Centre de Physique Théorique*, CNRS Luminy, Case 907
F-13288 Marseille – Cedex 9

An exactly soluble model with non-conventional and non-extensive condensations

J.-B. BRU[1] and V.A. ZAGREBNOV[2]

Abstract

We present a new exactly soluble model which is motivated by the Michoel-Schröder-Verbeure model [1] and our model [2]. It manifests two kinds of condensations which occur in two stages: for intermediate densities one has a *non-conventional* Bose condensation in the mode $k = 0$ due to a diagonal perturbation of the Imperfect Bose Gas Hamiltonian, whereas for large densities ρ it coexists with *conventional (generalized, non-extensive)* Bose-Einstein condensation corresponding to the standard mechanism of saturation.

Keywords : conventional/non-conventional Bose-condensations, generalized (non-extensive) Bose-Einstein condensation , (Weakly) Imperfect Bose Gas

anonymous ftp : ftp.cpt.univ-mrs.fr
web : www.cpt.univ-mrs.fr
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*Unité Propre de Recherche 7061
[1]Allocataire de recherche MRT - Email : bru@cpt.univ-mrs.fr
[2]Université de la Méditerranée (Aix-Marseille II) - Email : zagrebnov@cpt.univ-mrs.fr

1. Introduction

Let us consider first the Imperfect Bose Gas (IBG) model introduced by Huang [3]. It is the system of bosons of mass m enclosed in a cube $\Lambda \subset \mathbb{R}^d$ of volume $V = |\Lambda|$ centered at the origin defined by the Hamiltonian:

$$H_{\Lambda}^{IBG} = \sum_{k \in \Lambda^*} \varepsilon_k a_k^* a_k + \frac{\lambda}{V} N_{\Lambda}^2, \ \varepsilon_k \equiv \hbar^2 k^2 / 2m, \ \lambda > 0, \tag{1.1}$$

where $N_{\Lambda} = \sum_{k \in \Lambda^*} a_k^* a_k \equiv \sum_{k \in \Lambda^*} N_k$ is the particle-number operator and $\varepsilon_k$correspond to the one-particle kinetic-energy.

Here $\left\{ a_k^{\#} \right\}_{k \in \Lambda^*}$ are the boson creation/annihilation operators in the boson Fock space $\mathcal{F}_{\Lambda}$ over $L^2(\Lambda)$ corresponding to the second quantization in the box $\Lambda = \underset{\alpha=1}{\overset{d}{\times}} L$ with periodic boundary conditions, i.e. to the dual

$$\Lambda^* = \left\{ k \in \mathbb{R}^d : \ k_{\alpha} = \frac{2\pi n_{\alpha}}{L}, \ n_{\alpha} = 0, \pm 1, \pm 2, ..., \ \alpha = 1, 2, ..., d \right\}.$$

Then for $d > 2$, for a given temperature $\theta = \beta^{-1}$ and for a total particle density $\rho > \rho_c^P(\theta)$ (here $\rho_c^P(\theta) \equiv \rho^P(\theta^{-1}, \mu = 0)$ where $\rho^P(\beta, \mu)$ is the total particle density of the Perfect Bose Gas (PBG) in the grand-canonical ensemble) the IBG manifests a Bose-Einstein condensation of the *type I* [4, 5, 6], i.e. a macroscopic occupation of only the ground-state level $k = 0$, see [7, 8] for classification of *conventional* Bose-Einstein condensations.

However in the recent paper [1] it is shown that in the IBG perturbed by the repulsive diagonal interaction

$$\widetilde{U}_{\Lambda} = \frac{\lambda}{2V} \sum_{k \in \Lambda^*} N_k^2, \ \lambda > 0, \tag{1.2}$$

the Bose-Einstein condensation which occurs again for densities $\rho > \rho_c^P(\theta)$ or $\mu > 2\lambda \rho_c^P(\theta) \equiv \mu_c^I(\theta)$ is spread out into the Bose-Einstein condensation of the *type III* (or *non-extensive* condensation), i.e. a condensation, when *no* levels are macroscopically occupied. This model for $\lambda > 0$ was introduced in [9]. Below we call it the Michoel-Schröder-Verbeure (MSV) model:

$$H_{\Lambda}^{MSV} \equiv H_{\Lambda}^{IBG} + \widetilde{U}_{\Lambda}. \tag{1.3}$$

Then the result [1] says that

$$\lim_{\Lambda} \frac{\langle N_k \rangle_{H_{\Lambda}^{MSV}}}{V} = 0, \ k \in \Lambda^*,$$

for any ρ, whereas one has the double limit

$$\lim_{\delta \to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*, 0 \leq \|k\| \leq \delta\}} \langle N_k \rangle_{H_{\Lambda}^{MSV}} = \rho - \rho_c^P(\theta) > 0,$$

for $\mu > \mu_c^I(\theta)$.

We denote by $\langle - \rangle_{H_\Lambda^{MSV}}(\beta,\mu)$, $\beta \geq 0$, $\mu \in \mathbb{R}^1$, the grand-canonical Gibbs state for the Hamiltonian H_Λ^{MSV}. Notice that the model $H_\Lambda^{IBG} - \widetilde{U}_\Lambda$ is rigorously studied in [10], where it was shown that Bose-Einstein condensation is of the type I only in the zero mode $k = 0$.

The fact that a gentle repulsive interaction can produce a generalized *non-extensive* Bose-Einstein condensation without transforming the corresponding pressure of the non-perturbed Hamiltonian H_Λ^{IBG} has been also shown in our recent paper [2]. This was done in the context of a system:

$$H_\Lambda^0 = \sum_{k \in \Lambda^* \backslash \{0\}} \varepsilon_k a_k^* a_k + \varepsilon_0 a_0^* a_0 + \frac{g_0}{2V} a_0^* a_0^* a_0 a_0, \tag{1.4}$$

with $\varepsilon_0 \neq \varepsilon_{k=0} \in \mathbb{R}^1$ and $g_0 > 0$ perturbed by the interaction

$$U_\Lambda = \frac{1}{V} \sum_{k \in \Lambda^* \backslash \{0\}} g_k(V) \, a_k^* a_k^* a_k a_k, \; 0 < g_- \leq g_k(V) \leq \gamma_k V^{\alpha_k}, \tag{1.5}$$

with $\alpha_k \leq \alpha_+ < 1$ and $0 < \gamma_k \leq \gamma_+$. This perturbation U_Λ (similar to the interaction $\widetilde{U}_\Lambda$ when $g_k = \lambda$) leads to the Hamiltonian [2]:

$$H_\Lambda^{BZ} \equiv H_\Lambda^0 + U_\Lambda. \tag{1.6}$$

In contrast to the MSV model, the grand-canonical pressure for our model (1.6)

$$p_\Lambda^{BZ}(\beta,\mu) \equiv p_\Lambda \left[H_\Lambda^{BZ}\right] \equiv \frac{1}{\beta V} \ln Tr_{\mathcal{F}_\Lambda} e^{-\beta\left(H_\Lambda^{BZ} - \mu N_\Lambda\right)} \tag{1.7}$$

is defined in thermodynamic limit only in domain $Q = \{\mu \leq 0\} \times \{\theta \geq 0\}$ and it is equal to

$$p^{BZ}(\beta,\mu) \equiv \lim_\Lambda p_\Lambda^{BZ}(\beta,\mu) = \lim_\Lambda p_\Lambda\left[H_\Lambda^0\right] = p^P(\beta,\mu) - \inf_{\rho_0 \geq 0} \left[(\varepsilon_0 - \mu)\,\rho_0 + \frac{g_0 \rho_0^2}{2}\right], \tag{1.8}$$

see [2]. Here $p^P(\beta,\mu)$ is the pressure of the PBG in thermodynamic limit. Notice that the pressure (1.8) is independent of parameters $\{g_k(V)\}_{k \in \Lambda^* \backslash \{0\}}$, i.e. of the interaction (1.5).

Remark 1.1. *Let domain D_{ε_0} be defined by*

$$D_{\varepsilon_0} \equiv \left\{(\theta,\mu) \in Q : p^P(\beta,\mu) < p^{BZ}(\beta,\mu)\right\}. \tag{1.9}$$

Then the thermodynamic limit (1.8) says that to insure $D_{\varepsilon_0} \neq \{\emptyset\}$, the ε_0 must be negative, i.e.

$$D_{\varepsilon_0} = \{(\theta,\mu) \in Q : \varepsilon_0 < \mu \leq 0\}. \tag{1.10}$$

Below we consider only the case $\varepsilon_0 < 0$, $d > 2$.

We denote by $\rho_\Lambda^{BZ}(\beta,\mu)$, the total particle density in the grand-canonical ensemble for the model H_Λ^{BZ}:

$$\rho_\Lambda^{BZ}(\beta,\mu) \equiv \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^{BZ}} (\beta,\mu)\,. \tag{1.11}$$

Then $\rho^{BZ}(\beta,\mu) \equiv \lim_\Lambda \rho_\Lambda^{BZ}(\beta,\mu)$ is the corresponding thermodynamic limit which, according to [2], is equal for $\varepsilon_0 < 0$ to:

$$\rho^{BZ}(\beta,\mu) = \rho^P(\beta,\mu)\,, \tag{1.12}$$

for $(\theta, \mu \leq \varepsilon_0)$,

$$\rho^{BZ}(\beta,\mu) = \rho^P(\beta,\mu) + \frac{\mu - \varepsilon_0}{g_0}, \tag{1.13}$$

for $(\theta, \varepsilon_0 < \mu < 0)$. Notice that for $d > 2$ one has a critical density

$$\rho_c^{BZ}(\theta) \equiv \sup_{\mu \leq 0} \rho^{BZ}\left(\theta^{-1},\mu\right) = \rho^{BZ}\left(\theta^{-1},\mu=0\right) = \rho_c^P(\theta) - \frac{\varepsilon_0}{g_0} < +\infty. \tag{1.14}$$

Proposition 1.2. *[2] Let $\rho > \rho_c^{BZ}(\theta)$ $(d > 2)$ and $0 < g_- \leq g_k(V) \leq \gamma_k V^{\alpha_k}$ for $k \in \Lambda^* \backslash \{0\}$, with $\alpha_k \leq \alpha_+ < 1$ and $0 < \gamma_k \leq \gamma_+$. Then for any $\varepsilon_0 < 0$ we have:*
(i) (even if $d < 3$) a condensation in the mode $k = 0$, i.e.

$$\begin{aligned} \rho_0^{BZ}(\theta,\mu) &\equiv \lim_\Lambda \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^{BZ}} \\ &= \left\{ \begin{array}{c} 0, \text{ for } (\theta,\mu) \in Q \backslash D_{\varepsilon_0} \\ \left(\frac{\mu - \varepsilon_0}{2g_0}\right), \text{ for } (\theta,\mu) \in D_{\varepsilon_0} \end{array} \right\}; \end{aligned} \tag{1.15}$$

(ii) for any $\varepsilon_0 \in \mathbb{R}^1$

$$\lim_\Lambda \left\langle \frac{a_k^* a_k}{V} \right\rangle_{H_\Lambda^{BZ}} = 0,\ k \in \Lambda^* \backslash \{0\}\,, \tag{1.16}$$

i.e. there is no macroscopic occupation of modes $k \neq 0$ but we have a generalized (non-extensive) Bose-Einstein condensation:

$$\lim_{\delta \to 0^+} \lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^*:\ 0 < \|k\| < \delta\}} \langle N_k \rangle_{H_\Lambda^{BZ}} = \rho - \rho_c^{BZ}(\theta) > 0 \tag{1.17}$$

which for $\varepsilon_0 < 0$ coexists with the condensation in the mode $k = 0$ if $(\theta,\mu) \in D_{\varepsilon_0}$.

Therefore, Proposition 1.2 implies the *coexistence* of *two kinds* of Bose condensations in the model (1.6) for densities $\rho > \rho_c^{BZ}(\theta)$:
- a Bose condensation in the single mode $k = 0$ due to the term $(\varepsilon_0 a_0^* a_0)$ which mimics for $\varepsilon_0 < 0$ an *attraction* by an external potential [11] giving rise to a *non-conventional* condensation of the *type I*;
- a *conventional* Bose-Einstein condensation due to the *saturation* of the total particle density where (similar to the MSV model) the *type III* of this Bose-Einstein condensation is due to the

elastic repulsive interaction U_Λ (1.5) of bosons in the modes $k \neq 0$. Therefore in contrast to the value of the pressure (1.8) the interaction (1.5) is decisive for formation of the non-extensive Bose-Einstein condensation in the model (1.6).

The aim of the present paper is to study the model (1.6) stabilized by the IBG-interaction (1.1) as it is in the MSV-Hamiltonian (1.3):

$$H_\Lambda = \sum_{k\in\Lambda^*\backslash\{0\}} \varepsilon_k a_k^* a_k + \varepsilon_0 a_0^* a_0 + \frac{g_0}{2V} N_0^2 + \frac{\lambda}{V} N_\Lambda^2 + \frac{g}{2V} \sum_{k\in\Lambda^*\backslash\{0\}} N_k^2, \tag{1.18}$$

with $\lambda > 0$, $g_0 > 0$, $g > 0$ but $\varepsilon_0 < 0$. Notice that the model H_Λ for $\varepsilon_0 = 0$ and for $\lambda = g = g_0$ coincides with the MSV model (1.3).
In Section 2 we show that this generalization of the models (1.3) and (1.6) can be solved exactly. Bose condensations which occur in the model (1.18) are described in Section 3. We reserve Section 4 for concluding remarks and discussions.

2. Thermodynamic study

2.1 First we consider our model (1.18) in the *canonical ensemble* (β, ρ). This simplifies essentially the thermodynamic study of the model. We denote by $f_\Lambda\left(\beta, \rho = \frac{n}{V}\right)$ the free-energy density of the model H_Λ, i.e.

$$f_\Lambda(\beta,\rho) \equiv -\frac{1}{\beta V} \ln Tr_{\mathcal{H}^n_{\Lambda,S}}\left(e^{-\beta H_\Lambda}\right), \tag{2.1}$$

where $\mathcal{H}^n_{\Lambda,S} \equiv S\left(\overset{n}{\underset{i=1}{\otimes}} L^2(\Lambda)\right)$ is the symmetrized n-particle Hilbert space.

Theorem 2.1. *Let $\lambda > 0$, $g > 0$, $g_0 > 0$and $\varepsilon_0 < 0$, then we have*

$$\begin{aligned} f(\beta,\rho) &\equiv \lim_\Lambda f_\Lambda(\beta,\rho) \\ &= \lambda\rho^2 + \inf_{\rho_0\in[0,\rho]} \left\{\varepsilon_0\rho_0 + \frac{g_0}{2}\rho_0^2 + f^P(\beta,\rho-\rho_0)\right\}, \end{aligned} \tag{2.2}$$

i.e. it is independent of g provided $g > 0$. Here $f^P(\beta,\rho)$ is the free-energy of the PBG in thermodynamic limit, i.e.

$$f^P(\beta,\rho) \equiv \lim_\Lambda f^P_\Lambda(\beta,\rho) \tag{2.3}$$

with

$$f^P_\Lambda(\beta,\rho) \equiv -\frac{1}{\beta V} \ln \sum_{\{n_k=0,1,2,\ldots\}_{k\in\Lambda^*}} e^{-\beta\left(\sum_{k\in\Lambda^*}\varepsilon_k n_k\right)} \delta_{\sum_{k\in\Lambda^*} n_k = [\rho V]}, \tag{2.4}$$

and $[x]$ equals to the integer part of $x \geq 0$.

Proof. By (1.18) and (2.1) we get

$$f_\Lambda(\beta,\rho) = -\frac{1}{\beta V}\ln\left\{\sum_{n_0=0}^{[\rho V]} e^{-\beta V h\left(\rho,\frac{n_0}{V}\right)}\right\} + \lambda\rho^2, \tag{2.5}$$

where

$$h_\Lambda(\rho,\rho_0) \equiv \varepsilon_0\rho_0 + \frac{g_0}{2}\rho_0^2 - \frac{1}{\beta V}\ln\sum_{\{n_k=0,1,2,\ldots\}_{k\in\Lambda^*\setminus\{0\}}} e^{-\beta\left(\sum_{k\in\Lambda^*\setminus\{0\}}\left[\varepsilon_k n_k + \frac{g}{2V}n_k^2\right]\right)}\delta_{\sum_{k\neq 0} n_k = [\rho V]-[\rho_0 V]}. \tag{2.6}$$

By (2.5) one obtains the estimate

$$\lambda\rho^2 + \inf_{\rho_0\in[0,\rho]} h_\Lambda(\rho,\rho_0) - \frac{1}{\beta V}\ln([\rho V]+1) \leq f_\Lambda(\beta,\rho) \leq \lambda\rho^2 + \inf_{\rho_0\in[0,\rho]} h_\Lambda(\rho,\rho_0),$$

which gives in thermodynamic limit:

$$f(\beta,\rho) \equiv \lim_\Lambda f_\Lambda(\beta,\rho) = \lambda\rho^2 + \lim_\Lambda \inf_{\rho_0\in[0,\rho]} h_\Lambda(\rho,\rho_0). \tag{2.7}$$

Notice that (2.6) can be rewritten as

$$h_\Lambda(\rho,\rho_0) = \varepsilon_0\rho_0 + \frac{g_0}{2}\rho_0^2 - \frac{1}{\beta V}\ln\left\langle e^{-\frac{\beta g}{2V}\sum_{k\in\Lambda^*\setminus\{0\}} n_k^2}\right\rangle_{\widetilde{H}_\Lambda^P}(\beta,\rho-\rho_0), \tag{2.8}$$

where $\langle - \rangle_{\widetilde{H}_\Lambda^P}(\beta,\rho-\rho_0)$ is the canonical Gibbs state for the PBG with *excluded* mode $k=0$ at the density $\rho-\rho_0$, with $f_\Lambda^P(\beta,\rho)$ defined by (2.4). Then the Jensen inequality

$$\left\langle e^{-\frac{\beta g}{2V}\sum_{k\in\Lambda^*\setminus\{0\}} n_k^2}\right\rangle_{\widetilde{H}_\Lambda^P} \geq e^{-\frac{\beta g}{2V}\left\langle\sum_{k\in\Lambda^*\setminus\{0\}} n_k^2\right\rangle_{\widetilde{H}_\Lambda^P}}$$

and (2.8) imply the estimate

$$\lim_\Lambda h_\Lambda(\rho,\rho_0) \leq \varepsilon_0\rho_0 + \frac{g_0}{2}\rho_0^2 + f^P(\beta,\rho-\rho_0). \tag{2.9}$$

Moreover, since

$$e^{-\frac{\beta g}{2V}n_k^2} \leq 1,$$

by (2.6) we have

$$h_\Lambda(\rho,\rho_0) \geq \varepsilon_0\rho_0 + \frac{g_0}{2}\rho_0^2 + f_\Lambda^P(\beta,\rho-\rho_0),$$

which together with (2.9) gives (2.2). ■

Remark 2.2. *Let denote by $f_{\Lambda}^{BZ}(\beta,\rho)$ the free-energy density corresponding to H_{Λ}^{BZ} (1.6) with $g_k(V)=g/2$, i.e.*

$$f_{\Lambda}^{BZ}(\beta,\rho)\equiv -\frac{1}{\beta V}\ln Tr_{\mathcal{H}^n_{\Lambda,S}}\left(e^{-\beta H_{\Lambda}^{BZ}}\right).$$

Then (1.6) (1.18) and (2.1) imply that

$$f_{\Lambda}(\beta,\rho)=\lambda\rho^2+f_{\Lambda}^{BZ}(\beta,\rho),$$

from which by Theorem 2.1 we deduce

$$f^{BZ}(\beta,\rho)\equiv \lim_{\Lambda} f_{\Lambda}^{BZ}(\beta,\rho)=\inf_{\rho_0\in[0,\rho]}\left\{\varepsilon_0\rho_0+\frac{g_0}{2}\rho_0^2+f^P(\beta,\rho-\rho_0)\right\}, \tag{2.10}$$

and

$$f(\beta,\rho)=\lambda\rho^2+f^{BZ}(\beta,\rho). \tag{2.11}$$

Remark 2.3. *Since the pressure $p^{BZ}(\beta,\mu)$ is a Legendre transform of the corresponding free-energy density $f^{BZ}(\beta,\rho)$, we get from (2.10) that*

$$\begin{aligned}
p^{BZ}(\beta,\mu) &= \sup_{\rho\geq 0}\left\{\mu\rho-f^{BZ}(\beta,\rho)\right\}\\
&= \sup_{\rho_0\geq 0}\left\{\sup_{\rho\geq\rho_0}\left\{\mu\rho_0-\varepsilon_0\rho_0-\frac{g_0}{2}\rho_0^2+\mu(\rho-\rho_0)-f^P(\beta,\rho-\rho_0)\right\}\right\}\\
&= \sup_{\rho_0\geq 0}\left\{p^P(\beta,\mu)-(\varepsilon_0-\mu)\rho_0-\frac{g_0}{2}\rho_0^2\right\}
\end{aligned}$$

which coincides with (1.8) already announced in paper [2]. Notice that by explicit calculation one checks the strict convexity of the pressure $p^{BZ}(\beta,\mu)$ (1.8) for the chemical potential $\mu\leq 0$. Then $f^{BZ}(\beta,\rho)$ is a strictly convex function of ρ. Therefore, the same is true for $f(\beta,\rho)$, see (2.2) and (2.11).

2.2 Now we study our model (1.18) in the *grand-canonical ensemble* (β,μ), where $\mu\in\mathbb{R}^1$ is the chemical potential of this ensemble. Let

$$p_{\Lambda}(\beta,\mu)\equiv\frac{1}{\beta V}\ln Tr_{\mathcal{F}_{\Lambda}}e^{-\beta(H_{\Lambda}-\mu N_{\Lambda})}.$$

be the grand-canonical pressure corresponding to H_{Λ}.

Theorem 2.4. *Let $\lambda>0$, $g_0>0$, $g>0$,and $\varepsilon_0<0$, then*
(i) the domain of stability of H_{Λ}, i.e.

$$\widetilde{Q}\equiv\left\{\left(\theta\geq 0,\mu\in\mathbb{R}^1\right):\ \lim_{\Lambda}p_{\Lambda}(\beta,\mu)<+\infty\right\}, \tag{2.12}$$

is equal to $\widetilde{Q}=\{\theta\geq 0\}\times\{\mu\in\mathbb{R}^1\}$;
(ii) in the thermodynamic limit one gets

$$p(\beta,\mu)\equiv\lim_{\Lambda}p_{\Lambda}(\beta,\mu)=\inf_{\alpha\leq 0}\left\{p^{BZ}(\beta,\alpha)+\frac{(\mu-\alpha)^2}{4\lambda}\right\} \tag{2.13}$$

for $(\theta, \mu) \in \widetilde{Q}$, *where* $p^{BZ}(\beta, \mu)$ *is the pressure defined by (1.8). Therefore the pressure (2.13) is independent of the parameter* g *provided it is positive.*

Proof. (i) Notice that the Hamiltonian H_Λ (1.18) is superstable, i.e. there are $B = -\varepsilon_0$ and $C = \lambda$ such that

$$H_\Lambda \geq -N_\Lambda B + \frac{C}{V} N_\Lambda^2 \tag{2.14}$$

for any box Λ. Therefore by (2.14) we have the existence of the infinite volume limit for the pressure $p_\Lambda(\beta, \mu)$ for any $\mu \in \mathbb{R}^1$.

(ii) Since the pressure $p(\beta, \mu)$ is in fact a Legendre transform of the corresponding free-energy density $f(\beta, \rho)$ (2.2) or (2.11), by Theorem 2.1 we get

$$p(\beta, \mu) = \sup_{\rho \geq 0} \{\mu\rho - f(\beta, \rho)\} = \sup_{\rho \geq 0} \{\mu\rho - \lambda\rho^2 - f^{BZ}(\beta, \rho)\}, \tag{2.15}$$

with $f^{BZ}(\beta, \rho)$ defined by (2.10). By straightforward calculations one obtains that

$$\inf_{\alpha \leq 0} \left\{\alpha\rho + \frac{(\mu - \alpha)^2}{4\lambda} - f^{BZ}(\beta, \rho)\right\} = \mu\rho - \lambda\rho^2 - f^{BZ}(\beta, \rho)$$

and thus (2.15) takes the form:

$$p(\beta, \mu) = \sup_{\rho \geq 0} \left\{\inf_{\alpha \leq 0} \left\{\alpha\rho + \frac{(\mu - \alpha)^2}{4\lambda} - f^{BZ}(\beta, \rho)\right\}\right\} \tag{2.16}$$

Notice that the $\sup_{\rho \geq 0}$ and $\inf_{\alpha \leq 0}$ generally don't commute. However, convexity of the free-energy density $f^{BZ}(\beta, \rho)$ (cf. Remark 2.3) implies that

$$F(\rho, \alpha) \equiv \alpha\rho + \frac{(\mu - \alpha)^2}{4\lambda} - f^{BZ}(\beta, \rho) \tag{2.17}$$

is a strictly concave function of ρ and a strictly convex function of α, see Figure 1. This ensures the uniqueness of the stationary point $(\widetilde{\rho}, \widetilde{\alpha})$ such that

$$\begin{aligned} \partial_\alpha F(\widetilde{\rho}, \widetilde{\alpha}) &= 0, \\ \partial_\rho F(\widetilde{\rho}, \widetilde{\alpha}) &= 0, \end{aligned}$$

and

$$F(\widetilde{\rho}, \widetilde{\alpha}) = \sup_{\rho \geq 0} \left\{\inf_{\alpha \leq 0} \{F(\rho, \alpha)\}\right\} = \inf_{\alpha \leq 0} \left\{\sup_{\rho \geq 0} \{F(\rho, \alpha)\}\right\}. \tag{2.18}$$

Since

$$\sup_{\rho \geq 0} F(\rho, \alpha) = \left\{\frac{(\mu - \alpha)^2}{4\lambda} + p^{BZ}(\beta, \alpha)\right\},$$

(2.16)-(2.18) imply (2.13). ■

3. Bose condensations

3.1 Let $\rho_\Lambda(\beta,\mu)$ denote the grand-canonical total particle density corresponding to the model (1.18), i.e.

$$\rho_\Lambda(\beta,\mu) \equiv \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda} = \partial_\mu p_\Lambda(\beta,\mu), \tag{3.1}$$

where $\langle - \rangle_{H_\Lambda}(\beta,\mu)$ represents the grand-canonical Gibbs state for the Hamiltonian H_Λ (1.18).

Theorem 3.1. *For $(\theta,\mu) \in \widetilde{Q}$ (2.12) we have*

$$\rho(\beta,\mu) \equiv \lim_\Lambda \rho_\Lambda(\beta,\mu) = \rho^{BZ}(\beta, \widetilde{\alpha}(\beta,\mu)). \tag{3.2}$$

Here $\widetilde{\alpha}(\beta,\mu) \le 0$ is a unique solution of

$$\rho^{BZ}(\beta,\widetilde{\alpha}(\beta,\mu)) + \frac{(\widetilde{\alpha}(\beta,\mu)-\mu)}{2\lambda} = 0, \tag{3.3}$$

when $\mu \le \mu_c^{BZ}(\theta) \equiv 2\lambda\rho_c^{BZ}(\theta)$, whereas for $\mu > \mu_c^{BZ}(\theta)$ one gets $\widetilde{\alpha}(\beta,\mu) = 0$ (see Figure 2) and

$$\rho(\beta,\mu) \equiv \lim_\Lambda \rho_\Lambda(\beta,\mu) = \frac{\mu}{2\lambda}, \tag{3.4}$$

where $\rho_c^{BZ}(\theta)$ is defined by (1.14).

Proof. Let $\widetilde{\alpha}(\beta,\mu) \le 0$ be defined by (2.13), i.e.

$$p(\beta,\mu) = \inf_{\alpha\le 0}\left\{p^{BZ}(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda}\right\} = p^{BZ}(\beta,\widetilde{\alpha}(\beta,\mu)) + \frac{(\mu-\widetilde{\alpha}(\beta,\mu))^2}{4\lambda}. \tag{3.5}$$

Since

$$\partial_\alpha\left[p^{BZ}(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda}\right] = \rho^{BZ}(\beta,\alpha) + \frac{(\alpha-\mu)}{2\lambda}, \tag{3.6}$$

then for $\mu \le \mu_c^{BZ}(\theta)$ (cf. (1.14)) there is a unique solution $\widetilde{\alpha}(\beta,\mu) \le 0$ of (3.3) which corresponds to a minimum in (2.13). Since $\{p_\Lambda(\beta,\mu)\}_\Lambda$ are convex functions of $\mu \in \mathbb{R}^1$ then combining (3.1) and (3.5) with the Griffiths lemma [12, 13] we obtain the thermodynamic limit for the total particle density

$$\rho(\beta,\mu) = \partial_\mu p(\beta,\mu) = \frac{(\mu - \widetilde{\alpha}(\beta,\mu))}{2\lambda},$$

which together with (3.3) gives (3.2).
Now let for $\mu > \mu_c^{BZ}(\theta)$. Then by definition of $\rho_c^{BZ}(\theta)$ (see (1.14)) one gets

$$\partial_\alpha\left[p^{BZ}(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda}\right] = \rho^{BZ}(\beta,\alpha) + \frac{(\alpha-\mu)}{2\lambda} \le 0$$

which implies that

$$p(\beta,\mu) = \inf_{\alpha\leq 0}\left\{p^{BZ}(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda}\right\} = p^{BZ}(\beta,0) + \frac{\mu^2}{4\lambda}, \tag{3.7}$$

i.e. $\widetilde{\alpha}(\beta,\mu) = 0$. Therefore, by the Griffiths lemma [9, 10] and (3.1), (3.7) we get (3.4). ■

Theorem 3.2. *Let $\varepsilon_0 < 0$. Then we have:*

$$\begin{aligned}\rho_0(\theta,\mu) &\equiv \lim_{\Lambda}\left\langle\frac{a_0^* a_0}{V}\right\rangle_{H_\Lambda}(\beta,\mu) \\ &= \left\{\begin{array}{c} 0, \ \textit{for}\ (\theta,\mu)\in\widetilde{Q}\backslash\widetilde{D}_{\varepsilon_0} \\ \left(\frac{\widetilde{\alpha}(\beta,\mu)-\varepsilon_0}{g_0}\right), \ \textit{for}\ (\theta,\mu)\in\widetilde{D}_{\varepsilon_0}\end{array}\right\},\end{aligned} \tag{3.8}$$

with $\widetilde{\alpha}(\beta,\mu)$ defined by equation (3.5), see Figure 2. Here domain $\widetilde{D}_{\varepsilon_0}$ is defined by:

$$\begin{aligned}\widetilde{D}_{\varepsilon_0} &= \left\{(\theta,\mu)\in\widetilde{Q} : \varepsilon_0 < \widetilde{\alpha}(\beta,\mu)\right\} = \\ &= \left\{(\theta,\mu)\in\widetilde{Q} : \widetilde{\mu}_0(\theta) < \mu\right\},\end{aligned} \tag{3.9}$$

see Figures 2 and 4, where we denote by $\widetilde{\mu}_0(\theta)$ a unique solution of the equation

$$\widetilde{\alpha}(\beta,\mu) = \varepsilon_0. \tag{3.10}$$

Proof. Since $\{p_\Lambda(\beta,\mu)\}_\Lambda$ are convex functions of $\varepsilon_0\in\mathbb{R}^1$, then by

$$\left\langle\frac{a_0^* a_0}{V}\right\rangle_{H_\Lambda}(\beta,\mu) = -\partial_{\varepsilon_0} p_\Lambda(\beta,\mu), \tag{3.11}$$

and by the Griffiths lemma [12, 13] we obtain that

$$\lim_{\Lambda}\left\langle\frac{a_0^* a_0}{V}\right\rangle_{H_\Lambda} = -\partial_{\varepsilon_0} p(\beta,\mu). \tag{3.12}$$

For $\mu\leq\mu_c^{BZ}(\theta)\equiv 2\lambda\rho_c^{BZ}(\theta)$ there is a unique $\widetilde{\alpha}(\beta,\mu)\leq 0$ defined by (3.5) which verifies (3.3), whereas for $\mu > \mu_c^{BZ}(\theta)$ according to (3.7) we obtain $\widetilde{\alpha}(\beta,\mu) = 0$. Notice that by (1.8) for $\mu\leq\varepsilon_0$ we have

$$p^{BZ}(\beta,\mu) = p^P(\beta,\mu).$$

Therefore, by (3.9), (3.12) one gets from (3.5) and (3.7) that

$$\lim_{\Lambda}\left\langle\frac{a_0^* a_0}{V}\right\rangle_{H_\Lambda}(\beta,\mu) = \left\{\begin{array}{c} 0, \ \text{for}\ \widetilde{\alpha}(\beta,\mu)\leq\varepsilon_0<0 \\ \left(\frac{\widetilde{\alpha}(\beta,\mu)-\varepsilon_0}{g_0}\right), \ \text{for}\ \varepsilon_0\leq\widetilde{\alpha}(\beta,\mu)\end{array}\right\},$$

i.e. (3.8). ■

Hence by Theorem 3.2, the domain $\widetilde{D}_{\varepsilon_0}$ can be described as

$$\widetilde{D}_{\varepsilon_0} = \left\{ (\theta, \mu) \in \widetilde{Q} : \ \rho_0 (\theta, \mu) \equiv \lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda} > 0 \right\}. \tag{3.13}$$

Notice that in contrast to D_{ε_0}, see (1.9), (1.10), the domain $\widetilde{D}_{\varepsilon_0}$ has a temperature dependent boundary and lasts to the positive μ, see Figure 4. This macroscopic occupation of the mode $k = 0$ is a *non-conventional* Bose condensation which occurs due to the "attraction" term $\varepsilon_0 a_0^* a_0$, i.e. when $\varepsilon_0 < 0$. It is *similar* to the *first* stage of condensation manifested by the model H_Λ^{BZ} (1.6) with $g_k (V) = g/2)$ [2] but for $\mu > 0$. In particular, we again recover a saturation of the total particle density in the mode $k = 0$:

$$\sup_{\mu \in \mathbb{R}^1} \rho_0 (\theta, \mu) = \rho_0 \left(\theta, \mu \geq \mu_c^{BZ} (\theta)\right) = -\frac{\varepsilon_0}{g_0}, \tag{3.14}$$

cf. (1.15) and Figure 3. Notice that for any μ

$$\lim_{\beta \to 0^+} \widetilde{\alpha} (\beta, \mu) = -\infty.$$

Thus, in contrast to the model (1.6) (with $g_k (V) = g/2)$, the non-conventional condensation in the model (1.18) depends on the temperature and there is $\widetilde{\theta}_0 (\mu)$ (solution of the equation $\widetilde{\alpha} (\theta^{-1}, \mu) = \varepsilon_0$, (3.10)) such that

$$\rho_0 (\theta, \mu) = \frac{\widetilde{\alpha} (\beta, \mu) - \varepsilon_0}{g_0} > 0, \tag{3.15}$$

for $\theta \leq \widetilde{\theta}_0 (\mu)$ and

$$\rho_0 (\theta, \mu) = 0, \tag{3.16}$$

for $\theta > \widetilde{\theta}_0 (\mu)$. This is another way to describe the phase diagram of the Figure 4: $\widetilde{\theta}_0 (\mu)$ is simply the inverse function to $\widetilde{\mu}_0 (\theta)$.

Similar to the model (1.6), for $d > 2$ we encounter for large total particle densities *another kind* of condensation: a *conventional non-extensive* Bose-Einstein condensation in the vinicity of $k = 0$. In order to control this condensation we introduce an auxiliary Hamiltonian

$$H_{\Lambda,\gamma} \equiv H_\Lambda - \gamma \sum_{\{k \in \Lambda^* : \|k\| \geq \delta\}} a_k^* a_k, \tag{3.17}$$

for a fixed $\delta > 0$, and we set

$$p_\Lambda (\beta, \mu, \gamma) \equiv \frac{1}{\beta V} \ln Tr_{\mathcal{F}_\Lambda} e^{-\beta H_{\Lambda,\gamma}(\mu)}. \tag{3.18}$$

Remark 3.3. *Let $\gamma < \varepsilon_\delta \equiv \varepsilon_{\|k\|=\delta}$. Then the system with Hamiltonian $H_{\Lambda,\gamma}$ has the same properties as the model H_Λ modulo the free-particle spectrum transformation:.*

$$\varepsilon_k \to \varepsilon_{k,\gamma} \equiv \varepsilon_k - \gamma . \chi_{[\delta,+\infty)} (\|k\|) \tag{3.19}$$

where $\chi_A(x)$ is the characteristic function of domain A. In particular the results of Theorem 2.1 and 2.4 remain unchanged. For $(\theta,\mu)\in\widetilde{Q}$ and $\gamma<\varepsilon_\delta$ we have

$$p(\beta,\mu,\gamma)\equiv\lim_{\Lambda}p_\Lambda(\beta,\mu,\gamma)=\inf_{\alpha\leq 0}\left\{p^{BZ}(\beta,\alpha,\gamma)+\frac{(\mu-\alpha)^2}{4\lambda}\right\}, \tag{3.20}$$

where $p^{BZ}(\beta,\mu,\gamma)$ is the pressure (1.8) but with the free-particle spectrum (3.19):

$$\begin{aligned} p^{BZ}(\beta,\mu,\gamma) &= p^P(\beta,\mu,\gamma)-\inf_{\rho_0\geq 0}\left[(\varepsilon_0-\mu)\rho_0+\frac{g_0\rho_0^2}{2}\right] \\ &= \frac{1}{\beta(2\pi)^d}\int\limits_{k\in\mathbb{R}^d}\ln\left[\left(1-e^{-\beta\left(\varepsilon_{k,\gamma}-\mu\right)}\right)^{-1}\right]d^dk \\ &\quad-\inf_{\rho_0\geq 0}\left[(\varepsilon_0-\mu)\rho_0+\frac{g_0\rho_0^2}{2}\right]. \end{aligned} \tag{3.21}$$

Theorem 3.4. *For any $(\theta,\mu)\in\widetilde{Q}$ we have*

$$\lim_{\Lambda}\left\langle\frac{a_k^*a_k}{V}\right\rangle_{H_\Lambda}=0,\ k\in\Lambda^*\backslash\{0\}, \tag{3.22}$$

i.e. there is no macroscopic occupation of modes $k\neq 0$ whereas for $\mu\geq\mu_c^{BZ}(\theta)\equiv 2\lambda\rho_c^{BZ}(\theta)$ the model H_Λ (1.18) manifests a generalized (non-extensive) Bose-Einstein condensation for those modes:

$$\lim_{\delta\to 0^+}\lim_{\Lambda}\frac{1}{V}\sum_{\{k\in\Lambda^*:\,0<\|k\|<\delta\}}\langle N_k\rangle_{H_\Lambda}=\rho(\beta,\mu)-\rho_c^{BZ}(\theta)=\frac{1}{2\lambda}\left(\mu-\mu_c^{BZ}(\theta)\right)>0. \tag{3.23}$$

Here $\rho(\beta,\mu)$ is defined by (3.4). For $\varepsilon_0<0$ this condensation coexists with the non-conventional condensation in the mode $k=0$ (see Theorem 3.2).

Proof. Let $g>0$ and $\Delta g>0$ be such that $g-\Delta g>0$. Then by the Bogoliubov inequality (see e.g. [14]), one gets:

$$0\leq\frac{\Delta g}{2V^2}\sum_{k\in\Lambda^*\backslash\{0\}}\langle N_k^2\rangle_{H_\Lambda}\leq p_\Lambda\left[H_\Lambda-\frac{\Delta g}{2V}\sum_{k\in\Lambda^*\backslash\{0\}}N_k^2\right]-p_\Lambda[H_\Lambda]. \tag{3.24}$$

Notice that by Theorems 2.1 and 2.4 thermodynamic limits of pressures for two models (1.18) with $g>0$ and $g-\Delta g>0$ coincide with (2.13), i.e. one has

$$\lim_{\Lambda}\left\{p_\Lambda\left[H_\Lambda-\frac{\Delta g}{2V}\sum_{k\in\Lambda^*\backslash\{0\}}N_k^2\right]-p_\Lambda[H_\Lambda]\right\}=0. \tag{3.25}$$

Since for $k \in \Lambda^{*} \backslash \{0\}$ we have the estimate

$$0 \leq \left(\frac{\langle N_k \rangle_{H_\Lambda}}{V} \right)^2 \leq \frac{\langle N_k^2 \rangle_{H_\Lambda}}{V^2} \leq \frac{1}{V^2} \sum_{k \in \Lambda^{*} \backslash \{0\}} \langle N_k^2 \rangle_{H_\Lambda},$$

then combining it with (3.24) and (3.25) we obtain (3.22).
Let $\delta > 0$, then we have

$$\frac{1}{V} \sum_{\{k \in \Lambda^{*}:\, 0<\|k\|<\delta\}} \langle N_k \rangle_{H_\Lambda} = \rho_\Lambda (\beta, \mu) - \left\langle \frac{a_0^{*} a_0}{V} \right\rangle_{H_\Lambda} - \frac{1}{V} \sum_{\{k \in \Lambda^{*}: \|k\| \geq \delta\}} \langle N_k \rangle_{H_\Lambda}. \tag{3.26}$$

Now we can follow the same line of reasoning as in proofs of Theorems 3.1 and 3.2: we have the set $\{p_\Lambda (\beta, \mu, \gamma)\}_\Lambda$ of convex functions of $\gamma \in (-\infty, \varepsilon_\delta]$ with

$$\frac{1}{V} \sum_{\{k \in \Lambda^{*}: \|k\| \geq \delta\}} \langle N_k \rangle_{H_{\Lambda,\gamma}} = \partial_\gamma p_\Lambda (\beta, \mu, \gamma),$$

which by the Griffiths lemma [12, 13] and (3.20), (3.21) implies for $\gamma = 0$ that

$$\lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^{*}: \|k\| \geq \delta\}} \langle N_k \rangle_{H_\Lambda} = \partial_\gamma p (\beta, \mu, \gamma = 0). \tag{3.27}$$

Then by definitions (3.19), (3.20) and Theorem 3.1, see (3.2), (3.4), together with explicit formula (1.13) we get:

$$\partial_\gamma p (\beta, \mu, \gamma = 0) = \frac{1}{(2\pi)^d} \int_{\|k\| \geq \delta} \frac{d^d k}{e^{\beta(\varepsilon_k - \widetilde{\alpha}(\beta,\mu))} - 1} \tag{3.28}$$

for $\mu < \mu_c^{BZ}(\theta)$ and

$$\partial_\gamma p (\beta, \mu, \gamma = 0) = \frac{1}{(2\pi)^d} \int_{\|k\| \geq \delta} \frac{d^d k}{e^{\beta \varepsilon_k} - 1} \tag{3.29}$$

for $\mu \geq \mu_c^{BZ}(\theta)$. Now taking account (3.4), (3.14) and definition (1.14) we obtain (3.23) from (3.26), (3.27) and (3.29) by taking thermodynamic limit and the limit $\delta \to 0^+$. ■

4. Concluding remark

We have presented a new exactly soluble model (1.18) which is inspired by the MSV model [1] and our model [2]. Due to an "attractive"-type interaction in the mode $k = 0$ it belongs to the family of models which manifest two types of condensations: non-conventional one in the mode $k = 0$ and conventional (generalized) Bose-Einstein condensation in modes $k \neq 0$, see [2, 15, 16]. They coexist for large total particle densities.

The important feature of models (1.18) and [1, 2] (in contrast to [15]) is that the repulsion between bosons with $k \neq 0$ is strong enough to produce a generalized type III (i.e. non-extensive) Bose-Einstein condensation. Notice that in the Bogoliubov Weakly Imperfect Bose Gas [15, 16], the Bose-Einstein condensation is of the type I.

It seems that the model [1] and then [2] together with the present one give a first example when the non-extensive Bose-Einstein condensation is produced by the interaction *between particles* and not by geometry or by external fields, see [7, 8] and [17, 18, 19].

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FIGURE CAPTIONS

Figure 1. Illustration of the function $F(\rho,\alpha)$.

Figure 2. Illustration of the function $\widetilde{\alpha}(\beta,\mu)$ defined by the equation (2.13).

Figure 3. Non-conventional condensate density $\rho_0(\theta,\mu)$ as a function of the chemical potential μ and the temperature θ for the model H_Λ.

Figure 4. Domains D_{ε_0} and $\widetilde{D}_{\varepsilon_0}$ corresponding to the existence of a non-conventional condensation respectively for models H_Λ^{BZ} and H_Λ.