Path Crossing Exponents and the External Perimeter in 2D Percolation

Michael Aizenman[a], Bertrand Duplantier[b] and Amnon Aharony[c]

[a] *Departments of Physics and Mathematics, Jadwin Hall, Princeton University, Princeton, NJ 08544*

[b] *Service de Physique Théorique de Saclay, 91191 Gif-sur-Yvette Cedex, and Institut Henri Poincaré, 11 rue Pierre et Marie Curie, 75231 Paris Cedex 05, France*

[c] *School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences Tel Aviv University, Tel Aviv 69978, Israel*

(January 4, 1999)

Percolation path crossing exponents describe probabilities for ℓ non-overlapping traversing paths, each of either occupied sites or vacancies. We show, for collections including at least one of each, that in 2D the exponents are those of an $O(N=1)$ loop model. This extends the earlier identification by Saleur and Duplantier of k spanning cluster exponents, for which $\ell = 2k$. The results yield $D_{EP} = 4/3$ for the fractal dimension of the accessible external cluster perimeter, and explain the absence of narrow gate fjords, in agreement with the original findings of Grossman and Aharony.

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The fractal geometry of critical percolation clusters has been of interest both for intrinsic reasons and as a window on a range of phenomema. It is characterized by fractal dimensions of various sets [1,2], e.g., of the connected clusters, their backbones, the sets of pivotal (singly–connecting) bonds, the clusters' boundaries (hulls), and their external (accessible) perimeters. A set S is said here to be of fractal dimension D_S if the density of points in S within a box of linear size R decays as R^{-x_S}, with $x_S = d - D_S$ in d dimensions. The interest in these dimensions is enhanced by their apparent relation, in 2D, with the dimensions of polymers in different conformational states (as mentioned below).

For two–dimensional (2D) independent percolation many of the fractal dimensions have been found exactly [3–6], though most of these values have not yet been established at a rigorous level. In several cases, Saleur and Duplantier (SD) [5] identified the co–dimension x_S with the exponent $x_k^{\mathcal{C}}$ which describes the decay of the probability $P_k^{\mathcal{C}} \approx (r/R)^{x_k^{\mathcal{C}}}$ that in an annular region, $D(r,R) = \{x | r \le |x| \le R\}$, the small circle is connected to the outer one ($R >> r$) by k different clusters of occupied sites (or bonds). SD utilized the observation that the statistics of the $2k$ boundary lines of the connected clusters (i.e. lines running between the clusters and the dual clusters) are described by well recognized loop models. The loop models corresponding to the bond and site percolation models are a bit different, however the predicted exponent values coincide. For the bond percolation model the loops bounce off the Fortuin-Kasteleyn clusters of the $Q = 1$ Potts model at its critical point. For the site percolation on the triangular lattice the loops are lines on the hexagonal (dual) lattice separating the occupied from the vacant sites. The latter boundary lines obey the statistics of the $O(N=1)$ loop model of Domany *et al.* [7] – at its *low temperature phase*. Using the "Coulomb gas" representation for the corresponding ℓ-line exponents, $x_\ell^{O(N)}$, SD derived [5]:

$$x_k^{\mathcal{C}} = x_{2k}^{O(N=1)} = (4k^2 - 1)/12 \, , \tag{1}$$

where k *clusters* correspond to $2k$ *lines* in the loop model.

Among the noteworthy applications of the above formula are the "hull dimension", i.e., the dimension of the cluster's perimeter,

$$D_H = 2 - x_1^{\mathcal{C}} = 7/4, \tag{2}$$

and the dimension of the set of "red" (singly connecting) bonds, which are pinching points between two large clusters:

$$D_{SC} = 2 - x_2^{\mathcal{C}} = 3/4 = 1/\nu, \tag{3}$$

where ν is the correlation length exponent [8], in agreement with previously derived values [3]. Nevertheless, Eq. (1) does not yield all the dimensions of interest; in particular, it does not cover the dimension of the external (accessible) perimeter D_{EP} [9].

The present Letter presents several extensions of the above results for the independent site percolation model on the triangular lattice:

i. Basing the relation of percolation exponents with the $O(N=1)$ exponents on a somewhat different footing than that used in Ref. [5], we extend the list of exact values proposed for critical percolation in 2D. Instead of focusing on entire clusters, we consider the probability $P_\ell^{\mathcal{P}}(r,R;\tau_1,\ldots,\tau_\ell)$ that the annulus $D(r,R)$ is traversed by (at least) ℓ non–overlapping connected *paths*, which are monochromatic in the sense that each consists of either occupied sites ($\tau_j = +$) or of vacancies ($\tau_j = -$). We rigorously prove that such probabilities are also related to the $O(N=1)$ exponents. Assuming the validity of the exact values for the latter, we find that for all the $\pm$ color sequences which include at least one of *each* type,

$$P_\ell^{\mathcal{P}}(r,R;\tau_1,\ldots,\tau_\ell) \approx (r/R)^{x_\ell^{\mathcal{P}}} \tag{4}$$

with the *path crossing exponents* $x_\ell^{\mathcal{P}}$ satisfying

$$x_\ell^{\mathcal{P}} = x_\ell^{O(N=1)} = (\ell^2-1)/12. \tag{5}$$

Since the cluster exponents are $x_k^{\mathcal{C}} = x_{2k}^{\mathcal{P}}$, Eq. (5) may be viewed as an extension of the SD formula to *odd* values of ℓ, or *half integer* values of k.

ii. Using the newly acquired values we explain some of the quantitative and qualitative features of the *external perimeter* (or the "accessible perimeter") of critical clusters which were observed by Grossman and Aharony (GA) [9]: its dimension, which we identify as

$$D_{EP} = 2 - x_3^{O(1)} = 4/3 \ , \tag{6}$$

and the interesting fact that – unlike the hull – the external perimeter appears to be self–avoiding on the macroscopic scale.

iii. We consider also the analogous *boundary* or *"surface" exponents*, which describe the probability $\tilde{P}_\ell^{\mathcal{P}}(r,R;\tau_1,\ldots,\tau_\ell)$ that, *within the upper half space*, a semi-annular region $\tilde{D}(r,R)$ is traversed by ℓ paths. For the exponents defined by

$$\tilde{P}_\ell^{\mathcal{P}}(r,R;\tau_1,\ldots,\tau_\ell) \approx (r/R)^{\tilde{x}_k^{\mathcal{P}}} \tag{7}$$

we find

$$\tilde{x}_\ell^{\mathcal{P}} = \tilde{x}_{\ell+1}^{O(N=1)} = (\ell+1)\ell/6. \tag{8}$$

In this case the relation is valid with *no* restriction on the color sequence τ; however there is a shift: ℓ crossing paths correspond to $(\ell+1)$ $O(N=1)$ lines. Thus, with odd ℓ, one recovers the cluster boundary exponents $\tilde{x}_k^{\mathcal{C}} = \tilde{x}_{2k-1}^{\mathcal{P}} = k(2k-1)/3$, as in Refs. [10,11].

The above results are based on a rigorous relation of path crossing probabilities with amplitudes of a loop model which we shall now define, and on known exact values for the exponents of the latter which still remain to be proven at a rigorous level. The arguments are formulated for the special model of independent site (i.e., hexagon) percolation on the triangular lattice. For reasons of universality one may expect the conclusions to apply to other 2D percolation models, e.g., the bond percolation, and also to the statistics of the connected clusters of $(+)$ or $(-)$ spins in the 2D Ising model at all temperatures above T_c.

The loop-model configurations, Γ, are collections of nonoverlapping loops and lines, in suitable subsets of the plane, which are allowed to have end-points only within prescribed regions. The weight of a configuration is

$$W(\Gamma) \ = \ K^{\mathcal{N}_{\mathcal{B}}} N^{\mathcal{N}_{\mathcal{P}}} \ , \tag{9}$$

with $\mathcal{N}_{\mathcal{P}}$ the number of closed lines (or "polygons") and $\mathcal{N}_{\mathcal{B}}$ their total length (the number of bonds). For the particular case of hexagon percolation discussed here the fugacities are $K=1$, $N=1$, i.e., $W(\Gamma)=1$ for all Γ [5].

A probability distribution of loop and line configurations in a prescribed region is defined by means of the weights $W(\Gamma)/Z$, with Z a suitable normalizing factor. Let now $P_\ell^{O(N)}(r,R)$ denote the probability that such a systems of lines with no end points in the annular domain $D(r,R)$ contains at least ℓ lines traversing $D(r,R)$. For a representation of the surface exponents, we also let $\tilde{P}_\ell^{O(N)}(r,R)$ denote the corresponding event with the lines restricted to lie in the upper half plane, taken here with the "free boundary conditions".

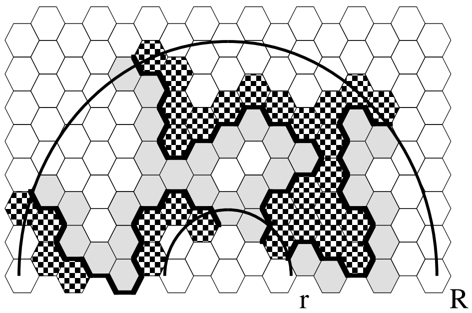

FIG. 1. A configuration with $\ell = 2$ paths traversing the semi-annular region $\tilde{D}(r,R)$, one of occupied (checkered) and the other (dual) of vacant (gray) hexagons. The configuration has $(\ell+1=3)$ $O(N=1)$ boundary lines (drawn thickly); the states of the hexagons not adjacent to these lines are left unspecified.

A close variant of the quantity $P_\ell^{O(N)}(r,R)$ is the $O(N)$ *amplitude* $G_\ell^{O(N)}(r,R)$ which is defined as the *sum* over ℓ lines $\{\gamma_1,\ldots,\gamma_\ell\}$ spanning the annulus $D(r,R)$ of the *probability* $\omega(\gamma_1,\ldots,\gamma_\ell \in \Gamma)$ that the lines are included in Γ. For $N=1$, that probability reduces to the *local* expression [12,13]:

$$\omega(\gamma_1,\ldots,\gamma_\ell) = 2^{-\mathcal{N}_H(\gamma_1,\ldots,\gamma_\ell)} 2^{\mathcal{K}(\gamma_1,\ldots,\gamma_\ell)}/2^{\#} \tag{10}$$

with $\mathcal{N}^H(\gamma_1,\ldots,\gamma_\ell)$ the number of hexagons touched by the lines, $\mathcal{K}(\gamma_1,\ldots,\gamma_\ell)$ the number of line clusters – two lines being regarded as in the same cluster if they touch a common hexagon, and $\#$ defined as taking the value 0 if the ℓ lines leave room for another curve to traverse the annulus and 1 otherwise. The amplitudes then read

$$G_\ell^{O(N=1)}(r,R) \ = \ \sum_{\gamma_1,\ldots,\gamma_\ell} \omega(\gamma_1,\ldots,\gamma_\ell), \tag{11}$$

the sum running over sets of ℓ nonoverlapping lines which traverse $D(r,R)$. It can be shown that the probabilities and amplitudes agree to the leading order [13]:

$$P_\ell^{O(N=1)}(r,R) \ = \ \Big(1+o(\frac{r}{R})\Big)\, G_\ell^{O(N=1)}(r,R) \ . \tag{12}$$

"Coulomb gas" and Bethe Ansatz methods [4,5,14–16] yield the conclusion that the loop model amplitudes, thus probabilities, decay by power laws, $G_\ell^{O(1)}(r,R) \approx$

$(r/R)^{x_\ell^{O(1)}}$, and $\tilde{G}_\ell^{O(1)}(r,R) \approx (r/R)^{\tilde{x}_\ell^{O(1)}}$, with the exponents taking the values given in Eq. (5) and Eq. (8). Our results rest now on the fact that the $O(N=1)$ line probabilities are of the same order of magnitude as the path crossing probabilities. Their compatibility is expressed in the following statement.

Proposition *In the site percolation model on the triangular lattice:*

1) For any "color sequence" $\{\tau_j = \pm\}_{j=1}^{\ell}$ which includes at least one of each kind (+ and −),

$$P_\ell^{\mathcal{P}}(r,R;\tau_1,\ldots,\tau_\ell) \ \lessgtr\ P_\ell^{O(N=1)}(r,R)\ , \tag{13}$$

where $A \lessgtr B$ means that there are constants $0 < c_1, c_2 < \infty$ with which $c_1 A \leq B \leq c_2 A$ uniformly in r and R.

2) The surface probabilities satisfy

$$\tilde{P}_\ell^{\mathcal{P}}(r,R;\tau_1,\ldots,\tau_\ell) \ =\ \tilde{P}_{\ell+1}^{O(N=1)}(r,R)\ , \tag{14}$$

without any restriction on the color sequence τ.

The simplest case of the above relation is in the example of the half-disk amplitude with alternating color paths (as in Fig. 1), which corresponds to $\tilde{P}_\ell^{\mathcal{P}}(r,R;+,-,+,-,\ldots)$. Equation (14) holds there since the statistics of the boundary lines is given exactly by the $O(N=1)$ loop model. The result is then extended by establishing independence on the color sequence. This is done by successively conditioning on the suitable "rightmost path" and flipping the site variables left of the line. Thus use is made of the Markov property combined with the spin flip symmetry, which are enabled by the independence and the self duality of the site percolation model on the triangular lattice. The argument is a bit more involved in the case of the full disk. There we need to have at least one traversing boundary line, which we employ to slit the annulus. The previous argument is then applied to the resulting simply connected domain. Equation (13) reflects the fact that the overcounting involved in the selection of the slit is by at most a finite factor. The complete argument will be spelled in Ref. [13].

Let us turn now to some of the conclusions which can be drawn from the exact values of the path exponents, as provided by Eqs. (5) and (8). The external perimeter is the boundary of the set obtained by filling in "fjords" which are connected to the cluster's complement through very narrow passages (or "gates"). GA [9] made the puzzling observation that while typical clusters do show many fjords with only a narrow passage to the complement, once one fills in fjords with passages of width two or three lattice spacings no fjords of broader microscopic passages, and depth comparable with that of the cluster, are left. This is clearly visible in Fig. 6 of the second Ref. [9]. The dimension of the external perimeter was found numerically to be $D_{EP} \approx 4/3$ [17]. Although there appeared conjectures relating this value to the fractal dimension of self–avoiding walks [5,9] (SAW's), this connection was never elucidated.

Our generalized path statistics enables us to find D_{EP} exactly: each point on the external perimeter is next to the end of three paths of lengths comparable with the diameter of the cluster – a path of occupied sites and in addition two distinct dual paths of vacancies (Fig. 2a), which guarantee that the point is not within a fjord of narrow opening (both paths must be able to exit the fjord via the narrow gate). This yields $D_{EP} \ =\ 2 - x_3^{\mathcal{P}} = 4/3$, in excellent agreement with the numerical results.

GA's observation is particularly striking from the perspective of the *scaling limit*, where one sends the lattice spacing to zero while keeping the sight on the curves observed on the macroscopic scale. (The limit can be constructed using the analysis of Ref. [18], which implies that the cluster hulls and external perimeters can still be described by means of Hölder continuous random curves.) While the external perimeter is self–avoiding on the lattice scale, like the hull it could have close encounters which appear as self–intersections when viewed from the macroscopic perspective. Their absence is also explained by the generalized path statistics: the occurence in an $L \times L$ box of a cluster with a fjord of depth $R = sL$ and neck width $r = \epsilon L$ requires there being *six* paths, two pairs of triplets as used in the derivation of Eq. (6), which meet in a region of size r and avoid each other up to a radius R (Fig. 2b). The probability of finding such six paths scales as $\epsilon^{-d} \times \left(\frac{\epsilon}{s}\right)^{x_6^{\mathcal{P}}} \ =\ O(\epsilon^{x_6^{\mathcal{P}} - d})$ where $d = 2$. Equation (5) yields the exponent value $x_6^{\mathcal{P}} = 2\frac{11}{12}$ and hence the probability for a randomly picked configuration to exhibit such a gate tends to zero, in the situation where s is fixed at some non-infinitesimal value $0 < s < 1$, and $\epsilon \to 0$. The crucial point here is that the fractal dimension of the set of these gates is negative:

$$D_G \ =\ 2 - x_6^{\mathcal{P}} \ <\ 0\ . \tag{15}$$

This explains the asymptotic absence of macroscopic size fjords of neck width r anywhere between few multiples of the lattice spacing and ϵL, for any fixed $\epsilon \ll 1$. The above argument also implies that the external perimeter will not exhibit *peninsulas* with narrow isthmuses, even though that condition was not built into the construction.

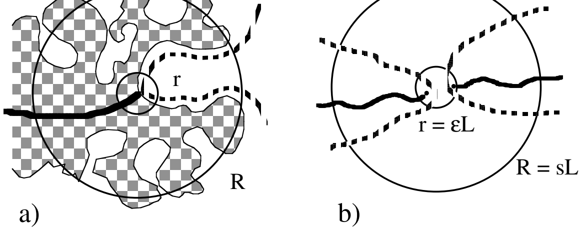

FIG. 2. Paths, and dual (dotted) paths characterizing a) the external perimeter ($\ell = 3$) and b) a gate ($\ell = 6$).

It is of interest to recall here the suggestion which was made on a theoretical as well as a numerical basis [19,12], that percolation's hull and external perimeter dimensions coincide with the dimensions of polymers, respectively at the θ-point (the onset of collapse), or in the SAW state:

$$D_H = D_\theta , \quad D_{EP} = D_{SAW} \ . \tag{16}$$

It is natural to conjecture that in the scaling limit the external perimeter *coincides*, in its local statistics, with a SAW. This may appear to be in conflict with the a-symmetry between the two sides of the external perimeter, however the absence of peninsulas in the scaling limit suggests that the symmetry is restored asymptotically.

As noted in [20,21], it is possible to obtain some selected path exponents by direct arguments. The values are consistent with the formulas given above. It is instructive to list specific values of $x_\ell^{\mathcal{P}} = (\ell^2-1)/12$:
$\ell = 2$: $x_2^{\mathcal{P}}$ yields $D_H = D_\theta = 7/4$.
$\ell = 3$: $x_3^{\mathcal{P}}$ yields $D_{EP} = D_{SAW} = 4/3$.
$\ell = 4$: $x_4^{\mathcal{P}}$ yields $D_{SC} = \nu^{-1} = 3/4$.
$\ell = 5$: $x_5^{\mathcal{P}} = 2$ can be derived directly.
$\ell = 6$: $x_6^{\mathcal{P}} > 2$ implies that the external perimeter is self–avoiding on the large scale ($D_G < 0$).

The relation (5) was not claimed for $\ell = 1$, or for paths of a single color ($\hat{x}_\ell^{\mathcal{P}}$). Concerning this let us note:
– It can be shown directly that $x_1^{O(N=1)} = 0$ ([13]) while $\hat{x}_1^{\mathcal{P}} > 0$ ([22]). The path exponent is related to the cluster dimension; its value appears to be $\hat{x}_1^{\mathcal{P}} = 5/48$ [3].
– The case of $\ell = 2$, with two paths of the same color, is of special interest since it relates to the *backbone dimension*. Numerically, $\hat{x}_2^{\mathcal{P}} = 0.3568 \pm 0.0008$ [23].

For the surface exponents, $\tilde{x}_\ell^{\mathcal{P}} = (\ell+1)\ell/6$ of Eq. (8), we note that for $\ell \to \infty$ $\tilde{x}_\ell^{\mathcal{P}}/x_\ell^{\mathcal{P}} \to 2$, as it should; and
$\ell = 1$: $\tilde{x}_1^{\mathcal{P}} = 1/3$ is consistent with Cardy's equation for the crossing probability [6].
$\ell = 2$: $\tilde{x}_2^{\mathcal{P}} = 1$ can be derived directly.
$\ell = 3$: $\tilde{x}_3^{\mathcal{P}} = 2$ is also directly derivable.
The last one ([21]) is related to a slit disc exponent which is attributed to J. van den Berg in Ref. [20].

Finally, we note that the SD formalism also yields predictions for the hull dimensions of Fortuin-Kasteleyn random clusters, describing the Q-state Potts model. These were recently confirmed in numerical simulations by Hovi and Mandelbrot [24]. In contrast, the values found in that work for the external perimeters do not agree with the generalizations of the SD formulas to odd ℓ. The results presented here were derived only for site percolation. It would be interesting to see generalizations.

This paper is dedicated to the memory of Tal Grossman. The work was started and carried out while the authors enjoyed the gracious hospitality of the Institut Henri Poincaré, the Institute for Advanced Studies (MA and BD), and of Tel Aviv University (MA). It was supported in part by the NSF Grant PHY-9512729 (MA), a grant from the German Israeli Foundation (AA), and by a grant to the IAS from the NEC Research Institute.

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