Symmetries of the Nonlinear Schrödinger Equation

B. Grébert[1], T. Kappeler[2]

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1. UMR 6629 CNRS, Universite de Nantes, 2 rue de la Houssière, BP 92208, 44322 Nantes cedex 3, France.

2. Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland.

Abstract: Fundamental symmetries of the defocusing nonlinear Schrödinger equation are expressed in action-angle coordinates and characterized in terms of periodic and Dirichlet spectrum of the associated Zakharov-Shabat system. As a main application we prove a conjecture, raised by several experts in field, that the periodic spectrum is symmetric iff the sequence of gap lengths $(\gamma_k)_{k\in\mathbb{Z}}$ or, equivalently, the sequence of actions $(I_k)_{k\in\mathbb{Z}}$ is symmetric with respect to $k=0$.

0 Introduction

The defocusing nonlinear Schrödinger equation NLS on the circle

$$i\partial_t\varphi = -\partial_x^2\varphi + 2 \mid \varphi \mid^2 \varphi \tag{0.1}$$

can be viewed as a completely integrable Hamiltonian system of infinite dimension. Indeed, on the phase $L^2(S^1;\mathbb{C})$, introduce the Poisson bracket

$$\{F,G\} := i\int_{S^1}\Big(\frac{\partial F}{\partial\varphi(x)}\,\frac{\partial G}{\partial\overline{\varphi}(x)} - \frac{\partial F}{\partial\overline{\varphi}(x)}\,\frac{\partial G}{\partial\varphi(x)}\Big)dx$$

Equation (0.1) can then be written in Hamiltonian form as follows

$$\begin{array}{rclcl} \frac{\partial \varphi}{\partial t} & = & \{\mathcal{H}, \varphi\} & = & -i\frac{\partial \mathcal{H}}{\partial \overline{\varphi}} \\ \frac{\partial \overline{\varphi}}{\partial t} & = & \{\mathcal{H}, \overline{\varphi}\} & = & i\frac{\partial \mathcal{H}}{\partial \varphi} \end{array}$$

where the Hamiltonian $\mathcal{H}$ is given by (cf [FT])

$$\mathcal{H}(\varphi) := \int_{S^1} \Big(\mid \frac{\partial \varphi}{\partial x} \mid^2 + \mid \varphi \mid^4 \Big) dx.$$

Consider the following symmetry operators, acting on $L^2(S^1; \mathbb{C})$,

$$\mathcal{S}_1(\varphi) := \overline{\varphi}; \quad \mathcal{S}_2(\varphi) = \check{\varphi}; \tag{0.2}$$

$$M_\alpha \varphi := e^{i\alpha}\varphi; \quad T_\tau \varphi := \varphi(\tau + \cdot) \tag{0.3}$$

where $\check{\varphi}$ is defined by $\check{\varphi}(x) = \varphi(-x)$. For convenience, we introduce $\mathcal{S}_3 := M_\pi$, i.e. $\mathcal{S}_3(\varphi) = -\varphi$. Notice that the Hamiltonian $\mathcal{H}$ is invariant under $\mathcal{S}_1, \mathcal{S}_2, M_\alpha$ and T_τ.
Denote by $U(t)$ the solution operator of (0.1) for initial data in $L^2(S^1; \mathbb{C})$ (or some Sobolev space $H^N(S^1; \mathbb{C})$) (cf [B]). It is immediate that $U(t)$ commutes with $\mathcal{S}_2, S_3, M_\alpha$ and T_τ and that

$$U(t)\mathcal{S}_1 = \mathcal{S}_1 U(-t). \tag{0.4}$$

Recall that NLS admits a Lax pair representation

$$\frac{dL}{dt} = [L, A]$$

where $L = L(\varphi)$ is the operator

$$L(\varphi) := i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & \varphi \\ \overline{\varphi} & 0 \end{pmatrix} \tag{0.5}$$

and A is a (rather complicated) operator given in [FT]. We remark that $L(\varphi)$ is unitarily equivalent to the well known AKNS-operator

$$H(\varphi) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} -q & p \\ p & q \end{pmatrix} \tag{0.6}$$

where $\varphi = -q + ip$, a fact which will be used throughout the paper. Denote by $spec_{per} L(\varphi)$ the periodic spectrum of $L(\varphi)$ when considered on the interval $[0, 2]$ and by $spec^{\pm}_{Dir} L(\varphi)$ the Dirichlet spectra of $L(\varphi)$ when considered on the interval $[0, 1]$ (cf definitions (1.5) and (1.6)). By elementary considerations one shows that

$$\begin{array}{lclclcl} spec_{per} L(\bar{\varphi}) & = & -spec_{per} L(\varphi); & spec_{per} L(\check{\varphi}) & = & -spec_{per} L(\varphi); \\ spec_{per} L(M_\alpha \varphi) & = & spec_{per} L(\varphi); & spec_{per} L(T_\tau \varphi) & = & spec_{per} L(\varphi); \end{array}$$

and expresses $spec^{+}_{Dir} L(\mathcal{S}_j \varphi)$ for $j = 1, 2, 3$ in terms of $spec^{-}_{Dir} L(\varphi)$. This is used to compute the symmetries in action-angle variables. Recall from [GK1] that NLS admits global Birkhoff coordinates $(z_j)_{j \in \mathbb{Z}}$

$$z_j := x_j + i y_j = \sqrt{2 I_j} e^{i \theta_j}$$

with $I_j = (x_j^2 + y_j^2)/2$ $(j \in \mathbb{Z})$ denoting the actions and for φ with $I_j \neq 0$, θ_j denoting the angle variable.

Theorem 0.1

(i) The actions are left invariant by M_α and T_τ $(\forall k \in \mathbb{Z})$

$$I_k(M_\alpha \varphi) = I_k(\varphi); \quad I_k(T_\tau \varphi) = I_k(\varphi);$$

for $\mathcal{S}_j$, $j = 1, 2$ $(\forall k \in \mathbb{Z})$

$$I_k(\mathcal{S}_j \varphi) = I_{-k}(\varphi).$$

(ii) For k with $I_k \neq 0$

$$\begin{array}{lcl} \theta_k(M_\alpha \varphi) & \equiv & \theta_k + \alpha \quad (mod\ 2\pi); \\ \theta_k(\check{\varphi}) & \equiv & \theta_{-k}(\varphi) \quad (mod\ 2\pi); \\ \theta_k(\overline{\varphi}) & \equiv & -\theta_{-k}(\varphi) \quad (mod\ 2\pi). \end{array}$$

As an immediate application of Theorem 0.1 one obtains

Corollary 0.1 *When evaluated at $I = (I_k)_{k \in \mathbb{Z}}$ with $I_k = I_{-k}$ $\forall k \in \mathbb{Z}$, the NLS frequencies $\omega = (\omega_k)_{k \in Z}$, $\omega_k = \frac{\partial \mathcal{H}}{\partial I_k}$, are symmetric, $\omega_k(I) = \omega_{-k}(I)$ $\forall k \in \mathbb{Z}$.*

Theorem 0.1 and Corollary 0.1 will be used for a KAM type theorem in [GK2]

Further we use Theorem 0.1 to prove the conjecture, raised by several experts in the field, that the periodic spectrum is symmetric if and only if the sequence of the gap lengths is symmetric. More precisely, let $specL(\varphi) = (\lambda_k^\pm(\varphi))_{k\in\mathbb{Z}}$ denote the periodic spectrum of $L(\varphi)$. $\lambda_k^\pm(\varphi)$ are real numbers satisfying $\lambda_k^-(\varphi) \le \lambda_k^+(\varphi) < \lambda_{k+1}^-(\varphi)$. Let $\gamma(\varphi) := (\gamma_k(\varphi))_{k\in\mathbb{Z}}$ be the sequence of gap lengths, $\gamma_k(\varphi) := \lambda_k^+(\varphi) - \lambda_k^-(\varphi)$. We prove

Theorem 0.2 *For $\varphi \in L^2(S^1;\mathbb{C})$, the following assertions are equivalent:*

(i) $\lambda_k^\pm(\varphi) = -\lambda_{-k}^\mp(\varphi)$ *for all* $k \ge 0$.

(ii) $\gamma_k(\varphi) = \gamma_{-k}(\varphi)$ *for all* $k \ge 1$.

1 Symmetries and spectra

1.1 Periodic spectrum

The periodic spectrum of the Zakharov-Shabat operator $L(\varphi)$ is given by

$$\begin{aligned} spec_{per}L(\varphi) := \{\lambda \in \mathbb{C} \quad | \quad & \exists F \in H^1_{loc}(\mathbb{R};\mathbb{C}^2), F \not\equiv 0 \ \text{ with } L(\varphi)F = \lambda F \\ & \text{and} \quad F(x+2) = F(x) \quad \forall x \in \mathbb{R}\}. \end{aligned}$$

By [GG] $spec_{per}L(\varphi)$ is a sequence of real numbers $(\lambda_k^\pm(\varphi))_{k\in\mathbb{Z}}$, which can be ordered in such a way that $(\forall k \in \mathbb{Z})$

$$\lambda_k^-(\varphi) \le \lambda_k^+(\varphi) < \lambda_{k+1}^-(\varphi). \tag{1.1}$$

We have the following

Proposition 1.1 *Let $\varphi \in L^2(S^1;\mathbb{C})$. Then, for any $k \in \mathbb{Z}$,*

(i) $\lambda_k^\pm(e^{i\alpha}\varphi) = \lambda_k^\pm(\varphi), \quad \lambda_k^\pm(\varphi) = \lambda_k^\pm(T_\tau\varphi) \qquad (\forall \alpha \in \mathbb{R}, \tau \in \mathbb{R})$;

(ii) $\lambda_k^\pm(\check{\varphi}) = -\lambda_{-k}^\mp(\varphi), \quad \lambda_k^\pm(\bar{\varphi}) = -\lambda_{-k}^\mp(\varphi)$.

Proof (i) For $\alpha \in \mathbb{R}$ arbitrary, define $V_\alpha = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$. One easily verifies that

$$L(e^{i\alpha}\varphi) = V_\alpha^{-1} L(\varphi) V_\alpha \tag{1.2}$$

and $L(T_\tau\varphi) = T_\tau L(\varphi)T_{-\tau}$. Thus both, $L(e^{i\alpha}\varphi)$ and $L(T_\tau\varphi)$, are unitarily equivalent to $L(\varphi)$ and the claimed statement follows. To prove (ii) notice that

$$L(-\check{\varphi}) = -W^{-1}L(\varphi)W \tag{1.3}$$

where W is the unitary operator defined by

$$W\begin{pmatrix} Y \\ Z \end{pmatrix} := \begin{pmatrix} \check{Y} \\ \check{Z} \end{pmatrix}, \quad \text{with} \quad \begin{pmatrix} Y \\ Z \end{pmatrix} \in L^2_{loc}(\mathbb{R};\mathbb{C}^2).$$

Thus

$$spec_{per}L(-\check{\varphi}) = -spec_{per}L(\varphi). \tag{1.4}$$

Combining (1.4) and (i) we obtain $\lambda_k^{\pm}(\check{\varphi}) = -\lambda_{-k}^{\mp}(\varphi)$ $(\forall k \in \mathbb{Z})$. Consider $\lambda \in spec_{per}L(\varphi)$ and choose $F \in H^1_{loc}(\mathbb{R};\mathbb{C}^2)$, satisfying $F(x+2) = F(x) \quad (\forall x \in \mathbb{R})$ and $L(\varphi)F = \lambda F$. As λ is real, $L(-\bar{\varphi})\overline{F} = -\lambda\overline{F}$ and thus $-\lambda \in spec_{per}L(-\bar{\varphi})$. Combined with (i), this leads to $\lambda_k^{\pm}(\bar{\varphi}) = -\lambda_{-k}^{\mp}(\varphi) \quad (\forall k \in \mathbb{Z})$. ■

1.2 Dirichlet spectra and divisors

Let $F_j(x,\lambda;\varphi) := \begin{pmatrix} Y_j(x,\lambda;\varphi) \\ Z_j(x,\lambda;\varphi) \end{pmatrix}$, $j = 1,2$, be the fundamental solutions of $H(\varphi)$, i.e. the solutions to $HF = \lambda F$ such that

$$F_1(0,\lambda;\varphi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; F_2(0,\lambda;\varphi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For each $x \in \mathbb{R}$ and $\varphi \in L^2(S^1;\mathbb{C})$, $F_1(x,\lambda;\varphi)$ and $F_2(x,\lambda;\varphi)$ are entire functions of λ. The two Dirichlet spectra are defined as follows

$$spec^+_{Dir}L(\varphi) = \{\lambda \in \mathbb{C} \mid Z_1(1,\lambda;\varphi) = 0\} \tag{1.5}$$

$$spec^-_{Dir}L(\varphi) = \{\lambda \in \mathbb{C} \mid Y_2(1,\lambda;\varphi) = 0\}. \tag{1.6}$$

It is proved in [GG] that $spec^+_{Dir}L(\varphi)$, respectively $spec^-_{Dir}L(\varphi)$, consists of simple, real eigenvalues $(\mu_k(\varphi))_{k\in\mathbb{Z}}$, respectively $(\nu_k(\varphi))_{k\in\mathbb{Z}}$. The numerotation is chosen in such a way that $(\mu_k(\varphi))_{k\in\mathbb{Z}}$ and $(\nu_k(\varphi))_{k\in\mathbb{Z}}$ are strictly increasing satisfying $\mu_k(\varphi) \sim k\pi$ and $\nu_k(\varphi) \sim k\pi$ for $|k|$ large. Further introduce the function $\delta(\lambda;\varphi)$, defined by

$$\delta(\lambda;\varphi) = Z_2(1,\lambda;\varphi) - Y_1(1,\lambda;\varphi). \tag{1.7}$$

Notice that $(\forall k \in \mathbb{Z})$

$$\delta(\mu_k(\varphi);\varphi)^2 = \Delta(\mu_k(\varphi);\varphi)^2 - 4, \tag{1.8}$$

where

$$\Delta(\lambda;\varphi) = Y_1(1,\lambda;\varphi) + Z_2(1,\lambda;\varphi) \tag{1.9}$$

is the discriminant.

Proposition 1.2 *Let $\varphi \in L^2(S^1;\mathbb{C})$. Then*

(i) $\mu_k(-\varphi) = \nu_k(\varphi), \quad \nu_k(-\varphi) = \mu_k(\varphi) \qquad (\forall\ k \in \mathbb{Z})$;

(ii) $\delta(\lambda;-\varphi) = -\delta(\lambda;\varphi)$.

Remark For $\alpha \not\equiv 0,\pi\ (mod\ 2\pi)$, $\mu_k(e^{i\alpha}\varphi)$ satisfies an equation involving $spec^{\pm}_{Dir}L(\varphi)$, the solution of which does not seem to be given in form of a closed expression.

Proof (Proposition 1.2) For $F = \begin{pmatrix} Y \\ Z \end{pmatrix} \in H^1_{loc}(\mathbb{R};\mathbb{R}^2)$ define

$$F^{\perp} = \begin{pmatrix} Z \\ -Y \end{pmatrix}. \tag{1.10}$$

If $H(\varphi)F = \lambda F$ we have

$$H(-\varphi)F^{\perp} = \lambda F^{\perp}.$$

Therefore

$$F_1(x,\lambda;-\varphi) = F_2^{\perp}(x,\lambda;\varphi) \tag{1.11}$$

and

$$F_2(x,\lambda;-\varphi) = -F_1^{\perp}(x,\lambda;\varphi). \tag{1.12}$$

Using (1.11) - (1.12) one easily gets (i) and (ii). $\blacksquare$

Proposition 1.3 *For* $\varphi \in L^2(S^1;\mathbb{C}), k \in \mathbb{Z}$

(i) $\mu_k(\bar\varphi) = -\nu_{-k}(\varphi), \quad \nu_k(\bar\varphi) = -\mu_{-k}(\varphi);$

(ii) $\delta(\mu_k(\bar\varphi), \bar\varphi) = -\delta(\nu_{-k}(\varphi), \varphi).$

Proof (i) For $F = \begin{pmatrix} Y \\ Z \end{pmatrix} \in H^1_{loc}(\mathbb{R}, \mathbb{R}^2)$ we define

$$\tilde F = \begin{pmatrix} Z \\ Y \end{pmatrix}. \tag{1.13}$$

If $H(\varphi)F = \lambda F,\ \tilde F$ satisfies $H(\bar\varphi)\tilde F = -\lambda \tilde F$. Therefore

$$F_1(x,\lambda;\bar\varphi) = \tilde F_2(x,-\lambda;\varphi); \quad F_2(x,\lambda;\bar\varphi) = \tilde F_1(x,-\lambda;\varphi). \tag{1.14}$$

Using (1.14) and the asymptotics of μ_k and ν_k, one easily obtains (i).
(ii) From (1.14) one deduces $\delta(\mu_k(\bar\varphi);\bar\varphi) = -\delta(-\mu_k(\bar\varphi);\varphi)$ and thus (ii) follows using (i). ∎

Proposition 1.4 *For* $\varphi \in L^2(S^1;\mathbb{C}), k \in \mathbb{Z}$,

(i) $\mu_k(\check\varphi) = -\nu_{-k}(\varphi), \quad \nu_k(\check\varphi) = -\mu_{-k}(\varphi);$

(ii) $\delta(\mu_k(\check\varphi), \check\varphi) = \delta(\nu_{-k}(\varphi), \varphi).$

Remark By Proposition 1.1, 1.3 and 1.4, $\check\varphi$ and $\bar\varphi$ have the same periodic and the same Dirichlet spectra. They are only distinguished by

$$\delta(\mu_k(\check\varphi), \check\varphi) = -\delta(\mu_k(\bar\varphi), \bar\varphi) \quad (\forall k \in \mathbb{Z}).$$

Proof (Proposition 1.4) For $F = \begin{pmatrix} Y \\ Z \end{pmatrix} \in H^1_{loc}(\mathbb{R};\mathbb{R}^2)$ we define

$$F^*(x) = F(1-x). \tag{1.15}$$

If $H(\varphi)F = \lambda F,\ F^*$ satisfies

$$H(-\check\varphi)F^* = -\lambda F^*. \tag{1.16}$$

By the definition of $\mu_k(\varphi)$,

$$F_1(1, \mu_k(\varphi); \varphi) = Y_1(1, \mu_k(\varphi); \varphi) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore ($\forall k \in \mathbb{Z}$)

$$F_1(x, -\mu_k(\varphi);\ -\check{\varphi}) = \frac{1}{Y_1(1, \mu_k(\varphi); \varphi)} F_1(1 - x, \mu_k(\varphi); \varphi). \tag{1.17}$$

Evaluated at $x = 1$, (1.17) leads to $Z_1(1, -\mu_k(\varphi); -\check{\varphi}) = 0$ ($\forall k \in \mathbb{Z}$). In view of the asymptotics of μ_k we conclude that $\mu_k(-\check{\varphi}) = -\mu_{-k}(\varphi)$. Statement (i) then follows from Proposition 1.2 (i).
From (1.17) and (i), we deduce that

$$Y_1(1, \mu_{-k}(-\check{\varphi}); -\check{\varphi}) = \frac{1}{Y_1(1, \mu_k(\varphi); \varphi)}\ . \tag{1.18}$$

Further, the Wronskian identity (see [GG])

$$Y_1(1, \lambda; \varphi) Z_2(1, \lambda; \varphi) - Y_2(1, \lambda; \varphi) Z_1(1, \lambda; \varphi) = 1 \tag{1.19}$$

implies that

$$Z_2(1, \mu_k(\varphi); \varphi) = \frac{1}{Y_1(1, \mu_k(\varphi); \varphi)}. \tag{1.20}$$

Hence (1.18) and (1.20) yield

$$\delta(\mu_{-k}(-\check{\varphi}); -\check{\varphi}) = -\delta(\mu_k(\varphi); \varphi). \tag{1.21}$$

Statement (ii) then follows by combining (1.21) and Proposition 1.2. ■

Remark The evolution of the Dirichlet spectra under the translation flow has been studied in [BGGK].

2 Symmetries in action-angle coordinates

2.1 Actions

Recall from [GK1] (see also [MV]) that, for $\varphi \in L^2(S^1; \mathbb{C})$, the actions, $I_k(\varphi), k \in \mathbb{Z}$, are defined by

$$I_k(\varphi) := \frac{-2}{\pi} \int_{\lambda_k^-(\varphi)}^{\lambda_k^+(\varphi)} (-1)^k \ \frac{\mu \dot{\Delta}(\mu; \varphi)}{\mid (\Delta^2(\mu; \varphi) - 4)^{1/2} \mid} \, d\mu \tag{2.1}$$

where $\Delta(\mu;\varphi)$ is the discriminant given by

$$\Delta(\mu;\varphi) = Y_1(1,\mu;\varphi) + Z_2(1,\mu;\varphi). \tag{2.2}$$

Proposition 2.1 *For $\varphi \in L^2(S^1;\mathbb{C})$ and $k \in \mathbb{Z}$*

(i) $I_k(e^{i\alpha}\varphi) = I_k(\varphi) \quad \forall \alpha \in \mathbb{R}$;

(ii) $I_k(\bar{\varphi}) = I_{-k}(\varphi)$;

(iii) $I_k(\check{\varphi}) = I_{-k}(\varphi)$;

(iv) $I_k(T_\tau \varphi) = I_k(\varphi) \quad \forall \tau \in \mathbb{R}$.

Proof Recall from [GG] that $\Delta^2(\mu;\varphi) - 4$ has a representation as an infinite product

$$\Delta^2(\mu;\varphi) - 4 = -4(\lambda_0^+(\varphi) - \mu)(\lambda_0^-(\varphi) - \mu) \times \prod_{k \in \mathbb{Z}^*} \frac{(\lambda_k^+(\varphi) - \mu)(\lambda_k^-(\varphi) - \mu)}{k^2\pi^2} \tag{2.3}$$

where, in order that the above infinite product $\prod_{k\in\mathbb{Z}^*} a_k$ be absolutely convergent, it has to be computed as $\prod_{k\geq 1} a_k \cdot a_{-k}$. Furthermore, for $\mu \in (\lambda_k^-(\varphi), \lambda_k^+(\varphi))$, sign $\Delta(\mu;\varphi) = (-1)^k$. Therefore $\Delta(\cdot;\varphi)$, and thus $(I_k(\varphi))_{k\in\mathbb{Z}}$, is uniquely determined by $spec_{per} L(\varphi)$. In particular, statements (i) and (iv) follow from Proposition 1.1. To prove (ii) and (iii) notice that, by Proposition 1.1, for $\psi \in \{\overline{\varphi}, \check{\varphi}\}$, $spec_{per} L(\psi) = -spec_{per} L(\varphi)$. Therefore $\Delta(\lambda;\psi) = \Delta(-\lambda;\varphi)$ and thus, by (2.1), one easily obtains for any $k \in \mathbb{Z}$, $I_k(\psi) = I_{-k}(\varphi)$. ■

2.2 Angles

Denote by $\mathcal{M}(\varphi)$ the hyperelliptic Riemann surface, $y = \sqrt{\Delta(\lambda;\varphi)^2 - 4}$, and by $\mathcal{P} = (\lambda, y)$ an arbitrary point on $\mathcal{M}(\varphi)$. Let $\beta_j = \beta_j(\varphi) \quad (\forall j \in \mathbb{Z})$ be Abelian differential 1-forms of the third kind on $\mathcal{M}(\varphi)$, uniquely determined by the normalization conditions ([GK1], [MV]),

$$\int_{a_k} \beta_j = 2\pi\ \delta_{kj},$$

where the cycle $a_k \equiv a_k(\varphi)$ $\quad(\forall k \in \mathbb{Z})$ is a contour on the upper sheet of $\mathcal{M}(\varphi)$ around $[\lambda_k^-, \lambda_k^+]$ with clockwise orientation (see [GK1]). For $k \in \mathbb{Z}$ with $I_k(\varphi) \neq 0$, the angle $\theta_k(\varphi)$ is given by

$$\theta_k(\varphi) = \int_{\lambda_k^-(\varphi)}^{\mathcal{P}_k(\varphi)} \sum_{j\in\mathbb{Z}} \beta_j(\varphi), \tag{2.4}$$

where $\mathcal{P}_k(\varphi)$ denotes the point $(\mu_k(\varphi), \delta(\mu_k(\varphi); \varphi))$ on $\mathcal{M}(\varphi)$ (cf. (1.8)). As path of integration in (2.4) one chooses the straight line on the sheet of $\mathcal{M}(\varphi)$ containing $\mathcal{P}_k(\varphi)$, connecting λ_k^- and $\mathcal{P}_k$.

Proposition 2.2 *For $\varphi \in L^2(S^1; \mathbb{C})$, $\alpha \in \mathbb{R}$, and $k \in \mathbb{Z}$ with $I_k(\varphi) \neq 0$*

$$\theta_k(e^{i\alpha}\varphi) \equiv \theta_k(\varphi) + \alpha \pmod{2\pi}.$$

Proof Denote by R_α the action of M_α expressed in Birkhoff coordinates, i.e. $R_\alpha = \Phi^{-1} M_\alpha \Phi$ where Φ is the Birkhoff map, $\varphi = \Phi\left((\sqrt{2I_j}e^{i\theta_j})_{j\in\mathbb{Z}}\right)$ (cf [GK1]). Notice that $R_{\alpha+\beta} = \Phi^{-1} M_\alpha M_\beta \Phi = R_\alpha R_\beta$, $R_0 = Id$, and

$$\frac{d}{d\alpha} R_\alpha = \lim_{\beta\to 0} \frac{R_{\alpha+\beta} - R_\alpha}{\beta} = \lim_{\beta\to 0} \frac{R_\beta - Id}{\beta} R_\alpha = i \cdot R_\alpha.$$

Hence $R_\alpha = e^{i\alpha} Id$. ■

Proposition 2.3 *For $\varphi \in L^2(S^1; \mathbb{C})$,*

(i) $\theta_k(\check{\varphi}) \equiv \theta_{-k}(\varphi) \pmod{2\pi}$ $\quad\forall\, k \in \mathbb{Z}$ *with* $I_{-k}(\varphi) \neq 0$;

(ii) $\theta_k(\bar{\varphi}) \equiv -\theta_{-k}(\varphi) \pmod{2\pi}$ $\quad\forall\, k \in \mathbb{Z}$ *with* $I_{-k}(\varphi) \neq 0$.

Proof For $k \in \mathbb{Z}$ with $I_k(-\check{\varphi}) \neq 0$ we have, by (2.4) and Proposition 1.1,

$$\theta_k(-\check{\varphi}) = \int_{\lambda_k^-(-\check{\varphi})}^{\mathcal{P}_k(-\check{\varphi})} \sum_{j\in\mathbb{Z}} \beta_j(-\check{\varphi}) = \int_{-\lambda_{-k}^+(\varphi)}^{\mathcal{P}_k(-\check{\varphi})} \sum_{j\in\mathbb{Z}} \beta_j(-\check{\varphi}). \tag{2.5}$$

To compute the latter integral, introduce the map $\sigma : \mathcal{M}(-\check{\varphi}) \to \mathcal{M}(\varphi)$ defined by

$$\sigma(\lambda, y) = (-\lambda, -y). \tag{2.6}$$

Notice that for $(\lambda, y) \in \mathcal{M}(-\check{\varphi})$, $\sigma(\lambda, y) \in \mathcal{M}(\varphi)$ as $\Delta^2(\lambda; -\check{\varphi}) = \Delta^2(-\lambda; \varphi)$ (cf formula (2.3) and Proposition 1.1).

In view of Proposition 1.1

$$\int_{\sigma(a_k(-\check{\varphi}))} \beta_{-j}(\varphi) = \int_{a_{-k}(\varphi)} \beta_{-j}(\varphi) = 2\pi\delta_{jk}.$$

On the other hand, changing coordinates according to σ,

$$\int_{\sigma(a_k(-\check{\varphi}))} \sigma^*\beta_j(-\check{\varphi}) = \int_{a_k(-\check{\varphi})} \beta_j(-\check{\varphi}) = 2\pi\delta_{kj}.$$

As $\sigma^*\beta_j(-\check{\varphi})$ is uniquely determined by its nomalization conditions we conclude that

$$\sigma^*\beta_j(-\check{\varphi}) = \beta_{-j}(\varphi).$$

Further, using Proposition 1.2 and Proposition 1.4 we get

$$\begin{aligned} \sigma(\mathcal{P}_k(-\check{\varphi})) &= \sigma(\mu_k(-\check{\varphi}), \delta(\mu_k(-\check{\varphi}); -\check{\varphi})) \\ &= \sigma(-\mu_{-k}(\varphi), -\delta(\mu_{-k}(\varphi); \varphi)) \\ &= (\mu_{-k}(\varphi), \delta(\mu_{-k}(\varphi); \varphi)) = \mathcal{P}_{-k}(\varphi). \end{aligned}$$

Therefore, by a change of variable of integration in (2.5) we obtain

$$\theta_k(-\check{\varphi}) = \int_{\lambda^+_{-k}(\varphi)}^{\mathcal{P}_{-k}(\varphi)} \sum_{j\in\mathbb{Z}} \beta_j(\varphi). \tag{2.7}$$

By contour integration,

$$\int_{\lambda_k^-}^{\mathcal{P}_k} \sum_{j\in\mathbb{Z}} \beta_j - \int_{\lambda_k^+}^{\mathcal{P}_k} \sum_{j\in\mathbb{Z}} \beta_j \equiv \int_{\lambda_k^-}^{\lambda_k^+} \sum_{j\in\mathbb{Z}} \beta_j \pmod{2\pi}, \tag{2.8}$$

where in the last integral the path of integration is chosen to be the straight line on the upper sheet connecting λ_k^- with λ_k^+. Notice that by the normalization conditions, $\int_{a_k} \beta_j = 2\pi\delta_{jk}$, we get $\forall k \in \mathbb{Z}$

$$\int_{\lambda_k^-}^{\lambda_k^+} \sum_{j\in\mathbb{Z}} \beta_j \equiv \pi \pmod{2\pi}. \tag{2.9}$$

Therefore combining (2.4), (2.7), (2.8) and (2.9) we obtain $\theta_k(-\check{\varphi}) \equiv \theta_{-k}(\varphi) + \pi$, which leads to (i) using Proposition 2.2 with $\alpha = \pi$.

In order to prove (ii), recall from Proposition 1.1 that $\lambda_k^\pm(\check{\varphi}) = \lambda_k^\pm(\bar{\varphi})$ and thus, in particular, $\beta_j(\check{\varphi}) = \beta_j(\bar{\varphi})$ $(\forall j \in \mathbb{Z})$. Further, by Proposition 1.3 and 1.4,

$$\mu_k(\check{\varphi}) = \mu_k(\bar{\varphi}); \quad \delta(\mu_k(\check{\varphi}); \check{\varphi}) = -\delta(\mu_k(\bar{\varphi}); \bar{\varphi}).$$

From (2.4) it thus follows that

$$\theta_k(\bar{\varphi}) = \int_{\lambda_k^-(\check{\varphi})}^{f(\mathcal{P}_k(\check{\varphi}))} \sum_{j\in\mathbb{Z}} \beta_j(\check{\varphi}) \tag{2.10}$$

where $f(\lambda, \delta) = (\lambda, -\delta)$ is the involution on $\mathcal{M}(\check{\varphi})$ exchanging the two sheets. As $-\beta_j(\varphi)$ satisfies the normalization conditions,

$$-\int_{f(a_k(\varphi))} \beta_j(\varphi) = 2\pi\delta_{kj},$$

we conclude that $f^*\beta_j(\varphi) = -\beta_j(\varphi)$. Therefore, by a change of variable of integration in (2.10), we get

$$\theta_k(\bar{\varphi}) = -\int_{\lambda_k^-(\check{\varphi})}^{\mathcal{P}_k(\check{\varphi})} \sum_{j\in\mathbb{Z}} \beta_j(\check{\varphi}) \equiv -\theta_k(\check{\varphi}). \tag{2.11}$$

Combining (2.11) and (i) we obtain (ii). ∎

Proposition 2.4 *Let $\varphi \in L^2(S^1; \mathbb{C}), \tau \in \mathbb{R}$ and $k \in \mathbb{Z}$ with $I_k(\varphi) \neq 0$. Then*

$$\theta_k(T_\tau\varphi) = \theta_k(\varphi) + 2\pi k\tau.$$

Proof By continuity, it suffices to consider $\varphi \in H^1(S^1; \mathbb{C})$. The translation flow $T_\tau\varphi(\cdot) = \varphi(\tau + \cdot)$ is the Hamiltonian flow associated with the Hamiltonian (see [FT] or [GG]).

$$\mathcal{H}_1(\varphi) = i/2 \int_{S^1} (\bar{\varphi}\varphi' - \varphi\overline{\varphi'})\, dx \tag{2.12}$$

which commutes with the NLS-Hamiltonian. Thus for $\varphi = \Phi((\sqrt{2I_j}e^{i\theta_j})_{j\in\mathbb{Z}})$ (Φ = Birkhoff map)

$$\theta_k(T_\tau\varphi) = \theta_k(\varphi) + w_k(I)\tau \tag{2.13}$$

where

$$w_k(I) := \frac{\partial \mathcal{H}_1}{\partial I_k}(I) \qquad (\forall k \in \mathbb{Z}) \tag{2.14}$$

are the frequencies of the translation flow. Since $\theta_k(T_1\varphi) = \theta_k(\varphi)$, there exists for any $k \in \mathbb{Z}$ and $I \in \ell^1(\mathbb{Z};\mathbb{C})$, an integer $n_k(I) \in \mathbb{Z}$ such that $w_k(I) = n_k(I)$. Furthermore, since $w_k(I)$ is continuous and $n_k(I)$ takes discrete values, $n_k(I)$ does not depend on I, i.e. $n_k(I) \equiv n_k$. From Lemma 2.5 below we deduce that for a 1-gap potential φ with $I_j(\varphi) = 0 \quad (\forall j \neq k)$ and $I_k(\varphi) > 0$ the frequency $w_k(I)$ is given by $w_k(I) = 2\pi k$. ∎

Lemma 2.5 *For $k \in \mathbb{Z}$*

$$\{\varphi \in L^2(S^1;\mathbb{C}) \mid \gamma_j(\varphi) = \gamma\delta_{kj},\ j \in \mathbb{Z}\} = \{\varphi(x) = ce^{2i\pi kx} \mid c \in \mathbb{C},\ \ |c| = \gamma/2\}.$$

Proof A straightforward computation proves that for $\varphi \in L^2(S^1;\mathbb{C})$ and $j, k \in \mathbb{Z}$ arbitrary,

$$\gamma_j(e^{-2i\pi kx}\varphi) = \gamma_{j-k}(\varphi). \tag{2.15}$$

Further one knows (cf [GG])

$$\{\varphi \in L^2(S^1;\mathbb{C}) \mid \gamma_j(\varphi) = \gamma\delta_{0j},\ \forall j \in \mathbb{Z}\} = \{\varphi(x) = c \mid c \in \mathbb{C},\ |c| = \gamma/2\}. \tag{2.16}$$

Combining (2.15) and (2.16), Lemma 2.5 follows. ∎

3 Applications

3.1 Symmetries of the Hamiltonian and its frequencies

As already mentioned in the introduction, the NLS Hamiltionian $\mathcal{H}$ is invariant under $\mathcal{S}_2$. When $\mathcal{H}$ is expressed with respect to action variables, $\mathcal{H} = \mathcal{H}(I)$, this invariance of $\mathcal{H}$ leads to our first application. For $I = (I_k)_{k\in\mathcal{Z}}$, denote by $\mathcal{J}(I)$ the sequence given by

$$\mathcal{J}(I)_k := I_{-k} \qquad (\forall k \in \mathbb{Z}).$$

Denote by $\omega_j := \frac{\partial \mathcal{H}}{\partial I_j}$ the j'th frequency.

Proposition 3.1

(i) $\mathcal{H}(\mathcal{J}(I)) = \mathcal{H}(I)$*;*

(ii) $\omega_j(I) = \omega_{-j}(I) \quad (\forall j \geq 1)$ *for* I *with* $\mathcal{J}(I) = I$.

Proof (i) follows from combining the two identities $\mathcal{H}(\check{\varphi}) = \mathcal{H}(\varphi)$ and $I(\check{\varphi}) = \mathcal{J}(I(\varphi))$ (Proposition 2.1).
(ii) Write $\mathcal{H}$ as a function of $r_k := \frac{I_k + I_{-k}}{2} \quad (k \geq 0)$ and $\rho_k := \frac{I_k - I_{-k}}{2} \quad (k \geq 1)$. By (i), $\mathcal{H}$ is an even function of the ρ_k's and thus $\frac{\partial \mathcal{H}}{\partial \rho_k} = 0$ at points where $\mathcal{J}(I) = I$. ∎

3.2 Symmetric phase spaces

For $\alpha \in \mathbb{R}$, introduce the subspace

$$P_\alpha := \{\varphi \in L^2(S^1; \mathbb{C}) \mid e^{i\alpha}\check{\varphi} = \varphi\}. \tag{3.1}$$

Notice that for $\alpha = \pi$, respectively $\alpha = 0$, $P_\alpha \cap C^\infty$ is the phase space consisting of elements $\varphi \in C^\infty$ satisfying a generalized Dirichlet respectively Neumann condition, i.e. for $\forall k \geq 0$,

$$\partial_x^{2k}\varphi(0) = \partial_x^{2k}\varphi(1) = 0 \quad \text{(Dirichlet)}; \quad \partial_x^{2k+1}\varphi(0) = \partial_x^{2k+1}\varphi(1) = 0 \quad \text{(Neumann)}.$$

Next, introduce the subspace

$$Q_\alpha := \{\varphi \in L^2(S^1; \mathbb{C}) \mid e^{i\alpha}\bar{\varphi} = \varphi\}.$$

Notice that $e^{i\alpha}\bar{\varphi} = \varphi$ means that φ is of the form $\varphi = e^{i\alpha/2}f(x)$ with $f(x)$ a real valued function.
Proposition 3.2 provides a charaterization of P_α and Q_α in terms of action-angle variables. Recall that Φ denotes the Birkhoff map, $\varphi = \Phi\left((\sqrt{2I_k}e^{i\theta_k})_{k\in\mathbb{Z}}\right)$.

Proposition 3.2 *The following statements hold:*

(i)
$$\begin{aligned} P_0 = \ & \{\Phi\left((\sqrt{2I_k}e^{i\theta_k})_{k\in\mathbb{Z}}\right) \mid I_k = I_{-k} \quad \forall k \geq 1; \\ & \theta_k \equiv \theta_{-k} \ (\mathrm{mod}\ 2\pi), \quad \forall k \geq 1 \textit{ with } I_k \neq 0\}; \end{aligned}$$

(ii) For $\alpha \not\equiv 0 \pmod{2\pi}$,
$$\begin{aligned} P_\alpha = \ & \{\Phi((\sqrt{2I_k}e^{i\theta_k})_{k\in\mathbb{Z}}) \mid I_0 = 0;\ I_k = I_{-k}, \quad \forall k \geq 1; \\ & \theta_k \equiv \theta_{-k} + \alpha \ (\mathrm{mod}\ 2\pi), \quad \forall k \geq 1 \textit{ with } I_k \neq 0\}; \end{aligned}$$

(iii)
$$\begin{aligned} Q_\alpha = \ & \{\Phi((\sqrt{2I_k}e^{i\theta_k})_{k\in\mathbb{Z}}) \mid I_k = I_{-k} \quad \forall k \geq 1 \\ & \theta_k \equiv -\theta_{-k} + \alpha \ (\mathrm{mod}\ 2\pi) \quad \forall k \geq 0 \textit{ with } I_k \neq 0\}. \end{aligned}$$

Proof (i) Let us denote by $\tilde{P}_0$ the set on the right side of equality (i). If $\varphi \in P_0$, Proposition 2.1 and 2.3 imply $\varphi \in \tilde{P}_0$. Conversely, if $\varphi \in \tilde{P}_0$ then by Proposition 2.1, 2.3 $I_k(\check{\varphi}) = I_k(\varphi) \quad (\forall k \in \mathbb{Z})$ and $\theta_k(\check{\varphi}) = \theta_k(\varphi) \quad \forall k \in \mathbb{Z}$ with $I_k \neq 0$. Thus $\check{\varphi} = \varphi$ since Φ is one to one.

(ii) Let us denote $\tilde{P}_\alpha$ the set on the right side of the equality (ii). If $\varphi \in \tilde{P}_\alpha$ then, by Proposition 2.1, 2.2, 2.3, $I_k(e^{i\alpha}\check{\varphi}) = I_k(\varphi)$ $(\forall k \in \mathbb{Z})$ and $\theta_k(e^{i\alpha}\check{\varphi}) = \theta_k(\varphi)$ $(\forall k \in \mathbb{Z}$ with $I_k \neq 0)$ and thus $\varphi \in P_\alpha$. Conversely, if $\varphi \in P_\alpha$ we have $I_k(\varphi) = I_{-k}(\varphi) \quad (\forall k \in \mathbb{Z})$ and $\theta_k(\varphi) \equiv \theta_{-k}(\varphi) + \alpha \pmod{2\pi}$ $(\forall k \in \mathbb{Z}$ with $I_k(\varphi) \neq 0)$. In particular, if $I_0(\varphi) \neq 0,\ \theta_0(\varphi) \equiv \theta_0(\varphi) + \alpha \pmod{2\pi}$. As $\alpha \not\equiv 0 \pmod{2\pi}$, it follows that $I_0(\varphi) = 0$ and thus $\varphi \in \tilde{P}_\alpha$.
The statement (iii) is proved in a similar way. ■

Further, introduce the subspace $E_\alpha := \{\varphi \in L^2(S^1;\mathbb{C}) \mid e^{i\alpha}\check{\varphi} = \bar{\varphi}\}$ and let $E := E_0$. Notice that E is also given by

$$E = \{\varphi \in L^2(S^1;\mathbb{C}) \mid \mathrm{Re}\varphi \text{ is even}; \quad \mathrm{Im}\varphi \text{ is odd}\} \tag{3.2}$$

and recall from [GG] that

$$\begin{aligned} E &= \{\varphi \in L^2(S^1;\mathbb{C}) \mid \mathit{spec}_{per}L(\varphi) = \mathit{spec}^+_{Dir}L(\varphi) \cup \mathit{spec}^-_{Dir}L(\varphi)\} \\ &= \{\varphi \in L^2(S^1;\mathbb{C}) \mid \forall k \in \mathbb{Z},\ \{\lambda_k^+(\varphi), \lambda_k^-(\varphi)\} = \{\mu_k(\varphi), \nu_k(\varphi)\}\}. \end{aligned} \tag{3.3}$$

Proposition 3.3

$$E_\alpha = \{\Phi((\sqrt{2I_k}e^{i\theta_k})_{k\in\mathbb{Z}}) \mid \theta_k \equiv \alpha/2 \pmod{\pi} \quad \forall k \in \mathbb{Z} \text{ with } I_k \neq 0\}.$$

Proof The statement is proved in a similar way as Proposition 3.2. ■

Remark In particular for $\alpha = 0$, Proposition 3.3, combined with (3.3), shows that

$$\begin{aligned} &\{\varphi \in L^2(S^1;\mathbb{C}) \mid \mathit{spec}_{per}L(\varphi) = \mathit{spec}^+_{Dir}L(\varphi) \cup \mathit{spec}^-_{Dir}L(\varphi)\} = \\ &\{\Phi((\sqrt{2I_k}e^{i\theta_k})_{k\in\mathbb{Z}}) \mid \theta_k \equiv 0 \pmod{\pi} \quad \forall k \in \mathbb{Z} \text{ with } I_k \neq 0\}. \end{aligned}$$

3.3 Symmetric spectrum

In this subsection we present two spectral results. Recall from [GG] that the sequence of the gap lengths, $(\gamma_k(\varphi))_{k\in\mathbb{Z}} = (\lambda_k^+(\varphi) - \lambda_k^-(\varphi))_{k\in\mathbb{Z}}$, uniquely determines the periodic spectrum of $L(\varphi)$. Similarly, the sequence of actions

$(I_k(\varphi))_{k\in\mathbb{Z}}$ determines uniquely the periodic spectrum of $L(\varphi)$ (cf [GK1], [MV], [BBGK]). In Theorem 3.1, and 3.2 below, we prove that the periodic spectrum is symmetric if and only if the sequence of the actions is symmetric (Theorem 3.1) or, if and only if the sequence of the gap lengths is symmetric (Theorem 3.2).

Theorem 3.1 *For any $\varphi \in L^2(S^1;\mathbb{C})$, the following two statements are equivalent:*

(i) $\lambda_k^{\pm}(\varphi) = -\lambda_{-k}^{\mp}(\varphi) \;\; \forall k \in \mathbb{Z}$;

(ii) $I_k(\varphi) = I_{-k}(\varphi) \;\; \forall k \in \mathbb{Z}$.

Proof In view of the formula for the actions, (i) implies (ii). Conversely, assume (ii) holds. As the Birkhoff map is bijective, there exists $\varphi_0 \in L^2(S^1;\mathbb{C})$ such that $I_k(\varphi_0) = I_k(\varphi)$ $\forall k \in \mathbb{Z}$ and $\theta_k(\varphi_0) \equiv \theta_{-k}(\varphi_0) \pmod{2\pi}$ $(\forall k \in \mathbb{Z}$ with $I_k \neq 0)$. By Proposition 3.2 (i), it follows that $\check{\varphi}_0 = \varphi_0$. By Proposition 1.1 (ii), this then implies that $spec_{per} L(\varphi_0)$ is symmetric. Since φ_0 and φ are isospectral, $spec_{per} L(\varphi)$ is symmetric. ■

Theorem 3.2 *For any $\varphi \in L^2(S^1;\mathbb{C})$, the following two statements are equivalent:*

(i) $\lambda_k^{\pm}(\varphi) = -\lambda_{-k}^{\mp}(\varphi), \;\; \forall k \in \mathbb{Z}$;

(ii) $\gamma_k(\varphi) = \gamma_{-k}(\varphi), \;\; \forall k \in \mathbb{Z}$.

Proof As we did not succeed to deduce the symmetry of the sequence of gap lengths, $\gamma_k = \gamma_{-k}$ $(\forall k \in \mathbb{Z})$, from the symmetry of the sequence of actions, $I_k = I_{-k}$ $(\forall k \in \mathbb{Z})$, we use, instead of Birkhoff coordinates, the coordinates defined in [BGGK]. In that paper, it is proved that there exist angles, $(\tilde{\theta}_k(\varphi))_{k\in\mathbb{Z}} \in (\mathbb{R}/\pi\mathbb{Z})^\infty$, depending continuously on $\varphi \in L^2(S^1;\mathbb{C})$ for φ with $\gamma_k(\varphi) \neq 0$, such that the map $\tilde{\Phi}(\varphi) := (\tilde{\Phi}(\varphi))_{k\in\mathbb{Z}}$ from $L^2(S^1;\mathbb{C})$ into $\ell^2(\mathbb{Z};\mathbb{R}^2)$, defined by $\tilde{\Phi}_k(\varphi) = \frac{\gamma_k(\varphi)}{2}(\cos 2\tilde{\theta}_k(\varphi), \sin 2\tilde{\theta}_k(\varphi))$, is a homeomorphism. Arguing in the same way as in the proof of Theorem 3.1, Theorem 3.2 then follows from Proposition 3.4 below. ■

Proposition 3.4 *For $\varphi \in L^2(S^1;\mathbb{C})$ and $k \in \mathbb{Z}$,* $\tilde{\theta}_k(\check{\varphi}) \equiv \tilde{\theta}_{-k}(\varphi) \pmod{\pi}$.

Proof The statement is proved in Appendix A. ■

3.4 Additional symmetries of the spectrum

Theorem 3.2 suggests that the 0's gap plays a special role for symmetry properties of the spectrum. Indeed, if we consider the action ρ of $\mathbb{Z}$ on $L^2(S^1;\mathbb{C})$ given by $\rho(j)\varphi(x) = e^{-2\pi ijx}\varphi(x)$, one verifies easily (cf (2.15)) that the following statements are equivalent

(i) $\lambda^{\pm}_{j+k}(\varphi) = -\lambda^{\mp}_{j-k}(\varphi) + 2j\pi, \quad \forall k \in \mathbb{Z}$;

(ii) $\gamma_{j+k}(\varphi) = \gamma_{j-k}(\varphi), \quad \forall k \in \mathbb{Z}$.

A Appendix

In this appendix, we prove Proposition 3.4. First, recall some notations and results from [GK1] (cf also [BGGK]). Let $\varphi \in L^2(S^1;\mathbb{C})$ and $k \in \mathbb{Z}$ with $\gamma_k(\varphi) \neq 0$. Denote by $F^{\pm}_k(\cdot) = F^{\pm}_k(\cdot;\varphi) = (F^{\pm}_{k,1}, F^{\pm}_{k,2})$ the eigenfunctions of norm 1 of the AKNS-operator $H(\varphi)$, corresponding to the eigenvalues $\lambda^{\pm}_k(\varphi)$ normalized as follows,

$$\begin{array}{ll} F^{\pm}_{k,1}(0) > 0 & (\text{if } F^{\pm}_{k,1}(0) \neq 0) \\ F^{\pm}_{k,2}(0) > 0 & (\text{if } F^{\pm}_{k,1}(0) = 0). \end{array}$$

Let $E_k(\varphi)$ be the two-dimensional subspace of $L^2(S^1;\mathbb{R}^2)$ generated by $F^+_k(\cdot;\varphi)$ and $F^-_k(\cdot;\varphi)$ and denote by $G^+_k(\cdot,\varphi), G^-_k(\cdot,\varphi)$ the basis of $E_k(\varphi)$ uniquely determined by

$$\begin{array}{ll} \text{(i)} & \| G^{\pm}_k(\cdot;\varphi) \|_{L^2} = 1; \\ \text{(ii)} & G^-_{k,1}(0;\varphi) = 0 \text{ and } G^-_{k,2}(0;\varphi) > 0; \\ \text{(iii)} & \langle G^+_k(\cdot;\varphi), G^-_k(\cdot;\varphi)\rangle_{L^2} = 0; \\ \text{(iv)} & G^+_{k,1}(0;\varphi) > 0. \end{array} \tag{A.1}$$

Let $\tilde{\theta}_k(\varphi) \in [0, 2\pi)$ be the counterclockwise oriented angle between $G^+_k(\varphi)$ and $F^+_k(\varphi)$ and introduce $\tilde{\Phi}_k(\varphi) := (\tilde{\Phi}_{k,1}(\varphi), \tilde{\Phi}_{k,2}(\varphi))$ with

$$\tilde{\Phi}_{k,1}(\varphi) := \langle G^+_k(\cdot;\varphi), (H(\varphi) - \tau_k(\varphi))G^+_k(\cdot,\varphi)\rangle \tag{A.2}$$

and

$$\tilde{\Phi}_{k,2}(\varphi) := \langle G^+_k(\cdot;\varphi), (H(\varphi) - \tau_k(\varphi))G^-_k(\cdot,\varphi)\rangle, \tag{A.3}$$

where $\tau_k(\varphi) := (\lambda_k^+(\varphi)+\lambda_k^-(\varphi))/2$. One verifies in a straightforward way that

$$\tilde{\Phi}_k(\varphi) = \frac{\gamma_k(\varphi)}{2}(\cos 2\tilde{\theta}_k(\varphi), \sin 2\tilde{\theta}_k(\varphi)). \tag{A.4}$$

From [BGGK] we learn that $\tilde{\Phi} = (\tilde{\Phi}_k)_{k\in\mathbb{Z}}$ is a homeomorphism from $L^2(S^1;\mathbb{C})$ onto $\ell^2(\mathbb{Z};\mathbb{R}^2)$.

Lemma A.1 *For $\varphi \in L^2(S^1;\mathbb{C})$ and $k \in \mathbb{Z}$ with $\gamma_k(-\check{\varphi}) \neq 0$,*

$$\tilde{\theta}_k(-\check{\varphi}) \equiv \tilde{\theta}_{-k}(\varphi) + \pi/2 \ (\mathrm{mod}\ \pi)$$

Proof By (1.16) and Proposition 1.1

$$\begin{aligned} E_k(-\check{\varphi}) &= span(F^+_{-k}(1-x;\varphi), F^-_{-k}(1-x;\varphi)) \\ &= span(G^+_{-k}(1-x;\varphi), G^-_{-k}(1-x;\varphi)). \end{aligned} \tag{A.5}$$

Hence, as $G^{\pm}_{-k}(1;\varphi) = (-1)^k G^{\pm}_{-k}(0;\varphi)$ (see [BGGK]), we obtain, using (A.1),

$$G_k^{\pm}(x, -\check{\varphi}) = (-1)^k G^{\pm}_{-k}(1-x;\varphi). \tag{A.6}$$

By (A.2), (A.3) (A.6) and Proposition 1.1 one concludes that

$$\tilde{\Phi}_k(-\check{\varphi}) = -\tilde{\Phi}_{-k}(\varphi). \tag{A.7}$$

In view of (A.4) and (A.7), Lemma A.1 follows. ■

Lemma A.2 *For $\varphi \in L^2(S^1;\mathbb{C})$, $k \in \mathbb{Z}$ with $\gamma_k(\varphi) \neq 0$, and $\alpha \in \mathbb{R}$,*

$$\tilde{\theta}_k(e^{i\alpha}\varphi) \equiv \theta_k(\varphi) + \frac{\alpha}{2} \ (\mathrm{mod}\ \pi).$$

Proof Let $\tilde{V}_\alpha$ be the following rotation matrix

$$\tilde{V}_\alpha := \begin{pmatrix} \cos\alpha/2 & -\sin\alpha/2 \\ \sin\alpha/2 & \cos\alpha/2 \end{pmatrix}. \tag{A.8}$$

One has (cf (1.2))

$$\tilde{V}_\alpha^{-1} H(e^{i\alpha}\varphi)\tilde{V}_\alpha = H(\varphi) \tag{A.9}$$

and therefore, with $\varepsilon_k = \pm 1$,

$$F_k^+(\cdot; e^{i\alpha}\varphi) = \varepsilon_k \tilde{V}_\alpha F_k^+(\cdot;\varphi) \tag{A.10}$$

and

$$E_k(e^{i\alpha}\varphi) = \tilde{V}_\alpha E_k(\varphi) = E_k(\varphi). \tag{A.11}$$

As a consequence,

$$G_k^\pm(e^{i\alpha}\varphi) = G_k^\pm(\varphi). \tag{A.12}$$

Using (A.10), (A.12) and the definition of the angle $\tilde{\theta}_k$, one obtains

$$\tilde{\theta}_k(e^{i\alpha}\varphi) = \tilde{\theta}_k(\varphi) + \frac{\alpha}{2} \pmod{\pi}. \quad \blacksquare \tag{A.13}$$

Proof (Proposition 3.4). Combining Lemma A.1 and A.2 (with $\alpha = \pi$) Proposition 3.4 follows. $\blacksquare$

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