# Phase Transition in the Semiclassical Regime

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**Abstract**

In continuation with [He1] and an unpublished paper by F. Daumer [Da], we would like to explore the variation of the splitting for a Schrödinger operator coming from field theory when the potential changes as a function of the temperature from a one-well situation to a double well situation. We are for the moment only able to treat the fixed dimension case. But, because the most interesting thing would be to analyze the dependence on the dimension (in the perspective of applications to statistical mechanics) we have tried to follow some of the steps in the proof with respect to the dimension. All this analysis was strongly motivated by a course by M. Kac where some heuristical discussion based on some Born-Oppenheimer analysis is presented. So our work can be considered as a first attempt for justifying this discussion.

## 1 Introduction

Our model is the following operator defined on $I\!R^m$ :

$$P(h,\nu) = -h^2\Delta + \sum_{i=1}^{m}\left[\nu x_i^2 + x_i^4\right] + \mathcal{J}\sum_{i=1}^{m}|x_i - x_{i+1}|^2 \ . \tag{1.1}$$

There are 4 parameters :

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- $m$ is the dimension. If we think of the origin to statistical mechanics, this corresponds to a family of potentials parametrized by a subset $\Lambda$ of $\mathbb{Z}^d$. Here $d = 1$ and $\Lambda$ is identified to a periodic sublattice $\mathbb{Z}/m\mathbb{Z}$. This choice is done through the convention that

$$x_{m+1} = x_1 \ , \tag{1.2}$$

in (1.1). The treatment of similar questions with other boundary conditions in the large dimension limit is an interesting largely open problem. By other boundary conditions we mean for example the "free case", corresponding to the potential :

$$V^{(m,free)} := \sum_{i=1}^{m} \left[\nu x_i^2 + x_i^4\right] + \mathcal{J} \sum_{i=1}^{m-1} |x_i - x_{i+1}|^2 \ ,$$

or, given some $\omega \in \mathbb{R}^{\mathbb{Z}}$, the problem with boundary $\omega$, corresponding to the potential

$$V^{m,\omega} := V^{(m,free)} + \mathcal{J} \left(|x_1 - \omega_0|^2 + |x_m - \omega_{m+1}|^2\right) \ .$$

- $\mathcal{J}$ is a positive parameter measuring the size of the interaction. The case $\mathcal{J} = 0$ is not the most interesting because the operator becomes a model with separate variables which is easily analyzed by direct semi-classical analysis of the one dimensional model

$$-h^2 \frac{d^2}{dt^2} + \nu t^2 + t^4 \ , \tag{1.3}$$

which was already analyzed in [He1] or in the pionneering work of [KaTh]. We shall recall some of the main results in the next section.
This case corresponds to the case without interaction. In the future, when $m > 1$, we will take $\mathcal{J} > 0$ fixed independently of the other parameters. This means that we will not treat the interaction perturbatively which is an independent interesting problem.

- The parameter $\nu$ represents the influence of the temperature. The value $\nu = 0$ is critical in the sense that

  - when $\nu > 0$ the electric potential

$$V^{(m)} := \sum_{i=1}^{m} \left[\nu x_i^2 + x_i^4\right] + \mathcal{J} \sum_{i=1}^{m} |x_i - x_{i+1}|^2 \ , \tag{1.4}$$

    has a unique non-degenerate minimum (and the potential is uniformly strictly convex) at $X = 0$,

  - when $\nu < 0$, $V^{(m)}$ has two non-degenerate minima which are symmetric through $x \mapsto -x$.

- $h$ is the semi-classical parameter which will tend to 0.

Our main problem will be to analyze the transition between $\nu > 0$ and $\nu < 0$. This means that we no more assume that $\nu$ is fixed and that we shall analyze more carefully how the limits $\nu \to 0$ and $h \to 0$ intertwin.

# 2 What is known?

## 2.1 The case $m = 1$.

The case $m = 1$ was analyzed in [He1]. We simply find the model (1.3). We know in this case from [Si1] that the splitting between the two lowest eigenvalues of the operator $\lambda_2(\nu, h) - \lambda_1(\nu, h)$ is a monotonic increasing function of $\nu$. When $\nu < 0$, we know that the splitting is exponentially small

$$\lambda_2(\nu, h) - \lambda_1(\nu, h) \sim h^{\frac{1}{2}} c_\nu(h) \exp -\frac{S_\nu}{h} , \tag{2.1}$$

with

$$S_\nu = |\nu|^{\frac{3}{2}} S_1 , \tag{2.2}$$

and

$$S_1 = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (\frac{1}{2} - t^2) dt = \frac{\sqrt{2}}{3} . \tag{2.3}$$

For the moment, the asymptotic is not uniform with respect to $\nu$ as $\nu \to 0$. But at least formally, it is clear that the nature of the splitting change as $S_\nu / h$ becomes bounded, and this strongly suggests to distinguish between three regions

- $$\nu \leq -C h^{\frac{2}{3}} , \tag{2.4}$$
- $$-C h^{\frac{2}{3}} \leq \nu \leq C h^{\frac{2}{3}} , \tag{2.5}$$
- $$\nu \geq C h^{\frac{2}{3}} , \tag{2.6}$$

where $C$ is some positive constant.
When $\nu > 0$ (fixed), by standard semiclassical analysis one can use an harmonic approximation of the operator and obtain :

$$\lambda_2(\nu, h) - \lambda_1(\nu, h) = 2\sqrt{\nu}\, h + r_\nu(h) , \tag{2.7}$$

with

$$r_\nu(h) = \mathcal{O}_\nu(h^2) .$$

If one want to control the uniformity of the remainder with respect to $\nu$, it is probably better to make the change of variable $t = \rho y$, and to choose

$$\rho = \sqrt{\nu} \ . \tag{2.8}$$

The initial operator becomes unitary equivalent to to the operator

$$\rho^4 \left( -\tilde{h}^2 \frac{d^2}{dy^2} + y^2 + y^4 \right) \tag{2.9}$$

with

$$\tilde{h} = h/\rho^3 \ . \tag{2.10}$$

This gives the remainder in the form

$$r_\nu(h) = \nu^2 \mathcal{O}(\tilde{h}^2) = \frac{1}{\nu} \mathcal{O}(h^2) \ . \tag{2.11}$$

This has a meaning only when $\tilde{h} \to 0$, that is when $\frac{h}{\nu^{\frac{3}{2}}}$ is small enough which is the condition we have met already.
As a side remark, we note that, when $\nu = 0$, which is inside the second zone, we meet the semi-classical quartic oscillator and the splitting is given by

$$\lambda_2(h,0) - \lambda_1(h,0) = c_0 h^{\frac{4}{3}} \ , \tag{2.12}$$

where $c_0$ is the gap between the first two eigenvalues of the quartic oscillator $-d^2/dx^2 + x^4$.

## 2.2 The case $m$, $\nu$ and $\mathcal{J}$ fixed.

When these three parameters are fixed the analysis of the splitting as $h \to 0$ is well understood.

Here we refer (for mathematical proofs) to the results by Helffer-Sjöstrand [HeSj1], [HeSj2] and B. Simon [Si2] and [Si3] when $\nu \neq 0$ and when $\nu = 0$ to Martinez-Rouleux [MaRo].

## 2.3 The convex case.

When the potential is strictly convex, there is an explicit lower bound for the splitting which was first observed by J.Sjöstrand [Sj1] (see also [HeSj3]). This gives in our particular case the following uniform lower bound :

$$\lambda_2(\nu,h) - \lambda_1(\nu,h) \geq 2\sqrt{\nu} h \ . \tag{2.13}$$

We observe that, if this estimate is rather optimal as $h \to 0$ for fixed $\nu$, this is no more the case when $\nu \to 0$ as can be seen from (2.12) in the case $m = 1$.

### 2.4 Large dimension results

Although there exist folk theorems for this case, we are not sure that a complete answer to the problem is known. Of course, in the convex case, (2.13) gives a reasonable answer when $\nu > 0$ is fixed. The estimate is indeed universal and independent of $m$.
When $\nu < 0$, it is thought that the splitting tends to 0 as $m \to +\infty$ and more optimistic but heuristic considerations lead to conjecture that there exists $C$, $h_0$ and $S > 0$ such that, for all $m > 0$ and all $h \leq h_0$, we have

$$\lambda_2(\nu, h, m) - \lambda_1(\nu, h, m) \leq C \exp -\frac{mS}{h} . \tag{2.14}$$

On the basis of formal WKB constructions, one can actually propose some asymptotic formula for the splitting in large dimension, of the type

$$\lambda_2(\nu, h, m) - \lambda_1(\nu, h, m) \sim A_m(h) \exp -\frac{mS}{h} , \tag{2.15}$$

where $A_m(h)$ is an elliptic semiclassical symbol.
This is done at a formal level in [DoKo] who propose an asymptotic formula for $A_m(h)$ based on instantons considerations.
Unfortunately, we have no effective reference for proofs of these estimates. When $(\mathbb{Z}/m\mathbb{Z})$ is replaced by $(\mathbb{Z}/m\mathbb{Z})^d$ with $d > 1$, considerations based on infrared estimates lead to a proof of the first assertion : the splitting tends to 0 like $1/m^d$ as $m \to +\infty$. But the estimate is no more valid in full generality when $d = 1$, but only under the condition that for sufficiently small enough $\epsilon > 0$, we have

$$m \leq \exp \frac{\epsilon}{h} . \tag{2.16}$$

The analysis (see [He5] [He6] and references therein) is based on a uniform control as $\beta \to +\infty$ of $\exp -\beta H$.
The common wisdom is that when $d = 1$, one has to analyze the problem through the Peierls method but we have no reference where it is done in detail (See [Re]).

If we consider the estimates where tunneling effect is measured, we know only, outside heuristical considerations by [Ka], the work by [HeSj3], [HeSj4] completed in [He3] which is done under the additional condition that there exists $N$ such that

$$m \leq h^{-N} . \tag{2.17}$$

What is basically missing is a good analog of the Agmon estimates in the large dimension context.

## 3 The sufficiently deep double well case

### 3.1 Description

We consider in this section the case when the following condition is satisfied

$$\nu \leq -C\sqrt{h} , \tag{3.1}$$

where $C$ is a sufficiently large positive constant.
We hope in this case to get a splitting which is of the same type as in the case $\nu < 0$ fixed. We do not try to have the control with respect to the dimension and, at least as a first step, we concentrate our analysis to the proof

of a rough upperbound for the splitting.
As it is now standard, this type of estimates is obtained by the conjonction of the formula

$$\lambda_2 - \lambda_1 = \inf_{\{\theta\,;\,\int \theta\, u_1^2\, dx = 0\}} \frac{\int |\nabla\theta|^2 u_1^2 dx}{\int \theta^2 u_1^2 dx}\ , \tag{3.2}$$

where $u_1$ is the first eigenfunction of the operator, and of the control of the decay of $u_1$. We observe that by parity of the potential the lowest eigenfunction is even and, as far as upper bounds are considered, it is enough to look for an odd function $\theta$ equal to $-1$ near one well and $+1$ near the other well. From the analysis of the previous sections we recall that the two wells are given by $\pm x_c$ with

$$x_c = (\frac{\sqrt{-\nu}}{\sqrt{2}}, \cdots, \frac{\sqrt{-\nu}}{\sqrt{2}})\ . \tag{3.3}$$

Moreover, there exists for the Agmon distance a unique geodesics joining the two wells which is the segment $[-x_c, +x_c] = \{-(1-t)x_c + tx_c \mid t \in [0,1]\}$.
This discussion is valid for fixed negative $\nu$ but may become dangerous as $\nu \to 0$.
So we have to analyze the arguments carefully.

## 3.2 Adapted coordinates

Let us first perform a change of variables $x = \rho y$ with $\rho$ like in (2.8):

$$\rho = \sqrt{-\nu}\ . \tag{3.4}$$

We obtain in this way, and dividing by $\rho^4$ the operator

$$-\tilde{h}^2\Delta_y + \sum_{j=1}^{m} (y_j^4 - y_j^2) + \frac{\mathcal{J}}{\rho^2} \sum_{j=1}^{m} |y_j - y_{j+1}|^2\ , \tag{3.5}$$

with

$$\tilde{h} = \frac{h}{\rho^3} = h|\nu|^{-\frac{3}{2}}\ . \tag{3.6}$$

After this change of operator the new eigenvalues are related to the previous ones by

$$\tilde{\lambda}_j = \frac{\lambda_j}{\rho^4} = \frac{\lambda_j}{\nu^2}\ . \tag{3.7}$$

Although we are mainly concerned by the operator on the form (3.5), we make a second change of coordinates (that will be usefull also in Section 4). Since the minima of the potential are always on the line $\{y_1 = y_2 = ... = y_m\}$, we can choose coordinates in such a way that $e_1 := m^{-1/2}(1, ..., 1)$ becomes the first vector of the new orthonormal basis. In fact, we can at the same time diagonalize the quadratic form $y? \mapsto \sum_{j=1}^{m} |y_j - y_{j+1}|^2$ by taking as the $j$-th basis vector :

$$e_j = \varepsilon_{j,m} m^{-1/2} \left(1, \cos\frac{2\pi(j-1)}{m}, \cos\frac{4\pi(j-1)}{m}, ..., \cos\frac{2\pi(m-1)(j-1)}{m}\right)$$

where $\varepsilon_{j,m} = 1$ if $2(j-1) \in m\mathbb{Z}$, and $\varepsilon_{j,m} = \sqrt{2}$ if $2(j-1) \notin m\mathbb{Z}$.

Then one can verify that $(e_1, ..., e_m)$ is an orthonormal basis of $\mathbb{R}^m$, in which the coordinates of $y = (y_1, ...y_m)$ become $(z_1, ..., z_m)$ with :

$$z_j = \varepsilon_{j,m} m^{-1/2} \sum_{k=1}^{m} y_k \cos\frac{2\pi(k-1)(j-1)}{m}. \tag{3.8}$$

In these new coordinates, our operator becomes :

$$-\tilde{h}^2 \Delta_z + \sum_{k=1}^{m} (\frac{1}{\sqrt{m}} z_1 + \ell_k(z'))^4 - |z|^2 + \frac{4\mathcal{J}}{\rho^2} q(z') \tag{3.9}$$

where

$$\ell_k(z') = m^{-1/2} \sum_{j=2}^{m} \varepsilon_{j,m} z_j \cos\frac{2\pi(k-1)(j-1)}{m} \ , \ k = 1, \cdot, m \ ,$$

and

$$q(z') = Q(z_2, ..., z_m) = \sum_{j=2}^{m} z_j^2 \sin^2\frac{\pi(j-1)}{m}.$$

## 3.3 Lower bound for the splitting

Under the assumption (3.1), the parameter $\tilde{h}$ is small (we recover the condition (2.4) ), and we shall perform a semiclassical analysis with respect to $\tilde{h}$.
This is a standard semiclassical problem outside the important point that the coefficient $\frac{\mathcal{J}}{\rho^2}$ of $\sum_{j=1}^{m} |y_j - y_{j+1}|^2$ is not bounded
independently of the parameters.
Before analyzing the decay of the distorted eigenfunction $\tilde{u}_1(y) := \rho^{-\frac{m}{2}} u_1(\rho y)$, let us first give a rough upperbound for the lowest eigenvalue. Rough gaussian quasimodes and the mini-max principle give the following lemma.

**Lemma 3.1** .
*There exists a constant $C_0$ and $h_0$ such that, for all $h \in ]0, h_0]$, $\nu$ and $C > 0$ satisfying (3.1), one has :*

$$\tilde{\lambda}_1 \leq -\frac{m}{4} + \frac{C_0}{C^2} \ . \tag{3.10}$$

**Proof.**
By monotonicity of the lowest eigenvalue with respect to $\nu$, it is enough to prove

(3.10) under the additional condition that $\nu \geq \nu_0$ for some fixed $\nu_0 < 0$. Let us indeed prove the following inequality :

$$\tilde{\lambda}_1 \leq -\frac{m}{4} + C_0 \frac{\tilde{h}}{\rho} . \tag{3.11}$$

It is easier to work in the $z$-variables. For the construction of the Gaussian quasi-modes, we now consider the quadratic approximation of (3.9) at the point $z' = 0$, $z_{1,c} = \sqrt{\frac{m}{2}}$. Using that $\sum_{k=1}^{m} \ell_k(z') = 0$ and $\sum_{k=1}^{m} \ell_k(z')^2 = |z'|^2$, this gives the operator with separate variables :

$$-\tilde{h}^2 \Delta_z + 2(z_1 - z_{1,c})^2 - \frac{m}{4} + \frac{4\mathcal{J}}{\rho^2} q(z') + 2|z'|^2 . \tag{3.12}$$

If we take as quasimode the eigenfunction of this operator corresponding to the lowest eigenvalue

$$\hat{\lambda}_1 = -\frac{m}{4} + \sqrt{2}\, \tilde{h} \left( 1 + \sum_{j=2}^{m} \sqrt{1 + \frac{2\mathcal{J}}{\rho^2} \sin^2(\frac{\pi(j-1)}{m}} \right) , \tag{3.13}$$

which takes the form

$$\hat{\lambda}_1 = -\frac{m}{4} + \mathcal{O}(\frac{\tilde{h}}{\rho}) ,$$

the error terms are easily seen (through the Laplace method) as contributions in

$$R(h, \rho) = \mathcal{O}(\tilde{h}^2) \tag{3.14}$$

This gives (3.11) and the lemma observing that

$$\frac{\tilde{h}}{\rho} = h(-\nu)^{-2} \leq \frac{1}{C^2} . \tag{3.15}$$

$\diamond$

Now we write down the energy relation giving traditionally the decay of the eigenfunction associated with $\tilde{\lambda}_1$. We get, for any $\phi$,

$$\int |\nabla \left( e^{\phi/\tilde{h}} \tilde{u}_1 \right)|^2 + \int (\tilde{V} - |\nabla \phi|^2 - \tilde{\lambda}_1) e^{2\phi/\tilde{h}} |\tilde{u}_1|^2 dy = 0 \tag{3.16}$$

where

$$\tilde{V} = \sum_{j=1}^{m} (y_j^4 - y_j^2) + \frac{\mathcal{J}}{\rho^2} \sum_{j=1}^{m} |y_j - y_{j+1}|^2 .$$

In particular we get

$$\int (\tilde{V} - |\nabla \phi|^2 - \tilde{\lambda}_1) e^{2\phi/\tilde{h}} \tilde{u}_1^2 dy \leq 0 . \tag{3.17}$$

Let us now specify our choice of $\phi$. The easiest way is to take as $\phi$ the Lipschitz function $(1-\eta)d_{\tilde{\mathcal{J}}}$ where $\eta > 0$ and $d_{\tilde{\mathcal{J}}}$ is the Agmon distance (see [HeSj1]) relative to the potential

$$V_{\tilde{\mathcal{J}}} = \sum_{j=1}^{m}(y_j^4 - y_j^2) + \tilde{\mathcal{J}} \sum_{j=1}^{m} |y_j - y_{j+1}|^2 \,. \tag{3.18}$$

Here we observe that

$$\tilde{V} \geq V_{\tilde{\mathcal{J}}} \,, \tag{3.19}$$

if

$$\frac{\mathcal{J}}{\rho^2} \geq \tilde{\mathcal{J}} \,.$$

We also observe that the minima and the Agmon distance relative to these two minima are actually independent of the parameter $\tilde{\mathcal{J}} > 0$ and that the minimum of the potential is $-\frac{m}{4}$.
Following the standard proof on the decay, we get that, for any $\epsilon > 0$, there exist $C_\epsilon > 0$, $C'_\epsilon$, $h_\epsilon$
such that if $C = C_\epsilon$, $h \leq h_\epsilon$, $C^{\frac{3}{2}}\, h^{\frac{3}{4}} \leq (-\nu)^{\frac{3}{2}} \leq (\frac{\mathcal{J}}{\tilde{\mathcal{J}}})^{\frac{3}{2}}$, then

$$|| \exp \frac{(1-\eta)d_{\tilde{\mathcal{J}}}}{\tilde{h}} \tilde{u}_1 ||_{L^2} \leq C'_\epsilon \exp \frac{\epsilon}{\tilde{h}} \,. \tag{3.20}$$

Coming back to the formula characterizing the splitting, we get, by chosing suitably $\eta$, that for any $\epsilon > 0$, there exists a constant $C'_\epsilon$ such that

$$\tilde{\lambda}_2 - \tilde{\lambda}_1 \leq C'_\epsilon \exp \left[ -(1-\epsilon) m S_1 (-\nu)^{\frac{3}{2}} / h \right] \,, \tag{3.21}$$

where $S_1 = \sqrt{2}/3$ is the Agmon distance relative to the one-dimensional potential $t \mapsto t^4 - t^2 + \frac{1}{4}$ between $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ (see (2.3)). Coming back to the initial operator, we obtain using (3.7) :

**Proposition 3.2** .
*For any $\epsilon > 0$ annd $\nu_0 < 0$, there exist $C > 0$, $C'$ and $h_0 > 0$, such that, for $(\nu, h)$ satisfying (3.1), $\nu \geq \nu_0$ and $h \leq h_0$, then*

$$\lambda_2 - \lambda_1 \leq C' \exp \left[ -(1-\epsilon) m S_1 (-\nu)^{\frac{3}{2}} / h \right] \,, \tag{3.22}$$

**Remark 3.3** .
Of course it would be interesting to get the lower bound for the splitting. But the most important point would be to control the upper bound uniformly with respect to $m$.

# 4 The exact Feshbach reduction

In this section we turn back to the operator $P(h,\nu)$ in the general case. We start by the second change of variables introduced in Subsection 3.2 given by :

$$x = (x_1, ...x_m) \mapsto (y_1, ..., y_m)$$

with :

$$y_j = \varepsilon_{j,m} m^{-1/2} \sum_{k=1}^{m} x_k \cos \frac{2\pi(k-1)(j-1)}{m}. \tag{4.1}$$

In these new coordinates, $P(h,\nu)$ becomes :

$$P(h,\nu) = -h^2\Delta_y \ + \ \sum_{k=1}^{m} \left( \frac{y_1}{\sqrt{m}} + \ell_k(y') \right)^4 \ + \ \nu|y|^2 + 4\mathcal{J}q(y') \tag{4.2}$$

that is, using again that $\sum_{k=1}^{m} \ell_k(y') = 0$ and $\sum_{k=1}^{m} \ell_k(y')^2 = |y'|^2$ :

$$\begin{aligned} P(h,\nu) \quad &= \quad -h^2\Delta_y + \frac{1}{m} y_1^4 + \frac{6}{m} y_1^2 |y'|^2 + \frac{4}{\sqrt{m}} y_1 \sum_{k=1}^{m} \ell_k(y')^3 \\ &\qquad + \sum_{k=1}^{m} \ell_k(y')^4 + \nu|y|^2 + 4\mathcal{J}q(y'). \end{aligned} \tag{4.3}$$

Then we make a change of scale by setting :

$$t = h^{-1/4} y_1 \ ; \ z = h^{-1/2} y'.$$

We get $P(h,\nu) = h\widetilde{P}(h,\nu)$ with :

$$\widetilde{P}(h,\nu) := h^{-1}P(h,\nu) = -h^{1/2}\partial_t^2 + \frac{1}{m}t^4 + \frac{\nu}{\sqrt{h}}t^2 + Q_0(z, D_z) + r_1(t,z;h) \tag{4.4}$$

where

$$\left\{ \begin{array}{l} Q_0(z, D_z) = -\Delta_z + 4\mathcal{J}q(z) + \nu|z|^2 \ ; \\ r_1(t,z;h) = \frac{6\sqrt{h}}{m} t^2 |z|^2 + h \sum_{k=1}^{m} \ell_k(z)^4 + \frac{4h^{3/4}}{\sqrt{m}} t \sum_{k=1}^{m} \ell_k(z)^3. \end{array} \right. \tag{4.5}$$

Now let us perform a Feshbach reduction on (4.4) by the use of a Grushin problem. We denote by $u_0 = u_0(z)$ the first normalized eigenfunction of $Q_0$ on $L^2(\mathbb{R}_z^{m-1})$, and $\mu_1$ its first eigenvalue, the value of which is :

$$\mu_1 = \sum_{j=2}^{m} \left( 4\mathcal{J} \sin^2 \frac{\pi(j-1)}{m} + \nu \right)^{1/2}.$$

Then, for any real number $\lambda$ close enough to

$$\lambda_0 := \mu_1 + \inf_{t \in \mathbb{R}} \left( \frac{t^4}{m} + \frac{\nu t^2}{\sqrt{h}} \right)$$

(which is equal to $\mu_1 - \nu^2 m/4h$ when $\nu < 0$), we consider on $\mathcal{H} := L^2(I\!R^m_{t,z}) \oplus L^2(I\!R_t)$ the following operator :

$$\mathcal{P}_\lambda = \begin{pmatrix} \widetilde{P}(h,\nu) - \lambda & u_0 \\ \langle \cdot, u_0 \rangle_z & 0 \end{pmatrix} \tag{4.6}$$

where $\langle \cdot, \cdot \rangle_z$ denotes the scalar product in $L^2(I\!R_z^{m-1})$. Then $\mathcal{P}_\lambda$ is self-adjoint with domain

$$\mathcal{H}_1 = \mathcal{D}_P \oplus L^2(I\!R_t)$$

where

$$\mathcal{D}_P = \{u \in H^2(I\!R^m_{t,z}) \; ; \; (t^4 + |z|^4)u \in L^2(I\!R^m_{t,z})\}$$

is the domain of $\widetilde{P}(h,\nu)$.

Denote also by $\mathcal{F} = \{v \in H^2(I\!R_t) \; ; \; t^4 v \in L^2(I\!R_t)\}$, $\mu_2$ the second eigenvalue of $Q_0$, by $\Pi$ the orthogonal projector defined on $L^2(I\!R^m_{t,z}) \ni \varphi$ as :

$$\Pi\varphi := \langle \varphi, u_0 \rangle_z u_0$$

and by $\Pi^\perp := 1 - \Pi$.

Then we have :

**Proposition 4.1** *For $\lambda < \lambda_0 + \mu_2 - \mu_1$ the operator*

$$\mathcal{P}_\lambda \; : \; \mathcal{D}_P \oplus L^2(I\!R_t) \; \to \; L^2(I\!R^m_{t,z}) \oplus \mathcal{F}$$

*is invertible, and its inverse is given by :*

$$\mathcal{P}_\lambda^{-1} = \begin{pmatrix} X_\lambda & u_0 - X_\lambda(\cdot r_1 u_0) \\ \langle (1 - r_1 X_\lambda)(\cdot), u_0 \rangle_z & \lambda - A(\lambda) \end{pmatrix} \tag{4.7}$$

*where $X_\lambda = \Pi^\perp(\widetilde{P}(h,\nu)' - \lambda)^{-1}\Pi^\perp$, $\widetilde{P}(h,\nu)'$ is the restriction of $\Pi^\perp \widetilde{P}(h,\nu)\Pi^\perp$ to the space $\Pi^\perp(L^2(I\!R^m_{t,z}))$, and $A(\lambda)$ is the self-adjoint operator on $L^2(I\!R_t)$ defined by :*

$$A(\lambda) = -h^{1/2}\partial_t^2 + \frac{1}{m}t^4 + \frac{\nu}{\sqrt{h}}t^2 + \langle r_1 u_0, u_0 \rangle_z + \mu_1 - \langle X_\lambda(\cdot r_1 u_0), r_1 u_0 \rangle_z \tag{4.8}$$

*with domain $\mathcal{D}_A = \mathcal{F}$. Moreover, in (4.7), we have used the notation $X_\lambda(\cdot r_1 u_0)$ for the operator : $L^2(I\!R_t) \ni v \mapsto X_\lambda(v\, r_1\, u_0)$.*

**Proof** - Although such a type of result is rather standard (see e.g. [Ma]), let us recall the procedure. The central observation here is that $r_1(t,z;h)$ is a non negative function.
In fact, since

$$\begin{aligned} \left| \tfrac{4h^{3/4}}{\sqrt{m}} t \textstyle\sum_{k=1}^m \ell_k(z)^3 \right| &\leq \textstyle\sum_{k=1}^m 2\left(\frac{\sqrt{5}h^{1/4}|t\ell_j(z)|}{\sqrt{m}}\right)\left(\frac{2}{\sqrt{5}}h^{1/2}\ell_j(z)^2\right) \\ &\leq \tfrac{5\sqrt{h}}{m}t^2|z|^2 + \tfrac{4h}{5}\textstyle\sum_{k=1}^m \ell_k(z)^4 \;, \end{aligned}$$

we even have :

$$r_1(t,z;h) \geq \frac{\sqrt{h}}{m} t^2|z|^2 + \frac{h}{5}\sum_{k=1}^{m} \ell_k(z)^4. \tag{4.9}$$

As a consequence, we can write :

$$\begin{aligned}
\Pi^{\perp}(\widetilde{P}(h,\nu)-\lambda)\Pi^{\perp} &\geq \Pi^{\perp}(-h^{1/2}\partial_t^2 + \frac{1}{m}t^4 + \frac{\nu}{\sqrt{h}}t^2 + Q_0(z,D_z) - \lambda)\Pi^{\perp} \\
&\geq \Pi^{\perp}(\frac{1}{m}t^4 + \frac{\nu}{\sqrt{h}}t^2 + \mu_2 - \lambda)\Pi^{\perp} \\
&\geq (\lambda_0 + \mu_2 - \mu_1 - \lambda)\Pi^{\perp}
\end{aligned}$$

so that $\widetilde{P}(h,\nu)' - \lambda$ is invertible from $\Pi^{\perp}(\mathcal{D}_P)$ to $\Pi^{\perp}(L^2(I\!R^m_{t,z}))$ if $\lambda < \lambda_0 + \mu_2 - \mu_1$. In particular, the operator $X_\lambda$ introduced in the proposition has a well defined meaning, and then it is straightforward to verify that the expression given for $\mathcal{P}_\lambda^{-1}$ does invert the operator $\mathcal{P}_\lambda$. $\diamond$

Now, let us make a few observations. The first one is that, by construction, we have the equivalence :

$$\lambda \in \sigma(\widetilde{P}(h,\nu)) \iff \lambda \in \sigma(A(\lambda)) \tag{4.10}$$

where, for a selfadjoint operator $P$, $\sigma(P)$ stands for the spectrum of $P$. Then we notice that, since $u_0$ is an even function of $z$, the quantity $\langle r_1 u_0, u_0\rangle_z$ appearing in the expression of $A(\lambda)$ actually reduces to :

$$\langle r_1 u_0, u_0\rangle_z = \frac{6\sqrt{h}}{m} t^2 \rho_1 + h\rho_2 \tag{4.11}$$

with

$$\rho_1 = \int_{I\!R^{m-1}} |z u_0(z)|^2 dz \ \ ; \ \ \rho_2 = \sum_{k=1}^{m} \int_{I\!R^{m-1}} \ell_k(z)^4 |u_0(z)|^2 dz.$$

(4.12)

Finally, the last term appearing in the expression of $A(\lambda)$ can be estimated on any function $v \in \mathcal{F}$ by :

$\|\langle X_\lambda(v r_1 u_0), r_1 u_0\rangle_z\|$

$\leq h$

-$\lambda$(4.13)where :

$$\tau_0 = \|\sum_{k=1}^{m} \ell_k(z)^4 u_0\|^2$$

$$\tau_1 = \frac{8}{\sqrt{m}} \|\sum_{k=1}^{m} \ell_k(z)^4 u_0\|.\|$$

$\sum_{k=1}^{m} \ell_k(z)^3 u_0\|$
$\tau_2 = \frac{12}{m}\|z^2u_0\|.\| \sum_{k=1}^{m} \ell_k(z)^4 u_0\| + \frac{16}{m}\| \sum_{k=1}^{m} \ell_k(z)^3 u_0\|^2$
$\tau_3 = \frac{48}{m^{3/2}}\|z^2u_0\|.\| \sum_{k=1}^{m} \ell_k(z)^3 u_0\|$
$\tau_4 = \frac{36}{m^2}\|z^2u_0\|^2$.In particular, we can conclude these remarks on the operator $A(\lambda)$, which permits by (4.10) to study the spectrum of $\widetilde{P}(h,\nu)$, by stating :

**Proposition 4.2** *For $\lambda < \lambda_0 + \mu_2 - \mu_1$, the operator $A(\lambda)$ (see (4.7), (4.8)) satisfies :*

$$A(\lambda) = -h^{1/2}\partial_t^2 + \frac{1}{m}t^4 + \frac{\nu}{\sqrt{h}}t^2 + \frac{6\sqrt{h}}{m}t^2\rho_1 + \mu_1 + h\rho_2 + R_\lambda$$

*with*

$$\|R_\lambda v\| \leq C_m\, h(\lambda_0 + \mu_2 - \mu_1 - \lambda)^{-1}(\|t^4 v\| + h\|v\|) \tag{4.14}$$

*where $C_m$ depends only on $m$ and not on $h$. In particular, if $\nu \geq -C\sqrt{h}$ for some positive constant $C$, then $R_\lambda$ has a relative bound of order $\mathcal{O}_m(h)$ with respect to*

$$A_0 := -h^{1/2}\partial_t^2 + \frac{1}{m}t^4 + \frac{\nu}{\sqrt{h}}t^2 + \frac{6\sqrt{h}}{m}t^2\rho_1 + \mu_1 + h\rho_2\ .$$

# 5 The semiclassical limit when $\nu \geq -Ch^{2/3}$

In this section we study the effective operator $A(\lambda)$ found in Proposition 4.1, in order to get informations on the gap between the first two eigenvalues of $P(h,\nu)$. Here we consider $m$ as fixed while $h$ is allowed to tend to 0, and for the moment we restrict our study to $\nu \geq -Ch^{2/3}$ where $C > 0$ is fixed arbitrarily (we postpone the case $\nu \leq -Ch^{2/3}$ to Section 6).

At first, let us consider the case $\nu > 0$. Then we make the change of variable :

$$t \mapsto s = \frac{\tilde{\nu}^{1/4}}{h^{1/8}}\, t$$

where as before $\tilde{\nu} = \nu/\sqrt{h}$, and we get :

$$A(\lambda) = \mu_1 + h^{1/4}\tilde{\nu}^{1/2}\left[-\partial_s^2 + s^2 + \tilde{h}\frac{s^4}{m} + 6\tilde{h}^2\tilde{\nu}^2\rho_1 s^2 + \tilde{h}^3\tilde{\nu}^4 R_2(\lambda,\tilde{h})\right]$$

still with

$$\tilde{h} = \frac{h^{1/4}}{\tilde{\nu}^{3/2}} = \frac{h}{\nu^{3/2}}. \tag{5.1}$$

Here $R_2(\lambda,\tilde{h})$ is uniformly relatively bounded with respect to $-\partial_s^2 + s^2 + \tilde{h}\frac{s^4}{m}$ and depends analytically on $\lambda$ in a $h$-independent neighborhood of $\lambda_0$ (which is equal to $\mu_1$ here). As a consequence, if $\tilde{h}$ is small enough one can prove in a standard way (see e.g. [Ma], Section 4) that the first two values $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of $\lambda$ which satisfy $\lambda \in \sigma(A(\lambda))$ are of the form :

$$\tilde{\lambda}_j(h) = \mu_1 + h^{1/4}\tilde{\nu}^{1/2}e_j + \mathcal{O}(\tilde{h}\, h^{1/4}\tilde{\nu}^{1/2}) \quad (j = 1, 2)$$

where $e_1$ and $e_2$ are the first two eigenvalues of $-\partial_s^2 + s^2$, that is : $e_1 = 1$ and $e_2 = 3$. If we transfer this result to $P(h, \nu)$ by using (4.10), (4.4) and (5.1), we get :

**Proposition 5.1** *There exists a positive constant $C_m$ depending only on $m$ such that if :*

$$0 < C_m h^{2/3} \leq \nu \leq \nu_0 \ ,$$

*for some fixed positive $\nu_0$, then the gap $\Delta\lambda$ between the first two eigenvalues of $P(h, \nu)$ satisfies :*

$$\Delta\lambda = 2h\sqrt{\nu} + \mathcal{O}\left(\frac{h^2}{\nu}\right)$$

*uniformly with respect to $h$ and $\nu$.*

**Remark 5.2** This is in complete accordance with (2.13), and actually specifies the parameters domain where (2.13) is optimal. Notice also that the result is the same as for the one dimensional case (see (2.7) and (2.11)).

To complete this section it remains to study the case where $|\nu| \leq Ch^{2/3}$ with $C \geq C_m$. Then we set :

$$t = h^{1/12} s \ ,$$

so that $A(\lambda)$ becomes :

$$A(\lambda) = \mu_1 + h^{1/3}\left[-\partial_s^2 + \frac{s^4}{m} + \frac{\nu}{h^{2/3}}s^2 + \frac{6\rho_1 h^{1/3}}{m}s^2 + h^{2/3}R_1(\lambda, h)\right]$$

where $R_1(\lambda, h)$ is relatively uniformly bounded with respect to $-\partial_s^2 + \dfrac{s^4}{m} + s^2$.

Denoting $\beta_1(\mu)$ and $\beta_2(\mu)$ the first two eigenvalues of $-\partial_s^2 + \dfrac{s^4}{m} + \mu s^2$, we see that :

**Proposition 5.3** *Fix $C > 0$ arbitrarily. Then, if $|\nu| \leq Ch^{2/3}$, the gap $\Delta\lambda$ between the first two eigenvalues of $P(h, \nu)$ satisfies :*

$$\Delta\lambda = \left(\beta_2(\nu h^{-2/3}) - \beta_1(\nu h^{-2/3})\right) h^{4/3} + \mathcal{O}(h^{5/3})$$

*uniformly for $h$ (and therefore $\nu$) small enough.*

**Remark 5.4** Since $\nu h^{-2/3}$ remains uniformly bounded here, this result allows to get uniform estimates on $\Delta\lambda$. In particular, $\Delta\lambda$ behaves like $h^{4/3}$ as $h$ tends to 0.

## 6 The weakly deep double well case

To study $\Delta\lambda$ when $\nu \leq -Ch^{2/3}$ and $h$ small enough ($m$ fixed), the idea is to try to re-write in a different spirit the Feshbach reduction of Section 4. More precisely, since the gap can now have an exponentially small behaviour, the estimates obtained on the remainder $R_\lambda$ of Proposition 4.2 are not sufficient here. Indeed, $R_\lambda$ has no more to be treated as a perturbation of $A(\lambda)$, but rather as a specification of it. In order to do so, one must study its operator structure, and our purpose here is to show that it is essentially a nice semiclassical pseudodifferential operator. To this aim, we use a technique similar to the one already used for the study of molecules in the Born-Oppenheimer approximation (see e.g. [Ma] and [KMSW]), and which consists in working in the framework of the semiclassical pseudodifferential operators with operator-valued symbols (see [BK], or the appendix of [GMS]).

First of all notice that, thanks to Sections 3 and 6, we can reduce our study to the case : $-C_1\sqrt{h} \leq \nu \leq -Ch^{2/3}$ where $C_1 > 0$ is arbitrary, and $C \geq 1$ can be taken large enough. With the notation $\tilde{\nu} = \nu/\sqrt{h}$, this reads :

$$-C_1 \leq \tilde{\nu} \leq -Ch^{1/6}. \tag{6.1}$$

Recalling (4.4), we first make the change of variable :

$$t \mapsto s = \frac{1}{\sqrt{|\tilde{\nu}|}}\, t\ ,$$

so that $\widetilde{P}(h,\nu)$ becomes :

$$\widetilde{P}(h,\nu) = \tilde{\nu}^2 \left[ -\frac{h^{1/2}}{|\tilde{\nu}|^3}\partial_s^2 + \frac{s^4}{m} - s^2 + \frac{1}{\tilde{\nu}^2}Q_0(z,D_z) + r_2(s,z) \right]$$

with

$$r_2(s,z) = \frac{6\sqrt{h}}{m|\tilde{\nu}|}s^2|z|^2 + \frac{h}{\tilde{\nu}^2}\sum_{k=1}^{m}\ell_k(z)^4 + \frac{4h^{3/4}}{|\tilde{\nu}|^{3/2}\sqrt{m}}\ s \sum_{k=1}^{m}\ell_k(z)^3.$$

Then we set :

$$A(h,\nu) = -\frac{h^{1/2}}{|\tilde{\nu}|^3}\partial_s^2 + \frac{s^4}{m} - s^2 + \frac{1}{\tilde{\nu}^2}Q_0(z,D_z) + \chi(s, \frac{h^{1/8}}{|\tilde{\nu}|^{3/4}}z)r_2(s,z) \tag{6.2}$$

where $\chi \in C_0^\infty(\mathbb{R}^m)$, $\chi \geq 0$ everywhere, $\chi(s,\theta) = 1$ for $s^2 + 4\mathcal{J}q(\theta) \leq 4mC$.

We first show :

**Lemma 6.1** *The spectra of $\tilde{\nu}^{-2}\widetilde{P}(h,\nu)$ and $A(h,\nu)$ in $(-\infty, C^2m]$ coincide up to an error of order $\mathcal{O}(e^{-Cm|\tilde{\nu}|^{3/2}/h^{1/4}})$ uniformly with respect to $h$ and $\nu$.*

**Proof** - We still denote by

$$\tilde{h} = \frac{h^{1/4}}{|\tilde{\nu}|^{3/2}} = \frac{h}{|\nu|^{3/2}} \in$$

$\left[\frac{h^{1/4}}{C_1^{3/2}}, \frac{1}{C^{3/2}}\right]$ .(6.3)The change of variables :

$$(s,z) \mapsto (s,\theta) = (s, \tilde{h}^{1/2} z)$$

transforms the two operators into :

$$\tilde{\nu}^{-2}\widetilde{P}(h,\nu) = -\tilde{h}^2\partial_s^2 + \frac{1}{m}s^4 - s^2 + \frac{1}{\tilde{\nu}^2}\left[-\tilde{h}\Delta_\theta + \frac{1}{\tilde{h}}(4\mathcal{J}q(\theta) + \nu|\theta|^2)\right] + r_3(s,\theta;\tilde{h})$$

$$A(h,\nu) = -\tilde{h}^2\partial_s^2 + \frac{1}{m}s^4 - s^2 + \frac{1}{\tilde{\nu}^2}\left[-\tilde{h}\Delta_\theta + \frac{1}{\tilde{h}}(4\mathcal{J}q(\theta) + \nu|\theta|^2)\right] + \chi(s,\theta)r_3(s,\theta;\tilde{h})$$

with

$$r_3(s,\theta;\tilde{h}) = \frac{6\tilde{h}\tilde{\nu}^2}{m}s^2|\theta|^2 + \tilde{h}^2\tilde{\nu}^4\sum_{k=1}^{m}\ell_k(\theta)^4 + \frac{\tilde{h}^{3/2}\tilde{\nu}^3}{\sqrt{m}}s\sum_{k=1}^{m}\ell_k(\theta)^3.$$

Then we can perform Agmon-type estimates on these two operators, with a weight of the form :

$$f(s,\theta;\tilde{h}) = \exp\left[\frac{\psi_1(s) + \psi_2(\theta)}{\tilde{h}}\right].$$

More precisely, since also $\chi$ and $r_3$ are both non-negative, we have with such an $f$ :

$$\begin{aligned}&\mathrm{Min}\left(\langle f\tilde{\nu}^{-2}\widetilde{P}(h,\nu)u, fu\rangle\ ,\ \langle fA(h,\nu)u, fu\rangle\right)\\ &\quad\geq \langle\left(\frac{s^4}{m} - s^2 - (\psi_1'(s))^2 + \frac{1}{\tilde{h}\tilde{\nu}^2}\left(4\mathcal{J}q(\theta) + \nu|\theta|^2 - |\nabla\psi_2(\theta)|^2\right)\right) fu, fu\rangle\end{aligned}$$

for all $u \in \mathcal{D}_P$. In particular, taking $\psi_2$ supported in $\{q(\theta) \geq Cm\}$ such that :

$$|\nabla\psi_2(\theta)|^2 \leq 2\mathcal{J}Cm$$

and $\psi_1$ supported in $\{s^2 \geq 2Cm\}$ such that :

$$(\psi_1'(s))^2 \leq \frac{1}{2}C^2 m$$

we get for $\tilde{h}$ small enough :

$$\begin{aligned}&\mathrm{Min}\left(\langle f\tilde{\nu}^{-2}\widetilde{P}(h,\nu)u\ ,\ fu\rangle\ ,\ \langle fA(h,\nu)u, fu\rangle\right)\\ &\quad\geq \langle\frac{3}{2}C^2 m fu, fu\rangle + \langle\left(\frac{s^4}{m} - s^2\right) fu\ ,\ fu\rangle.\end{aligned}$$

As a consequence, if $u$ is a normalized eigenfunction of $\tilde{\nu}^{-2}\widetilde{P}(h,\nu)$ or of $A(h,\nu)$ associated to an eigenvalue $\lambda \leq C^2 m$, it satisfies :

$$\|fu\| = \mathcal{O}(1)$$

uniformly with respect to $\tilde{h}$. Then, specifying the choice of $\psi_1$ and $\psi_2$, we can manage so that :

$$\psi_1(t) + \psi_2(\theta) \geq Cm \quad \text{for} \quad s^2 + 4\mathcal{J}q(\theta) \geq 4Cm$$

and thus we obtain that $u$ (together with all its derivatives) is $\mathcal{O}(e^{-Cm/\tilde{h}})$ on $\{\chi(s,\theta) \neq 1\}$, and we can conclude e.g. by applying the spectral theorem. $\diamond$

In particular, for our study of the gap between the first two eigenvalues of $P(h,\nu)$, the previous lemma allows us to work with $A(h,\nu)$ instead of $\tilde{\nu}^{-2}\widetilde{P}(h,\nu)$. The gain is that now, the remainder $\chi(s,\tilde{h}^{1/2}z)r_2(s,z)$ is uniformly $\mathcal{O}(\tilde{h}\tilde{\nu}^2)$ so that it can be treated as a regular perturbation of $\tilde{\nu}^{-2}Q_0(z,D_z)$.

Noticing that the domain of $A(h,\nu)$ is :

$$\mathcal{D}_A = \{u \in H^2(\mathbb{R}^m_{s,z})\ ;\ (s^4 + |z|^2)u \in L^2(\mathbb{R}^m_{s,z})\}$$

we now consider the operator :

$$\mathcal{A}_E = \begin{pmatrix} A(h,\nu) - E & u_0 \\ \langle \cdot, u_0\rangle_z & 0 \end{pmatrix} \quad : \quad \mathcal{D}_A \oplus L^2(\mathbb{R}_s) \to$$

$\mathrm{L}^2(\mathbb{R}^m_{s,z}) \oplus \mathcal{F}$(6.4)where $u_0$ and $\mathcal{F}$ are as in Section 4. Then $\mathcal{A}_E$ can be interpreted as a semiclassical operator with small parameter $\tilde{h}$ and with operator-valued symbol given by :

$$a_E(s,\sigma;\tilde{h}) = \begin{pmatrix} \sigma^2 + \frac{s^4}{m} - s^2 + \tilde{\nu}^{-2}Q_0(z,D_z) + \chi(s,\tilde{h}^{1/2}z)r_2(s,z) - E & u_0 \\ \langle \cdot, u_0\rangle_z & 0 \end{pmatrix} \tag{6.5}$$

considered as an operator from $\mathcal{D}_Q \oplus \mathbb{C}$ to $L^2(\mathbb{R}^{m-1}_z) \oplus \mathbb{C}$ , with $\mathcal{D}_Q := \{u \in H^2(\mathbb{R}^{m-1}_z)\ ;\ |z|^2 u \in L^2(\mathbb{R}^{m-1}_z)\}$. This means that we have :

$$\mathcal{A}_E = a_E(s,\tilde{h}D_s;\tilde{h}) = \mathrm{Op}^W_{\tilde{h}}(a_E)$$

where $\mathrm{Op}^W_{\tilde{h}}$ denotes the usual Weyl-quantization of symbols with small parameter $\tilde{h}$. Since $\chi(s,\tilde{h}^{1/2}z)r_2(s,z)$, together with all its derivatives, is $\mathcal{O}(\tilde{h})$ uniformly, we also have a notion of principal symbol for $\mathcal{A}_E$, given by :

$$a^0_E(s,\sigma) = \begin{pmatrix} \sigma^2 + \frac{s^4}{m} - s^2 + \tilde{\nu}^{-2}Q_0(z,D_z) - E & u_0 \\ \langle \cdot, u_0\rangle_z & 0 \end{pmatrix}. \tag{6.6}$$

Notice that, since $Q_0$ is a non-negative operator, the presence of $\tilde{\nu}^{-2}$ in front of it does not create problems (and actually makes it even 'more elliptic'). Then it is straightforward to verify that for all $(s,\sigma) \in \mathbb{R}^2$ and for $E < \tilde{\nu}^{-2}(\lambda_0 + \mu_2 - \mu_1)$, the operator $a_E^0(s,\sigma)$ is a bijection from $\mathcal{D}_Q \oplus \mathbb{C}$ to $L^2(\mathbb{R}_z^{m-1}) \oplus \mathbb{C}$. Moreover, its inverse is given by (using the same notations as in Section 4, apart $\Pi$ which here acts on $L^2(\mathbb{R}_z^{m-1})$) :

$$a_E^0(s,\sigma)^{-1}$$

$$= \begin{pmatrix} \Pi^\perp(\sigma^2 + \frac{s^4}{m} - s^2 + \tilde{\nu}^{-2}Q_0' - E)^{-1}\Pi^\perp & u_0 \\ \langle \cdot, u_0\rangle_z & E - (\sigma^2 + \frac{s^4}{m} - s^2 + \tilde{\nu}^{-2}\mu_1) \end{pmatrix}$$

where $Q_0'$ denotes the restriction of $Q_0$ on $\mathrm{Ran}\Pi^\perp$. In particular, we see on these formulas that both $a_E^0(s,\sigma)$ and $a_E^0(s,\sigma)^{-1}$ admit good symbol estimates (e.g. in the sense of [GMS]). As a consequence, the semiclassical pseudodifferential calculus with operator-valued symbols can be applied (see e.g. [BK] or the appendix of [GMS]), and show that $\mathcal{A}_E$ is invertible and its inverse can be written as :

$$\mathcal{A}_E^{-1} = \mathrm{Op}_{\tilde{h}}^W(b_E) + \mathcal{O}(\tilde{h}^\infty) \ \text{ in } \mathcal{L}(L^2(\mathbb{R}^m \oplus \mathcal{F}; L^2(\mathbb{R}^m) \oplus L^2(\mathbb{R})) \ ,$$

where the principal part of $b_E(s,\sigma;\tilde{h})$ is $a_E^0(s,\sigma)^{-1}$.
Moreover, for all $s \in \mathbb{R}$, $b_E(s,\sigma;\tilde{h})$ can be extended holomorphically with respect to $\sigma$ in :

$$\{\sigma \in \mathbb{C} \ ; \ |\mathrm{Im}\,\sigma|^2 < \tilde{\nu}^{-2}\mu_2 + \frac{s^4}{m} - s^2 - E\}$$

and this also permits to perform as in [Ma] (Section 3) or [KMSW] a *formal* pseudodifferential calculus acting on formal WKB-type series of the form :

$$e^{-\psi(s)/\tilde{h}}c(s,\theta;\tilde{h}) = e^{-\psi(s)/\tilde{h}}\sum_{k\geq 0} c_k(s,\theta)\tilde{h}^{k/2} \tag{6.7}$$

where $\psi$ is any smooth function satisfying :

$$(\psi'(s))^2 < \tilde{\nu}^{-2}\mu_2 + \frac{s^4}{m} - s^2 - E.$$

In particular, writing :

$$\mathrm{Op}_{\tilde{h}}^W(b_E) = \begin{pmatrix} H_E & H_E^+ \\ H_E^- & E - H_E^\pm \end{pmatrix}$$

one has at a formal level :

$$\begin{aligned} &(A(h,\nu) - E)(e^{-\psi(s)/\tilde{h}}c(s,\theta;\tilde{h})) = 0 \\ \Longleftrightarrow &\left\{ \begin{array}{l} \text{There exists } \alpha = \alpha(s;\tilde{h}) \text{ such that} \\ (H_E^\pm - E)(e^{-\psi(s)/\tilde{h}}\alpha(s;\tilde{h})) = 0 \ \text{ and} \\ e^{-\psi(s)/\tilde{h}}c(s,\theta;\tilde{h}) = H_E^+(e^{-\psi(s)/\tilde{h}}\alpha(s;\tilde{h})). \end{array} \right. \end{aligned}$$

Notice also that the principal part of $H_E^{\pm}$ is a Schrödinger operator with symmetric double well potential : $\frac{1}{m}s^4 - s^2 + \tilde{\nu}^{-2}\mu_1$. Then, since the two wells are $\pm\sqrt{\frac{m}{2}}$, if we denote $A_0(h,\nu)$ the Dirichlet realization of $A(h,\nu)$ e.g. on $(-\infty, \frac{\sqrt{m}}{2}]$ (so that $A_0(h,\nu)$ is a single-well problem), and if we use the same arguments as in [Ma] Sections 4 and 5, one can show the following properties :

- The lowest eigenfunction of $A_0(h,\nu)$ admits, as $\tilde{h} \to 0$, a WKB asymptotics of the form (6.7) with $\psi(s) = \left| \int_{-\sqrt{m/2}}^{s} \left( \frac{1}{m}t^4 - t^2 + \frac{m}{4} \right)^{1/2} dt \right| = \left| \frac{m}{3\sqrt{2}} - \frac{s}{\sqrt{m}} \left( \frac{s^2}{3} - \frac{m}{2} \right) \right|$, for $s$ in any compact subset of $(-\infty, \frac{\sqrt{m}}{2})$ ;

- The gap $\Delta E$ between the first two eigenvalues of $A(h,\nu)$ admits, as $\tilde{h} \to 0$, an asymptotic expansion of the form :

$$\Delta E \sim \tilde{h}^{1/2} e^{-S_m/\tilde{h}} \sum_{k=0}^{\infty} E_k \tilde{h}^{k/2} \tag{6.8}$$

  where $S_m = \psi(\sqrt{m/2}) = mS_1$, $S_1$ is as in (3.21), and $E_k$ $(k \geq 0)$ are $\tilde{h}$-independent real numbers with $E_0 > 0$.

Summing up, using Lemma 6.1 we can conclude this section by stating :

**Proposition 6.2** *If $C_1 > 0$ is fixed arbitrarily and $C \geq 1$ is fixed large enough, then for $-C_1\sqrt{h} \leq \nu \leq -Ch^{2/3}$ the gap $\Delta\lambda$ between the first two eigenvalues of $P(h,\nu)$ satisfies :*

$$\Delta\lambda = |\nu|^{5/4}\sqrt{h}E_0(m)e^{-mS_1|\nu|^{3/2}/h}\left[1 + \mathcal{O}\left(\frac{\sqrt{h}}{|\nu|^{3/4}}\right)\right]$$

*uniformly with respect to $h$ and $\nu$, where $E_0(m) > 0$ depends only on $m$.*

**Remark 6.3** Taking $\nu = -Ch^{2/3}$ in this result, one recovers the behaviour in $h^{4/3}$ of $\Delta\lambda$ that we had in the previous section.

# 7 A model in large dimension

In this section we assume $\nu \geq 0$, and in view of Proposition 4.2, we decide to forget the remainder term $R_\lambda$ in the expression of $A(\lambda)$, and therefore to limit our study to the operator $A_0$. Of course, when the dimension becomes large, it is not clear that $R_\lambda$ remains small. However (and since not much can be said about it), the study of $A_0$ should permit at least to make ourselves an idea of which kind of problems can be encountered as $m \to +\infty$.

Since we are mainly interested in the behaviour of the gap between the first two eigenvalues of $A_0$, let us first observe that the values of $\mu_1$ and $\rho_2$ are completely irrelevant. Therefore, it is worth-while to work directly with :

$$A_1 := A_0 - \mu_1 - h\rho_2 = -h^{1/2}\partial_t^2 + \frac{1}{m}t^4 + \frac{\nu}{\sqrt{h}}t^2 + \frac{6\sqrt{h}}{m}t^2\rho_1. \tag{7.1}$$

At first, let us compute $\rho_1$ more precisely. If we denote by

$$\lambda_j = \lambda_j(m,\nu) := \left(\nu + 4\mathcal{J}\sin^2\frac{\pi(j-1)}{m}\right)^{1/2}$$

then, for $z = (z_2, ..., z_m) \in I\!R^{m-1}$, we know that

$$u_0(z) = c_m \prod_{j=2}^{m} e^{-\lambda_j z_j^2/2}$$

with

$$c_m = \pi^{-(m-1)/4}\prod_{j=2}^{m}\lambda_j^{1/4}.$$

Therefore, using (4.12) we find :

$$\rho_1 = c_m^2\sum_{k=2}^{m}\int_{I\!R} z_k^2 e^{-\lambda_k z_k^2}dz_k \prod_{j\neq k}\int_{I\!R} e^{-\lambda_j z_j^2}dz_j = \pi^{-1/2}\sum_{k=2}^{m}\sqrt{\lambda_k}\int_{I\!R} z_k^2 e^{-\lambda_k z_k^2}dz_k$$

that is :

$$\rho_1 = \frac{1}{2}\sum_{k=2}^{m}\frac{1}{\lambda_k}. \tag{7.2}$$

In particular as $m \to +\infty$, the ratio $\rho_1/m$ behaves like :

$$\frac{\rho_1}{m} \sim \frac{1}{2}\int_{\pi/m}^{(m-1)\pi/m}\frac{dt}{\sqrt{\nu + 4\mathcal{J}\sin^2 t}} = \int_{\pi/m}^{\pi/2}\frac{dt}{\sqrt{\nu + 4\mathcal{J}\sin^2 t}} \quad (m \to +\infty)$$

that is, more precisely :

$$\frac{\rho_1}{m} = \int_{\pi/m}^{\pi/2}\frac{dt}{\sqrt{\nu + 4\mathcal{J}\sin^2 t}} + \mathcal{O}(1)$$

uniformly with respect to $m \geq 2$ and $\nu \geq 0$. Therefore, we also have :

$$\begin{aligned}\frac{\rho_1}{m} &= \int_{\pi/m}^{\pi/2}\frac{\cos t}{\sqrt{\nu + 4\mathcal{J}\sin^2 t}}dt + \mathcal{O}(1)\\ &= \left[\frac{1}{2\sqrt{\mathcal{J}}}\ln\left(2\sqrt{\mathcal{J}}\sin t + \sqrt{\nu + 4\mathcal{J}\sin^2 t}\right)\right]_{\pi/m}^{\pi/2} + \mathcal{O}(1)\end{aligned}$$

and thus :

$$\frac{\rho_1}{m} = \frac{1}{2\sqrt{\mathcal{J}}}\ln\left(\frac{m}{2\pi\sqrt{\mathcal{J}} + \sqrt{\nu m^2 + 4\pi^2\mathcal{J}}}\right) + \mathcal{O}(1). \tag{7.3}$$

If one studies the expression appearing in (7.3) by separating the two cases $\nu m^2 \geq 1$ and $\nu m^2 \leq 1$, one can easily see that for some constant $C_{\mathcal{J}} \geq 1$ depending only on $\mathcal{J}$, one has :

$$\frac{1}{C_{\mathcal{J}}}\ln\left(\mathrm{Min}(\frac{1}{\sqrt{\nu}}\ ,\ m)\right) \leq \frac{\rho_1}{m} \leq C_{\mathcal{J}}\ln\left(\mathrm{Min}(\frac{1}{\sqrt{\nu}}\ ,\ m)\right) \tag{7.4}$$

for all $m \geq 2$ and $\nu \geq 0$. In particular, $\rho_1/m$ behaves like $\ln m$ if $\nu = 0$, and like $|\ln\nu|$ if $\nu > 0$ (possibly depending on $h$) and $m$ is large enough.

Now, let us come back to the operator $A_1$ of (7.1), and denote by :

$$\tilde{\rho} = \frac{6\rho_1}{m} \quad ; \quad \tilde{\nu} = \frac{\nu}{\sqrt{h}}$$

(7.5)

so that $A_1$ becomes :

$$A_1 = -h^{1/2}\partial_t^2 + \frac{1}{m}t^4 + (\tilde{\nu} + \tilde{\rho}\sqrt{h})t^2.$$

Making the change of variable :

$$t = \left(m(\tilde{\nu} + \tilde{\rho}\sqrt{h})\right)^{1/2} s$$

we get :

$$A_1 = m(\tilde{\nu} + \tilde{\rho}\sqrt{h})^2\left[\frac{-\sqrt{h}}{m^2(\tilde{\nu} + \tilde{\rho}\sqrt{h})^3}\partial_s^2 + s^4 + s^2\right]. \tag{7.6}$$

Finally, applying a well-known semiclassical result (see e.g. [HeSj1]), and noticing that, by (7.4) and (7.5), the quantity $\tilde{\nu} + \tilde{\rho}\sqrt{h}$ behaves like :

$$h^{-1/2}\left(\nu + h\ln(\mathrm{Min}(\nu^{-1/2}, m))\right),$$

we deduce from the previous analysis :

**Proposition 7.1** *There exists a constant $C_{\mathcal{J}} > 0$ such that if $m \geq 2$ and :*

$$\frac{h^2}{m^2\left(\nu + h\ \ln(\mathrm{Min}(\nu^{-1/2}, m))\right)^3} \leq \frac{1}{C_{\mathcal{J}}} \tag{7.7}$$

*then the gap $\Delta E$ between the first two eigenvalues of the operator $A_0$ introduced in Proposition 4.2 satisfies :*

$$\frac{1}{C_{\mathcal{J}}}\left(\nu + h\ln(\mathrm{Min}(\nu^{-1/2}, m))\right)^{1/2} \leq \Delta E \leq C_{\mathcal{J}}\left(\nu + h\ln(\mathrm{Min}(\nu^{-1/2}, m))\right)^{1/2}. \tag{7.8}$$

**Remark 7.2** Using Proposition 4.2, the previous result suggests that the gap between the first two eigenvalues of $P(h,\nu)$ should behave like :

$$h\sqrt{\nu + h\ln(\min(\nu^{-1/2}, m))}.$$

This seems in rather good accordance with (2.13), although most probably some restrictions has to be made on the dimension $m$ in order to make such a result true (see also the next remark).

**Remark 7.3** If for instance we try to apply the previous result with $\nu = 0$, then by (7.4) the condition (7.7) means that $m^2h(\ln m)^3$ must be large enough. In this case, (7.8) says that the gap behaves like $\sqrt{h\ln m}$. In particular, in the parameters region $m = \mathcal{O}(h^{-N})$ for some $N \geq 1$ (or equivalently $h = \mathcal{O}(m^{-1/N})$), this gap goes to 0 as $m \to +\infty$. On the contrary, if $m$ goes to infinity while $h$ is keeped fixed, then the gap increases like $\sqrt{\ln m}$. In this latter case, however, most probably the approximation we have made (by forgetting the remainder $R_\lambda$ appearing in Proposition 4.2) has no reason at all to be valid.

**Remark 7.4** On the contrary, if $\nu$ is positive (possibly depending on $h$), our result says that if $m$ is large enough (in the sense that $m \geq \nu^{-1/2}$ and $m^2(\nu + h\,|\ln\nu|)^3 >> h^2$), then the gap behaves like $\sqrt{\nu + h\,|\ln\nu|}$. For $\nu > 0$ fixed, this does not tell us more than the universal estimate (2.13). However, if $\nu = \nu(h)$ is such that $h\,|\ln\nu| >> \nu$ (e.g. if $\nu \leq h$), then our result suggests that (2.13) is no more optimal in this case.

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