widetilde{R}}} \mu_{3,\widetilde{R}}^{\alpha_{\widetilde{R}},\beta_{\widetilde{R}}}(\gamma_{\widetilde{R}}) \prod_{i=1}^k \xi_{b_i}(\alpha_{\widetilde{R}},\beta_{\widetilde{R}},\gamma_{\widetilde{R}}) \tag{3.58}$$

where $\alpha_{\widetilde{R}},\beta_{\widetilde{R}},\gamma_{\widetilde{R}}$ denote the α,β,γ variables in $\widetilde{R}$;

$$\mu_{3,\widetilde{R}}^{\alpha_{\widetilde{R}},\beta_{\widetilde{R}}}(\gamma) = \prod_{k_3 : C_{k_3} \subset \widetilde{R}} \mu_{C_{k_3}}^{\alpha,\beta}(\gamma_{k_3}) \tag{3.59}$$

and so on.

Going back to the specific structure of our multi–canonical model it is immediately seen that the activity of a polymer R is a function of the renormalized variables n_i ($\equiv$ number of particles fixing the constraint in the block $Q_L(i)$) *only* for $Q_L(i) \in \widetilde{R}$. To make explicit this dependence we write

$$\zeta_R = \zeta_R(n_{\widetilde{R}}) \tag{3.60}$$

where $n_{\widetilde{R}} = \{n_i\}_{Q_L(i) \subset \widetilde{R}}$.

From (3.56), (3.58) we get the desired expression:

$$\Xi^{(\ell)}_{\Lambda_p,\underline{n}} \;=\; 1 + \sum_{k \ge 1} \sum_{\substack{R_1,\dots,R_k : \widetilde{R}_i \subset \Lambda_p, \\ \widetilde{R}_i \cap \widetilde{R}_j = \emptyset, i<j=1,\dots,n}} \prod_{i=1}^k \zeta_{R_i}(n_{\widetilde{R}_i}) \tag{3.61}$$

Now we state a Proposition referring to a general class of polymer systems. Its proof, which is based on the standard methods of the theory of the cluster expansion, can be found in [O] (see also [GMM], [KP], [D3], [NOZ]).

Proposition 3.2. *Consider a general polymer system (see [GrK], [KP], [D]) where the only interaction is a hard core exclusion forbidding overlap of the supports $\widetilde{R}$ of the polymers R. Its partition function is:*

$$\Xi_\Lambda \;=\; 1 + \sum_{k \ge 1} \sum_{\substack{R_1,\dots,R_k : \widetilde{R}_i \subset \Lambda, \\ \widetilde{R}_i \cap \widetilde{R}_j = \emptyset, i<j=1,\dots,n}} \prod_{i=1}^k \zeta_{R_i} \tag{3.62}$$

Suppose that:

i) $\exists\, \kappa > 0$ such that the number of different polymers R with m bonds (we write $|R| = m$) and support $\widetilde{R}$ containing a fixed point (say the origin) is bounded by κ^m;

ii) $\exists\, \varepsilon > 0$ *such that* $|\zeta_R| < \varepsilon^{|R|}$.

Let

$$\varphi_T(R_1, \dots, R_n) = \frac{1}{n!} \sum_{g \in G(R_1,\dots,R_n)} (-1)^{\#\text{ edges in } g} \tag{3.63}$$

where $G(R_1, \dots, R_n)$ *is the set of connected graphs with* n *vertices* $(1, \dots, n)$ *and edges* i, j *corresponding to pairs* R_i, R_j *such that* $\widetilde{R}_i \cap \widetilde{R}_j \neq \emptyset$ *(we set the sum equal to zero if* G *is empty and one if* $n = 1$*). If*

$$\varepsilon < \frac{1}{\kappa} \frac{x}{1+x} e^{-x} \Big|_{x = (5^{\frac{1}{2}} - \frac{1}{2})} \tag{3.64}$$

then there exists a positive constant $C(\varepsilon)$ *such that*

$$\sum_{\substack{R_1,\dots,R_n : \widetilde{R}_i \subset \Lambda \\ \exists R_i = R}} |\varphi_T(R_1, \dots, R_n)| \prod_{i=1}^{n} |\zeta_{R_i}| \leq C(\varepsilon) \left(\varepsilon \exp\left\{ \frac{\sqrt{5}-1}{2} \right\} \right)^{|R|} \tag{3.65}$$

$$\Xi_\Lambda = \exp\left\{ \sum_{n \geq 1} \sum_{R_1,\dots,R_n : \widetilde{R}_i \subset \Lambda} \varphi_T(R_1, \dots, R_n) \prod_{i=1}^{n} \zeta_{R_i} \right\} \tag{3.66}$$

In our context, it is clear that we can find a constant κ so that the hypothesis *(i)* of Proposition 3.2 holds. It is also clear from (3.17), (3.25), (3.31), (3.39), (3.47), (3.49) that there exists a universal constant C such that hypothesis *(ii)* holds with $\varepsilon = C\delta(\ell)$ (recall that $\delta(\ell) \to 0$ as $\ell \to \infty$) so that (3.64) holds for any ℓ sufficiently large. In fact in the two–dimensional case we use a weaker condition: we do not need, in the left hand side of (3.5) to take the supremum over $V \in P_L^{(i)}(j)$, but only the analogous condition only for the squares Q_L and for the $L \times 3L$ rectangles.

Then, using the results of Proposition 3.2, we can compute the renormalized potential and perform the thermodynamic limit. Suppose, instead of considering periodic boundary conditions, we had a generic b.c. τ outside our cube Λ_p. It is clear that we can apply the same procedure (block decimation and cluster expansion) that we have used above in the case of periodic boundary conditions and get very similar results. Let us briefly sketch the differences.

Recall that our square Λ_p has a side being an integer multiple of the elementary square Q_L with side $L = 2\ell$; then certainly we will have a horizontal edge of $\partial\Lambda_p$ adjacent to a row (of thickness L) made by C and B blocks (a CB row) and a horizontal edge adjacent to a DA row. Similarly we will have a vertical edge adjacent to an AB column and one adjacent to a CD column.

It is easy to convince ourselves that even with generic τ b.c. we can repeat the same sequence of splitting and gluing, following the same "path" joining the 4 sub–lattices of $\mathcal{L}_L$ namely $D \to C \to B \to A$. In the bulk, namely where the sets $D, \widetilde{C}, \widetilde{B}, \widetilde{A}$ do not touch the boundary, we get the same results as in the case of periodic b.c. For the blocks close to $\partial\Lambda_p$ we get the following modifications:

i) The various sets $\widetilde{C}, \widetilde{B}, \widetilde{A}$ of the bulk are substituted by their "truncations in Λ_p" namely by $\widetilde{C} \cap \Lambda_p, \widetilde{B} \cap \Lambda_p, \widetilde{A} \cap \Lambda_p$ with the proper τ b.c. on their part touching $\partial\Lambda_p$ and 0, like in the bulk, otherwise.

ii) The various probability measures $\mu^{\alpha,\beta}_{C_{k_3}}, \mu^{\alpha}_{B_{k_2}}$ are defined similarly to what is done in the bulk with the difference that, in their definitions, the terms corresponding to partition functions on regions lying totally (resp. partially) outside Λ_p are absent (resp. truncated); moreover the configuration on which they depend parametrically: α, β in $\mu^{\alpha,\beta}_{C_{k_3}}$; α in $\mu^{\alpha}_{B_{k_2}}$ may contain τ; notice that $\mu_{A_{k_1}}$ stays unchanged.

iii) Some of the bonds, close to $\partial\Lambda_p$, are consequently modified and their weights can depend on τ. By an abuse of notation, we still denote them by $D^{(\Phi)}_{k_4}$, $C^{(\Phi)}_{k_3}$, $B^{(\Phi)}_{k_2}$, $D^{(\Psi)}_{k_4}$, $A^{(\Psi)}_{k_1}$.

Indeed the splitting operation is very similar in the bulk and close to the boundary; the true difference is the following. When we have some term produced by a splitting that, following the "bulk rule", we would like to glue with some other term outside Λ_p or coming from Λ_p, simply we omit the gluing and in this way we construct some new domains just consisting in the parts of the corresponding bulk domains $(\widetilde{C}, \widetilde{B}, \widetilde{A})$, lying inside Λ_p.

Let us describe an example. Suppose that the upper horizontal side of $\partial\Lambda_p$ is adjacent from the exterior to a CB row (which, indeed, is the case with our choice of the location of Λ_p). After integrating over δ variables and splitting like in (3.16) we do not glue on the blocks C_{k_3} sitting on the top row like in (3.19) but we make an analogous operation combining the term $Z_{D_{k_3}}$ (coming from the splitting on the D_{k_3} block in Λ_p) with the self–interaction in C_{k_3} and its interaction with the exterior configuration τ. In other words we use a formula analogous to (3.19) but without the term $Z_{D_{k_3+e_2}}$ which, now, is absent. In this way the set corresponding to $\widetilde{C}_{k_3}$ in the bulk, just consists, now, of $C_{k_3} \cup D_{k_3}$. Accordingly we define $\mu^{\alpha,\beta}_{C_{k_3}}$ by omitting the factor $Z_{D_{k_3+e_2}}(\gamma_{k_3})$ in its definition. When we continue with the splitting on the horizontal direction and the gluing, say, on $B_{k_3+e_1}$ we end up with the construction of a set, playing the role of $\widetilde{B}_{k_3+e_1}$, obtained by removing from $\widetilde{B}_{k_3+e_1}$ the two D blocks exterior to Λ_p where the "top" b.c. are given by τ whereas the other b.c are still given by the reference configuration 0 like in the bulk. Of course also the error terms (of Φ or Ψ type) are, accordingly, modified.

In this way we can repeat the transformation of our system into a polymer gas. We just have to introduce the obvious modifications in the terms appearing in the expression of the partition function of the reference system $\bar{Z}^{(\ell)}_{\Lambda_p,\underline{n}}$ (see (3.55)) as well as in the bonds $D^{(\Phi)}_{k_4}$, $C^{(\Phi)}_{k_3}$, $B^{(\Phi)}_{k_2}$, $D^{(\Psi)}_{k_4}$, $A^{(\Psi)}_{k_1}$ close to the boundary and in the measures $\mu^{\alpha,\beta}_{C_{k_3}}$, $\mu^{\alpha}_{B_{k_2}}$ when C_{k_3}, B_{k_2} happen to be adjacent to the boundary $\partial\Lambda_p$; then, accordingly, we modify the definition of the polymers and of their activity, $\zeta^{\tau}_R = \zeta^{\tau}_R(\underline{n}_{\widetilde{R}})$ (see (3.57)) which, now, will in general depend on the location of the polymer and on the b.c. τ. Anyway if $d_\ell\left(\widetilde{R}, I^c_p\right) > d$ the activity ζ^{τ}_R of R is the same as in the bulk and does not depend on τ.

Proof of Theorem 3.1. Let us take the logarithm of (3.54). By using (3.55) (3.61) and (3.66) we get the following expression for the renormalized Hamiltonian.

$$\begin{aligned} H^{(\ell,\tau)}_{I_p}(\underline{n}) := \log\left[Z^{(\ell,\tau)}_{\Lambda_p,\underline{n}}\right] &= \sum_{k_1} \log\left[Z_{\widetilde{A}_{k_1}}((0))\right] - \sum_{k_2} \log\left[Z_{F_{k_2}}(0)\right] - \sum_{k_3} \log\left[Z_{\widetilde{C}_{k_3}}((0))\right] \\ &+ \sum_{k_4} \log\left[Z_{D_{k_4}}((0))\right] + \sum_{k\ge 1} \sum_{R_1,\dots,R_k:\widetilde{R}_i\subset\Lambda_p} \varphi_T(R_1,\dots,R_n) \prod_{i=1}^{k} \zeta^{\tau}_{R_i}(n_{\widetilde{R}_i}) \end{aligned} \tag{3.67}$$

We have

$$H_{I_p}^{(\ell,\tau)}(\underline{n}) = \text{const} + \sum_{X \subset I_p} \Phi_X^{(\ell,\tau),sr}(m_X) + \sum_{X \subset I_p} \Phi_{I_p,X}^{(\ell,\tau),lr}(m_X) \tag{3.68}$$

where, with $A_{k_1} \in \mathcal{A}_p$, $B_{k_2} \in \mathcal{B}_p$, $C_{k_3} \in \mathcal{C}_p$, $D_{k_4} \in \mathcal{D}_p$ and $d_\ell(X, I_p^c) > d$ (see the above discussion for $d_\ell(X, I_p^c) \le d$), we set

$$\Phi_X^{(\ell,\tau),sr}(m_X) := \begin{cases} \log\left[\mu^0_{\widetilde{A}_{k_1},z}(M_i = m_i, Q_\ell(i) \subset \widetilde{A}_{k_1})\right] & \text{if } X : \cup_{i \in X} Q_\ell(i) = \widetilde{A}_{k_1} \\ -\log\left[\mu^0_{F_{k_2},z}(M_i = m_i, Q_\ell(i) \subset F_{k_2})\right] & \text{if } X : \cup_{i \in X} Q_\ell(i) = F_{k_2} \\ -\log\left[\mu^0_{\widetilde{C}_{k_3},z}(M_i = m_i, Q_\ell(i) \subset \widetilde{C}_{k_3})\right] & \text{if } X : \cup_{i \in X} Q_\ell(i) = \widetilde{C}_{k_3} \\ \log\left[\mu^0_{D_{k_4},z}(M_i = m_i, Q_\ell(i) \subset D_{k_4})\right] & \text{if } X : \cup_{i \in X} Q_\ell(i) = D_{k_4} \\ 0 & \text{otherwise} \end{cases} \tag{3.69}$$

and

$$\Phi_{I_p,X}^{(\ell,\tau),lr}(m_X) := \sum_{R_1,\ldots,R_k : \cup_i \widetilde{R}_i = X} \varphi_T(R_1,\ldots,R_n) \prod_{i=1}^{k} \zeta^\tau_{R_i}(n_{\widetilde{R}_i}) \tag{3.70}$$

By the above discussion on the dependence of the activity on the boundary condition, for each $X \subset\subset \mathcal{L}_\ell$ such that $d_\ell(X, I_p^c) > d$, $\Phi_X^{(\ell,\tau)}$ is independent of τ. Therefore the limit in (3.6) exists and is actually reached for a finite p. Finally the estimate (3.7) is a direct consequence of (3.70) and Proposition 3.2. □

4. The multi–canonical measure.

Given a positive integer ℓ and a volume $\Lambda \subset\subset \mathcal{L}$ of the form (2.10) we want to study the *multi–canonical* state which is obtained from the multi–grancanonical one by fixing the total number of particles in each cube $Q_\ell(i)$, $i \in I$. Let thus $\underline{N} = \{N_i, i \in I\}$ be the random variables defined in (2.12) and, given $\underline{n} = \{n_i = 0, \cdots, |Q_\ell|, \ i \in I\}$, the multi–canonical state $\nu^\tau_{\Lambda,\underline{n}}$ is given by

$$\nu^\tau_{\Lambda,\underline{n}}(\cdot) := \mu^\tau_{\Lambda,\underline{z}}\left(\cdot \,|\underline{N} = \underline{n}\right)$$

which, in the RG context, represents the *constrained* model. Note that $\nu^\tau_{\Lambda,\underline{n}}$ is independent on $\underline{z}$.

4.1. Thermodynamic relationships.

We need to compare the multi–canonical and multi–grancanonical state. We start here by discussing some thermodynamic relationships between them. With respect to the usual treatment we work in finite volume and take advantage of the strong mixing condition to obtain explicit bounds.

Let the volume Λ be of the form (2.10) for some $I \subset\subset \mathcal{L}_\ell$ and $\mu^\tau_{\Lambda,\underline{z}}$ be a multi–grancanonical state satisfying Condition MUSM($\mathcal{A}$). Introduce the map $\mathcal{A}^I \ni \underline{z} \mapsto \underline{\rho}^\tau(\underline{z}) \in [0,1]^I$ defined by

$$\rho^\tau_i(\underline{z}) = \rho^{\tau,(\ell)}_i(\underline{z}) := \frac{1}{|Q_\ell|}\mu^\tau_{\Lambda,\underline{z}}(N_i), \quad i \in I \tag{4.1}$$

Proposition 4.1. *For each $I \subset\subset \mathcal{L}_\ell$ and each closed $\mathcal{C} \subset \mathcal{A}$ there is a constant $C > 0$ such that for any boundary condition τ, any $\underline{z} \in \mathcal{C}^I$ and all ℓ multiple of ℓ_0*

$$\frac{1}{C} \le \frac{\partial}{\partial z_i}\rho^\tau_i(\underline{z}) \le C \tag{4.2}$$

$$\left|\frac{\partial}{\partial z_j}\rho^\tau_i(\underline{z})\right| \le C z_i \frac{1 + \left|\overline{Q}^r_\ell(i) \cap \overline{Q}^r_\ell(j)\right|}{|Q_\ell|} e^{-d(Q_\ell(i),Q_\ell(j))/C}, \quad i \ne j \tag{4.3}$$

$$\left|\rho^{\tau^x}_i(\underline{z}) - \rho^\tau_i(\underline{z})\right| \le C\frac{z_i}{|Q_\ell|}e^{-d(x,Q_\ell(i))/C} \tag{4.4}$$

The proof of the lower bound in (4.2) is based on the following Gaussian bound on the characteristic function (see [DS4,§2.3] and [Y,§9]) which will be extensively used in the sequel. For $\underline{t} \in \mathbb{R}^{|I|}$, we use the notation $\langle \underline{t}, \underline{N}\rangle := \sum_{i\in I} t_i N_i$.

Lemma 4.2. *For each $I \subset\subset \mathcal{L}_\ell$ there is a constant $C > 0$ such that for any ℓ and $\underline{t} \in [-\pi,\pi]^{|I|}$*

$$\left|\mu^\tau_{\Lambda,\underline{z}}\left(\exp\{i\,\langle \underline{t}, \underline{N}\rangle\}\right)\right| \le \exp\left\{-\frac{1}{C}\frac{1}{2}|Q_\ell|\sum_{i\in I} z_i |t_i|^2\right\} \tag{4.5}$$

Proof. Before starting we stress that the proof is based only on the finite range and boundedness of the interaction and does not use Condition MUSM($\mathcal{A}$).

Let $\Lambda' \subset \Lambda$ be a subset of a sub–lattice of $\mathcal{L}$ with spacing parameter larger than the range r of the interaction. This means that for any $x, y \in \Lambda'$ we have $d(x,y) > r$ but nonetheless $|\Lambda'| \geq |\Lambda|/C$ for some constant $C = C(r) \geq 1$. If we set $Q'_\ell(j) := \Lambda' \cap Q_\ell(j)$, we then have

$$\left|\mu^\tau_{\Lambda,\underline{z}}\left(e^{i\langle \underline{t},\underline{N}\rangle}\right)\right| = \left|\int \mu^\tau_{\Lambda,\underline{z}}(d\zeta)\mu^\zeta_{\Lambda',\underline{z}}\left(\prod_{j=1}^{|I|}\prod_{x_j\in Q_\ell(j)} e^{it_j\eta_{x_j}}\right)\right|$$

$$\leq \sup_\zeta \left|\mu^\zeta_{\Lambda',\underline{z}}\left(\prod_{j=1}^{|I|}\prod_{x_j\in Q'_\ell(j)} e^{it_j\eta_{x_j}}\right)\right| = \prod_{j=1}^{|I|}\prod_{x_j\in Q'_\ell(j)} \sup_\zeta \left|\mu^\zeta_{\Lambda',\underline{z}}\left(e^{it_j(\eta_{x_j})}\right)\right|$$

since $\mu^\zeta_{\Lambda',\underline{z}}$ is a product measure.

Let $p_x(\zeta) := \mu^\zeta_{\Lambda',\underline{z}}(\eta_x = 1)$. Since the interaction is bounded we get, for some constant $C = C(\|U\|) > 0$ independent on x, $\underline{z}$ and ζ, $z_j/C \leq p_x(\zeta) \leq Cz_j$ for $x \in Q_\ell(j)$. A simple computation on Bernoulli variables shows now that for $|t| \leq \pi$, $x_j \in Q_\ell(j)$

$$\left|\mu^\zeta_{\Lambda',\underline{z}}\left(e^{it_j\eta_{x_j}}\right)\right| \leq \exp\left\{-\frac{1}{C}\frac{1}{2}z_j t_j^2\right\}$$

the bound (4.5) follows. $\square$

Proof of Proposition 4.1. We first note that

$$\frac{\partial}{\partial z_j}\rho^\tau_i(\underline{z}) = \frac{1}{z_j|Q_\ell|}\mu^\tau_{\Lambda,\underline{z}}(N_i; N_j)$$

Let

$$v_{i,j} = v^{\tau,(\ell)}_{i,j}(\underline{z}) := \mu^\tau_{\Lambda,\underline{z}}(N_i; N_j)$$

the lower bound in (4.2) follows by noticing that Lemma 4.2 implies the quadratic form estimate

$$\sum_{i,j\in I} t_i t_j v_{i,j} \geq \frac{1}{C}|Q_\ell|\sum_{i\in I} z_i t_i^2 \tag{4.6}$$

To prove the upper bound in (4.2) and (4.3) we instead use Condition MUSM($\mathcal{A}$) to get

$$|v_{i,j}| \leq \sum_{x\in Q_\ell(i)}\sum_{y\in Q_\ell(j)}\left|\mu^\tau_{\Lambda,\underline{z}}(\eta_x,\eta_y)\right| \leq \sum_{x\in Q_\ell(i)\cap Q_\ell(j)}\mu^\tau_{\Lambda,\underline{z}}(\eta_x,\eta_x) + Cz_iz_j\sum_{\substack{x\in Q_\ell(i)\\ y\in Q_\ell(j), y\neq x}} e^{-d(x,y)/C}$$

and that for $x \in Q_\ell(i)$, by the same argument as in Lemma 4.2, $\mu^\tau_{\Lambda,\underline{z}}(\eta_x,\eta_x) \leq Cz_i$. The proof of (4.4) is analogous and we omit it. $\square$

Let μ_z be the infinite volume Gibbs state associated to the (translation invariant) interaction (z, U) satisfying Condition MUSM($\mathcal{A}$). We introduce the (one dimensional) map $\mathcal{A} \ni z \mapsto \rho(z) \in [0,1]$ by $\rho(z) = \mu_z(\eta_x)$ and denote by $\rho \mapsto z(\rho)$ the inverse map which is analytical as a consequence of the strong mixing assumption. Let finally $\mathcal{B} \subseteq [0,1]$ be defined by $\mathcal{B} := \rho(\mathcal{A})$ where $\mathcal{A}$ is as given in Condition MUSM($\mathcal{A}$); we note $\mathcal{B} = [0,1]$ if $\mathcal{A} = [0,\infty)$.

Recall that the map $\underline{z} \mapsto \underline{\rho}^{\tau,(\ell)}(\underline{z})$ has been defined in (4.1). We need an inverse map $\underline{\rho} \mapsto \underline{z}^{\tau,(\ell)}(\underline{\rho})$ defined for all possible boundary condition τ. When $\mathcal{B}$ is a proper subset of $[0,1]$ we take ℓ large enough and define it on a subset of $\mathcal{B}$. By using strong mixing and Proposition 4.1 it is easy to deduce that for each closed $\mathcal{C} \subset \mathcal{A}$

$$\lim_{\ell\to\infty} \rho_i^{\tau,(\ell)}(\underline{z}) = \rho(z_i), \quad \text{uniformly for } \tau \in \Omega, \underline{z} \in \mathcal{C}^I \tag{4.7}$$

and that for each closed set $\mathcal{D} \subset \mathcal{B}$ and any ℓ large enough (depending on $\mathcal{D}$) we have

$$\mathcal{D}^I \subset \bigcap_{\tau} \underline{\rho}^{\tau,(\ell)}(\mathcal{A}^I)$$

Finally, by (4.6), the Jacobian of the map $\underline{z} \mapsto \underline{\rho}^{\tau,(\ell)}(\underline{z})$ is not degenerate uniformly in τ and ℓ. Let $\mathcal{D} \subset \mathcal{B}$ be a closed set and ℓ large enough; we can therefore define the inverse map on the set $\mathcal{D}^I$, i.e. the map $\mathcal{D}^I \ni \underline{\rho} \mapsto \underline{z}^{\tau}(\rho) = \underline{z}^{\tau,(\ell)}(\rho)$ such that

$$\underline{\rho}^{\tau}\left(\underline{z}^{\tau}(\underline{\rho})\right) = \underline{\rho}$$

for any $\underline{\rho} \in \mathcal{D}^I$, $\tau \in \Omega$.

When $\mathcal{B} = [0,1]$ we can instead define the inverse map for any ℓ. Indeed we have

$$\lim_{z_i\to 0} \rho_i^{\tau,(\ell)}(\underline{z}) = 0, \quad \lim_{z_i\to +\infty} \rho_i^{\tau,(\ell)}(\underline{z}) = 1, \quad \text{uniformly for } \tau \in \Omega, \{z_j \in [0,\infty), j \neq i\}$$

which, together with (4.6), implies

$$\underline{\rho}^{\tau,(\ell)}\left([0,\infty)^I\right) = [0,1]^I$$

We prove below some estimate on the Jacobian of the map $\underline{\rho} \mapsto \underline{z}^{\tau,(\ell)}(\underline{\rho})$; in order to describe them we need some more notation. Let $\{\omega_h, h = 0,\cdots,k\}$ be a path on the rescaled lattice $\mathcal{L}_\ell$ such that $d_\ell(\omega_{h-1},\omega_h) = 1$, $h = 1,\cdots,k$. We introduce $q(\omega_{h-1},\omega_h) := |\overline{Q}_\ell^{\tau}(\omega_{h-1}) \cap \overline{Q}_\ell^{\tau}(\omega_h)|/|Q_\ell|$.

Proposition 4.3. *For each $k \in \mathbb{Z}^+$, $I \subset\subset \mathcal{L}_\ell$ and each closed $\mathcal{D} \subseteq \mathcal{B}$ there is a constant $C > 0$ such that for any $\tau \in \Omega$, $\underline{\rho} \in \mathcal{D}^I$, $x \in \partial_r \Lambda$ and all ℓ large enough*

$$\frac{1}{C} \leq \frac{\partial}{\partial \rho_i} z_i^{\tau}(\underline{\rho}) \leq C \tag{4.8}$$

$$\left|\frac{\partial}{\partial \rho_j} z_i^{\tau}(\underline{\rho})\right| \leq C\rho_i \left\{ \sup_{1\leq k'\leq k} \sup_{\substack{\omega: \\ \omega_0=i, \\ \omega_{k'}=j}} \prod_{h=1}^{k'} q(\omega_{h-1},\omega_h) + \frac{1}{\ell^{k+1}} \right\}, \quad i \neq j \tag{4.9}$$

Moreover

$$\left|z_i^{\tau^x}(\underline{\rho}) - z_i^{\tau}(\underline{\rho})\right| \le \frac{C\rho_i}{|Q_\ell|}\left(e^{-d(x,Q_\ell(i))/C} + \sup_{j:x\in\partial_r Q_\ell(j)}\ \sup_{1\le k'\le k}\ \sup_{\substack{\omega:\\ \omega_0=i,\\ \omega_{k'}=j}} \prod_{h=1}^{k'} q(\omega_{h-1},\omega_h) + \frac{1}{\ell^{k+1}}\right) \tag{4.10}$$

Proof. Let

$$\mathbb{J}_{i,j} = \mathbb{J}_{i,j}^{\tau,(\ell)}(\underline{z}) := \frac{\partial}{\partial z_j}\rho_i^\tau(\underline{z})$$

be the Jacobian of the map $\underline{z} \mapsto \underline{\rho}^\tau(\underline{z})$. We split in in its diagonal and off diagonal part; $\mathbb{J} = \mathbb{D} + \mathbb{A}$ where

$$\mathbb{D}_{i,j} := \delta_{i,j}\frac{\partial}{\partial z_i}\rho_i^\tau(\underline{z})$$

and note that from (4.2), (4.3) it follows $\mathbb{D} \ge 1/C$, $\|\mathbb{A}\| \le C\ell^{-1}$.

In order to prove the bounds (4.8), (4.9) we need to invert the Jacobian $\mathbb{J}$. We use the above splitting and Neumann series to get

$$\mathbb{J}^{-1} = \mathbb{D}^{-1}\left(\mathbb{1} + \mathbb{A}\mathbb{D}^{-1}\right)^{-1} = \mathbb{D}^{-1}\left(\sum_{h=0}^{k}(-1)^h(\mathbb{A}\mathbb{D}^{-1})^h + \mathbb{R}_{k+1}\right)$$

where $\|\mathbb{R}_{k+1}\| \le C\ell^{-(k+1)}$. Since $\mathbb{D}$ is bounded from below and $\mathbb{A}_{i,j}$ is exponentially small for $d_\ell(i,j) > 1$, (4.9) follows easily from (4.3).

To prove (4.10) we note that, by definition of the map $\underline{\rho} \mapsto \underline{z}^\tau(\underline{\rho})$ we have

$$\rho_i^{\tau^x}\left(\underline{z}^{\tau^x}(\underline{\rho})\right) = \rho_i^\tau\left(\underline{z}^\tau(\underline{\rho})\right), \qquad i \in I \tag{4.11}$$

By using the invertibilty (uniform in $\tau \in \Omega$ and ℓ) of $\underline{z} \mapsto \underline{\rho}^{\tau,(\ell)}(\underline{z})$ and (4.7), it is not diffucult to see that (4.11) implies that, for ℓ large enough, $\underline{z}^{\tau^x}(\underline{\rho})$ and $\underline{z}^\tau(\underline{\rho})$ are in the same connected component of $\underline{z}^{\tau^x}\left(\mathcal{D}^I\right) \cup \underline{z}^\tau\left(\mathcal{D}^I\right)$.

On the other hand, by Lagrange theorem

$$\rho_i^\zeta(\underline{z}^2) - \rho_i^\zeta(\underline{z}^1) = \sum_{j\in I}\frac{\partial}{\partial z_j}\rho_i^\zeta(\bar{\underline{z}})\cdot[z_j^2 - z_j^1]$$

where $\bar{\underline{z}} \in \mathcal{A}^I$ if $\underline{z}^1$, $\underline{z}^2$ are the same connected component of $\mathcal{A}^I$. Whence, by usig (4.11),

$$z_i^{\tau^x}(\underline{\rho}) - z_i^\tau(\underline{\rho}) = \sum_{j\in I}\left(\mathbb{J}^\tau(\bar{\underline{z}})\right)_{ij}^{-1}\cdot\left[\rho_j^\tau\left(\underline{z}^\tau(\underline{\rho})\right) - \rho_j^{\tau^x}\left(\underline{z}^\tau(\rho)\right)\right]$$

and (4.10) follows from (4.4) and (4.9). □

4.2. Comparison of ensembles in finite volumes.
We here discuss the equivalence of multi–grancanonical and multi–canonical ensembles. We shall work in finite volume with the aim of obtaining explicit bounds as a consequence of the strong mixing assumption.

Let $I \subset\subset \mathcal{L}_\ell$, and Λ as in (2.10). We want to compare the measures $\mu^\tau_{\Lambda,\underline{z}}$ and $\nu^\tau_{\Lambda,\underline{n}}$ where the activity $\underline{z}$ is chosen, depending on $\underline{n}$, Λ *and* τ, as (recall that the fuction $\underline{\rho} \mapsto \underline{z}^\tau(\rho)$ as been defined above) $\underline{z} = \underline{z}^\tau(\underline{n}/|Q_\ell|)$, i.e. so that $\mu^\tau_{\Lambda,\underline{z}}(\underline{N}) = \underline{n}$. We have the following result. Recall that $\mathcal{B} = \rho(\mathcal{A})$.

Theorem 4.4. *Assume $\mu^\tau_{\Lambda,\underline{z}}$ satisfies Condition MUSM($\mathcal{A}$). Then for each closed $\mathcal{D} \subseteq \mathcal{B}$, each $I \subset\subset \mathcal{L}_\ell$ and each local function f, there is a constant C depending on the constants in Condition MUSM($\mathcal{A}$), $\mathcal{D}$, $|I|$, $\operatorname{diam}(S(f))$, $\|f\|$, such that for any b.c. τ, any $\underline{n} \in \mathcal{D}^I$ and all ℓ multiple of ℓ_0 the following bound holds*

$$\left|\nu^\tau_{\Lambda,\underline{n}} f - \mu^\tau_{\Lambda,\underline{z}} f\right| \le C \frac{1}{|Q_\ell|} \quad . \tag{4.12}$$

The proof of this theorem is based on the DS complete analyticity conditions [DS1], [DS2], [DS3]. Although originally formulated for arbitrary volumes their theory carries over to our strong mixing for regular domains as already remarked.

More precisely we need the following condition [DS3, Condition Ib] which is equivalent to SM(ℓ_0). There is a constant $\varepsilon > 0$ such that for all complex interactions $\widetilde{\Phi}$ in an ε–neighborhood of Φ, i.e.

$$\widetilde{\Phi} \in \mathcal{O}_\varepsilon(\Phi) := \{\|\widetilde{\Phi} - \Phi\|_0 < \varepsilon\}$$

and all finite volumes Λ as in (2.2) the analytic functions $Z^\tau_\Lambda(\widetilde{\Phi})$ are non–vanishing. Moreover, there is another constant $A' < \infty$ such that for all $\widetilde{\Phi}_1, \widetilde{\Phi}_2 \in \mathcal{O}_\varepsilon(\Phi)$ we have the bound

$$\sup_{\tau\in\Omega} \left|P^\tau_\Lambda(\widetilde{\Phi}_1) - P^\tau_\Lambda(\widetilde{\Phi}_2)\right| < A' \left|\overline{\Lambda}^\tau \cap \operatorname{supp}\left(\widetilde{\Phi}_1 - \widetilde{\Phi}_2\right)\right| \tag{4.13}$$

where the pressure P is defined by

$$P^\tau_\Lambda(\widetilde{\Phi}) := \log Z^\tau_\Lambda(\widetilde{\Phi}) \tag{4.14}$$

and $\operatorname{supp}(\Phi) := \cup_{\Delta\,:\,\Phi_\Delta \neq 0} \Delta$.

Proof of Theorem 4.4. Since the b.c. τ is kept fixed we drop it from the notation. We also assume, without loss of generality, that $\|f\|$ is small enough.

Step 1. We express here the difference between multi–grancanonical and multi–canonical states by introducing the Fourier transform of the indicator $1\!\!\mathrm{I}_{\underline{N}=\underline{n}}$.

By definition of the multi–canonical state $\nu_{\Lambda,\underline{n}}$, we have

$$\nu_{\Lambda,\underline{n}}(f) - \mu_{\Lambda,\underline{z}}(f) = \frac{\mu_{\Lambda,\underline{z}}\left((f - \mu_{\Lambda,\underline{z}}(f))\, 1\!\!\mathrm{I}_{\underline{N}=\underline{n}}\right)}{\mu_{\Lambda,\underline{z}}(\underline{N} = \underline{n})} = \frac{\mu_{\Lambda,\underline{z}}\left((1+u) 1\!\!\mathrm{I}_{\underline{N}=\underline{n}}\right)}{\mu_{\Lambda,\underline{z}}(\underline{N} = \underline{n})} - 1 \tag{4.15}$$

where we introduced $u := f - \mu_{\Lambda,\lambda}(f)$ which has the same support as f and is mean zero w.r.t. $\mu_{\Lambda,\underline{z}}$.

We next introduce the perturbed probability measure $d\mu^u_{\Lambda,\underline{z}} := (1+u)d\mu_{\Lambda,\underline{z}}$. We regard it as the Gibbs measure w.r.t. an interaction Φ^u. Since f is a local function, we have that Φ^u has range bounded by $\max\{r, \mathrm{diam}\,(\mathrm{supp}(f))\}$. Moreover, by taking $\|f\|$ small (depending on ε) we have that $\Phi^u \in \mathcal{O}_\varepsilon(\Phi)$.

By taking the Fourier transform on the r.h.s. of (4.15), we have (recall that $\mu_{\Lambda,\underline{z}}(\underline{N}) = \underline{n}$ by the choice of $\underline{z}$)

$$\begin{aligned}
\nu_{\Lambda,\underline{n}}(f) - \mu_{\Lambda,\underline{z}}(f) &= \frac{\int_{|\underline{t}|\le\pi} d\underline{t}\, e^{-i\langle \underline{t},\mu_{\Lambda,\underline{z}}\underline{N}\rangle}\, \mu^u_{\Lambda,\underline{z}}\left(e^{i\langle\underline{t},\underline{N}\rangle}\right)}{\int_{|\underline{t}|\le\pi} d\underline{t}\, e^{-i\langle \underline{t},\mu_{\Lambda,\underline{z}}\underline{N}\rangle}\, \mu_{\Lambda,\underline{z}}\left(e^{i\langle\underline{t},\underline{N}\rangle}\right)} - 1 \\
&= \frac{\int_{|\underline{t}|\le\pi} d\underline{t}\, e^{\psi_\Lambda(\underline{t},\underline{z}) - i\langle \underline{t},\mu_{\Lambda,\underline{z}}\underline{N}\rangle}\left[e^{\psi^u_\Lambda(\underline{t},\underline{z}) - \psi_\Lambda(\underline{t},\underline{z})} - 1\right]}{\int_{|\underline{t}|\le\pi} d\underline{t}\, e^{\psi_\Lambda(\underline{t},\underline{z}) - i\langle \underline{t},\mu_{\Lambda,\underline{z}}\underline{N}\rangle}}
\end{aligned} \tag{4.16}$$

where, indicating with a superscript the dependence on the perturbation u and inside the parentheses the dependence on the complex activity, we introduced

$$\psi_\Lambda(\underline{t},\underline{z}) := \log \mu_{\Lambda,\underline{z}}\left(e^{i\langle\underline{t},\underline{N}\rangle}\right) = P_\Lambda\left(\{z_j e^{it_j}\}_{j\in I}\right) - P_\Lambda(\underline{z}) \tag{4.17}$$

where the second identity holds by expressing the l.h.s. in terms of ratio of partition functions. The definition of $\psi^u_\Lambda(\underline{t},\underline{z})$ is analogous, it is enough to consider the pressure of the perturbed interaction.

Step 2. Here we estimate from below the denominator on the r.h.s. of (4.16).

Let us introduce the variances

$$v_i^2 = v_i^{\tau,(\ell)}(\underline{z})^2 := \mu^\tau_{\Lambda,\underline{z}}(N_i; N_i)$$

and note that from Proposition 4.1 we have $C^{-1} z_i |Q_\ell| \le v_i^2 \le C z_i |Q_\ell|$. This bound will be used extensively in the sequel.

We shall prove the following bound. There is a constant C independent on τ, ℓ and $\underline{z}$ such that for ℓ large enough

$$\mu_{\Lambda,\underline{z}}(\underline{N} = \underline{n}) = \frac{1}{(2\pi)^{|I|}} \int_{|\underline{t}|\le\pi} d\underline{t}\, e^{\psi_\Lambda(\underline{t},\underline{z}) - i\langle \underline{t},\mu_{\Lambda,\underline{z}}\underline{N}\rangle} \ge \frac{1}{C}\frac{1}{\prod_{i\in I} v_i} \tag{4.18}$$

where we recall $\underline{z}$ has been chosen so that $\mu_{\Lambda,\underline{z}}(\underline{N}) = \underline{n}$.

By a change of variables we get

$$\mu_{\Lambda,\underline{z}}(\underline{N} = \underline{n}) = \frac{1}{\prod_{j\in I} 2\pi v_j} \int_{|s_j|\le\pi v_j} d\underline{s}\, e^{\psi_\Lambda(\underline{s}/v,\underline{z}) - i\langle \underline{s}/v,\mu_{\Lambda,\underline{z}}\underline{N}\rangle}$$

where we used the notation $\underline{s}/v$ to denote the variables $\{s_j/v_j,\ j\in I\}$

Let K be a large constant. We take advantage of the Gaussian bound in Lemma 4.2 to get

$$\left| \int_{\exists j : K \wedge (\pi v_j) \leq |s_j| \leq \pi v_j} d\underline{s}\, e^{\psi_\Lambda (\underline{s}/v, \underline{z}) - i \langle \underline{s}/v, \mu_{\Lambda, \underline{z}} \underline{N} \rangle} \right| \leq \int_{\exists j : |s_j| \geq K \wedge (\pi v_j)} d\underline{s}\ \exp \left\{ -\frac{1}{2} \frac{1}{C} |Q_\ell| \sum_{i \in I} z_i \frac{s_i^2}{v_i^2} \right\}$$
$$\leq C e^{-K^2/C} \tag{4.19}$$

By the above bound we can restrict ourselves to bounded $\underline{s}$. We need however to treat separately the Gaussian scaling in which v_i diverges with ℓ and the very low density case in which it remains bounded. Let M be another large constant ($1 \ll K \ll M \ll \ell$) and introduce $I_g := \{ i \in I : v_i^2 \geq M \}$, $I_p := I \setminus I_g$. Let also $\underline{s}_g := \{ s_i, i \in I_g \}$ (resp. $\underline{s}_p := \{ s_i, i \in I_p \}$); we use an analogous notation for $\underline{z}$. We shall prove the following expansion on the logarithm of the characteristic function.

$$\begin{aligned} &\psi_\Lambda (\underline{s}/v, \underline{z}) - i \langle \underline{s}/v, \mu_{\Lambda, \underline{z}} \underline{N} \rangle \\ &\quad = \sum_{j \in I_p} \left(e^{i s_j / v_j} - 1 - i s_j / v_j \right) \mu_{\Lambda, \underline{z}} N_j - \frac{1}{2} \sum_{j j' \in I_g} \mu_{\Lambda, \underline{z}} (N_j ; N_{j'}) \frac{s_j}{v_j} \frac{s_{j'}}{v_{j'}} + R_\Lambda (\underline{s}, \underline{z}) \end{aligned} \tag{4.20}$$

where

$$\sup_{|\underline{s}| \leq K} |R_\Lambda (\underline{s}, \underline{z})| \leq C \left(\frac{K^3}{\sqrt{M}} + \frac{M^2}{|Q_\ell|} + \frac{KM}{\sqrt{|Q_\ell|}} + \frac{K^2 M}{|Q_\ell|} \right)$$

Note that on the r.h.s. of (4.20) the first term corresponds to a Poisson limit for N_j, $j \in I_p$ and to a (joint) Gaussian limit for N_j, $j \in I_g$.

Postponing the proof of (4.20), let us first show that, together with (4.19), it implies the bound (4.18). It is enough to notice that if Z is a Poisson r.v. with mean $\lambda \in \mathbb{Z}^+$ we have

$$\frac{1}{2\pi} \int_{|s| \leq \pi u} ds\ e^{\left(e^{is/u} - 1 - is/u \right) \lambda} = u\ \mathrm{Prob}\,(Z = \lambda) = u \frac{e^{-\lambda} \lambda^\lambda}{\lambda!}$$

By using the bounds $v_i^2 \geq z_i |Q_\ell| / C$, $\mu_{\Lambda, \underline{z}} N_i \leq C z_i |Q_\ell|$, Stirling's formula and estimating the Gaussian integral (recall (4.6)) we thus get

$$\int_{|s_i| \leq K \wedge (\pi v_i)} d\underline{s}\ \exp \left\{ \sum_{j \in I_p} \left(e^{i s_j / v_j} - 1 - i s_j / v_j \right) \mu_{\Lambda, \underline{z}} (N_j) - \frac{1}{2} \sum_{j j' \in I_g} \mu_{\Lambda, \underline{z}} (N_j ; N_{j'}) \frac{s_j}{v_j} \frac{s_{j'}}{v_{j'}} \right\} \geq \frac{1}{C}$$

and (4.18) follows since we can make the remainder as small as we want.

In order to prove (4.20) let us first expand ψ_Λ in power series of $\underline{s}_g$ and get

$$\begin{aligned} \psi_\Lambda (\underline{s}/v, \underline{z}) = &\psi_\Lambda \left(0, \underline{s}_p / v, \underline{z} \right) + \sum_{i \in I_g} \frac{\partial}{\partial t_i} \psi_\Lambda \left(0, \underline{s}_p / v, \underline{z} \right)\ \frac{s_i}{v_i} \\ &+ \frac{1}{2} \sum_{i, i' \in I_g} \frac{\partial^2}{\partial t_i \partial t_{i'}} \psi_\Lambda \left(0, \underline{s}_p / v, \underline{z} \right)\ \frac{s_i}{v_i} \frac{s_{i'}}{v_{i'}} + R^1_\Lambda (\underline{s}, \underline{z}) \end{aligned} \tag{4.21}$$

We note that by Condition [DS3,Ic], still equivalent to $\mathrm{SM}(\ell_0)$,

$$\left|\frac{\partial}{\partial t_i}\psi(\underline{t},\underline{z})\right| = \left|\mu^{\tau}_{\Lambda,\underline{z},\underline{t}}\left(N_i\right)\right| \leq C z_i |Q_\ell| \tag{4.22}$$

here $\mu^{\tau}_{\Lambda,\underline{z},\underline{t}}$ denotes the complex measure defined by

$$\mu^{\tau}_{\Lambda,\underline{z},\underline{t}}(f) := \frac{\mu^{\tau}_{\Lambda,\underline{z}}\left(e^{i\langle \underline{t},\underline{N}\rangle} f\right)}{\mu^{\tau}_{\Lambda,\underline{z}}\left(e^{i\langle \underline{t},\underline{N}\rangle}\right)}$$

We remark that in [DS3,Ic] does not include z_i on the r.h.s. of (4.22). However, by the remark following Condition $\mathrm{USM}(\ell_0)$, we can easily verify that (4.22) holds.

Recall that the pressure $P_\Lambda(\underline{z})$ is holomorphic in an ε–neighborhood of $\underline{z}$. Therefore (see (4.17)) $\psi_\Lambda(\underline{t},\underline{z})$ is holomorphic in a neighborhood of $\underline{t}=0$. By taking $K/\sqrt{M}$ small enough we can thus use Cauchy integral formula and bound the third order derivatives (w.r.t. to $\underline{t}$) of $\psi(\underline{t},\underline{z})$ in terms of the first one. By applying (4.22) we get

$$\sup_{|\underline{s}|\leq K}\left|R^1_\Lambda\left(\underline{s},\underline{z}\right)\right| \leq C \sup_{|\underline{s}|\leq K}\sum_{i,j,k\in I_g}\min\{z_i,z_j,z_k\}|Q_\ell|\frac{|s_i s_j s_k|}{v_i v_j v_k} \leq C K^3 \frac{1}{\sqrt{M}}$$

We next expand the other terms on the r.h.s. of (4.21) in power series of $\underline{z}_p$. Note in fact that for $i\in I_p$ we have $z_i \leq CM/|Q_\ell|$. Let us consider the first one. We get

$$\psi_\Lambda\left(0,\underline{s}_p/v,\underline{z}\right) = \psi_\Lambda\left(0,\underline{s}_p/v,\underline{z}_g,0\right) + \sum_{j\in I_p} z_j \frac{\partial}{\partial z_j}\psi_\Lambda\left(0,\underline{s}_p/v,\underline{z}_g,0\right) + R^2_\Lambda\left(\underline{s},\underline{z}\right)$$

Noticing that $|\psi_\Lambda(\underline{t},\underline{z})| \leq C|Q_\ell|$ and using again the Cauchy integral formula, we can bound the remainder as follows

$$\left|R^2_\Lambda\left(\underline{s},\underline{z}\right)\right| \leq C \sum_{j,j'\in I_p} z_j z_{j'} |Q_\ell| \leq C\frac{M^2}{|Q_\ell|}$$

We next observe that $\psi_\Lambda\left(0,\underline{s}_p/v,\underline{z}_g,0\right)=0$. On the other hand, by (4.17)

$$\frac{\partial}{\partial z_j}\psi_\Lambda\left(0,\underline{s}_p/v,\underline{z}_g,0\right) = \left(e^{i s_j/v_j}-1\right)\frac{\partial}{\partial z_j}P_\Lambda\left(\underline{z}_g,0\right)$$

By the analyticity of the pressure (see (4.22)) we also have

$$\left|\frac{\partial}{\partial z_j}P_\Lambda\left(\underline{z}_g,0\right) - \frac{\partial}{\partial z_j}P_\Lambda\left(\underline{z}_g,\underline{z}_p\right)\right| \leq C|Q_\ell|\sum_{i\in I_p} z_i \leq CM$$

Since $z_j \frac{\partial}{\partial z_j} P_\Lambda\left(\underline{z}\right) = \mu_{\Lambda,\underline{z}}\left(N_j\right)$, we thus get

$$\psi_\Lambda\left(0,\underline{s}_p/v,\underline{z}\right) = \sum_{j\in I_p}\left(e^{i s_j/v_j}-1\right)\mu_{\Lambda,\underline{z}}\left(N_j\right) + R^3_\Lambda(\underline{s},\underline{z}), \qquad \left|R^3_\Lambda(\underline{s},\underline{z})\right| \leq C\frac{M^2}{|Q_\ell|}$$

We expand similary the other two terms in (4.21). For the second one we have

$$\frac{\partial}{\partial t_i}\psi_\Lambda\left(0, \underline{s}_p/v, \underline{z}\right) = \frac{\partial}{\partial t_i}\psi_\Lambda\left(0, \underline{s}_p/v, \underline{z}_g, 0\right) + R^4_{\Lambda,i}\left(\underline{s}, \underline{z}\right)$$

where, by using again (4.22) and the analyticity of $\psi_\Lambda\left(\underline{t}, \underline{z}\right)$, we have

$$\left|R^4_{\Lambda,i}\left(\underline{s}, \underline{z}\right)\right| \le Cz_i|Q_\ell|\sum_{j\in I_p} z_j \le CMz_i$$

Furthermore, since by setting $z_i = 0$ ψ_Λ becomes independent of s_i,

$$\left|\frac{\partial}{\partial t_j}\psi_\Lambda\left(0, \underline{s}_p/v, \underline{z}_g, 0\right) - \frac{\partial}{\partial t_j}\psi_\Lambda\left(0, 0, \underline{z}_g, \underline{z}_p\right)\right| = \left|\frac{\partial}{\partial t_j}\psi_\Lambda\left(0, \underline{s}_p/v, \underline{z}_g, 0\right) - i\mu_{\Lambda,\underline{z}}\left(N_j\right)\right| \le CMz_j$$

so that

$$\sum_{j\in I_g}\frac{\partial}{\partial t_j}\psi_\Lambda\left(0, \underline{s}_p/v, \underline{z}\right)\ \frac{s_j}{v_j} = i\sum_{j\in I_g}\frac{s_j}{v_j}\mu_{\Lambda,\underline{z}}\left(N_j\right) + R^5_\Lambda\left(\underline{s}, \underline{z}\right), \qquad \sup_{|\underline{s}|\le K}\left|R^5_\Lambda\left(\underline{s}, \underline{z}\right)\right| \le C\frac{KM}{\sqrt{|Q_\ell|}}$$

By the same argument we finally have

$$\frac{\partial^2}{\partial t_i \partial t_{i'}}\psi_\Lambda\left(0, \underline{s}_p/v, \underline{z}_g, \underline{z}_p\right) = \frac{\partial^2}{\partial t_i \partial t_{i'}}\psi_\Lambda\left(0, \underline{s}_p/v, \underline{z}_g, 0\right) + R^6_{\Lambda,i,i'}\left(\underline{s}, \underline{z}\right), \quad \left|R^6_{\Lambda,i,i'}\left(\underline{s}, \underline{z}\right)\right| \le CMz_i \wedge z_{i'} \tag{4.23}$$

Moreover, as before,

$$\left|\frac{\partial^2}{\partial t_i \partial t_{i'}}\psi_\Lambda\left(0, \underline{s}_p/v, \underline{z}_g, 0\right) - \frac{\partial^2}{\partial t_i \partial t_{i'}}\psi_\Lambda\left(0, 0, \underline{z}_g, \underline{z}_p\right)\right|$$
$$= \left|\frac{\partial^2}{\partial t_i \partial t_{i'}}\psi_\Lambda\left(0, \underline{s}_p/v, \underline{z}_g, 0\right) + \mu_{\Lambda,\underline{z}}\left(N_i; N_{i'}\right)\right| \le CMz_i \wedge z_{i'}$$

which gives us

$$\sum_{i,i'\in I_g}\frac{\partial^2}{\partial t_i \partial t_{i'}}\psi_\Lambda\left(0, \underline{s}_p/v, \underline{z}\right)\ \frac{s_i}{v_i}\frac{s_{i'}}{v_{i'}} = -\frac{1}{2}\sum_{i,i'\in I_g}\mu_{\Lambda,\underline{z}}\left(N_i; N_{i'}\right)\ \frac{s_i}{v_i}\frac{s_{i'}}{v_{i'}} + R^7_\Lambda\left(\underline{s}, \underline{z}\right)$$

where

$$\sup_{|\underline{s}|\le K}\left|R^7_\Lambda\left(\underline{s}, \underline{z}\right)\right| \le C\frac{K^2M}{\sqrt{|Q_\ell|}}$$

The proof of (4.20) is now complete.

Step 3. We finally here estimate from above the numerator on the r.h.s. of (4.16).

Let $K_\ell := \log|Q_\ell|$. We make the change of variables $\underline{t} = \underline{s}/v$ and use Lemma 4.2 (which holds also for the perturbed measure $\mu^u_{\Lambda,\underline{z}}$) to get

$$\left|\int_{\exists j: K_\ell\wedge(\pi v_j)\le|s_j|\le\pi v_j} d\underline{s}\, e^{\psi_\Lambda(\underline{s}/v,\underline{z})-i\left\langle\underline{s}/v,\mu_{\Lambda,\underline{z}}\underline{N}\right\rangle}\left[e^{\psi^u_\Lambda(\underline{s}/v,\underline{z})-\psi_\Lambda(\underline{s}/v,\underline{z})}-1\right]\right|$$
$$\le \int_{\exists j: K_\ell\wedge(\pi v_j)\le|s_j|\le\pi v_j} d\underline{s}\,\left[\left|e^{\psi_\Lambda(\underline{s}/v,\underline{z})}\right|+\left|e^{\psi^u_\Lambda(\underline{s}/v,\underline{z})}\right|\right]\le Ce^{-K_\ell^2/C}\le C\frac{1}{|Q_\ell|}$$

We can thus consider the case $|s_j|\le K_\ell\wedge(\pi v_j)$.

Since $z_i|Q_\ell|/C\le v_i^2\le Cz_i|Q_\ell|$, either s_i/v_i or z_i is small. We can therefore apply the bound (4.13). We get

$$\left|P^u_\Lambda(\{e^{is_j/v_j}z_j\})-P_\Lambda(\{e^{is_j/v_j}z_j\})\right|\le C$$

We next expand the difference $\psi^u_\Lambda(\underline{s}/v,\underline{z})-\psi_\Lambda(\underline{s}/v,\underline{z})$ in power series of $\underline{s}$. Since $\mu^u_{\Lambda,\underline{z}}(N_k)-\mu_{\Lambda,\underline{z}}(N_k)=\mu_{\Lambda,\underline{z}}(f;N_k)$, we get

$$\psi^u_\Lambda(\underline{s}/v,\underline{z})-\psi_\Lambda(\underline{s}/v,\underline{z})=i\sum_{k\in I}\mu_{\Lambda,\underline{z}}(f;N_k)\,\frac{s_k}{v_k}+R^1_\Lambda(\underline{s},\underline{z})$$

where

$$R^1_\Lambda(\underline{s},\underline{z})=\frac{1}{2}\sum_{i,j\in I}\frac{\partial^2}{\partial t_i\partial t_j}\left[\psi^u_\Lambda(\underline{t},\underline{z})-\psi_\Lambda(\underline{t},\underline{z})\right]\Big|_{\underline{t}=\overline{s}/v}\frac{s_is_j}{v_iv_j}$$

We note that, by (4.17),

$$\frac{\partial}{\partial t_k}\left[\psi^u_\Lambda(\underline{t},\underline{z})-\psi_\Lambda(\underline{t},\underline{z})\right]=iz_ke^{it_k}\,\frac{\partial}{\partial z'_j}\left[P^u_\Lambda(\underline{z}')-P_\Lambda(\underline{z}')\right]\Big|_{z'_j=z_je^{it_j}}$$

By the analyticity of $P^u_\Lambda(\underline{z}')-P_\Lambda(\underline{z}')$, for $\underline{t}=\underline{s}/v$ we can bound the r.h.s. above by Cz_k. We thus have

$$\left|R^1_\Lambda(\underline{s},\underline{z})\right|\le C\sum_{i,j\in I}z_i\wedge z_j\frac{s_is_j}{v_iv_j}\le C\frac{|\underline{s}|^2}{|Q_\ell|}$$

As $\left|\mu_{\Lambda,\underline{z}}(f;N_j)\right|\le Cz_j$, for $|\underline{s}|\le K_\ell$ we finally have

$$\exp\left\{\psi^u_\Lambda(\underline{s}/v,\underline{z})-\psi_\Lambda(\underline{s}/v,\underline{z})\right\}-1=i\sum_{k\in I}\mu_{\Lambda,\underline{z}}(f;N_k)\,\frac{s_k}{v_k}+R^2_\Lambda(\underline{s},\underline{z}),\qquad\left|R^2_\Lambda(\underline{s},\underline{z})\right|\le C\frac{|\underline{s}|^2}{|Q_\ell|}\tag{4.24}$$

By Lemma 4.2, we have

$$\left|\int_{|s_j|\le K_\ell\wedge(\pi v_j)} d\underline{s}\;e^{\psi_\Lambda(\underline{s}/v,\underline{z})-i\left\langle\underline{s}/v,\mu_{\Lambda,\underline{z}}(\underline{N})\right\rangle}R^2_\Lambda(\underline{s},\underline{z})\right|\le C\frac{1}{|Q_\ell|}$$

To conclude the proof we consider separately each of the other terms on the r.h.s of (4.24). We want to show that, with a small error, the function $\psi(\underline{s}/v,\underline{z}) - i\left\langle \underline{s}/v, \mu_{\Lambda,\underline{z}}\left(\underline{N}\right)\right\rangle$ is even in s_k; hence the integral vanishes by symmetry. We thus expand $\psi(\underline{s}/v,\underline{z})$ as follows

$$\psi(\underline{s}/v,\underline{z}) - i\left\langle \underline{s}/v, \mu_{\Lambda,\underline{z}}\left(\underline{N}\right)\right\rangle = \frac{1}{2}\sum_{j,j'} \frac{\partial^2}{\partial t_j \partial t_{j'}}\psi(\underline{\bar{s}}/v,\underline{z})\frac{s_j s_{j'}}{v_j v_{j'}}$$

by letting $\underline{s}^{(k)} := \{s_i,\ i\in I\setminus\{k\}\}$, we have

$$\frac{\partial^2}{\partial t_j \partial t_{j'}}\psi(\underline{\bar{s}}/v,\underline{z}) = -B_{j,j'}(\underline{\bar{s}}^{(k)}) + R^3_{\Lambda,j,j'}(\underline{s},\underline{z}), \qquad B_{j,j'}(\underline{\bar{s}}^{(k)}) := -\frac{\partial^2}{\partial t_j \partial t_{j'}}\psi(0,\underline{\bar{s}}^{(k)}/v,\underline{z})$$

and, by (4.22) and the analyticity of ψ_Λ,

$$\left|R^3_{\Lambda,j,j'}(\underline{s},\underline{z})\right| \le C z_j \wedge z_{j'}|Q_\ell|\frac{s_k}{v_k}$$

Whence

$$\psi(\underline{s}/v,\underline{z}) - i\left\langle \underline{s}/v, \mu_{\Lambda,\underline{z}}\left(\underline{N}\right)\right\rangle = -\frac{1}{2}\sum_{j,j'\in I} B_{j,j'}(\underline{\bar{s}}^{(k)})\frac{s_j s_{j'}}{v_j v_{j'}} + R^4_{\Lambda,k}(\underline{s},\underline{z}), \qquad \left|R^4_{\Lambda,k}(\underline{s},\underline{z})\right| \le C\frac{|\underline{s}|^3}{v_k} \tag{4.25}$$

We next use the bound

$$\left|e^{R^4_{\Lambda,k}(\underline{s},\underline{z})} - 1\right| \le \left(1 + \left|e^{R^4_{\Lambda,k}(\underline{s},\underline{z})}\right|\right)\left|R^4_{\Lambda,k}(\underline{s},\underline{z})\right|$$

and (4.25) to get

$$\begin{aligned} &\left|\int_{|s_j|\le K_\ell\wedge(\pi v_j)} d\underline{s}\ e^{\psi_\Lambda(\underline{s}/v,\underline{z}) - i\left\langle \underline{s}/v, \mu_{\Lambda,\underline{z}}(\underline{N})\right\rangle}\mu_{\Lambda,\underline{z}}\left(f;N_k\right)\frac{s_k}{v_k}\right| \\ &\quad\le \int_{|s_j|\le K_\ell\wedge(\pi v_j)} d\underline{s}\ \left(\left|e^{-\frac{1}{2}\sum_{j,j'\in I} B_{j,j'}(\underline{\bar{s}}^{(k)})\frac{s_j s_{j'}}{v_j v_{j'}}}\right| + \left|e^{\psi_\Lambda(\underline{s}/v,\underline{z})}\right|\right)\left|R^5_{\Lambda,k}(\underline{s},\underline{z})\right| \end{aligned} \tag{4.26}$$

where, recalling that $\left|\mu_{\Lambda,\underline{z}}\left(f;N_k\right)\right| \le C z_k$ and $v_k^2 \ge z_k|Q_\ell|/C$,

$$R^5_{\Lambda,k}(\underline{s},\underline{z}) := \mu_{\Lambda,\underline{z}}\left(f;N_k\right)\frac{s_k}{v_k}R^4_{\Lambda,k}(\underline{s},\underline{z}), \qquad \left|R^5_{\Lambda,k}(\underline{s},\underline{z})\right| \le C\frac{|\underline{s}|^4}{|Q_\ell|}$$

By applying again Lemma 4.2 we have

$$\int_{|s_j|\le K_\ell\wedge(\pi v_j)} d\underline{s}\ \left|e^{\psi_\Lambda(\underline{s}/v,\underline{z})}\right|\left|R^5_{\Lambda,k}(\underline{s},\underline{z})\right| \le C\frac{1}{|Q_\ell|}$$

It now remains only to estimate the other term on the r.h.s. of (4.26). Let $M_\ell := \ell^{1/4}$ and introduce $I_g^{(k)} := \{i \in I \setminus \{k\} : v_i^2 \geq M_\ell\}$, $I_p^{(k)} := I \setminus \left(\{k\} \cup I_g^{(k)}\right)$. We have

$$B_{j,j'}(\underline{\bar{s}}^{(k)}) = -\frac{\partial^2}{\partial t_j \partial t_{j'}} \psi(0, \underline{\bar{s}}_p^{(k)}/v, \underline{z}) + R^6_{\Lambda,k,j,j'}(\underline{s}, \underline{z})$$

where

$$\sup_{|\underline{s}| \leq K_\ell} \left| R^6_{\Lambda,k,j,j'}(\underline{s}, \underline{z}) \right| \leq C z_j \wedge z_{j'} |Q_\ell| \sum_{i \in I^(k)_g} \frac{|s_i|}{v_i} \leq C z_j \wedge z_{j'} \frac{K_\ell}{\sqrt{M_\ell}}$$

so that, by using also (4.23),

$$\sum_{j,j' \in I} B_{j,j'}(\underline{\bar{s}}^{(k)}) \frac{s_j s_{j'}}{v_j v_{j'}} = \sum_{j,j' \in I} \mu_{\Lambda,\underline{z}}(N_j, N_{j'}) \frac{s_j s_{j'}}{v_j v_{j'}} + R^7_{\Lambda,k}(\underline{s}, \underline{z})$$

where

$$\sup_{|\underline{s}| \leq K_\ell} \left| R^7_{\Lambda,k}(\underline{s}, \underline{z}) \right| \leq C \left(\frac{K_\ell^3}{\sqrt{M_\ell}} + \frac{K_\ell^2 M_\ell}{|Q_\ell|} \right)$$

Hence, recalling (4.6),

$$\int_{|s_j| \leq K_\ell \wedge (\pi v_j)} d\underline{s} \left| e^{-\frac{1}{2} \sum_{j,j' \in I} B_{j,j'}(\underline{\bar{s}}^{(k)}) \frac{s_j s_{j'}}{v_j v_{j'}}} \right| \left| R^5_{\Lambda,k}(\underline{s}, \underline{z}) \right| \leq C \frac{1}{|Q_\ell|}$$

which concludes the proof. □

4.3. Local Central Limit Theorem with multiplicative error

In order to obtain the convergence of the short range part of the renormalized potential to the one of independent harmonic oscillators we need a local central limit theorem which will allow us to compute the asymptotic behaviour (as $\ell \to \infty$) of the r.h.s. of (3.69). Since we are interested in the logarithm of the partition function we do need a local CLT in which the error appears in a multiplicative way. It can be proven by applying the theory of moderate deviations as developed in [DS4]; alough these results are stated only for very high temperature, the proof is based only on the analyticity properties of the thermodynamic functions which hold under Condition MUSM($\mathcal{A}$).

Let us recall that $\mu^\tau_{\Lambda,\underline{z}}$ is the multi–grancanonical state in a volume $\Lambda \subset\subset \mathcal{L}$ of the form (2.10). We denote by $v^{(\ell)} = v^{\tau,(\ell)}(\underline{z})$ the covariance matrix of the total number of particles in each cube $Q_\ell(i)$, i.e. $v^{\tau,(\ell)}(\underline{z})_{i,j} := \mu^\tau_{\Lambda,\underline{z}}(N_i; N_j)$, where N_i has been defined in (2.12). We have the following local central limit theorem.

Theorem 4.5. *Let U satisfy MUSM($\mathcal{A}$) and $\underline{\rho}^{(\ell)} = \underline{\rho}^{\tau,(\ell)}(\underline{z}) := \mu^\tau_{\Lambda,\underline{z}}(\underline{N})/|Q_\ell|$. For each Λ of the form (2.10) and $\underline{z} \in \mathcal{A}$, $\varepsilon > 0$ there are constants $\delta = \delta(\underline{z}, I, \varepsilon) > 0$, $C = C(\underline{z}, I, \varepsilon) < \infty$ such that for any integer ℓ we have*

$$\mu^\tau_{\Lambda,\underline{z}}(\underline{N} = \underline{n}) = \left[(2\pi)^{|I|} \det v^{(\ell)} \right]^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \left\langle \left(\underline{n} - \underline{\rho}^{(\ell)} |Q_\ell| \right), \left(v^{(\ell)} \right)^{-1} \left(\underline{n} - \underline{\rho}^{(\ell)} |Q_\ell| \right) \right\rangle \right\} \times \left\{ 1 + R^\tau_\Lambda(\underline{n}) \right\} \tag{4.27}$$

where

$$\sup_{\tau\in\Omega} \sup_{\substack{\underline{n}: \\ \left|\underline{n}-\underline{\rho}^{(\ell)}|Q_\ell|\right|\le |Q_\ell|^{2/3-\varepsilon}}} |R^\tau_\Lambda(\underline{n})| \le C\frac{1}{|Q_\ell|^\delta} \tag{4.28}$$

This Theorem is essentially contained in [DS4]; however to make the paper selfcontained we give below a brief sketch of the proof. Given $\underline{n}$ we let $\underline{\zeta} = \underline{\zeta}^{\tau,(\ell)}(\underline{n})$ be defined by $\underline{\zeta} := \underline{z}^{\tau,(\ell)}\left(\underline{n}/|Q_\ell|\right)$ where we recall the function $\underline{\rho} \mapsto \underline{z}^{\tau,(\ell)}(\underline{\rho})$ has been defined in Section 4.1. We also recall the pressure has been defined in (4.14). We have the following Lemma.

Lemma 4.6. *Under the same hypotheses of the previous Theorem, there are constants $\varepsilon_0 = \varepsilon_0(\underline{z}, I) > 0$, $C = C(\underline{z}, I, \varepsilon_0) < \infty$ such that*

$$\mu^\tau_{\Lambda,\underline{z}}\left(\underline{N} = \underline{n}\right) = \left[(2\pi)^{|I|}\det v^{(\ell)}(\underline{\zeta})\right]^{-\frac{1}{2}} \exp\left\{-I_\Lambda(\underline{n})\right\}\left(1 + \hat{R}^\tau_\Lambda(\underline{n})\right) \tag{4.29}$$

where

$$I_\Lambda(\underline{n}) = I^\tau_{\Lambda,\underline{z}}(\underline{n}) := \sum_{i\in I} n_i \log\frac{\zeta_i}{z_i} - \left[P^\tau_\Lambda(\underline{\zeta}) - P^\tau_\Lambda(\underline{z})\right]$$

and

$$\sup_{\tau\in\Omega} \sup_{\substack{\underline{n}: \\ \left|\underline{n}-\underline{\rho}^{(\ell)}|Q_\ell|\right|\le \varepsilon_0|Q_\ell|}} \left|\hat{R}^\tau_\Lambda(\underline{n})\right| \le C\frac{1}{|Q_\ell|} \tag{4.30}$$

Sketch of the proof. By definition of the multi–grancanonical state $\mu^\tau_{\Lambda,\underline{z}}$ we have

$$\mu^\tau_{\Lambda,\underline{z}}\left(\underline{N} = \underline{n}\right) = \prod_{i\in I}\left(\frac{z_i}{\zeta_i}\right)^{n_i} \cdot \frac{Z^\tau(\underline{\zeta})}{Z^\tau(\underline{z})} \cdot \mu^\tau_{\Lambda,\underline{\zeta}}\left(\underline{N} = \underline{n}\right) = e^{-I_\Lambda(\underline{n})}\frac{1}{(2\pi)^{|I|}}\int_{|\underline{t}|\le\pi} d\underline{t}\; e^{-i\langle\underline{t},\underline{n}\rangle}\mu^\tau_{\Lambda,\underline{\zeta}}\left(e^{i\langle\underline{t},\underline{N}\rangle}\right)$$

If we take ε small enough, $\left|\underline{n} - \underline{\rho}^{(\ell)}|Q_\ell|\right| \le \varepsilon|Q_\ell|$ implies that $(\underline{\zeta}, U)$ satisfy SM(ℓ_0) for some $\ell_0 = \ell_0(\underline{z}, \varepsilon_0)$. In order to conclude the proof it is then enough to make the change of variables $t_i = s_i/\sqrt{v^{(\ell)}_{i,i}}$, use Lemma 4.2 to estimate the tail and expand $\log\mu^\tau_{\Lambda,\underline{\zeta}}\left(e^{i\langle\underline{t},\underline{N}\rangle}\right)$ up to the third order, using analyticity to estimate the remainder (see Section 4.2 for analogous computations). Note in fact that, by the definition of $\underline{\zeta}$ we have $\mu^\tau_{\Lambda,\underline{\zeta}}\left(\underline{N}\right) = \underline{n}$. □

Sketch of the proof of Theorem 4.5. By applying Proposition 4.1 we have

$$\sup_{\tau\in\Omega} \sup_{\substack{\underline{n}: \\ \left|\underline{n}-\underline{\rho}^{(\ell)}|Q_\ell|\right|\le |Q_\ell|^{2/3-\varepsilon}}} \left\|v^{(\ell)}(\underline{z}) - v^{(\ell)}(\underline{\zeta})\right\| \le C|Q_\ell|^{2/3-\varepsilon}$$

which, together with the bound (4.6), implies

$$\left(\det v^{(\ell)}(\underline{\zeta})\right)^{-\frac{1}{2}} = \left(\det v^{(\ell)}(\underline{z})\right)^{-\frac{1}{2}}\left(1 + R^{\tau,(1)}_\Lambda(\underline{n})\right)$$

where

$$\sup_{\tau\in\Omega} \sup_{\substack{\underline{n}: \\ \left|\underline{n}-\underline{\rho}^{(\ell)}|Q_\ell|\right|\le |Q_\ell|^{2/3-\varepsilon}}} \left|R^{\tau,(1)}_\Lambda(\underline{n})\right| \le C\frac{1}{|Q_\ell|^{1/3}}$$

On the other hand, by the analyticity (uniform in ℓ) of the thermoduynamic functions, we have (see [DS4 Eq. 1.2.15])

$$I_\Lambda(\underline{n}) = \frac{1}{2}\left\langle \left(\underline{n} - \underline{\rho}^{(\ell)}|Q_\ell|\right), \left(v^{(\ell)}(\underline{z})\right)^{-1}\left(\underline{n} - \underline{\rho}^{(\ell)}|Q_\ell|\right)\right\rangle \left(1 + R_\Lambda^{\tau,(2)}(\underline{n})\right)$$

where

$$\sup_{\tau\in\Omega} \sup_{\substack{\underline{n}: \\ \left|\underline{n}-\underline{\rho}^{(\ell)}|Q_\ell|\right| \le |Q_\ell|^{2/3-\varepsilon}}} \left|R_\Lambda^{\tau,(2)}(\underline{n})\right| \le C\frac{1}{|Q_\ell|^{3\varepsilon}}$$

in which we have used again that $\mu^\tau_{\Lambda,\underline{\zeta}}(\underline{N}) = \underline{n}$. □

5. Gibbsianess and convergence

In this section we conclude the proof of the main results. First, by applying the comparison of ensembles, we show the constrained models satisfy a finite size effective condition uniformly in the constraints. Secondly, by applying the local central limit theorem, we prove the short range part of the renormalized potential converges to the potential of independent harmonic oscillators. Finally, when the global condition GMUSM holds, we verify that the renormalized measure $\mu_z^{(\ell)}$ (defined directly in infinite volume) is Gibbs w.r.t. the potential constructed in Section 3 (obtained via a thermodynamic limit).

5.1. Finite size condition for the constrained models

We consider the BAT obtained by partitioning the original lattice $\mathcal{L}$ into cubes of side ℓ, $\mathcal{L} = \cup_{i\in\mathcal{L}_\ell} Q_\ell(i)$. Let μ_z be the (infinite volume) Gibbs state of the *original* system at activity z. We then introduce the *constrained* system by fixing the total number of particles in each cube; it is described by the conditional (multi–canonical) measure we introduced in the previous section.

We want to show that, provided Condition MUSM($\mathcal{A}$) is satisfied, the local specification associated to the multi–canonical state $\nu^\tau_{\Lambda,\underline{n}}$ satisfies (3.5) with $\delta(\ell) = C/\ell$. We shall consider ℓ to be an integer multiple of ℓ_0. Recall that $\mathcal{B} = \rho(\mathcal{A})$, $L = d\ell$ and $\mathcal{D}^{(\ell)}_{\hat\Lambda} = (|Q_\ell|\mathcal{D})^{\hat\Lambda} \cap \Omega^{(\ell)}_{\hat\Lambda}$ (see Theorem 3.1).

Proposition 5.1. *Assume the interaction U satisfies MUSM($\mathcal{A}$). Then for each closed set $\mathcal{D} \subseteq \mathcal{B}$ there is a constant C such that for all L the following bound holds.*

$$\sup_{i\in\mathcal{L}_\ell} \sup_{k=1,\cdots,d} \sup_{\Lambda\in P_L^{(k)}(i)} \sup_{\underline{n}\in\mathcal{D}^{(\ell)}_{\hat\Lambda}} \sup_{\sigma,\zeta,\tau} \left| \frac{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\sigma^{(k,-)},\tau\right) Z_{\Lambda,\underline{n}}\left(\zeta^{(k,+)},\zeta^{(k,-)},\tau\right)}{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\zeta^{(k,-)},\tau\right) Z_{\Lambda,\underline{n}}\left(\zeta^{(k,+)},\sigma^{(k,-)},\tau\right)} - 1 \right| \le \frac{C}{\ell} \quad (5.1)$$

Lemma 5.2. *In the same setting and notation of the above theorem, there is a constant C such that for any $\Delta \subset \Lambda$ for which $d\left(\Delta, \partial^{(k,-)}\Lambda\right) \le r$, $\mathrm{diam}(\Delta) \le r$*

$$\sup_{i\in\mathcal{L}_\ell} \sup_{k=1,\cdots,d} \sup_{\Lambda\in P_L^{(k)}(i)} \sup_{\underline{n}\in\mathcal{D}^{(\ell)}_{\hat\Lambda}} \sup_{\sigma,\zeta,\tau} \mathrm{Var}\left(\nu^{\sigma^{(k,+)},\tau,\tau}_{\Lambda,\underline{n};\Delta}, \nu^{\zeta^{(k,+)},\tau,\tau}_{\Lambda,\underline{n};\Delta}\right) \le C\frac{1}{|Q_\ell|} \quad (5.2)$$

Postponing the proof of the Lemma, we show how it implies the main estimate.

Proof of Proposition 5.1. Let us first show that (5.2) implies the following condition

$$\sup_{i\in\mathcal{L}_\ell}\ \sup_{k=1,\cdots,d}\ \sup_{\Lambda\in P_L^{(k)}(i)}\ \sup_{x\in\partial^{(k,-)}\Lambda}\ \sup_{\underline{n}\in\mathcal{D}_{\hat\Lambda}^{(\ell)}}\ \sup_{\sigma,\zeta,\tau}\left|\frac{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\tau^x,\tau\right)Z_{\Lambda,\underline{n}}\left(\zeta^{(k,+)},\tau,\tau\right)}{Z_{\Lambda,\underline{n}}\left(\zeta^{(k,+)},\tau^x,\tau\right)Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\tau,\tau\right)}-1\right|\le C\frac{1}{|Q_\ell|} \tag{5.3}$$

We have in fact

$$\begin{aligned}
&\frac{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\tau^x,\tau\right)Z_{\Lambda,\underline{n}}\left(\zeta^{(k,+)},\tau,\tau\right)}{Z_{\Lambda,\underline{n}}\left(\zeta^{(k,+)},\tau^x,\tau\right)Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\tau,\tau\right)}-1\\
&\quad=\frac{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\tau^x,\tau\right)}{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\tau,\tau\right)}\left[\frac{Z_{\Lambda,\underline{n}}\left(\zeta^{(k,+)},\tau,\tau\right)}{Z_{\Lambda,\underline{n}}\left(\zeta^{(k,+)},\tau^x,\tau\right)}-\frac{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\tau,\tau\right)}{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\tau^x,\tau\right)}\right]\\
&\quad=\frac{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\tau^x,\tau\right)}{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\tau,\tau\right)}\left[\nu_{\Lambda,\underline{n}}^{\zeta^{(k,+)},\tau^x,\tau}\left(h_x^\tau\right)-\nu_{\Lambda,\underline{n}}^{\sigma^{(k,+)},\tau^x,\tau}\left(h_x^\tau\right)\right]
\end{aligned} \tag{5.4}$$

where

$$h_x^\tau(\eta):=e^{-[H_\Lambda(\eta\circ_\Lambda\tau^x)-H_\Lambda(\eta\circ_\Lambda\tau)]}$$

is a local function with support contained in an r neighborhood of x. Since the first factor on the r.h.s. of (5.4) is bounded uniformly and the same holds for $\|h_x^\tau\|$, (5.3) follows from (5.2).

An easy telescopic argument shows (5.3) implies (5.1). Indeed, for any two configurations $\zeta^{(k,-)}$, $\sigma^{(k,-)}$, differing only on $\partial^{(k,-)}\Lambda$, we can find a path $\{\eta_l\}_{l=0,\cdots,M}$ of length $M\le r\cdot(3L)^{d-1}$ such that $\eta_0=\sigma^{(k,-)}$, $\eta_M=\zeta^{(k,-)}$ and η_l differs from η_{l-1} at most in one single site $x\in\partial^{(k,-)}\Lambda$. We then write

$$\begin{aligned}
&\frac{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\sigma^{(k,-)},\tau\right)Z_{\Lambda,\underline{n}}\left(\zeta^{(k,+)},\zeta^{(k,-)},\tau\right)}{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\zeta^{(k,-)},\tau\right)Z_{\Lambda,\underline{n}}\left(\zeta^{(k,+)},\sigma^{(k,-)},\tau\right)}\\
&\quad=\prod_{l=1}^{M}\frac{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\eta_{l-1},\tau\right)Z_{\Lambda,\underline{n}}\left(\zeta^{(k,+)},\eta_l,\tau\right)}{Z_{\Lambda,\underline{n}}\left(\sigma^{(k,+)},\eta_l,\tau\right)Z_{\Lambda,\underline{n}}\left(\zeta^{(k,+)},\eta_{l-1},\tau\right)}
\end{aligned}$$

and use (5.3) to get (5.1). □

Proof of Lemma 5.2. Let us recall that $\mathrm{Var}\,(\mu,\nu)=\sup_{\|f\|=1}|\mu f-\nu f|$. Let f be a local function with support contained in Δ. By Theorem 4.4 we have

$$\left|\nu_{\Lambda,\underline{n}}^{\zeta_1}f-\nu_{\Lambda,\underline{n}}^{\zeta_2}f\right|\le C\frac{1}{|Q_\ell|}+\left|\mu_{\Lambda,\underline{z}^1}^{\zeta_1}f-\mu_{\Lambda,\underline{z}^2}^{\zeta_2}f\right|$$

where $\underline{z}^\alpha=\underline{z}^\alpha(\Lambda,\underline{n},\zeta_\alpha)$, $\alpha=1,2$ is chosen so that $\mu_{\Lambda,\underline{z}^\alpha}^{\zeta_\alpha}(\underline{N})=\underline{n}$. Since ζ_1 differs from ζ_2 only on $\partial^{(k,+)}\Lambda$, by Condition MUSM($\mathcal{A}$) we now have

$$\left|\mu_{\Lambda,\underline{z}^1}^{\zeta_1}f-\mu_{\Lambda,\underline{z}^1}^{\zeta_2}f\right|\le Ce^{-d\left(\Delta,\partial^{(k,+)}\Lambda\right)/C}$$

On the other hand, by Lagrange theorem, for a suitable $\underline{\bar{z}}$,

$$\left|\mu^{\zeta_2}_{\Lambda,\underline{z}^1} f - \mu^{\zeta_2}_{\Lambda,\underline{z}^2} f\right| \le \sum_{i\in\hat{\Lambda}} \frac{1}{\bar{z}_i} \left|\mu^{\zeta_2}_{\Lambda,\underline{\bar{z}}}(f;N_i)\right| \cdot \left|z_i^2 - z_i^1\right|$$

By the exponential decay of correlations we have

$$\left|\mu^{\zeta_2}_{\Lambda,\underline{\bar{z}}}(f;N_i)\right| \le C\bar{z}_i e^{-d(\Delta,Q_\ell(i))/C}$$

the bound (5.2) is thus obtained by applying Proposition 4.3 to estimate $\left|z_i^2 - z_i^1\right|$. Note in fact that $d\left(\Delta, \partial^{(k,+)}\Lambda\right) \ge d\ell - r$. □

5.2. Short range renormalized potential.

In this section we consider the limit $\ell \to \infty$ of the short range part of the renormalized potential. By applying Theorem 4.5, we prove the necessary estimates. This would also allow us to conclude the proof of Theorem 2.3.

Proposition 5.3. *Recall that the short range part of the renormalized potential $\Phi^{(\ell),sr}_X$ have been defined in (3.69). We introduce*

$$\Psi^{(\ell),sr}_X(m_X) := S(X)\frac{1}{2}\sum_{i\in X} m_i^2 + \Phi^{(\ell),sr}_X(m_X) \tag{5.5}$$

where

$$S(X) := \begin{cases} +1 \text{ if } X = \widetilde{A}_{k_1}, D_{k_4} \\ -1 \text{ if } X = \widetilde{C}_{k_3}, F_{k_2} \\ \ \ 0 \text{ otherwise} \end{cases} \tag{5.6}$$

Then the renormalized Hamiltonian can be written as

$$H^{(\ell,\tau)}_{I_p}(\underline{n}) = -\frac{1}{2}\sum_{i\in I_p} m_i^2 + \sum_{X\subset I_p} \Psi^{(\ell),sr}_X(m_X) + \sum_{X\subset I_p} \Phi^{(\ell,\tau),lr}_X(m_X) \tag{5.7}$$

Moreover there is a constant $a > 0$ such that

$$\lim_{\ell\to\infty} \sup_{\substack{m_X\in\bar{\Omega}^{(\ell)}_X \\ |m_X|\le\ell^a}} \left|\Psi^{(\ell),sr}_X(m_X)\right| = 0 \qquad \text{for any } \ X \subset\subset \mathcal{L}_\ell,\ |X| \ge 2 \tag{5.8}$$

Note that Theorem 2.3 follows directly from Theorem 3.1 and Propositions 5.1 and 5.3.

Proof of Proposition 5.3. By using (4.27), for each $V = \cup_{i\in X} Q_\ell(i)$, recalling that $m_i = (n_i - \rho|Q_\ell|)/\sqrt{\chi|Q_\ell|}$, we have

$$\begin{aligned} \log \mu^\tau_{V,z}\left(M_i = m_i,\ i \in X\right) =& \text{const } - \frac{1}{2}\sum_{i\in X} m_i^2 \\ & - \left\{\frac{1}{2}\left\langle \left(\underline{n} - \underline{\rho}^{(\ell)}|Q_\ell|\right), \left(v^{(\ell)}\right)^{-1} \left(\underline{n} - \underline{\rho}^{(\ell)}|Q_\ell|\right)\right\rangle - \frac{1}{2}\sum_{i\in X} m_i^2\right\} \\ & + \log\left[1 + R^\tau_V(m_X)\right] \end{aligned} \tag{5.9}$$

Therefore, by (3.69) (where the boundary condition is $\tau = 0$ and $d_\ell(X, I_p^c) > d$), we have

$$\Psi_X^{(\ell),sr}(m_X) = -S(X)\left\{\frac{1}{2}\left\langle\left(\underline{n} - \underline{\rho}^{(\ell)}|Q_\ell|\right), \left(v^{(\ell)}\right)^{-1}\left(\underline{n} - \underline{\rho}^{(\ell)}|Q_\ell|\right)\right\rangle - \frac{1}{2}\sum_{i\in X} m_i^2\right\} + \log\left[1 + R_V^0(m_X)\right] \tag{5.10}$$

Indeed, to get (5.7), it is sufficient to observe that

i) given a block $Q_\ell(i)$ contained in $\mathcal{A}_p$, the corresponding one–body renormalized interaction $-\frac{1}{2}m_i^2$ appears only in one term $\Phi_X^{(\ell),sr}(m_X)$ with $X = \widetilde{A}_{k_1}$ for one and only one $A_{k_1} \in \mathcal{A}_p$ with $S(X) = +1$

ii) given a block $Q_\ell(i)$ contained in $\mathcal{B}_p$, the corresponding one–body renormalized interaction $-\frac{1}{2}m_i^2$ appears in two terms $\Phi_X^{(\ell),sr}(m_X)$ with $X = \widetilde{A}_{k_1}$ with $S(X) = +1$ and in one term with $X = F_{k_2}$ with $B_{k_2} \in \mathcal{B}_p$ and $S(X) = -1$

iii) given a block $Q_\ell(i)$ contained in $\mathcal{C}_p$, the corresponding one–body renormalized interaction $-\frac{1}{2}m_i^2$ appears in four terms $\Phi_X^{(\ell),sr}(m_X)$ with $X = \widetilde{A}_{k_1}$ with $S(X) = +1$, in two terms with $X = F_{k_2}$ with $B_{k_2} \in \mathcal{B}_p$ and $S(X) = -1$ and in one term $X = \widetilde{C}_{k_3}$ with $C_{k_3} \in \mathcal{C}_p$ and $S(X) = -1$

iv) given a block $Q_\ell(i)$ contained in $\mathcal{D}_p$, the corresponding one–body renormalized interaction $-\frac{1}{2}m_i^2$ appears in two terms $\Phi_X^{(\ell),sr}(m_X)$ with $X = \widetilde{A}_{k_1}$ with $S(X) = +1$, in two terms with $X = \widetilde{C}_{k_3}$ with $C_{k_3} \in \mathcal{C}_p$ and $S(X) = -1$ and in one term $X = D_{k_4}$ with $D_{k_4} \in \mathcal{D}_p$ and $S(X) = +1$

Performing the different cancellations in the four sub–lattices $\mathcal{A}_p, \mathcal{B}_p, \mathcal{C}_p, \mathcal{D}_p$ we easily get (5.7).

Finally, to prove (5.8), we note that by Proposition 4.3 we have

$$\left| |Q_\ell| \left(v^{(\ell)}\right)^{-1}_{i,j} - \delta_{i,j} \right| \leq \frac{C}{\ell}$$

and, by strong mixing,

$$\left|\rho_i^{(\ell)}(\underline{z}) - \rho(z_i)\right| \leq C\ell^{-1}$$

Hence the bound (5.8) follows from (5.10) and Theorem 4.5. □

5.3. Gibbsianess of renormalized potential

We show here that, provided Condition GMUSM holds and ℓ is large enough, the renormalized measure $\mu^{(\ell)}$ is Gibbsian w.r.t. the potential $\Phi^{(\ell)}$ which has been constructed in Section 3. We have in fact the following result.

Proposition 5.4. *Assume Condition GMUSM holds and define the renormalized potential $\Phi^{(\ell)}$ as in Section 3. Then the renormalized measure $\mu_z^{(\ell)}$ is Gibbsian w.r.t. $\Phi^{(\ell)}$, i.e.*

$$\mu_z^{(\ell)}\left(m_I | m_{I^c}\right) = \frac{\exp\left\{\sum_{X\cap I\neq\emptyset} \Phi_X^{(\ell)}\left(m_I \circ m_{I^c}\right)\right\}}{\sum_{m_I \in \bar{\Omega}_I^{(\ell)}} \exp\left\{\sum_{X\cap I\neq\emptyset} \Phi_X^{(\ell)}\left(m_I \circ m_{I^c}\right)\right\}}, \quad \mu_z^{(\ell)} \text{ a.s.} \tag{5.11}$$

Note that Theorem 2.2 follows directly from Theorem 3.1 and Propositions 5.1, 5.3 and 5.4. Indeed GMUSM implies $\mathcal{B} = \rho([0,\infty)) = [0,1]$

Proof of Proposition 5.4. We recall the random variables $M_i = M_i(\eta)$ have been defined in (1.3). We introduce the two families of σ–algebras: $\mathcal{F}_\Lambda := \sigma\{\eta_x,\, x \in \Lambda\}$, $\Lambda \subset \mathcal{L}$, and $\mathcal{F}_I^{(\ell)} := \sigma\{M_i,\, i \in I\}$, $I \subset \mathcal{L}_\ell$. For $I \subset\subset \mathcal{L}_\ell$ and $F : \bar\Omega_I^{(\ell)} \mapsto \mathbb{R}$ let us first prove that

$$\mu_z^{(\ell)} \left(F(m_I) \big| \mathcal{F}_{I^c}^{(\ell)} \right) = \mu_z \left(F(M_I) \big| \mathcal{F}_{I^c}^{(\ell)} \right), \qquad \mu_z^{(\ell)} \text{ a.s.} \tag{5.12}$$

let G be a local function measurable w.r.t. $\mathcal{F}_{I^c}^{(\ell)}$; by definition of the measure $\mu_z^{(\ell)}$ we have

$$\mu_z \left(F(M_I)G(M_{I^c})\right) = \mu_z^{(\ell)} \left(F(m_I)G(m_{I^c})\right) = \int d\mu_z^{(\ell)}(\underline{m})\; G(m_{I^c})\mu_z^{(\ell)} \left(F(m_I) \big| \mathcal{F}_{I^c}^{(\ell)} \right)$$

on the other hand,

$$\begin{aligned} \mu_z \left(F(M_I)G(M_{I^c})\right) &= \int d\mu_z(\eta)\; G\left(M_{I^c}(\eta)\right) \mu_z \left(F\left(M_I(\eta)\right) \big| \mathcal{F}_{I^c}^{(\ell)} \right) \\ &= \int d\mu_z^{(\ell)}(\underline{m})\; G\left(m_{I^c}\right) \mu_z \left(F\left(M_I(\eta)\right) \big| \mathcal{F}_{I^c}^{(\ell)} \right) \end{aligned}$$

which proves (5.12).

Let $V = \cup_{i \in \hat V} Q_\ell(i) \subset\subset \mathcal{L}$; we note that for $I \subset \hat V$ we have

$$\begin{aligned} \mu_z \left(M_I = m_I \big| \mathcal{F}_{I^c}^{(\ell)} \right) &= \mu_z \left(\mu_z \left(M_I = m_I \big| \mathcal{F}_{V^c} \vee \mathcal{F}_{I^c}^{(\ell)} \right) \Big| \mathcal{F}_{I^c}^{(\ell)} \right) \\ &= \mu_z \left(\mu_z \left(M_I = m_I \big| \mathcal{F}_{\hat V \setminus I}^{(\ell)} \vee \mathcal{F}_{V^c} \right) \Big| \mathcal{F}_{I^c}^{(\ell)} \right) \end{aligned} \tag{5.13}$$

on the other hand, by definition of the renormalized Hamiltonian and the corresponding potential, see Section 3

$$\mu_{V,z}^\tau \left(M_I = m_I \big| M_{\hat V \setminus I} = m_{\hat V \setminus I} \right) = \frac{\exp\left\{ \sum_{\substack{X \subset \hat V \\ X \cap I \neq \emptyset}} \Phi_X^{(\ell,\tau)} \left(m_I \circ m_{\hat V \setminus I} \right) \right\}}{\sum_{m_I \in \bar\Omega_I^{(\ell)}} \exp\left\{ \sum_{\substack{X \subset \hat V \\ X \cap I \neq \emptyset}} \Phi_X^{(\ell,\tau)} \left(m_I \circ m_{\hat V \setminus I} \right) \right\}} \tag{5.14}$$

Since Condition GMUSM holds, by Proposition 5.1, (3.5) is satisfied with $\mathcal{D} = [0,1]$ and therefore, by Theorem 3.1, the r.h.s. of (5.14) converges, as $V \uparrow \mathcal{L}$, to the r.h.s. of (5.11) uniformly in τ and $\underline{m}$. By using also (5.13) and (5.12) we thus conclude the proof. □

A.1 Proof of USM($\mathcal{A}$) $\Longrightarrow$ MUSM($\mathcal{A}$) in dimension 2.

Let $R_{L,3L}(i)$ be the rectangle with vertical and horizontal sides $L, 3L$, respectively, and which is centered at $Q_{(L)}(i)$.

The fact that we only consider this rectangle with longer horizontal side does not represent, of course, a loss of generality and is made only to fix notation.

For M an even integer, $M/2$ and L_0 odd integers, we write:

$$\bar{L} = ML_0; \qquad R_{\bar{L},3\bar{L}} = R_{\bar{L},3\bar{L}}((\frac{L_0-1}{2}, \frac{L_0-1}{2}));$$

again the choice of the center is made to fix notation and does not constitute a loss of generality. Recall that since M is even and L_0 is odd the center of $Q_{\bar{L}}((\frac{L_0-1}{2}, \frac{L_0-1}{2}))$ is in $(\frac{L_0}{2}, \frac{L_0}{2})$.

We set $R_{\bar{L},3\bar{L}} = Q^l_{\bar{L}} \cup Q^c_{\bar{L}} \cup Q^r_{\bar{L}}$ where by $Q^l_{\bar{L}}$, $Q^c_{\bar{L}}$, $Q^r_{\bar{L}}$ we denote the left, central and right $\bar{L} \times \bar{L}$ squares, respectively, contained in $R_{\bar{L},3\bar{L}}$.

Consider a 2D lattice gas with an interaction satisfying USM $(\mathcal{A})$ for some $\mathcal{A} \subseteq [0,\infty)$. We start noticing that from the validity of USM$(\mathcal{A})$ it is immediate to deduce that for each $z \in \mathcal{A}$ there exists an integer L_0 such that the following condition

$$\sup_{\sigma,\tau\in\Omega} \sup_{i\in\{1,2\}} \left| \frac{Z_V\left(\sigma^{(i,+)}, \sigma^{(i,-)}, \tau\right) Z_V\left(\tau^{(i,+)}, \tau^{(i,-)}, \tau\right)}{Z_V\left(\sigma^{(i,+)}, \tau^{(i,-)}, \tau\right) Z_V\left(\tau^{(i,+)}, \sigma^{(i,-)}, \tau\right)} - 1 \right| < \varepsilon(2) \qquad (A1.1)$$

is verified for $V = Q_{L_0}(i)$, $R_{L_0,3L_0}(i)$ in the homogeneous activity case. This, together with the results of [O], [OP] establishes the equivalence of USM and C1 in the homogeneous activity case; this result is valid in any dimension.

Now, given a closed set $\mathcal{C} \subseteq \mathcal{A}$ suppose that we are able to prove the existence of $\bar{L}$ such that: for all $z, z' \in \mathcal{C}$, if we consider our lattice gas enclosed in $V = R_{\bar{L},3\bar{L}}$ with activity z' in $Q^l_{\bar{L}}$ and z in $Q^c_{\bar{L}} \cup Q^r_{\bar{L}}$ (i.e. we take the same activity both in $Q^c_{\bar{L}}$ and $Q^r_{\bar{L}}$), then, calling $Z_{V,z,z'}(\tau)$ the corresponding partition function with τ boundary condition, we have:

$$\sup_{\sigma,\tau} \sup_{i\in\{1,2\}} \sup_{\substack{y\in\partial^{(i,+)}V \\ y'\in\partial^{(i,-)}V}} \left| \frac{Z_{V,z,z'}(\sigma_y, \sigma_{y'}, \tau) Z_{V,z,z'}(\tau_y, \tau_{y'}, \tau)}{Z_{V,z,z'}(\sigma_y, \tau_{y'}, \tau) Z_{V,z,z'}(\tau_y, \sigma_{y'}, \tau)} - 1 \right| < \frac{\varepsilon(\bar{L})}{\bar{L}^{2(d-1)}} \qquad (A1.2)$$

with $\varepsilon(\bar{L})$ going to 0 as $\bar{L}$ goes to infinity; then, using methods and results of [O], [OP] it is easy to get MUSM$(\mathcal{A})$. Indeed in the two–dimensional, multi–grancanonical case, to get strong mixing condition using effectiveness of some finite–size conditions for volumes of the form (2.10) with ℓ sufficiently large, it is sufficient to verify:

(i) (A1.2) for $V = Q_{\bar{L}}(i)$ and $V = R_{\bar{L},3\bar{L}}(i)$ with *uniform* activity in V arbitrarily chosen in $\mathcal{C}$, and

(ii) (A1.2) for $V = R_{\bar{L},3\bar{L}}(i)$ and activity z' in $Q^l_{\bar{L}}$ and z in $Q^c_{\bar{L}} \cup Q^r_{\bar{L}}$ uniformly for z, z' in $\mathcal{C}$.

In the homogeneous case *(i)*, as we noticed before, if, given $\mathcal{C}$, L_0 is the size for which SM(L_0) holds uniformly in $\mathcal{C}$, as prescribed by USM$(\mathcal{A})$, then, for $\bar{L}$ sufficiently large (A1.2) holds for $V = Q_{\bar{L}}(i)$ and $V = R_{\bar{L},3\bar{L}}(i)$ for each (constant in V) activity $z \in \mathcal{C}$. Then 2.1 will follow from next Proposition A1.1

Proposition A1.1. *Suppose that Condition* C1$^{(2)}(V)$ *holds for any* $V = Q_{L_0}(i), R_{L_0,3L_0}(i)$ *contained in one of the three squares* $Q^l_{\bar{L}}$, $Q^c_{\bar{L}}$ *or* $Q^r_{\bar{L}}$*; then, for* $M \equiv \frac{\bar{L}}{L_0}$ *sufficiently large, (A1.2) holds for* $V = R_{\bar{L},3\bar{L}}$.

Proof.

We make a geometrical construction similar to the one introduced in [O], [OP] and used in Section 3 to compute, via cluster expansion, the renormalized potential. We recall that we denote by $\mathcal{L}$ our original lattice $\mathbb{Z}^2$ whereas we denote by $\mathcal{L}_{L_0}$ the L_0–rescaled lattice: we partition $\mathcal{L}$ into cubes of side L_0. We write:

$$\mathcal{L} = \cup_{i\in\mathcal{L}_{L_0}} \; Q_{L_0}(i)$$

From now on we will mainly consider the L_0–rescaled lattice; our unit length will be L_0. In other words we will use the distance d_{L_0}. The "bricks" of our construction will be the blocks Q_{L_0} or $R_{L_0,3L_0}$ and the original length–scale will enter only when considering some properties of the partition functions in the regions Q_{L_0} or $R_{L_0,3L_0}$ that we use as input of our perturbative theory.

Let e_1, e_2 denote, respectively, the horizontal and vertical lattice unit vectors in $\mathcal{L}_{L_0}$: $e_1 = (1,0), e_2 = (0,1)$. Following definitions and notation of Section 3 we further partition $\mathcal{L}_{L_0}$ into four sub–lattices:

$$\mathcal{L}_{L_0} = \mathcal{L}^A_{2L_0} \cup \mathcal{L}^B_{2L_0} \cup \mathcal{L}^C_{2L_0} \cup \mathcal{L}^D_{2L_0}$$

where:

$$\begin{aligned}
&\mathcal{L}^A_{2L_0} := \{i=(i_1,i_2) \in \mathcal{L}_{L_0} : i_1 = 2j_1, x_2 = 2j_2, \text{for some integers } \; y_1,y_2\}\\
&\mathcal{L}^B_{2L_0} := \mathcal{L}^A_{2L_0} + e_2\\
&\mathcal{L}^C_{2L_0} := \mathcal{L}^A_{2L_0} + e_1 + e_2 = \mathcal{L}^B_{2L_0} + e_2\\
&\mathcal{L}^D_{2L_0} := \mathcal{L}^A_{2L_0} + e_1 = \mathcal{L}^C_{2L_0} + e_2 = \mathcal{L}^B_{2L_0} + e_1 + e_2
\end{aligned} \tag{A1.3}$$

We also set, for $i \in \mathcal{L}_{L_0}$:

$$A_i := Q_{L_0}(2i), \quad B_i := Q_{L_0}(2i+e_2) \quad C_i := Q_{L_0}(2i+e_1+e_2) \quad D_i := Q_{L_0}(2i+e_1). \tag{A1.4}$$

Then we can partition $V \equiv R_{\bar{L},3\bar{L}}$ into the union of the L_0–blocks of the four types: A,B,C,D:

$$V = \mathcal{A}_V \cup \mathcal{B}_V \cup \mathcal{C}_V \cup \mathcal{D}_V$$

where

$$\mathcal{A}_V := \{A_i : i=(i_1,i_2) \in \mathcal{L}_{L_0} \; : \; |i_2| \le (M/2-1)/2\,, \; |i_1| \le (3M/2-1)/2\}$$

and similarly for $\mathcal{B}_V, \mathcal{C}_V, \mathcal{D}_V$.

We have that the left block on the bottom is an A–block whereas the right one on the top is a C–block.

We denote by α_i a generic spin configuration in A_i: $\alpha_i \in \{-1,+1\}^{L_0^2}$. Similarly for $\beta_i, \gamma_i, \delta_i$. We simply denote by $\alpha,\beta,\gamma,\delta$ the configurations in $\mathcal{A}_V, \mathcal{B}_V, \mathcal{C}_V, \mathcal{D}_V$, respectively.

Notice that we have used the same notation (with a very similar meaning) as the one we used in Section 3 to describe "multi–canonical" block variables.

Consider the "column" V_l namely the rectangle with basis L_0 and height $\bar{L}$ placed at the left–hand of $Q^c_{\bar{L}}$, adjacent, from the exterior, to $Q^l_{\bar{L}}$:

$$V_l = \{(x_1,x_2) \in \mathcal{L} : -\frac{\bar{L}}{2} + \frac{L_0+1}{2} \le x_1 \le -\frac{\bar{L}}{2} + \frac{L_0+1}{2} + L_0, \;\; -\frac{\bar{L}}{2} + \frac{L_0+1}{2} \le x_2 \le +\frac{\bar{L}}{2} + \frac{L_0-1}{2}\}$$

we decompose V_l as disjoint union of A and B blocks:

$$V_l = \mathcal{A}_l \cup \mathcal{B}_l$$

where

$$\mathcal{A}_l := \mathcal{A}_V \cap V_l, \quad \mathcal{B}_l := \mathcal{B}_V \cap V_l$$

We have:

$$\mathcal{A}_l = \cup_{i \in I_A^l} A_i$$

where

$$I_A^l = \{(i_1, i_2) \ : \ i_1 = -(M/2 - 1)/2, \ |i_2| \leq (\frac{M}{2} - 1)/2\}$$

similarly for $\mathcal{B}_l$

We write $\mathcal{A}_V = \hat{\mathcal{A}} \cup \mathcal{A}_l$, $\mathcal{B}_V = \hat{\mathcal{B}} \cup \mathcal{B}_l$; in other words $\hat{\mathcal{A}}, \hat{\mathcal{B}}$ denote the union of A and B blocks, respectively, which belong to $V = R_{\bar{L},3\bar{L}}$ but not to V_l.

We will repeat almost the same computation that we made , in the multi–canonical framework, to compute the renormalized potential. Namely we adopt the same strategy based on a block decimation procedure over the sequence of sub–lattices D, C, B, A.

The main difference here is that we will treat in a different manner the region in $V \equiv R_{\bar{L},3\bar{L}}$ adjacent to the boundary between $Q_{\bar{L}}^c$ and $Q_{\bar{L}}^l$. Here we will exploit the fact that this boundary is one–dimensional.

Indeed we will see that the system of the surviving α–variables in $\mathcal{A}_l$, after decimation on δ, γ, β and α in $\hat{\mathcal{A}}$, gets an effective interaction which is exponentially decaying with the distance and uniformly bounded in norm. The resulting one–dimensional system, regarded on a sufficiently large scale, is in the weak coupling region and from this it easily follows a weak coupling between opposite horizontal sides of $V \equiv R_{\bar{L},3\bar{L}}$ so that condition C3 with an infinitesimal ε_3 is satisfied for V.

We want to perturbatively treat, similarly to what we did in Section 3, the partition function:

$$Z_V^\tau := \sum_{\eta \in \Omega_V} \exp\left(H_V^\tau(\eta)\right)$$

where

$$H_V^\tau(\eta) := \sum_{\Delta : \Delta \cap V \neq \emptyset} \Phi_\Delta(\eta \circ_V \tau)$$

and we recall that we are using the notation:

$$V := R_{\bar{L},3\bar{L}}, \quad \tau \ \in \Omega_{V^c} \quad \equiv \quad \text{boundary condition outside } V$$

Given $\tau, \tau' \in \Omega_{V^c}$ and x, y belonging to the set of conditioning sites above the upper side and below the lower side of V, respectively, we want to consider the ratio

$$\frac{Z_{V,z,z'}(\tau'_x, \tau'_y, \tau) Z_{V,z,z'}(\tau)}{Z_{V,z,z'}(\tau'_x, \tau) Z_{V,z,z'}(\tau'_y, \tau)} \tag{A1.5}$$

where (τ,τ'_x,τ'_y), (τ,τ'_x),(τ,τ'_y) are the configurations obtained from τ by substituting τ with τ' in $\{x,y\}$, $\{x\}$ and $\{y\}$, respectively.

The perturbative expression that we will obtain for Z_V^τ will show an almost factorized dependence on boundary conditions in opposite horizontal faces so that we will be able to show that the quantity

$$\left|\frac{Z_{V,z,z'}(\tau'_x,\tau'_y,\tau)Z_{V,z,z'}(\tau)}{Z_{V,z,z'}(\tau'_x,\tau)Z_{V,z,z'}(\tau'_y,\tau)}-1\right| \tag{A1.6}$$

can be made arbitrarily small for $\bar{L}$ sufficiently large so that condition C2 is satisfied.

It is easily seen, using DLR structure of the multi–grancanonical Gibbs field, that the case when x,y are close from the exterior to the two opposite vertical faces (at distance $3\bar{L}$) can be treated exactly like in the homogeneous (constant activity) case; thus we will only consider the above mentioned case of x,y belonging to upper and lower sets of conditioning spins.

Sometimes, just for the sake of simplicity of notation, we will actually drop the explicit dependence on the boundary condition τ (even though this dependence is crucial). We express H_V^τ exactly as we did in (3.13):

$$\begin{aligned}H_V(\eta)=&\sum_{k_1:A_{k_1}\in\mathcal{A}_V}H_{A_{k_1}}(\alpha_{k_1})+\sum_{k_2:B_{k_2}\in\mathcal{B}_V}H_{B_{k_2}}(\beta_{k_2})+W_{B_{k_2},V\setminus B_{k_2}}(\beta_{k_2}|\alpha)\\&+\sum_{k_3:C_{k_3}\in\mathcal{C}_V}H_{C_{k_3}}(\gamma_{k_3})+W_{C_{k_3},V\setminus C_{k_3}}(\gamma_{k_3}|\beta,\alpha)\\&+\sum_{k_4:D_{k_4}\in\mathcal{D}_V}H_{D_{k_4}}(\delta_{k_4})+W_{D_{k_4},V\setminus D_{k_4}}(\delta_{k_4}|\gamma,\beta,\alpha)\end{aligned} \tag{A1.7}$$

where, as in (3.12),

$$W_{\Lambda_1,\Lambda_2}(\eta_{\Lambda_1}|\eta_{\Lambda_2}):=W(\eta_{\Lambda_1}|\eta_{\Lambda_2})=H_{\Lambda_1\cup\Lambda_2}(\eta_{\Lambda_1},\eta_{\Lambda_2})\;-\;H_{\Lambda_1}(\eta_{\Lambda_1})\;-\;H_{\Lambda_2}(\eta_{\Lambda_2}) \tag{A1.8}$$

We now proceed to the summation over the δ,γ,β variables; we repeat exactly the same operations of splitting and gluing that we performed in section 3. We get:

$$\begin{aligned}Z_V^\tau=&\sum_\alpha\prod_{k_1:A_{k_1}\in\mathcal{A}_V}\exp\{H(\alpha_{k_1})\}\left[Z_{D_{k_1}}((0),(\alpha_{k_1}),(0))\,Z_{D_{k_1-e_1}}((0),(0),(\alpha_{k_1}))\right]^{-1}\\&\times\prod_{k_2:B_{k_2}\in\mathcal{B}_V}Z_{\widetilde{B}_{k_2}}\left((0),(\alpha)^u,(\alpha)^d\right)\sum_\beta\mu_2^\alpha(\beta)\\&\times\prod_{k_3:C_{k_3}\in\mathcal{C}_V}\left[Z_{\widetilde{C}_{k_3}}((0))\right]^{-1}\prod_{k_3:C_{k_3}\in\mathcal{C}_V}\left(1+\Phi^{(3)}_{C_{k_3}}(\alpha,\beta)\right)\sum_\gamma\mu_3^{\alpha,\beta}(\gamma)\\&\times\prod_{k_4:D_{k_4}\in\mathcal{D}_V}\left(1+\Phi^{(4)}_{D_{k_4}}(\alpha,\beta,\gamma)\right)\prod_{k_4:D_{k_4}\in\mathcal{D}_V}\left(1+\Psi^{(4)}_{D_{k_4}}(\alpha)\right)\prod_{k_4:D_{k_4}\in\mathcal{D}_V}\left[Z_{D_{k_4}}((0))\right]^{-1}\end{aligned} \tag{A1.9}$$

where the terms $[Z_{D_{k_1-e_1}}((0),(\alpha_{k_1}),(0))]^{-1}$, $[Z_{D_{k_1}}((0),(0),(\alpha_{k_1}))]^{-1}$ (defined in (3.29)) come from the splitting described in (3.30): in (A1.9), by an abuse of notation, we still denote by $\widetilde{C}$, $\widetilde{B}$ and $\widetilde{A}$

their truncation in $R_{\bar L,3\bar L}$. Indeed, since we have generic and not periodic b.c., we have to introduce the modifications described in Section 3 (below Proposition 3.2) in $\mu^{\alpha\beta}_{C_{k_3}}$, $\mu^{\alpha}_{B_{k_2}}$ as well as in the Φ and Ψ error terms. Moreover notice that in the expression in (A1.7) above, we continue to denote by $\alpha,\beta,\gamma,\delta$ also the configurations on the A,B,C,D blocks outside V; in other words we continue to denote by $\alpha,\beta,\gamma,\delta$ also the part of the τ (exterior) configuration in A,B,C,D sub–lattices. We did not have them in (3.34) since, there, we were using periodic boundary conditions. Now we continue with the same operations of splitting as in (3.38) (and gluing as in (3.50)) only for the B (and A) blocks in $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$ namely outside the two vertical column V_l. It is clear that we cannot perform the gluing operation described in (3.38) for the B blocks in V_l and obtain a small value for the term $\Phi^{(2)}_{B_{k_2}}(\alpha)$. Indeed to get a good upper bound for $\sup_\alpha|\Phi^{(2)}_{B_{k_2}}(\alpha)|$ we need the validity of condition C1, with a sufficiently small ε_1 for horizontal $R_{L_0,3L_0}$ rectangles and this condition is supposed to hold only for $R_{L_0,3L_0}$ rectangles *completely contained* in one of the three squares $Q^l_{\bar L}$, $Q^c_{\bar L}$ or $Q^r_{\bar L}$. For $R_{L_0,3L_0}$ rectangles centered at B block in $\mathcal{B}_l$ we cannot use condition C1 since these rectangles have simultaneously non–empty overlap with two of the big squares namely $Q^l_{\bar L}$, $Q^c_{\bar L}$; the rectangles $R_{L_0,3L_0}$ having non–empty overlap with $Q^c_{\bar L}$, $Q^r_{\bar L}$ behave exactly like in the homogeneous case since the activity in $Q^c_{\bar L}\cup Q^r_{\bar L}$ is supposed to be constant.

In this way we obtain the following expression

$$\begin{aligned} Z^\tau_V = \bar Z^\tau_V \sum_\alpha \widetilde Z_{V_l}(\alpha) \prod_{k_1:A_{k_1}\in\hat{\mathcal{A}}} \mu_{A_{k_1}}(\alpha_{k_1}) \\ \times \prod_{k_1:A_{k_1}\in\hat{\mathcal{A}}}\Big(1+\Psi^{(1)}_{A_{k_1}}(\alpha_{k_1})\Big) \prod_{k_2:B_{k_2}\in\hat{\mathcal{B}}}\Big(1+\Phi^{(2)}_{B_{k_2}}(\alpha)\Big) \prod_{k_4:D_{k_4}\in\mathcal{D}_V}\Big(1+\Psi^{(4)}_{D_{k_4}}(\alpha)\Big) \\ \times \sum_\beta \mu^\alpha_2(\beta) \prod_{k_3:C_{k_3}\in\mathcal{C}_V}\Big(1+\Phi^{(3)}_{C_{k_3}}(\alpha,\beta)\Big) \\ \times \sum_\gamma \mu^{\alpha,\beta}_3(\gamma) \prod_{k_4:D_{k_4}\in\mathcal{D}_V}\Big(1+\Phi^{(4)}_{D_{k_4}}(\alpha,\beta,\gamma)\Big) \end{aligned} \tag{A1.10}$$

where $\mu_{A_{k_1}}(\alpha_{k_1})$ is defined in (3.52), $\bar Z^\tau_V$ is given by

$$\bar Z^\tau_V = \prod_{k_1:A_{k_1}\in\hat{\mathcal{A}}} Z_{\widetilde A_{k_1}}((0)) \prod_{k_2:B_{k_2}\in\hat{\mathcal{B}}}\left[Z_{F_{k_2}}(0)\right]^{-1} \prod_{k_3:C_{k_3}\in\mathcal{C}_V}\left[Z_{\widetilde C_{k_3}}((0))\right]^{-1} \prod_{k_4:D_{k_4}\in\mathcal{D}_V} Z_{D_{k_4}}((0))$$

and

$$\begin{aligned} \widetilde Z_{V_l}(\alpha) := \prod_{\kappa_1:A_{k_1}\in V_l} \exp(H(\alpha_{k_1}))[Z_{D_{k_1}}((0),(\alpha_{k_1}),(0))]^{-1}[Z_{D_{k_1+e_1}}((0),(0),(\alpha_{k_1}))]^{-1} \\ \times \prod_{\kappa_2:B_{k_2}\in V_l} Z_{\widetilde B_{k_2}}((0),(\alpha_{k_2+e_2}),(\alpha_{k_2})) \end{aligned} \tag{A1.11}$$

Let us call α_l the complex of α variable in $\mathcal{A}_l$. If we perform, in the r.h.s. of (A1.10) the sum over the γ,β variables and over the α variables in $\hat{\mathcal{A}}_V$, we get:

$$Z^\tau_V = \bar Z^\tau_V \sum_{\alpha_l\in\Omega_{\mathcal{A}_l}} \widetilde Z_{V_l}(\alpha_l)\Xi^\tau_V(\alpha_l) \tag{A1.12}$$

where, of course,

$$\begin{aligned}\Xi_V^\tau(\alpha_l) = &\sum_{\alpha\in\Omega_{\hat{\mathcal{A}}_V}} \prod_{k_1:A_{k_1}\in\hat{\mathcal{A}}} \mu_{A_{k_1}}(\alpha_{k_1})\\ &\times \prod_{k_1:A_{k_1}\in\hat{\mathcal{A}}} \Big(1+\Psi^{(1)}_{A_{k_1}}(\alpha_{k_1})\Big) \prod_{k_2:B_{k_2}\in\hat{\mathcal{B}}} \Big(1+\Phi^{(2)}_{B_{k_2}}(\alpha)\Big) \prod_{k_4:D_{k_4}\in\mathcal{D}_V} \Big(1+\Psi^{(4)}_{D_{k_4}}(\alpha)\Big)\\ &\times \sum_\beta \mu_2^\alpha(\beta) \prod_{k_3:C_{k_3}\in\mathcal{C}_V} \Big(1+\Phi^{(3)}_{C_{k_3}}(\alpha,\beta)\Big)\\ &\times \sum_\gamma \mu_3^{\alpha,\beta}(\gamma) \prod_{k_4:D_{k_4}\in\mathcal{D}_V} \Big(1+\Phi^{(4)}_{D_{k_4}}(\alpha,\beta,\gamma)\Big)\end{aligned} \tag{A1.13}$$

Like in Section 3 we can write:

$$\Xi_V^\tau(\alpha_l) \;=\; 1+\sum_{n\geq 1} \sum_{\substack{R_1,\ldots,R_n:\widetilde{R}_i\subset V,\\ \widetilde{R}_i\cap\widetilde{R}_j=\emptyset, i<j=1,\ldots,n}} \prod_{i=1}^n \zeta^\tau_{R_i}(\alpha_l) \tag{A1.14}$$

where the polymers R_i are defined like in Section 2 with the obvious changes. In this way we are reduced to one–dimensional system on V_l, with finite norm, rapidly decaying interaction. Indeed we can write:

$$Z_V^\tau = \sum_{\alpha_l\in\Omega_{\mathcal{A}_l}} \exp\Big(\hat{H}(\alpha_l)\Big) \tag{A1.15}$$

Where

$$\begin{aligned}\hat{H}(\alpha_l) := \text{const.} &+ \sum_{k_1\in V_l} H(\alpha_{k_1}) - \log Z^\tau_{D_{k_1}}(\alpha_{k_1}) - \log Z^\tau_{D_{k_1-e_1}}(\alpha_{k_1-e_1})\\ &+ \sum_{\substack{k_2:B_{k_2}\in V_l\\ k_2\neq k_l^*}} \log Z_{\widetilde{B}_{k_2}}(\alpha_{k_2},\alpha_{k_2+e_2}) + \log Z_{\widetilde{B}_{k_l^*}}(\alpha_{k_l^*},\alpha^{(\tau)}_{k_l^*+e_2}) + \sum_{\Gamma\subset\mathcal{A}_l} \bar{\Phi}^\tau_\Gamma(\alpha_\Gamma)\end{aligned} \tag{A1.16}$$

where

1) $\bar{\Phi}^\tau_\Gamma(\alpha_\Gamma) := \sum^\Gamma_{R_1,\ldots,R_n} \varphi_T(R_1,\ldots,R_n)\prod_{i=1}^n \zeta^\tau_{R_i}(\alpha_\Gamma)$,

2) the sum $\sum^\Gamma_{R_1,\ldots,R_n}$ runs over the clusters of (incompatible) polymers "touching" the whole set Γ of A–blocks in the sense that the product of the activities of the polymers $R_1,\ldots,R_n$ explicitly depend on all the α–variables corresponding to the A–blocks in Γ and does not depend on any other α.

3) we introduced $Z_{D_k}(\alpha_k) = Z_{D_k}((0),(\alpha_k),(0))$, $Z_{\widetilde{B}_k}(\alpha_k,\alpha_{k+e_2}) = Z_{\widetilde{B}_k}((0),(\alpha_{k+e_2}),(\alpha_k))$

4) k_l^* is the index of the uppermost B–block in V_l:

$$k_l^* := -\left(M/2-1\right)/2, \left(M/2-1\right)/2$$

and$\alpha^{(\tau)}_{k_l^*+e_2}$ is the configurations in the A–blocks immediately outside (on the top) of V_l.

Notice that the dependence on the boundary condition τ external to V is really present (beyond the term $Z_{\widetilde{B}_{k_l^*}}(\alpha_{k_l^*}, \alpha^{(\tau)}_{k_l^*+e_2})$), only in Z_{D_k} with D_k adjacent to the boundary ∂V (upper and lower side).

From (A1.15),(A1.16) and the general theory of cluster expansion (see Proposition 3.2) it follows that,for M sufficiently large, there exist positive constants c_1, m_1, m_2, m_3 such that:

$$\sum_{\Gamma \ni A_0} \|\Phi^\tau_\Gamma\|_\infty e^{m_1|\Gamma|} e^{m_2 \text{diam } \Gamma} < \infty, \tag{A1.17}$$

and, for any $y \in \partial^V, \Gamma \subset \mathcal{A}_l$,:

$$\sup_{\alpha_l} \sup_{\tau,\tau' : \tau_x = \tau_{x'} \ \forall x \neq y} |\bar{\Phi}^\tau(\alpha_l) - \bar{\Phi}^{\tau'}(\alpha_l)| \leq c_1 e^{-m_3 \text{ dist}(\Gamma, y)} \tag{A1.18}$$

We are now reduced to a one–dimensional system with finite norm, rapidly decreasing potential. We can then apply the theory developed in [CO2] and especially in [CCO] (see also [CEO],[CO1]).

Let us summarize the strategy of [CO2], [CCO] to find good mixing properties of the Gibbs states for the one–dimensional systems like ours. Consider the system of $M/2$ variables α_k on V_l. Suppose that the integers p, n are such that $M/2$ is a multiple of pn. We divide the interval $[1, \dots, M/2]$ into $m = \frac{M}{2pn}$ intervals $I_1, \dots, I_m$ of length pn. We call long range the contribution to the interaction coming from the terms with range larger than p. We decompose the potential Φ as:

$$\Phi = \Phi^{sr} + \Phi^{lr} \quad \text{ with } \Phi^{sr}_\Gamma = 0 \text{ if } \text{diam}\,\Gamma > p; \qquad \Phi^{lr}_\Gamma \neq 0 \text{ only if } \text{diam}\,\Gamma > p \tag{A1.19}$$

The idea is to treat Φ^{lr} as a small perturbation. Indeed given a single block I_j, a uniform upper bound on the sum of the absolute values of the contributions of the long–range terms involving I_j is of the order of $n \exp(-cp)$ for a suitable positive constant c. On the other hand for the "reduced" system with only short range interactions we can exploit the one–dimensionality and the uniform boundedness of the interaction. Indeed the short range transfer matrix has a uniform positive gap in its spectrum. This would imply an exponential clustering of the short range Gibbs measure: the truncated correlations at the extrema of an interval I_j would decay as $\exp(-c'n)$ with c' depending only on the gap of the transfer matrix. In the perturbative expansions in [CO2],[CCO], the intervals I_j involved in at least one long range term are treated separately from the other ones and they happen to be very rare; on the other ones, where only the short range terms are present the mechanism of strictly positive gap of the transfer matrix is active, inducing exponential decay of correlations. We refer to [CO2], [CCO] for more details; in these articles, (actually in a more complicated situation), analyticity of the free energy and decay of truncated correlations are proved. In our case, as a consequence of the methods of [CO2], [CCO] we get exponential decay of truncated correlations. This, together with (A1.18) allows to conclude the proof of Proposition 3.2 □

A.2 A counterexample to USM $\Longrightarrow$ MUSM in dimension 3.

We give here an example that, in general, the implication USM($\mathcal{A}$) $\Longrightarrow$ MUSM($\mathcal{A}$) does not hold. We stress that our example is *ad hoc*, in particular the interaction is translation invariant only by even shifts. We believe however it sheds some light on the pathologies that may happen.

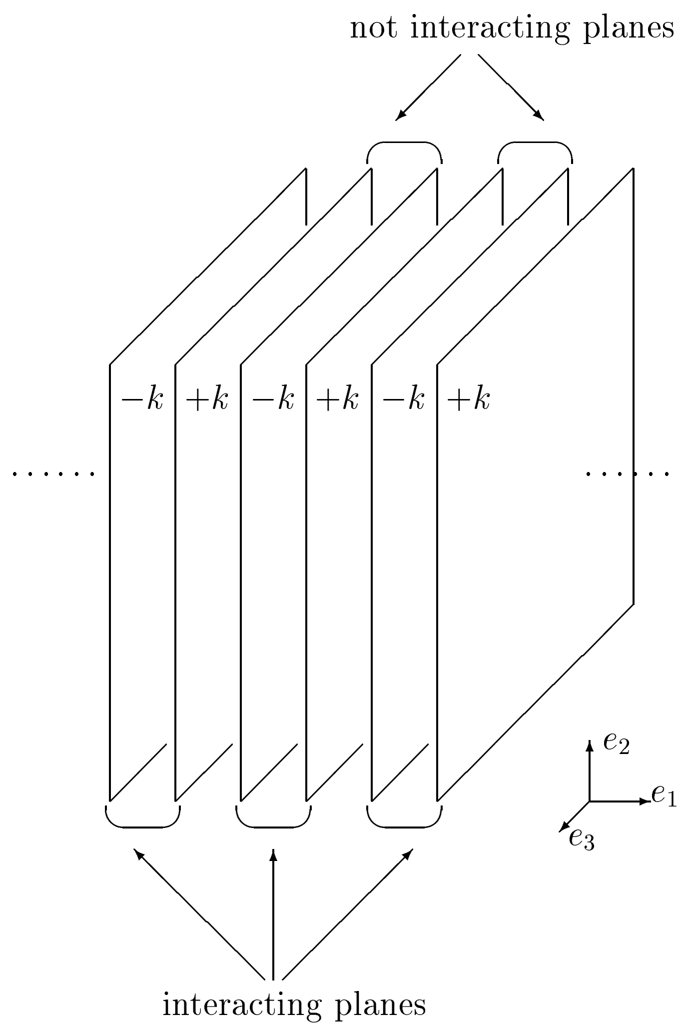

Fig. 9a

It is convenient to describe the example (see Fig. 9) by using spin variables, $\sigma \in \{-1,1\}^{\mathbb{Z}^3}$. We denote by (e_1, e_2, e_3) the canonical basis in $\mathbb{Z}^3$. The one body potential (magnetic field) is given as follows

$$\Phi_{\{x\}}(\sigma_x) = \begin{cases} -k\sigma_x & \text{if } x_1 \text{ is even} \\ k\sigma_x & \text{if } x_1 \text{ is odd} \end{cases}$$

the two body potential is instead given by

$$\Phi_{\{x,y\}}(\sigma_x, \sigma_y) = \begin{cases} -J\sigma_x\sigma_y & \text{if } x_1 \text{ is even, } y_1 \text{ is odd and } y = x + e_1 + ae_2 + be_3 \\ & \quad \text{for some } (a,b) \in \{(0,0),(1,0),(0,1),(0,-1)\} \\ 0 & \text{otherwise} \end{cases}$$

where $J > 0$. All the other potentials vanish, i.e. $\Phi_\Lambda = 0$ for $|\Lambda| > 2$.

Note that the layer $\{x : x_1 = a, a \text{ even}\}$ interacts only with the layer $\{x : x_1 = a+1\}$; in particular each double–layer is independent of everything else. Furthermore we claim that each double–layer is isomorphic to a standard two dimensional Ising model with staggered magnetic field (see Fig. 9b).

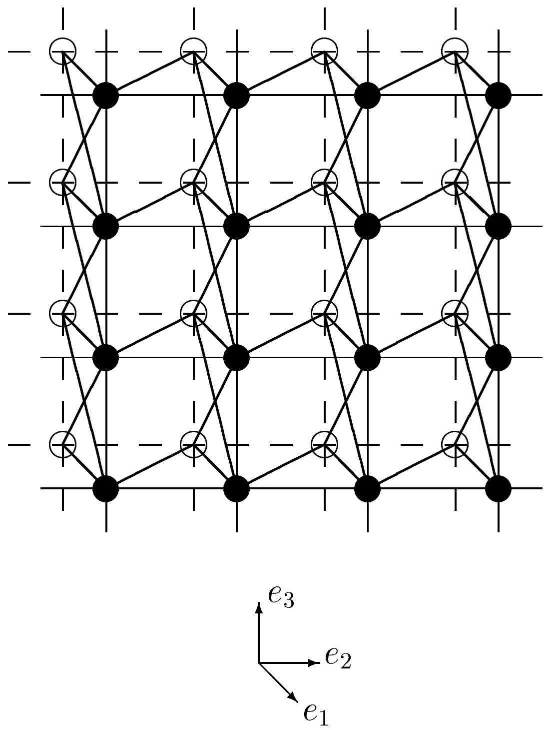

Fig. 9b

We can in fact map the layer $\{x : x_1 = a, a \text{ even}\}$ to the even sub–lattice of $\mathbb{Z}^2$, as follows

$$(a, x_2, x_3) \mapsto \begin{cases} (2x_2, x_3) & \text{if } x_3 \text{ is even} \\ (2x_2 - 1, x_3) & \text{if } x_3 \text{ is odd} \end{cases}$$

and the layer $\{x : x_1 = a+1\}$ to the odd sub–lattice of $\mathbb{Z}^2$, as follows

$$(a+1, x_2, x_3) \mapsto \begin{cases} (2x_2 - 1, x_3) & \text{if } x_3 \text{ is even} \\ (2x_2, x_3) & \text{if } x_3 \text{ is odd} \end{cases}$$

It is easy to verify that under the above mapping the double layer $\{x : x_1 = a, a \text{ even}\} \cup \{x : x_1 = a+1\}$ is mapped onto the two dimensional Ising model with the following interaction

$$\Phi_\Lambda(\sigma) = \begin{cases} -k\sigma_x & \text{if } \Lambda = \{x\} \text{ and } x_1 + x_2 \text{ is even} \\ k\sigma_x & \text{if } \Lambda = \{x\} \text{ and } x_1 + x_2 \text{ is odd} \\ -J\sigma_x\sigma_y & \text{if } \Lambda = \{x, x+e_1\},\ \Lambda = \{x, x+e_2\} \\ 0 & \text{otherwise} \end{cases} \tag{A2.1}$$

and we are left with studying the strong mixing properties of such a model.

Let us denote by μ_Λ^τ the Gibbs local specification associated to the interaction (A2.1) and by $\mu_{\Lambda,h}^\tau$ the measure obtained from μ_Λ^τ by adding a (constant) magnetic field $h \in \mathbb{R}$, i.e. $-k$ on the first line of (A2.1) becames $-k-h$ whereas k on the second line of (A2.1) becames $k-h$. We claim that, if k is chosen large enough (depending on J) such a measure does satisfy condition GUSM. Roughly speaking, we have a large magnetic field in either the odd or the even sub–lattice, therefore the phase is determined on that sub–lattice; since the other sub–lattice is conditionally independent (given the first sub–lattice) we get the strong mixing condition. Indeed one can verify that the finite size condition C1 holds on squares of side 2 with constants uniform in h.

On the other hand it is very easy to show that there is no ℓ_0 such that the 3 dimensional model we started from satisfies GMUSM. Let ℓ be an odd integer and consider $\Lambda = Q_\ell\left((-(\ell-1)/2,0,0)\right) \cup Q_\ell\left((\ell+1)/2,0,0\right)$; put a magnetic field $h_1 = -k$ (resp. $h_2 = +k$) on the first (resp. second) cube. The image, under above mapping, of the double–layer $\{x : x_1 = 0\} \cup \{x : x_1 = 1\}$ is now the standard two–dimensional Ising with zero magnetic field. If J is chosen large enough we then have a long range order, hence (2.3) fails to hold.

The pathology that has occurred is the following. Even if the local specification does satisfy the strong mixing condition separately in each one of the two cubes $Q_\ell\left((-(\ell-1)/2,0,0)\right)$, $Q_\ell\left((\ell+1)/2,0,0\right)$, when we put them together we have a long range order which propagates inside the double–layer which sits across the interface between the two cubes.

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