CIRCLE PACKING IN THE HYPERBOLIC PLANE

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By a *circle packing* we will mean is a finite collection of circles of some fixed diameter in a Riemannian manifold such that no two circles overlap except possibly in their boundaries. A circle packing is *optimal* if it maximizes the number of touches (between circles) for a fixed number of circles. Heitmann and Radin [1] showed, following Harborth [2], that any optimal circle packing in the Euclidean plane has all of its centers located on the vertices of a triangulation of the Euclidean plane by congruent equilateral triangles. This paper will show that the same result holds in the hyperbolic plane if the diameter d is such that an equilateral triangle in the hyperbolic plane of side length d has each of its angles is equal to $\frac{2\pi}{N}$ for some $N \in \{7, 8, 9, \dots\}$.

From now on fix such a diameter d. Let ρ be the distance function in $\mathbb{H}^2$ (the hyperbolic plane). An *admissible graph* G, is a graph in $\mathbb{H}^2$ such that if v, w are vertices of G then $\rho(v, w) \geq d$ and the set of edges of G is the set of geodesic arcs between any two vertices v, w such that $\rho(v, w) = d$. An *optimal graph* G is an admissible graph such that the number of edges of G is at least as large as the number of edges of H for any admissible graph H with the number of vertices of G equal to the number of vertices of H. There is a bijection between circle packings (of diameter d) in $\mathbb{H}^2$ and admissible graphs which maps the centers of a given circle packing bijectively onto the vertices of an admissible graph. This bijection restricts to a bijection between optimal

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circle packings and optimal graphs. Let α be an interior angle of an equilateral triangle of side length d. The main result follows immediately from

Theorem 1. *If* $\frac{2\pi}{\alpha} \in \{7, 8, \dots\}$ *then*

1. *All the faces of any optimal graph are triangles.*
2. *All spiral graphs are optimal.*

Here, a *face* of an admissible graph G is the closure of a bounded component of $\mathbb{H}^2 - G$. Spiral graphs are defined later.

Notation. Let G be a finite graph of $\mathbb{H}^2$. Define $V(G) = \{\text{vertices of } G\}$, $E(G) = \{\text{edges of } G\}$, $F(G) = \{\text{faces of } G\}$. Let $f \in F(G)$. Define $A(f)$ to be the hyperbolic area of f, and $A(G) = \sum_{f \in F(G)} A(f)$. ∂G will be the subgraph of G contained in the closure of the unbounded component of $\mathbb{H}^2 - G$. For any finite set S, let $|S|$ be the number of elements of S. Let $v \in \partial G$. If $|\{e \in E(G) \mid e \cap \{v\} \neq \emptyset\}| = 1$ then define $a(v) = 0$. Otherwise let $\{e_1, e_2\} = \{e \in E(\partial G) \mid e \cap \{v\} \neq \emptyset\}$ and define $a(v)$ to be the angle between e_1 and e_2 interior to the face of G containing v. Define α to be an angle interior to an equilateral triangle of side length d. Let $N = 2\pi/\alpha$. From now on, assume $N \in \{7, 8, 9, \dots\}$. For any sets $A, B \in \mathbb{H}^2$, let $\rho(A, B) = \inf\{\rho(a, b) \mid a \in A,\ b \in B\}$.

Spiral Graphs.

Let S_n denote the *spiral* graph on n vertices constructed as follows:

Let $v_1, \dots, v_{N+1}$ be distinct points in $\mathbb{H}^2$ such that $\rho(v_1, v_i) = d$ for $2 \le i \le N+1$ and $\rho(v_i, v_{i+1}) = d$ for $2 \le i \le N$. Let S_n (for $1 \le n \le N+1$) be the admissible graph with vertex set $\{v_i\}_{i=1}^n$. Suppose S_n ($n \ge N+1$) has been constructed in such a way that properties a–g (below) are satisfied.

a) All the faces of S_n are triangles.

$V(S_n)$ may be ordered $(v_1, \dots, v_n)$ such that

b) For all $m < n$, $V(S_m) = \{v_1, \dots, v_m\}$
c) For $1 \le i \le n-1$ there exists an edge in $E(S_n)$ with endpoints v_i and v_{i+1}.

Let m be the smallest integer such that there exists $x \in \mathbb{H}^2 - S_n$ with $d = \rho(v_m, x)$ and

$\rho(v, x) \geq d$ for all $v \in V(S_n)$. Then

d) $\{V_m, \ldots, V_n\} = V(\partial S_n)$.
e) $2\alpha \leq a(v_m) \leq 2\pi - 2\alpha$.
f) For $m < i < n$, $2\alpha \leq a(v_i) \leq 3\alpha$.
g) $\alpha \leq a(v_n) \leq 2\alpha$ and $a(v_n) = 2\alpha$ if and only if $a(v_n) = 2\pi - 2\alpha$.

$|v_n - v_m| = d$ by properties c) and d).

Let r_n, r_{m+1} be the half lines with endpoint v_m which contain v_n, v_{m+1} respectively. By f) and g), S_n is contained in a single component of $\mathbb{H}^2 - (r_n \cup r_{m+1})$. By e), there exists a unique point $v_{n+1} \in \mathbb{H}^2$ such that $\rho(v_m, v_{n+1}) = \rho(v_n, v_{n+1}) = \rho(V(S_n), v_{n+1}) = d$. Let S_{n+1} be the admissible graph with vertex set $\{v_1, \ldots, v_{n+1}\}$. a)–g) are seen to hold.

Before proving Theorem 1 we require two technical lemmas.

Lemma 1. *Let G be an admissable graph. Let $\ell_G = \min\{\rho(e, v) \mid e \in E(G),\, v \in V(G)$ and $v \notin e\}$. Let $\ell = \inf\{\ell_G \mid G$ is an admissable graph$\}$. Then ℓ is the length of the arc depicted in Figure 1:*

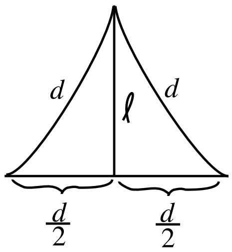


FIGURE 1

Proof. Let G be an admissible graph. Suppose that there exists $(v, e) \in V(G) \times E(G)$ such that $v \notin e$ and $\rho(v, e) < \ell$. Let v_1, v_2 be the endpoints of e. Without loss of generality, assume $\rho(v_1, v) \leq \rho(v_2, v)$. Let $p \in e$ such that $\rho(p, v) \leq \rho(x, v)$ for all $x \in e$. Then $\rho(p, v) < \ell < d < \rho(v, v_2)$. So there exists at point $v_2' \in e$ such that p is in the arc from v_1 to v_2' and $\rho(v, v_2') = \rho(v, v_1)$. See Figure 2. Let m be the midpoint of the arc from v_1 to v_2'.

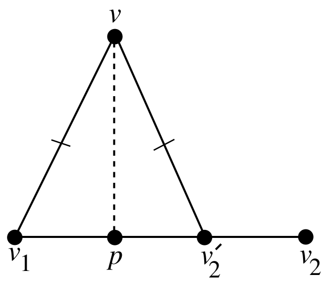


FIGURE 2

Without loss of generality assume p is in the arc from v_1 to m. By symmetry there exists a point p' on the arc from m to v_2' such that $\rho(v,p) = \rho(v,p')$. See Figure 3. By the law of sines,

$$(*) \qquad \frac{\sinh \rho(m,v)}{\sin a(vpm)} = \frac{\sinh \rho(v,p)}{1}$$

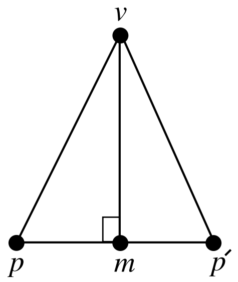


FIGURE 3

$$\Longrightarrow \quad \frac{\sinh \rho(m,v)}{\sinh \rho(v,p)} = \sin a(vpm) < 1$$
$$\Longrightarrow \quad \rho(m,v) < \rho(v,p) \Longrightarrow m = p\ .$$

By the law of cosines,

$$0 = \cos\frac{\pi}{2} = \frac{\cosh \rho(v,p)\cosh \rho(p,v_2') - \cosh \rho(v,v_2')}{\sinh \rho(v,p)\sinh \rho(p,v_2')}$$
$$\Rightarrow \cosh \rho(v,p) = \frac{\cosh \rho(v,v_2')}{\cosh \rho(p,v_2')}\ .$$

So $\rho(v,p)$ is minimized when $\rho(p,v_2')$ is maximized and $\rho(v,v_2')$ is minimized. This occurs precisely when $\rho(p,v_2') = d/2$ and $\rho(v,v_2') = d$, i.e., when $v_2 = v_2'$, vv_1v_2 form an equilateral triangle and $\rho(v,p) = \ell$. $\square$

Lemma 2. *Let G be an admissible graph. Let $f \in F(G)$. If f has n sides then $A(f) \geq (\frac{n}{3})(\pi - 3\alpha)$.*

Proof. Let $e_1, e_2 \in E(f)$ be different edges with endpoints v_1, v_2 and w_1, w_2 respectively. Let $v_3, w_3 \in \mathbb{H}^2$ such that the triangles $v_1v_2v_3$, $w_1w_2w_3$ have interior angles at v_1, v_2, w_1 and w_2 equal to $\alpha/2$.

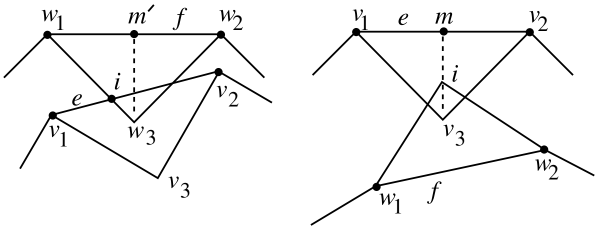


FIGURE 4

Suppose for a contradiction that the triangles $v_1v_2v_3$ and $w_1w_2w_3$ intersect in their interiors. See Figure 4. Since $\text{Int } e_1 \cap \text{Int } e_2 = \emptyset$ this implies that the triangle $v_1v_2v_3$ intersects $w_1w_2w_3$ in the arcs w_1w_3 and w_2w_3. Let m, m' be the midpoints of e_1 and e_2 respectively. Then $v_1v_2v_3$ intersects the arc $m'w_3$. If v_1v_2 intersects $m'w_3$ then without loss of generality $v_1v_2 \cap w_1w_3 \neq \emptyset$. Let $\{i\} = v_1v_2 \cap w_1w_3$. Then $\rho(w_1, e_1) \leq \rho(w_1, i) < \rho(w_1, w_3) < \ell$. This contradicts the previous lemma. So $v_1v_2 \cap m'w_3 = \emptyset$. So either v_1v_3 or v_2v_3 intersects $m'w_3$. Without loss of generality v_1v_3 intersects $m'w_3$. Let $\{i\} = v_1v_3 \cap m'w_3$. Then $\rho(v_1, e_2) \leq \rho(v_1, i) + \rho(i, m') < \rho(v_1, v_3) + \rho(w_3, m') = \ell$ contradicting the previous lemma. Thus $v_1v_2v_3$ and $w_1w_2w_3$ do not intersect in their interiors.

For each edge e of f form the triangle $v_1v_2v_3$ as above so that this triangle lies in the interior of f. Since none of these triangles overlap and each has area $= \frac{1}{3}(\pi - 3\alpha)$, the lemma is proved.

Proof of Theorem 1. If G is a graph of one vertex, then G is spiral and each face of G is a triangle. Let $n \geq 1$. Assume for induction that if G is an admissible graph such that $|V(G)| \leq n$ then if G is spiral then G is optimal and if G is optimal then each face of G is a triangle.

Let O be an optimal graph and let S be a spiral graph such that $|V(O)| = |V(S)| = n + 1$. Define F_k as the number of k-gon faces of O. Then $\sum_{k-3}^{\infty} kF_k = 2(|E(O)| - |E(\partial O)|) + |E(\partial O)|$. Combined with the Euler characteristic formula $V - E + F = 1$, this gives:

$$3|V(O)| - 3 - \left(|E(\partial O) + \sum_{k=4}^{\infty} (k-3) F_k \right) \tag{1}$$
$$= |E(O)| \geq |E(S)| = 3|V(S)| - 3 - |E(\partial S)| \, .$$

The last equality holds because all faces of S are triangles. Thus $|E(\partial O)| \le |E(\partial S)|$ with equality if and only if S is optimal and every face of O is a triangle. Assume for a contradiction that $|E(\partial O)| < |E(\partial S)|$.

Let O' be the admissible graph with $V(O') = V(O) - V(\partial O)$. If O' is empty then by (1) the theorem is true. So assume O' is not empty. Order the vertices of S, $(v_1, \dots, v_n)$ as in the definition of spiral. Let S' be the admissible graph with $V(S') = \{v_1, \dots, v_s\}$ and $s = |V(O')|$. Then S' is spiral. By induction, S' is optimal so $|E(O')| \le |E(S')|$.

By Gauss-Bonet,

$$A(O) - A(S) = \sum_{v \in \partial O} \pi - a(v) - \sum_{w \in \partial S} \pi - a(w) \tag{2}$$

$$\sum_{v \in \partial O} \pi - a(v) \le |E(\partial O)|\pi - (|E(O)| - |E(O')|)\alpha \tag{3}$$

because there are at least $|E(O)| - |E(O')|$ pairs of adjacent edges touching ∂O and each pair contributes at least α to the sum.

$$\sum_{w \in \partial S} \pi - a(w) = \sum_{w \in \partial S - \partial S'} \pi - a(w) + \sum_{w \in \partial S \cap \partial S'} \pi - a(w) \tag{4}$$

$$= \pi|E(\partial O)| - (|E(S)| - |E(S')| - 1)\alpha + \sum_{w \in \partial S \cap \partial S'} \pi - a(w) \tag{5}$$

because $|V(\partial S) - V(\partial S')| = |E(\partial O)|$, there are exactly $(|E(S)| - |E(S')| - 1)$ pairs of adjacent edges radiating from the vertices in $V(\partial S) - V(\partial S')$ (since if $a(v_1) = \alpha$ for $v_i \in \partial S$ then $i = n$ and the edge connecting v_{n-1} to v_m is not counted twice), and each pair contributes exactly α to the sum since every face of S is a triangle.

$$\sum_{w \in \partial S \cap \partial S'} \pi - a(w) \ge \pi - (2\pi - 2\alpha) + (\pi - 3\alpha)\Big[|E(\partial S)| - |E(\partial O)| - 1\Big] \tag{6}$$

since $a(v_m) \le (2\pi - 2\alpha)$ and $a(v_i) \le 3\alpha$ for $m < i < n$, and $|V(\partial S) \cap V(\partial S')| = |E(\partial S)| - |E(\partial O)|$.

By lemma 2,

$$\begin{aligned} A(O) - A(S) &= \sum_{f \in F(O)} A(f) - \sum_{f \in F(S)} A(f) \\ &\ge (|F(O)| - |F(S)|)(\pi - 3\alpha) \\ &= (|E(O)| - |E(S)|)(\pi - 3\alpha) \end{aligned} \tag{7}$$

and equality holds if and only if all faces of O are triangles. The last equality holds from the Euler characteristic formula.

Equations (2)–(7) and $\alpha = 2\pi/N$ imply

$$\begin{aligned}(8)\quad & |E(O)| - |E(S)| + |E(S')| - |E(O')| \\ & \leq N - 6 - \left(\frac{N-6}{2}\right)(|E(\partial S)| - |E(\partial O)|) - \left(\frac{N-6}{2}\right)(|E(O)| - |E(S)|)\end{aligned}$$

with equality holding only if all faces of O are triangles. Hence $|E(\partial S)| - |E(\partial O)| \leq 2$ with equality holding only if $|E(O)| = |E(S)|$ and all faces of O are triangles. This contradicts (1). So we may assume $|E(\partial S)| - |E(\partial O)| = 1$.

By (8),

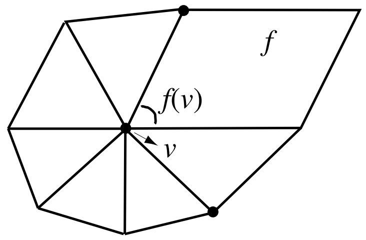
$$(9) \qquad (|E(O)| - |E(S)|)\left(\frac{N-4}{2}\right) \leq \frac{N-6}{2} + |E(O')| - |E(S')| \; .$$

since $|E(O')| \leq |E(S')|$, $|E(O)| = |E(S)|$, i.e., S is optimal.

By (1), there exists exactly one nontriangular face, f of O and it is a 4-gon. Suppose, for a contradiction, that there exists a vertex v in f which is not in ∂O. Let k be the number of triangular faces of O containing v. Since v is in the interior of O,

$$(10) \qquad f(v) = 2\pi - k\alpha = (N-k)\alpha$$

where $f(v) =$ the interior angle of f at v. But this implies



that two opposite vertices of f are distance less than or equal to d apart, contradicting that O is optimal. Thus all vertices of f are in ∂O. Thus any vertex in the interior of O is contained in N edges of O. For any admissible graph G, let $\hat{E}(G) = \{e \in$

$E(G)|e \cap V(\partial G) \neq \emptyset\}$. Then,

$$N(|V(O) - V(\partial O)|) = 2(|E(O) - \hat{E}(O)|) + |\hat{E}(O) - E(\partial O)| \tag{11}$$
$$= 2|E(O)| - |\hat{E}(O)| - |E(\partial O)|$$
$$= 2|E(S)| - |\hat{E}(O)| - (|E(\partial S)| - 1)$$
$$N(|V(O)| - |V(\partial O)|) = N(|V(S)| - (|V(\partial S)| - 1)) \tag{11a}$$
$$= 2|E(S)| - |\hat{E}(S)| - |E(\partial S)| + N \ .$$

So,

$$|\hat{E}(O)| = |\hat{E}(S)| - N + 1 \ . \tag{12}$$

Every edge of $\hat{E}(S)$ which is in $E(S')$ is connected to the unique vertex $v \in V(\partial S) \cap V(\partial S')$. Since $v \in \partial S$, at least 2 edges radiating from v touch other vertices on ∂S and are therefore not in S'. Hence,

$$|\hat{E}(S)| \leq |E(S)| - |E(S')| + N - 3 \ . \tag{13}$$

(12) and (13) imply

$$|E(O')| = |E(O)| - |\hat{E}(O)| = |E(S)| - |\hat{E}(S)| + N - 1 \geq |E(S')| + 2 \ .$$

This contradicts that S' is optimal. Thus $|E(\partial O)| = |E(\partial S)|$ and by (1), the theorem is proved. □

Acknowledgements

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2. H. Harborth, Elem. Math. **29** (1974), 14–15.