SCALING PROPERTIES OF HYPERBOLIC MEASURES

J. SCHMELING AND S. TROUBETZKOY

ABSTRACT. In this article we consider a class of maps which includes $C^{1+\alpha}$ diffeomorphisms as well as invertible and nonivertible maps with piecewise smooth singularities. We prove a general scaling result for any hyperbolic measure which is invariant for a map from our class. The existence of the pointwise dimension and the Brin-Katok local entropy formula are special cases of our scaling result.

1. INTRODUCTION

Let

$$d_m(x) := \lim_{r\to 0} \frac{\log m(B(x,r))}{\log r} \tag{1}$$

whenever the limit exists. For ergodic measures, if the limit exists, then it is almost everywhere constant and is called the pointwise dimension of the measure m. Recently Barreira, Pesin and Schmeling proved that if f is a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold then the pointwise dimension exists almost everywhere for any f-invariant hyperbolic measure m [1]. The proof of this result uses the local geometric structure of hyperbolic measures (local product structure).

Let $f\colon (X,d) \to (X,d)$ a continuous mapping of a compact metric space preserving the Borel probability nonatomic measure m. Define $B_n^f(x,\delta) := \{y\colon d(f^i x, f^i y) \le \delta,\ 0 \le i \le n-1\}$ and

$$h_m(f,x) := \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{\log m(B_n^f(x,\delta))}{-n} \tag{2}$$

whenever the limit exists. We call $B_n^f(x,\delta)$ a Bowen ball and the limit, when it exists, the local entropy. Local entropy can also be defined through backwards or two sides Bowen balls. Brin and Katok have shown that if m is ergodic with finite entropy, then the local entropy exists and $h_m(f,x) = h_m(f)$ for almost every $x \in X$ [3]. The proof of this formula is based on probabilistic methods (Shannon-McMillan-Breiman).

In this article we show that for hyperbolic measures the existence of the pointwise dimension and the Brin-Katok formula are connected and are included in a more general statement. Instead of a ball $B(x,r)$ in equation (1) or the Bowen ball in equation (2) we consider a

1991 *Mathematics Subject Classification.* 58F11, 28D05.

This paper was written while ST was visiting FUB supported by DANSE. We would like to thank Yasha Pesin for useful remarks. This paper generalizes an earlier preprint titled "Pointwise dimension for regular hyperbolic measures for endomorphisms".

rectangle (or ellipsoid) $R_n(x)$ whose axes are in the directions of the hyperbolic splitting at x. The length of the ith axis scales like $e^{a_i n}$ where $a_i < 0$. Our main result is that for any fixed scaling factors $a_1, \ldots, a_k$, the scaled pointwise dimension of the hyperbolic measure m,

$$\rho_m(x) = \rho_m(a_1, \ldots, a_k; x) := \lim_{n\to\infty} \frac{\log m(R_n(x))}{-n}$$

exists m-almost everywhere. In the case of smooth maps preserving hyperbolic measures if we set $a_i = \chi_i$, where χ_i is the ith Lyapunov exponent, then we recover the Brin–Katok theorem [3, 7]. If all the a_i's are equal then we recover the result of [1]. This means that hyperbolic measures exhibit a parameter family of scaling laws connecting the classical scalings along cylinders (Brin–Katok formula) or balls (pointwise dimension). We prove our result in the more general setting of hyperbolic maps with piecewise smooth singularities [5].

Finally we give an application of our scaling result to hyperbolic endomorphisms. Using the scaling result and the solenoidal construction developed in [15] we prove the existence of pointwise dimension for hyperbolic measures preserved by endomorphisms with piecewise smooth singularities. We remark that there is no natural way to define other scaling properties of hyperbolic measures for endomorphisms due to the non-uniqueness of unstable splittings.

The existence of the pointwise dimension has several important implications, for endomorphisms as well as for diffeomorphisms. The Hausdorff dimension and the box dimension of the measure as well as some other characteristics of dimension type of the measure coincide. We refer the reader to the reference [9] for a survey of these characteristics.

The proof of our theorem in the invertible case follows the proof of the existence of pointwise dimension in [1] closely. The only change that needs to be made is to prove the Borel density lemma for the special class of rectangles considered instead of balls. To show the existence of pointwise dimension for endomorphisms we use a solenoidal construction to get an invertible map with singularities and then show that the existence result still holds when projecting back to our original system.

2. Lyapunov charts and regular hyperbolic measures for endomorphisms

We consider the following class of endomorphisms with singularities. This class includes local diffeomorphisms with (or without) singularities that are finite unions of proper submanifolds. Let M be a p-dimensional Riemannian manifold, $K \subset M$ an open, bounded connected subset with compact closure and $N \subset K$ a closed subset. Let $K \backslash N = K_1 \cup \ldots \cup K_r$ where the K_i are disjoint open sets ($r < \infty$) with piecewise smooth boundary $\partial K_i \subset N \cup \partial K$, $\cup \partial K_i = N \cup \partial K$ and $f : K\backslash N \to K$. We assume the following condition holds:

$(A1)$ f can be extended to a $C^{1+\alpha}$–diffeomorphism $f : \overline{K_i} \to \overline{f(K_i)}$, $i = 1, 2, \ldots r$ for some $\alpha > 0$ and $f(K_i) \cap f(K_j) = \emptyset$ if $i \neq j$. We sometimes write f_i for $f|_{\overline{K_i}}$.

This is not the most general condition under which the results may be developed. But for sake of simplicity and because most of the applications fulfill $(A1)$ we restrict to this case. For the more general setup see [5]. In order to deal with diffeomorphisms without singularities we simply assume $N = \emptyset$ and $K = M$.

We let

$$\begin{aligned} N^+ &= N \cup \partial K, \\ N^- &= \{y \in K : \exists z \in N^+,\ z_n \in K \backslash N^+ \text{ s.t. } z_n \to z, f(z_n) \to y\}. \end{aligned}$$

The set N^+ consists of the singularity set and the boundary of K and N^- is the full "preimage" of N^+. Since we are interested in the asymptotic behavior of infinite orbits we have to exclude those points which are eventually mapped onto the singularities or the boundary. Thus we let

$$\begin{aligned} K^+ &= \{x \in K :\ f^n x \notin N^+,\ n = 0, 1, 2, ...\}, \\ D &= \bigcap_{n \geq 0} f^n(K^+), \\ \Lambda &= \overline{D}. \end{aligned}$$

Λ is called the attractor for f. Obviously, D is invariant: $f(D) = D$. Although no trajectory in K^+ hits the singularities N^+ it may happen that some of the orbits come arbitrarily close to N^+. This makes it difficult to control their behavior in comparison to nearby points. In order to get more control we define the following filtration:

$\forall \varepsilon > 0,\ l = 1, 2, ...$ we define

$$\begin{aligned} &K^+_{\varepsilon,l} = \{z \in K^+ : d(f^n z, N^+) \geq l^{-1} e^{-\varepsilon n},\ n = 0, 1, 2, \cdots\}, \\ &D^+_{\varepsilon,l} = K^+_{\varepsilon,l} \cap \Lambda. \end{aligned}$$

${D^+}_{\varepsilon,l}$ is closed (as a subset of Λ) and $D^+_\varepsilon = \bigcup_{l \geq 1} D^+_{\varepsilon,l}$ is f–invariant. Λ is called *regular* if $D^+_\varepsilon \neq \emptyset$ for all $\varepsilon > 0$ small enough.

We define

$$\begin{aligned} D^-_{\varepsilon,l,n} &:= \left\{x \in D : d(f^{-k}(x), N^+) \geq l^{-1} e^{-\varepsilon k}, k = 0, 1, \ldots, n\right\}; \\ D^-_{\varepsilon,l} &:= \left\{x \in D : d(f^{-n}(x), N^+) \geq l^{-1} e^{-\varepsilon n}, n = 0, 1, \ldots\right\}. \end{aligned}$$

Then

$$D^-_{\varepsilon,l} = \bigcap_{n \geq 0} D^-_{\varepsilon,l,n}, \quad D^-_\varepsilon = \bigcup_l D^-_{\varepsilon,l}$$

and

$$D_\varepsilon = D^+_\varepsilon \cap D^-_\varepsilon.$$

Definition 2.1. *We call a point $z \in D_\varepsilon$ regular if there exist numbers $\chi_1(z) < \cdots < \chi_{k(z)}(z)$ (called Lyapunov exponents) and a decomposition $T_z M = \oplus_{i=1}^{k(z)} E^i(z)$ composed of the vector spaces*

$$E^i(z) = \left\{v \in T_z M \setminus \{0\} : \lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n(z)v\| = \chi_i(z)\right\} \cup \{0\}$$

such that

$$\lim_{n\to\pm\infty}\frac{1}{n}\log|\det Df(z)| = \sum_{i=1}^{k(z)}\chi_i(z)\dim E^i(z).$$

Note that $\chi_i(fz) = \chi_i(z)$ and $DfE^i(z) = E^i(fz)$ for each i. Let $s(z) = \max\{i : \chi_i(z) < 0\}$ and $u(z) = \min\{i : \chi_i(z) > 0\}$.

Remark 2.2. *If f is a diffeomorphism of a compact manifold - and m is an invariant measure, Oseledec's theorem tells us that m-almost every point is regular. In general this statement is no longer true in the presence of singularities. Under some additional assumptions one can prove an analogous statement for the set D_ε* [5]*. Regular points always have Lyapunov charts, defined below.*

For a regular point z let $e_i(z) = \dim E^i(z)$. Let $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$, $|x| = \max|x_i|$ and $R(\rho) = \{x \in \mathbb{R}^p : \ |x| \le \rho\}$. We fix $\varepsilon > 0$ small. Then for δ sufficiently small there exists a measurable function $r : D_\varepsilon \to (1,\infty)$ with $r(f^{\pm 1}z) \le e^\delta r(z)$ and an embedding $\Phi_z : R(r(z)^{-1}) \to M$ such that the following conditions hold:

(i) $\Phi_z 0 = z$ and $D\Phi_z(0)$ maps $\{0\}\times\cdots\times\{0\}\times\mathbb{R}^{e_i}\times\{0\}\times\cdots\times\{0\}$ to $E^i(z)$;
(ii) $\exp_z^{-1}\circ\Phi_z$ coincides with $D\Phi_z(0)$ on $R(r(z)^{-1})$;
(iii) For $\tilde{f}_z = \Phi_{\tilde{f}_z}^{-1}\circ f\circ\Phi_z$ and $v \in \{0\}\times\cdots\times\{0\}\times\mathbb{R}^{e_i}\times\{0\}\times\cdots\times\{0\}$

$$e^{\chi_i-\delta}|v| \le |D\tilde{f}_z(0)v| \le e^{\chi_i+\delta}|v|$$

(iv) The Lipschitz constants L satisfies

$$L\left(\tilde{f}_z - D\tilde{f}_z(0)\right) \le \delta,$$
$$L\left(\tilde{f}_z^{-1} - D\tilde{f}_z^{-1}(0)\right) \le \delta \quad \text{and}$$
$$L(D\tilde{f}_z) \le r(z), \quad L(D\tilde{f}_z)^{-1} \le r(z);$$

(v) For all $y, y' \in R(r(z)^{-1})$

$$C^{-1}dist(\Phi_z y, \Phi_z y') \le |y - y'| \le r(z)dist(\Phi_z y, \Phi_z y')$$

for some constant C.
(vi) There is an $r = r(\varepsilon, l)$ such that $r(z) < r$ on $D_{\varepsilon,l}$.

The system of local charts $\{\Phi_z\}$, z a regular point, is called the Lyapunov chart system.

Definition 2.3. *We say that an invariant Borel probability measure m is* regular hyperbolic *if $m(D_\varepsilon) = 1$ for ε sufficiently small and all of its Lyapunov exponents are non-zero– i.e. $s(z) + 1 = u(z)$ a.e.*

Remark 2.4. *$m(D_\varepsilon) = 1$ implies the existence of Lyapunov charts for a.e. $z \in D_\varepsilon$* [5].

Remark 2.5. *One of the main examples of a regular hyperbolic measure is the SBR-measure. It has been shown to exist for a large class of hyperbolic maps* [8]*. In particular it has been shown to exist if the map is additionally C^2, has a hyperbolic cone field, has control how it can blow up near the singularities and satisfies the assumption.*

$$(A2) \qquad \exists C>0, q>0 \text{ such that } \forall \varepsilon>0\ \forall n>0, \quad \mathrm{Leb}(f^{-n}U(\varepsilon,\, N^+)) \le C\varepsilon^q.$$

We described the notion of Lyapunov charts in detail because they are the main tool in developing stable and unstable manifold theory. Lyapunov charts give control over stretching and contracting in the first step of iterating f while Lyapunov exponents are effective only asymptotically. Unfortunately, we have only little control over the size of the charts and in general the metric in the charts derived from the Euclidean metric is only measurable. Also we have to pay an exponentially small error in the exact stretching and contracting exponents. However it is possible to show the following stable manifold theorem.

For $C>0, z \in D_\varepsilon$ let

$$W^i(z) = \left\{ y \in \Lambda : d(f^{-n}y, f^{-n}z) \le C \cdot e^{-\chi_i(z)n},\ n=0,1,\dots \right\} \quad u(z) \le i \le k(z),$$
$$W^i(z) = \left\{ y \in K^+ : d(f^{n}y, f^{n}z) \le C \cdot e^{\chi_i(z)n},\ n=0,1,\dots \right\} \quad 1 \le i \le s(z).$$

For $\beta>0$ and $i \notin (s(z), u(z))$ let $W^i_\beta(z)$ be the connected component of $W^i(z) \cap B(x,\beta)$ containing z.

Theorem 2.6 ([5], [8], [12]). *Let f be as above and m a hyperbolic measure. Then the following assertions hold for m-a.e. z: there exists $C>0$ so that if $z \in D_{\varepsilon,l}$*

(i) There is a $\beta_0(\varepsilon,l)$ such that for $0<\beta<\beta_0(\varepsilon,l)$ the sets $W^i_\beta(z)$ are embedded disks.
(ii) For $y \in W^i_\beta(z)$ the tangent space $T_yW^i_\beta(z)$ coincides with $\oplus_{j\le i}E^j(y)$ if $i \le s(z)$ or $\oplus_{j\ge i}E^j(y)$ if $i \ge u(z)$.
(iii) $W^i_\beta(z)$ contains a ball of radius β inside $W^i(z)$ with the intrinsic metric.
(iv) $W^i_\beta(z)$ depends continuously on z on the set $D_{\varepsilon,l}$.
(v) For β_1 sufficiently small

$$f^{-1}W^i_{\beta_1}(z) \subset W^i_\beta(f^{-1}z) \quad i \ge u(z)$$
$$fW^i_{\beta_1}(z) \subset W^i_\beta(f\,z) \quad i \le s(z).$$

These facts will be used in the following sections. They are the main tools in analyzing hyperbolic systems with singularities.

3. The main theorem

In this section we give the main definition and results from [7] and produce a measurable partition of our system which is needed for the proof of our main results: theorems 3.9 and 5.8.

Remark 3.1. *In this paper the results are stated for C^2-diffeomorphisms but they depend only on the existence of Lyapunov charts with standard properties and the regularity of the invariant foliations (for the $C^{1+\alpha}$ case see [2]).*

By the regularity of the measure we are in the situation that the set of regular points - i.e. the points for which Lyapunov charts exist - has full m–measure. Thus we are able to apply the results of [7]. The basic idea in [7] is to consider conditional measures on the manifolds

$W_\beta^{(u)}(x)$, and $W_\beta^{(s)}(x)$. Unfortunately, the partitions into stable and unstable manifolds is not in general measurable. Thus we have to construct a measurable partition which gives us the information we want to have.

Let m be an ergodic regular hyperbolic measure for f. In the non-ergodic case there are standard techniques to reduce to the ergodic case, see [1] for details. Then $s(z)$, $u(z)$ are constant a.e. We denote these a.s. constants by s, u, respectively.

Definition 3.2. (see [7] or [11]) *A partition ξ is m-measurable iff for m–a.e. x there is a normalized measure m_x^ξ living on the partition element $\xi(x)$ containing x such that for the sub-σ-algebra $\mathcal{B}_\xi$ consisting entirely of unions of atoms of the partition ξ and a measurable set A the function $x \to m_x^\xi(A)$ is $\mathcal{B}_\xi$ - measurable and $m(A) = \int m_x^\xi(A)dm$. The measures m_x^ξ are called the conditional measures of m w.r.t. ξ. They are uniquely defined up to a set of measure 0.*

Definition 3.3. *A partition ξ is subordinate to the W^i foliation if for m.-a.e. x*

(i) $\xi(x) \subset W^i(x)$
(ii) $\xi(x)$ *contains a neighborhood of x in $W^i(x)$*

Partitions like this were considered by several authors ([16], [6], [7]). Their existence is ensured by:

Proposition 3.4. [7] *For $1 \le i \le k$ there exist measurable partitions ξ^i with the following properties:*

(i) ξ^i *is subordinate to* W^i
(ii) ξ^i *is generating - i.e.* $\bigvee_{n=0}^{\infty} f^{-n}\xi^i$ *if* $u \le i$ *or* $\bigvee_{n=0}^{\infty} f^{n}\xi^i$ *if* $i \le s$ *is the partition into points.*

Let $\{m_x^i\}$, $1 \le i \le k$ be fixed versions of conditional measures associated to m and ξ^i. For $x \in M$, $\gamma > 0$, $1 \le i \le s$ we define

$$\underline{h}_i(x,\gamma,\xi^i) = \liminf_{n\to\infty} -\tfrac{1}{n}\log m_x^i V^i(x,n,\gamma) \quad \text{and}$$
$$\overline{h}_i(x,\gamma,\xi^i) = \limsup_{n\to\infty} -\tfrac{1}{n}\log m_x^i V^i(x,n,\gamma)$$

with $V^i(x,n,\gamma) = \left\{y \in W^i(x) : dist^i(f^{-k}x, f^{-k}y) < \gamma, 0 \le k \le n\right\}$ where $dist^i$ is the induced metric on W^i.

The above definitions are motivated by the Shannon-McMillan-Breiman theorem on the pointwise entropy of a measure. They serve as characterizations of entropy type along the strong stable, stable and unstable direction, respectively. In [7] these definition are justified by a refinement of the Shannon-McMillan-Breiman theorem. Then ([7], Prop. 7.2.1 and Cor. 7.2.2)

$$\underline{h}_i(x) = \lim_{\gamma\to 0} \underline{h}_i(x,\gamma,\xi^i) = \lim_{\gamma\to 0} \overline{h}_i(x,\gamma,\xi^i) \equiv h_i, \quad 1 \le i \le s$$

for m–a.e. x. Analogous definitions are made for $i = u, \dots k$ by substituting f^{-1} for f. Also, if $h_m(f)$ is the Kolmogorov-Sinai entropy of f with respect to m, then

$$h_u = h_s = h_m(f) \tag{3}$$

holds.

Our next step is to define the dimension of the measure along the stable, strong stable and unstable direction. Again this is done pointwise. We consider for the ball $B^i(x,\gamma)$ in $W^i(x)$ centered at x of radius γ the quantities $(1 \le i \le k)$

$$\underline{d}^i_m(x,\xi^i) = \liminf_{\gamma\to 0} \frac{\log m^i_x B^i(x,\gamma)}{\log \gamma} \quad \text{and}$$
$$\overline{d}^i_m(x,\xi^i) = \limsup_{\gamma\to 0} \frac{\log m^i_x B^i(x,\gamma)}{\log \gamma}.$$

For these dimension values we also have a theorem corresponding to the Shannon-McMillan-Breiman theorem for entropies. Then ([7], Prop. 7.3.1) for m–a.e. x

$$\underline{d}^i_m(x,\xi^i) = \overline{d}^i_m(x,\xi^i) \equiv d^i_m. \tag{4}$$

Moreover, the following holds:

Proposition 3.5 (Ledrappier, Young [7]). *Let $a_1, \dots, a_i$, $i \le s$ be negative numbers and $R^s_t = [-e^{a_1 t}, e^{a_1 t}]^{e_1} \times \cdots \times [-e^{a_i t}, e^{a_i t}]^{e_i} \times \{0\} \times \cdots \times \{0\}$. Then for m–a.e. z the following limit exists*

$$\rho^i_m(a_1,\dots,a_i;z) := \rho^i_m(z) := \lim_{t\to\infty} \frac{\log m^i_z(\Phi_z R^s_t)}{-t} = -\sum_{j\le i} a_j (d^j_m - d^{j-1}_m) \tag{5}$$

where $d^0_m := 0$. If m is in addition ergodic then $\rho^s_m(z)$ is constant m–a.e.

A similar result holds for $a_j, \dots, a_k$ $(j \ge u)$ and $R^u_t = \{0\} \times \cdots \times \{0\} \times [-e^{a_j t}, e^{a_j t}]^{e_j} \times \cdots \times [-e^{a_k t}, e^{a_k t}]^{e_k}$ where $d^{k+1}_m := 0$.

Actually, in [7] the above proposition is only stated for two special cases, $a_i = \chi_i$, and $a_i = -1$. In the first case equation (5) is known as the Ledrappier–Young formula and reads

$$h_i = -\sum_{j\le i} \chi_j (d^j_m - d^{j-1}_m) \quad 1 \le i \le s. \tag{6}$$

The second case yields the existence of the stable (or unstable) dimension. Inspection of the proof of theorem C in [7] yields the proof of the proposition simply by rescaling the balls by the proper coefficients. Note that the rectangle $\Phi_z R^s_n$ is defined for n sufficiently large that $R^s_n \subset R(r^{-1}(z))$. The R^s_n, R^u_n are rectangles, but by changing the metric on our Lyapunov charts they could have been chosen to be ellipsoids equally well.

Definition 3.6. *We call ρ^s_m, ρ^u_m the scaled stable and unstable dimensions of m.*

In [1] it was recently shown that the dimensions d^s_m and d^u_m sum:

Theorem 3.7 (Barreira, Pesin, Schmeling [1]). *For a regular hyperbolic measure m the limit*

$$d_m(z) := \lim_{\gamma\to 0} \frac{\log mB(z,\gamma)}{\log\gamma} = d_m^s(x) + d_m^u(x)$$

exists m–a.e. If m is in addition ergodic $d_m(z) = d_m$ is constant a.e.

Remark 3.8. *In fact the above theorem was proven for hyperbolic measures of a $C^{1+\alpha}$-diffeomorphism without singularities. But for the same reasons as we explained in remark 3.1 this result can be carried over to the situation of regular hyperbolic measures of a diffeomorphism with singularities.*

Using the ideas of [1] we will show that the scaled dimensions sum:

Theorem 3.9. *Suppose m is a regular hyperbolic measure. Let $a_1,\dots,a_k$ be negative numbers and $R_t := [-e^{a_1 t}, e^{a_1 t}]^{e_1} \times \cdots \times [-e^{a_k t}, e^{a_k t}]^{e_k}$. Then for m–a.e. z the following limit exists*

$$\begin{aligned} \rho_m(z) &:= \rho_m(a_1,\dots,a_k;z) := \lim_{t\to\infty} \frac{\log m(\Phi_z R_t)}{-t} \\ &= -\sum_{j\le s} a_j(d_m^j - d_m^{j-1}) - \sum_{j\ge u} a_j(d_m^j - d_m^{j+1}) = \rho_m^s(z) + \rho_m^u(z). \end{aligned}$$

If m is in addition ergodic then $\rho_m(z)$ is constant m–a.e.

Remark 3.10. *We note that $\lim_{t\to\infty} \frac{\log m(\Phi_z R_t)}{-t}$, $\lim_{t\to\infty} \frac{\log m_z^s(\Phi_z R_t^s)}{-t}$, and $\lim_{t\to\infty} \frac{\log m_z^u(\Phi_z R_t^u)}{-t}$ (or limsup and liminf if the limits do not exist) can be evaluated as a limit along the natural numbers.*

Theorem 3.7 is a special case of this theorem. To see this choose all $a_i = -1$. Property (v) of Lyapunov charts implies

$$B\left(z, \frac{e^{-n}}{r(z)}\right) \subset \Phi_z R_n(z) \subset B(z, e^{-n})$$

which immediately implies theorem 3.7.

4. Proof of theorem 3.9

The idea of the proof is to match the assumption of the proofs in [1]. The main difference to our situation is that we deal with exponentially thin rectangles rather than balls. The proof in [1] is more general than stated in the main theorem. It works as long as similar conditions (3) to (19) in [1] hold. Our goal is now to modify our situation and verify these conditions.

Lemma 4.1.

$$\limsup_{n\to\infty} \frac{\log m(\Phi_z R_n)}{-n} \quad \textit{and} \quad \liminf_{n\to\infty} \frac{\log m(\Phi_z R_n)}{-n}$$

are a.e. constant.

Proof. By condition $(A1)$ the norms of the derivatives of f are bounded. Hence there is a $k_0 > 0$ and a $n_0 > k_0$ such that

$$\Phi_{fz} R_{n-k_0} \supset f(\Phi_z R_n) \supset \Phi_{fz} R_{n+k_0}$$

for all $n > n_0$. This together with the ergodicity of m implies the assertion of the lemma. $\square$

We use the following notations. Let $\mathcal{P}$ be a partition. For every integers $k,\ l \geq 0$, we define the partition $\mathcal{P}_k^l = \bigvee_{n=-k}^{l} f^{-n}\mathcal{P}$. We observe that $\mathcal{P}_k^0(x) \cap \mathcal{P}_0^l(x) = \mathcal{P}_k^l(x)$.

Consider $a_1, \ldots, a_k$ as fixed throughout the proof. In [7], Ledrappier and Young constructed a special countable partition $\mathcal{P}$ of M of finite entropy satisfying the following properties. Given $0 < \eta < 1$, there exists a set $\Gamma' \subset M$ of measure $m(\Gamma') > 1 - \eta/3$, an integer $n_0 \geq 1$, and a number $C > 1$ such that for every $x \in \Gamma'$ and any integer $n \geq n_0$, the following statements hold:

(a) for all integers $k,\ l \geq 1$ we have

$$C^{-1}e^{-(l+k)h-(l+k)\eta} \leq m(\mathcal{P}_k^l(x)) \leq Ce^{-(l+k)h+(l+k)\eta},$$
$$C^{-1}e^{-kh-k\eta} \leq m_x^s(\mathcal{P}_k^0(x)) \leq Ce^{-kh+k\eta},$$
$$C^{-1}e^{-lh-l\eta} \leq m_x^u(\mathcal{P}_0^l(x)) \leq Ce^{-lh+l\eta},$$

where $h = h_m(f)$ is the Kolmogorov–Sinai entropy of f with respect to m;

(b)

$$\xi^s(x) \cap \bigcap_{n\geq 0} \mathcal{P}_0^n(x) \supset B^s(x, e^{-n_0}),$$
$$\xi^u(x) \cap \bigcap_{n\geq 0} \mathcal{P}_n^0(x) \supset B^u(x, e^{-n_0});$$

(c')

$$e^{-\rho^s n - n\eta} \leq m_x^s(\Phi_z R_n) \leq e^{-\rho^s n + n\eta},$$
$$e^{-\rho^u n - n\eta} \leq m_x^u(\Phi_z R_n) \leq e^{-\rho^u n + n\eta};$$

Furthermore we may assume that $\Gamma' \subset D_{\varepsilon,l_0}$ for some fixed l_0 large enough. Assumption (a) follows from the Shannon-McMillan-Breiman theorem while assumption (c') follows from equation (5).

One of the important ingredients of the proofs in [1] is to use that almost all points in Γ' are density points. Since we work with rectangles instead of balls we have to do some additional work to get an according density result. It is well known that rectangles do not form a density basis in general – i.e. for rectangles R centered at x the expression $\frac{1}{m(R)}\int_R \mathbf{1}_{R\cap A}\, dm$ may not converge to 1 as $\operatorname{diam} R$ tends to 0 for a.e. point $x \in A$ [4]. This might fail even for "nice" measures. However for special classes of rectangles density theorems do hold. We must modify our rectangles to obtain such a result.

Let $z \in \Gamma'$ be a density point that is we have a ball $B = B(0, r) \subset \mathbb{R}^p$ of radius less than $r^{-1}(z)$ such that $2m(\Phi_z(B \cap \Phi_z^{-1}\Gamma') > m(\Phi_z(B)) > 0$. For a point $y \in B \cap \Phi_z^{-1}\Gamma'$ we consider the collection of rectangles contained in the Lyapunov chart of z defined by $\tilde{R}(n, y) := y + R_n$. We observe that all these "straight" rectangles are comparable in the sense when translated to the origin one is contained in the other. Covers of this kind of rectangles satisfy the Besicovitch covering lemma ([4] theorem 3.2.2.). Standard arguments imply then that for any $g \in L^1(m \circ \Phi_z)$

$$\lim_{n\to\infty} \frac{1}{m(\Phi_z(\tilde{R}(n,y)))} \int_{\tilde{R}(n,y)} g\,dm = g(y) \tag{7}$$

for m–a.e. y.

Next we note that the two different classes of rectangles behave the same:

Lemma 4.2. *For $x \in \Phi_z B \cap \Gamma$ the limits $\lim_{n\to\infty} \frac{\log \mu(\Phi_z \tilde{R}(n,x))}{-n}$ and $\lim_{n\to\infty} \frac{\log \mu(\Phi_x R_n)}{-n}$ coincide for each of the measures $\mu = m, m_x^s$, and m_x^u.*

Proof. The proof is similar to the one of the previous lemma.

Fix x and z according to the assumption. The map $\Phi_x^{-1} \circ \Phi_z(\cdot)$ is smooth. Hence there is a $k_1 = k_1(x,z) > 0$ and a $n_1 = n_1(x,z) > k_1$ such that

$$\Phi_x R_{n-k_1} \supset \Phi_z \tilde{R}(n,x) \supset \Phi_x R_{n+k_1}$$

for all $n > n_1$. This implies the assertion of the lemma. $\square$

Remark 4.3. *We have $k_1(x,z) < k_2$ on Γ' since $\Gamma' \subset D_{\varepsilon,l_0}$.*

There is a set $\Gamma \subset \Gamma' \cap \Phi_z B$ with $m(\Gamma \cap \Phi_z B) > 0$ and a number $n_1 > n_0$ satisfying the following two assumptions. For all $n > n_1$ and $x \in \Gamma$

(*c*)

$$e^{-\rho^s n - n\eta} \le m_x^s(\Phi_x \tilde{R}(n,x)) \le e^{-\rho^s n + n\eta},$$
$$e^{-\rho^u n - n\eta} \le m_x^u(\Phi_x \tilde{R}(n,x)) \le e^{-\rho^u n + n\eta};$$

(*d*)

$$\mathcal{P}_{an}^{an}(x) \subset \Phi_z \tilde{R}(n,x) \subset \mathcal{P}(x),$$
$$\mathcal{P}_{an}^{0}(x) \cap \xi^s(x) \subset \Phi_z \tilde{R}(n,x) \cap \xi^s(x) \subset \mathcal{P}(x) \cap \xi^s(x),$$
$$\mathcal{P}_{0}^{an}(x) \cap \xi^u(x) \subset \Phi_z \tilde{R}(n,x) \cap \xi^u(x) \subset \mathcal{P}(x) \cap \xi^u(x)$$

where a is the absolute value of the integer part of $k_2 \min_i a_i$.

Assumption (*c*) follows from equation (5) and lemma 4.2, while assumption (*d*) follows from (*b*). It immediately follows from equation (7) that one can choose an integer $n_2 > n_1$ and a set $\tilde{\Gamma} \subset \Gamma$ of positive measure such that for every $n \ge n_2$ and $y \in \Phi_z^{-1}\tilde{\Gamma} \cap B$,

(*e*)

$$m(\Phi_z \tilde{R}(n,x) \cap \Gamma) \ge \frac{1}{2} m(\Phi_z \tilde{R}(n,x));$$
$$m_x^s(\Phi_z \tilde{R}(n,x) \cap \Gamma) \ge \frac{1}{2} m_x^s(\Phi_z \tilde{R}(n,x))$$
$$m_x^u(\Phi_z \tilde{R}(n,x) \cap \Gamma) \ge \frac{1}{2} m_x^u(\Phi_z \tilde{R}(n,x)).$$

We now want to approximate our "straight" rectangles. For any $c > 0$ and set $A \subset \mathbb{R}^n$ let $cA = \{cx : x \in A\}$ and let $c\tilde{R}(n, y) = y + cR_n$.

For $x \in \Phi_z B$ let $Q_n(x)$ be defined by

$$\left\{y \in M \mid \mathcal{P}_0^{an}(y) \cap \xi^u(x) \cap \Phi_z 2\tilde{R}(n, x) \cap \Gamma \neq \emptyset \text{ and } \mathcal{P}_{an}^0(y) \cap \xi^s(x) \cap \Phi_z 2\tilde{R}(n, x) \cap \Gamma \neq \emptyset\right\}$$

Then by (d) we have

(f)
$$\Phi_z \tilde{R}(n, x) \cap \Gamma \subset Q_n(x) \cap \Gamma \subset \Phi_z 4\tilde{R}(n, x).$$

Under the conditions (a) to (f) a repetition of the proofs in [1] with the *only* changes being replacing balls by our "straight" rectangles and replacing d_m^i by ρ_m^i in the appropriate places gives

$$\lim_{n\to\infty} \frac{\log m(\Phi_z \tilde{R}(n, x))}{-n} = -\sum_{j\leq s} a_j(d_m^j - d_m^{j-1}) - \sum_{j\geq u} a_j(d_m^j - d_m^{j+1}) = \rho_m^s + \rho_m^u \tag{8}$$

for a.e. $x \in \Phi_z B$ – i.e. for a set of positive measure.

Now equation (8) with lemma 4.2 applied to the measure $\mu = m$ yields the desired result on a set of positive measure. Lemma 4.1 tells us that then the result holds on a set of full measure. This completes the proof of theorem 3.9.

5. Endomorphisms

Now we turn to endomorphisms with singularities. The only change to the previous setup is that we instead of $(A1)$ we only require

$(A1')$ f can be extended to a $C^{1+\alpha}$–diffeomorphism $f : \overline{K_i} \to \overline{f(K_i)}$, $i = 1, 2, \dots r$ for some $\alpha > 0$.

The difference to condition $(A1)$ is that the images $f(K_i)$ need not be disjoint. Again, this is not the most general condition under which the results may be developed. For a more general setup see [15, 13, 14]. In order to deal with endomorphisms without singularities we can add artificial singularities (cuts along proper submanifolds). For example for the expanding map of the circle $f(x) = 2x \mod 1$ we can choose $x = 0$, and $x = 1/2$ as the artificial singularities.

The definitions of the sets $N^+, N^-, K^+, D, \Lambda, K_{\varepsilon,l}^+$ and $D_{\varepsilon,l}^+$ are the same as in section 2.

We lift the map f to a map $\hat{f}$ which is invertible. Namely, we set for $I = [0, 1], 0 < \tau < \min\left\{r^{-1}, \inf_{x,y\in K\setminus N} \frac{\|Df(x)\|}{\|(Df(y))^{-1}\|^{-1}}\right\}$

$$\hat{M} := M \times I,\ \hat{K} := K \times I,\ \hat{K}_i := K_i \times I,\ \hat{x} := (x, \omega) \in \hat{M};$$

$$\hat{f}(x, \omega) := \left(fx, \tau\omega + \frac{i-1}{r}\right),\ \text{for } x \in K_i,\ \omega \in I,\ i = 1, ..., r.$$

Let $\pi : \hat{M} \to M$ be the projection of $\hat{M}$ onto the first coordinate. Then $f\pi = \pi\hat{f}$ and $\hat{f}$ is invertible. We use this solenoidal construction rather than the inverse limit space of the map f (see [10]) since the inverse limit space is not a manifold and we cannot apply the theory of diffeomorphisms with singularities to it.

Let $\hat{D}^+_{\varepsilon,l} := \pi^{-1}(D^+_{\varepsilon,l})$, $\hat{D}^+_{\varepsilon} := \pi^{-1}(D^+_{\varepsilon})$, and $\hat{D} := \pi^{-1}D$. In general we will use all notations from section 2 with hats when referring to them for the map $\hat{f}$. Since $\hat{f}$ is invertible there is no confusion about preimages and we are also interested in having control over the inverse map. We define

$$\hat{N}^+ := N^+ \times [0,1];\ \hat{N}^- := N^- \times [0,1];$$
$$\hat{D}^-_{\varepsilon,l,n} := \left\{ \hat{x} \in \hat{D} : d(\hat{f}^{-k}(\hat{x}), N^+) \geq l^{-1}e^{-\varepsilon k}, k = 0, 1, \ldots, n \right\};$$
$$\hat{D}^-_{\varepsilon,l} := \left\{ \hat{x} \in \hat{D} : d(\hat{f}^{-n}(\hat{x}), \hat{N}^+) \geq l^{-}1e^{-\varepsilon n}, n = 0, 1, \ldots \right\}.$$

Then

$$\hat{D}^-_{\varepsilon,l} = \bigcap_{n \geq 0} \hat{D}^-_{\varepsilon,l,n}, \quad \hat{D}^-_{\varepsilon} = \cup_l \hat{D}^-_{\varepsilon,l}$$

and

$$\hat{D}_\varepsilon = \hat{D}^+_\varepsilon \cap \hat{D}^-_\varepsilon.$$

Definition 5.1. *We call a point $\hat{z} \in \hat{D}_\varepsilon$ regular if it is regular for $\hat{f}$.*

Remark 5.2. *For $\hat{z} \in \hat{D}$ we have*

$$\lim_{n \to \pm\infty} \frac{1}{n} \log ||D\hat{f}^n_{\hat{z}} v|| = \log \tau = \hat{\chi}_1 \quad \textit{for } v \in \hat{E}^1(\hat{z}).$$

For regular points the numbers $\hat{\chi}_{\hat{s}}(\hat{z})$ coincide with the Lyapunov exponents $\chi_s(x)$ for the projected non-invertible system (Note that $\log \tau < \chi_s(x)$).

Since the strong stable direction is special (linear) we define our Lyapunov charts on rectangles of the form $\hat{R}(\rho) = I \times \{x \in \mathbb{R}^p : \ |x| \leq \rho\} = I \times R(\rho)$ with the same properties as before. We can choose the embedding $\hat{\Phi}_{\hat{z}}$ to satisfy $\hat{\Phi}_{\hat{z}} = id \times \Phi_{\pi\hat{z}}$.

Definition 5.3. *Let m be an invariant measure. A lift is a measure $\hat{m}$ on $\hat{M}$ such that $m = \hat{m} \circ \pi^{-1}$.*

Remark 5.4. *Since the solenoidal construction, equipped with a lift, is measure theoretically isomorphic to the inverse limit space (natural extension) all lifted measures are equal. Hence if the measure m is invariant (ergodic) the lift is invariant (ergodic).*

Definition 5.5. *We say that an invariant Borel probability measure m is* regular hyperbolic *if its lift is such that $\hat{m}(\hat{D}_\varepsilon) = 1$ for ε sufficiently small and all of its Lyapunov exponents are non-zero– i.e. $s(\hat{z}) + 1 = u(\hat{z})$ a.e. We also call $\hat{m}$ regular hyperbolic.*

Remark 5.6. *$\hat{m}(\hat{D}_\varepsilon) = 1$ implies the existence of Lyapunov charts for a.e. $\hat{z} \in \hat{D}_\varepsilon$* [15, 5].

Remark 5.7. *One of the main examples of a regular hyperbolic measure is the SBR-measure. It has been shown to exist for a large class of hyperbolic endomorphisms* [15]. *In particular it has been shown to exist if the endomorphism is additionally* C^2, *has a hyperbolic cone field, has control how it can blow up near the singularities and satisfies the assumption.*

$$(A2) \qquad \exists C>0, q>0 \text{ such that } \forall\varepsilon>0\ \forall n>0, \quad \mathrm{Leb}(f^{-n}U(\varepsilon,\, N^+)) \le C\varepsilon^q.$$

There is no canonical way to define rectangles in M, thus we define rectangles first in $\hat{M}$ and project them back to M. More precisely, suppose $\pi(\hat{x}) = x$. Fix $a_2, \ldots, a_{\hat{k}}$ negative numbers and consider rectangles $\hat{R}_n(\hat{x}) := I \times [-e^{a_2 n}, e^{a_2 n}]^{\hat{e}_2} \times \cdots \times [-e^{a_{\hat{k}} n}, e^{a_{\hat{k}} n}]^{\hat{e}_{\hat{k}}}$ and let $R_n(\hat{x}) = \pi\hat{\Phi}_{\hat{x}}\hat{R}_n$. Note that because of the nonuniqueness of unstable manifolds of f, the set R_n depends on the preimage $\hat{x}$ and not just on x, unless the projection is essentially one to one. A criterion for when this happens was given in [15]. If we consider a ball $B(x, e^{-n})$ on M, and the cylinder $\hat{B} := I \times B$ in $\hat{M}$ then it is to be expected that for typical preimages the rectangles $\hat{R}_n(\hat{x})$ with all $a_i = const$ are comparable in size with $\hat{B}$. We will use this property to prove the following theorem.

Theorem 5.8. *Suppose f is an endomorphism with singularities satisfying* $(A1')$ *and that m is a regular hyperbolic measure for f. Then the limit*

$$d_m(x) := \lim_{\varepsilon\to 0} \frac{\log mB(x,\varepsilon)}{\log\varepsilon} = d^{\hat{s}}_{\hat{m}}(\hat{x}) + d^{\hat{u}}_{\hat{m}}(\hat{x}) - d^{1}_{\hat{m}}(\hat{x})$$

exists m–almost everywhere. If m is additionally ergodic them $d_m(x) = d_m$ is constant a.e.

6. Proof of theorem 5.8

Let $\hat{m}$ be a regular hyperbolic lift of m. As in section 3 we may assume w.l.o.g. that m, and hence $\hat{m}$ is ergodic. We are going to prove that

$$d_{\hat{m}\circ\pi^{-1}}(x) := \lim_{\varepsilon\to 0} \frac{\log \hat{m}\circ\pi^{-1}B(x,\varepsilon)}{\log\varepsilon}$$

exists on a set of positive measure. The ergodicity of $\hat{m}$ then implies that this limit exists a.e. Since $\hat{m}\circ\pi^{-1} = m$ the assertion follows.

Throughout this section we assume $\pi(\hat{x}) = x$ and similarly for points denoted by other letters. Consider the cylinder

$$\hat{Z}_n(\hat{x}) = \hat{Z}_n(x) := I \times B(x, e^{-n}).$$

Then $\hat{m}(\hat{Z}_n(\hat{x})) = m(B(x, e^{-n}))$.

Choose $b \in \mathbb{N}$ so that $b > \hat{\chi}_{\hat{k}}$. Fix $\vec{a} = (\log\tau, \hat{\chi}_2 - b, \ldots, \hat{\chi}_{\hat{s}} - b, \hat{\chi}_{\hat{u}} - b, \ldots, \hat{\chi}_{\hat{k}} - b)$. For $\eta \in (0, (b-\hat{\chi}_{\hat{k}})/\hat{\chi}_{\hat{k}})$ there is an $n_3 > 0$ such that there is a set $\hat{\Gamma}$ with $\hat{m}(\hat{\Gamma}) > 1-\eta$ such that for all $n > n_3$ and all $\hat{x} \in \hat{\Gamma}$

($\hat{a}$)

$$e^{-\rho_{\hat{m}}(\vec{a})n - n\eta} \le \hat{m}(\hat{R}_n(\hat{x})) \le e^{-\rho_{\hat{m}}(\vec{a})n + n\eta}.$$

$(\hat{b})$
$$\hat{\Phi}_{\hat{x}}\hat{R}_{(1-\eta)n}(\hat{x}) \subset \hat{f}^n\hat{Z}_{bn}(\hat{f}^{-n}\hat{x}) \subset \hat{\Phi}_{\hat{x}}\hat{R}_{(1+\eta)n}(\hat{x}).$$

Assumption $(\hat{a})$ follows from theorem 3.9, while $(\hat{b})$ is a consequence of the choice of $\vec{a}$, the definition of Lyapunov exponents, and $dist(\hat{f}^{-n}\hat{x}, N^+) \geq l^{-1}e^{-\varepsilon n}$.

We are going to estimate the lower and upper bound for the pointwise dimension of the measure $m = \hat{m} \circ \pi^{-1}$. We set for $\hat{x} \in \hat{M}$
$$\underline{d}_m(\hat{x}) = \liminf_{n\to\infty} \frac{\log \hat{m} \circ \pi^{-1}B(\hat{x}, e^{-n})}{-n},$$
$$\overline{d}_m(\hat{x}) = \limsup_{n\to\infty} \frac{\log \hat{m} \circ \pi^{-1}B(\hat{x}, e^{-n})}{-n}.$$

Since m is ergodic the upper and lower pointwise dimensions are constant almost everywhere:

$(\hat{c})$
$$\underline{d}_m = \underline{d}_m(\hat{y}), \qquad \overline{d}_m = \overline{d}_m(\hat{y}) \quad m\text{–a.e.}$$

We want to sieve out the radii of points where they approach the upper limit. We set for $\kappa > 0$
$$\hat{\Omega}^n_\kappa := \left\{\hat{x} \in \hat{D}_\varepsilon : e^{-bn(\overline{d}_m-\kappa)} \geq m(B(x, e^{-bn}))\right\}.$$

Since $\overline{d}_m(\hat{x}) \geq \overline{d}_m - \kappa$ if and only if $\hat{x} \in \hat{\Omega}^n_\kappa$ for infinitely many n it follows that

(9) $$\hat{m}(\cap_{N=0}^\infty \cup_{n\geq N} \hat{\Omega}^n_\kappa) = 1.$$

In particular $\sum_{n=1}^\infty \hat{m}(\hat{\Omega}^n_\kappa) = +\infty$.

Claim 6.1. *For all $N > 0$ there is $n > N$ such that $\hat{m}\left(\hat{\Omega}^n_{2\kappa} \cap \hat{f}^{-n}\hat{\Gamma}\right) > 0$.*

Proof. Since for $\alpha < \frac{\overline{d}_m - 2\kappa}{\overline{d}_m - \kappa} - 1$ we have
$$e^{-bn(1+\alpha)(\overline{d}_m-\kappa)} \geq e^{-bn(\overline{d}_m-2\kappa)} \geq mB(x, e^{-bn}) \geq mB(x, e^{-(1+\alpha)bn})$$
it follows that $\hat{\Omega}^n_\kappa \subset \bigcap_{k=n}^{(1+\alpha)n} \hat{\Omega}^l_{2\kappa}$.

Let us consider the sets $\hat{\Theta}^n := \bigcap_{l=n}^{(1+\alpha)n}(\hat{M} \setminus \hat{f}^{-l}\hat{\Gamma})$. By lemma 7.1 from the appendix we have that $\hat{m}$–a.e. $\hat{x}$ is contained in only a finite number of the $\hat{\Theta}^n$. In view of equation (9) the measure of $\hat{\Omega}^n_\kappa \setminus \hat{\Theta}^n$ has to be non–zero for infinitely many n. By the above arguments we can choose $n \leq i = i(n) \leq (1+\alpha)n$ such that the measure of $\hat{\Omega}^i_{2\kappa} \setminus \hat{\Theta}^i$ has to be positive for infinitely many i. □

By claim 6.1 $n > n_2/(1-\eta)$ can be chosen such that $\hat{m}\left(\hat{\Omega}^n_{2\kappa} \cap \hat{f}^{-n}\hat{\Gamma}\right) > 0$. Let $\hat{y} = \hat{f}^{-n}\hat{x}$ be a point of this intersection. Then by $(\hat{a})$ and $(\hat{b})$
$$\hat{m}(\hat{f}^n\hat{Z}_{bn}(\hat{y})) \geq \hat{m}(\hat{\Phi}_{\hat{x}}\hat{R}_{n(1-\eta)}(\hat{x})) \geq e^{(-\rho_{\hat{m}}(\vec{a})-\eta)n(1-\eta)}.$$

By the definition of $\hat{Z}_{bn}$, and the invariance of the measure we have
$$m(B(y,e^{-bn})) = \hat{m}(\hat{Z}_{bn}(y)) = \hat{m}(\hat{f}^n \hat{Z}_{bn}(\hat{y})) \geq e^{(-\rho_{\hat{m}}(\vec{a})-\eta)n(1-\eta)}.$$
Since $\hat{y} \in \hat{\Omega}^n_{2\kappa}$ we have
$$e^{-bn(\overline{d}_m - 2\kappa)} \geq m(B(x,e^{-bn})) \geq e^{(-\rho_{\hat{m}}(\vec{a})-\eta)n(1-\eta)}.$$
Hence
$$\overline{d}_m \leq (\rho_{\hat{m}}(\vec{a}) + \eta)(1+\eta)/b + 2\kappa.$$
Since η and κ are arbitrary we have $\overline{d}_m \leq \rho_{\hat{m}}(\vec{a})/b$.

A similar proof by changing Ω^n_κ to $\left\{\hat{x} \in \hat{D}_\varepsilon : e^{-bn(\underline{d}_m+\kappa)} \leq m(B(x,e^{-bn}))\right\}$ and observing that $\hat{\Omega}^n_\kappa \subset \bigcap_{k=(1-\alpha)n}^n \hat{\Omega}^l_{2\kappa}$ yields the analogous inequality $\underline{d}_m \geq \rho_{\hat{m}}(\vec{a})/b$. Combining these two inequalities gives $d_m(x) = \rho_{\hat{m}}(\vec{a})/b$ for m–a.e. x.

Finally we are going to evaluate the dimension. We have
$$\rho_{\hat{m}}(\vec{a}) = -\hat{\chi}_1 d^1_{\hat{m}} - \sum_{2\leq j\leq \hat{s}} (\hat{\chi}_j - b)(d^j_{\hat{m}} - d^{j-1}_{\hat{m}}) - \sum_{j\geq \hat{u}} (\hat{\chi}_j - b)(d^j_{\hat{m}} - d^{j+1}_{\hat{m}}).$$
Using the Ledrappier-Young formula (6): $h_{\hat{m}}(\hat{f}) = -\sum_{j\leq \hat{s}} \hat{\chi}_j(d^j_{\hat{m}} - d^{j-1}_{\hat{m}}) = \sum_{j\geq\hat{u}} \hat{\chi}_j(d^j_{\hat{m}} - d^{j+1}_{\hat{m}})$ the above formula reduces to
$$\rho_{\hat{m}}(\vec{a}) = b(d^{\hat{s}}_{\hat{m}} + d^{\hat{u}}_{\hat{m}} - d^1_{\hat{m}}).$$
This completes the proof of the theorem.

7. Appendix

Lemma 7.1. *Let $(X,\mathcal{B},\mu,T)$ be an ergodic measure preserving system, $Y \subset X$ a set of positive measure, and $\alpha > 0$. Then*
$$\mu\left(\cap_{N=0}^\infty \cup_{n\geq N} \left((X\backslash f^{-n}Y) \cap \cdots \cap (X\backslash f^{-[n\alpha]+n}Y)\right)\right) = 0.$$

Proof. We call the set in question A. If $x \in A$ then there is an infinite sequence $\{n_k\}$, such that for each k, each of the points $f^i x \notin Y$ for $n_k \leq i \leq [n_k\alpha] + n$. Let $S_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_Y(f^k x)$. Thus, for $x \in A$ we have
$$\begin{aligned} S_{n_k}(x) - S_{[n_k\alpha]+n_k}(x) &= \left(\frac{1}{n_k} - \frac{1}{[n_k\alpha]+n_k}\right) \sum_{k=0}^{n_k-1} \mathbf{1}_y(f^k x) \\ &\geq \left(1 - \frac{\alpha}{1+\alpha/2}\right) S_{n_k}(x) \end{aligned}$$
for n sufficiently large. In particular, $S_n(x)$ can not converge to a non-zero number for any $x \in A$. Since $S_n(x)$ converges to $\mu(Y) > 0$ for almost every x we have $\mu(A) = 0$. $\square$

References

[1] Barreira, L., Pesin, Ya. and Schmeling, J., *Dimension of hyperbolic measures - a proof of the Eckmann-Ruelle conjecture*, WIAS Preprint 245/1996, announcement in ERA-AMS, Vol. 2, Number 1, 1996.

[2] Barreira, L., Pesin, Ya. and Schmeling, J., *Regularity of foliations for $C^{1+\alpha}$ diffeomorphisms*, in preparation

[3] Brin, M. and Katok, A., *On local entropy* Geometric dynamics (Rio de Janeiro, 1981), 30–38, Lecture Notes in Math., 1007, Springer, Berlin-New York, 1983.

[4] de Guzmán, M.,"Real variable methods in Fourier analysis", North Holland Mathematics Studies, 46, North Holland, Amsterdam, 1981.

[5] Katok, A. and Strelcyn, J.-M., "Invariant manifolds, entropy and billiards; smooth maps with singularities", Lecture Notes in Mathematics, 1222, Springer, Berlin, 1986.

[6] Ledrappier, F. and Strelcyn, J.-M., *A proof of the estimation from below in Pesin's entropy formula* Erg. Th. Dyn. Syst., 1982, 2, 203–219.

[7] Ledrappier, F. and Young, L.-S., *The metric entropy of diffeomorphisms, Part I and II*, Ann. Math., 1985, 122, 509–574.

[8] Pesin, Ya.B., *Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties* Erg. Th. Dyn. Syst., 1992, 12, 123–151.

[9] Pesin, Ya., *Dimension theory in dynamical systems, rigorous results and applications*, Chicago Univ. Press (to appear).

[10] Przytycki, F., *Anosov endomorphisms*, Studia Math., 1976, 58, 249–185.

[11] Rohlin, V.A., *On the fundamental ideas of measure theory*, Amer. Math. Soc. Transl., 1962, 1, 1–52.

[12] Sataev, E.A., *Invariant measures for hyperbolic maps with singularities*, Russian Math. Surveys, 1992, 47, 191–251.

[13] Schmeling, J., *A dimension formula for endomorphisms - The Belykh family*, to appear in Erg. Th. Dyn. Sys.

[14] Schmeling, J., *Ergodic and dimension theory of hyperbolic endomorphisms with singularities* Habilitationschrift 1995.

[15] Schmeling, J. and Troubetzkoy, S., *Dimension and invertibility of hyperbolic endomorphisms with singularities* to appear in Erg. Th. Dyn. Sys.

[16] Sinai, Ya.G., *Classical systems with countable Lebesgue spectrum II* Amer. Math. Soc. Transl., 1968, 68, 34–88.

Freie Universität Berlin, FB Mathematik und Informatik, Arnimallee 2-6, D–14195 Berlin

E-mail address: `schmeling@math.fu-berlin.de`

Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294

E-mail address: `troubetz@math.uab.edu`