LIE-POINT SYMMETRIES AND STOCHASTIC DIFFERENTIAL EQUATIONS

Giuseppe Gaeta[♯]
Departamento de Matemáticas, Universidad Carlos III de Madrid,
Avenida de la Universidad 30, E-28911 Leganés (Spain)
gaeta@math.uc3m.es

Niurka Rodríguez Quintero[♭]
Grupo Interdisciplinar de Sistemas Complicados,
Departamento de Matemáticas, Universidad Carlos III de Madrid,
Avenida de la Universidad 30, E-28911 Leganés (Spain)
kinter@math.uc3m.es

Abstract. We discuss Lie-point symmetries of stochastic (ordinary) differential equations, and the interrelations between these and analogous symmetries of the associated Fokker-Planck equation for the probability measure.

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♯ Address after 1/10/99: Dipartimento di Fisica, Universitá di Roma, 00185 Roma (Italy); gaeta@roma1.infn.it

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Introduction

Symmetry methods for differential equations are by now a popular and widely employed method in the study of ODEs and of PDEs [1-8]; this popularity is due not only to the aesthetical appeal of the method but, what's more, to its utility in lowering the order of (or solving) ODEs and in determining explicit solutions and reductions of the PDEs under study.

However, such methods have been only used, as far as we know, to study *deterministic* differential equations, and not in the study of *stochastic* differential equations (SDEs in the following) [9-13]; a partial exception to this is provided by the work of Misawa [14], focusing on conserved quantities, and implicitly by the recent work of Arnold and Imkeller on normal forms for SDEs [13,15].

The purpose of this note is indeed to point out that symmetry methods are potentially useful also in the stochastic context. As we hope our discussion will be of interest both to readers working on stochastic differential equations (dynamical systems) and to those working on symmetries of differential equations, we will briefly recall some general features of both these topics in sections 1 and 2; we also restrict our discussion to *projectable symmetries*, and briefly discuss the physical basis for this restriction.

The cognizant reader will find that we devote a relatively large attention to rather introductive considerations, either on symmetries of differential equations or on stochastic (Ito) equations and their associate diffusion equations; we felt this was needed, as we aim at these two rather different audiences above, and at bridging between them.

The first obstacle to overcome for applying symmetry methods to SDEs is just to provide a suitable definition of symmetry for a SDE, which (after recalling in section 1 the basics of the theory for deterministic differential equations) we do in sections 2 for symmetry which do not act on the time variable, and in section 3 for those acting on t as well. We also obtain explicit *determining equations* for the symmetries. Our definition of symmetry is less restrictive than the one used by Misawa [14] and proves to be indeed useful.

To a SDE is intimately associated the corresponding Fokker-Planck equation for the evolution of the probability measure; in section 4 we discuss the projectable symmetries of this. As a SDE and the associated Fokker-Planck equations carry – except in degenerate cases – the same statistical information, one would expect that symmetries of a SDE correspond more and less closely to symmetries of the associated Fokker-Planck equation. This is indeed the case, as we show in section 5; there we also discuss how this correspondence can be implemented computationally in order to obtain symmetries of the Fokker-Planck [which is an equation for $\rho(x,t)$] from symmetries of a SDE [which is an equation for $x(t)$ depending on a Wiener process $W(t)$], or the other way round.

The final section 6 is devoted to illustration of examples in one, two and more space dimensions. Symmetries of the Fokker-Planck equation in one dimension have been studied and classified by Cicogna and Vitali [16] and by Shtelen and Stogny [17], and in the first three examples of section 6 we check our results against theirs; similarly the symmetries of the two dimensional case (with constant diffusion matrix) has been studied and classified by Finkel [18], and we check our results against his in subsequent examples. In the final examples of section 6 we consider some simple higher-dimensional case.

In appendix A we derive the formula (used in section 3) for the change of a Wiener process $w(t)$ under a near-identity change of the time variable, $t \to s = t + \varepsilon\tau(t)$. The appendix B is devoted to the question of normalization of solutions to the Fokker-Planck equation (only correctly normalized solutions can be given a probabilistic interpretation), and how this can change under a symmetry transformation. In particular, we identify the condition a transformation has to satisfy in order to preserve the correct normalization of solutions.

Although our original physical motivation to endeavour this study was provided by KPZ theory [19,20,21], and thus by stochastic PDEs, we were able to develop the theory in a rigorous way only for stochastic ODEs. We hope extension to PDEs will be possible (maybe by some reader of the present work) in the future; this could help in determining invariance properties of the measure for stochastic PDEs.

1. Symmetry of a deterministic differential equation

We will first recall the geometrical ideas lying behind the Lie-point symmetry approach to differential equations [1-8], and in order to do this let us consider the simplest case, i.e. a first order ODE in R^1,

$$\Delta \equiv \frac{dx}{dt} - f(t,x) = 0 \ . \tag{1.1}$$

If we consider the space $M = R^3$ with coordinates $\{t, x, \dot{x}\}$, then (1.1) identifies a manifold in M, called the solution manifold for Δ,

$$S_\Delta = \{ (t,x,y) : \ y = f(t,x) \} \ . \tag{1.2}$$

If now we consider vector fields in M, which we write as

$$X = \tau(t,x,y)\partial_t + \xi(t,x,y)\partial_x + \psi(t,x,y)\partial_y \ , \tag{1.3}$$

the problem of determining the X – that is, the τ, ξ, ψ – which leave S_Δ invariant is a classical one. It is important to stress that, as the invariance of S_Δ under X amounts to requiring that

$$X : S_\Delta \to \mathrm{T}S_\Delta \ , \tag{1.4}$$

we are reduced to a problem on the tangent space, and actually one ends up with *linear* PDEs.

However, as y is in our case the time derivative of x, we cannot change independently the three coordinates in M, neither we like – in view of the physical interpretation – that the change in the x and t can depend on y. Thus, we are naturally led to restrict the class of transformations: we require to have point transformations in the (t,x) plane, i.e. $\tau=\tau(t,x)$, $\xi=\xi(t,x)$, and that the transformation in y corresponds to the transformation on $\dot{x}$ induced by the transformation acting in the (t,x) plane.

Thus we will consider Lie-point vector fields in the space of dependent and independent variables, in this case

$$X_0=\tau(t,x)\partial_t+\xi(t,x)\partial_x \ , \tag{1.5}$$

and the X acting on M should then be obtained as the "prolongation" of X_0 to this space.

The X_0 whose prolongation X leaves S_Δ invariant represent the (Lie-point) *symmetry generators* for the equation Δ.

It is clear how to generalize this approach to the case of a system of first (or higher) order ODEs in $\mathbf{R}^n$, or even to a PDE of order N for the fields $\mathbf{u}\equiv\{u^1,...,u^p\}$ depending on $\{t;\mathbf{x}\}\equiv\{t;x^1,...,x^n\}$. The *prolongation formula*, giving the action induced by a vector field

$$X_0=\tau(t,\mathbf{x},\mathbf{u})\frac{\partial}{\partial t}+\xi^i(t,\mathbf{x},\mathbf{u})\frac{\partial}{\partial x^i}+\phi^\alpha(t,\mathbf{x},\mathbf{u})\frac{\partial}{\partial u^\alpha} \ , \tag{1.6}$$

(summation over the repeated index $i=1,...,n$ and $\alpha=1,...,p$) on the space $J=R\times R^n\times R^p\times U^{(1)}\times...\times U^{(N)}$ (where $U^{(k)}$ is the space of partial derivatives of order k of the u's; the space J is also called the *jet space* of order N), is explicitly derived, e.g., in [1], and is given – with $\mathcal{J}$ a multi-index – by $X_0^*=X_0+\sum_{\alpha,\mathcal{J}}\phi^\alpha_\mathcal{J}\partial/\partial u^\alpha_\mathcal{J}$, with

$$\phi^\alpha_\mathcal{J} \ = \ D_\mathcal{J}\left(\phi_\alpha-\sum_{i=1}^n\xi^i u^\alpha_i\right) \ + \ \sum_{i=1}^n\xi^i u^\alpha_{\mathcal{J},i} \ . \tag{1.7}$$

On physical terms, as already mentioned, it is preferable to consider only transformations such that the independent variables are transformed independently of the values assumed by the fields, i.e. such that τ and ξ^i do not depend on the $\mathbf{u}$; also, we are primarily interested in evolution equations, and it is worth remarking that, in order to guarantee the considered transformations take evolution equations into evolution equations (i.e. the time keeps its distinguished role), we should also ask that τ does not depend on $\mathbf{x}$. Thus, in the end we want to consider transformations of the form [4]

$$X_0=\tau(t)\frac{\partial}{\partial t}+\xi^i(t;\mathbf{x})\frac{\partial}{\partial x^i}+\phi^\alpha(t,\mathbf{x},\mathbf{u})\frac{\partial}{\partial u^\alpha} \ , \tag{1.8}$$

for which the prolongation formula is somehow simpler than the general one (and will be given explicitly below in section 4 for the case of interest); we will actually deal only with $p=1$ [the field $u(\mathbf{x},t)$ representing a probability density].

Definition. *The vector fields – and in particular symmetry generators – of the form (1.8) will be called "fiber-preserving", or simply* **projectable**.

We will also, with (a common) abuse of language, write simply "symmetry" to mean symmetry generators or infinitesimal symmetries; this should not cause any confusion, as in this paper we will not consider discrete or finite symmetries at all.

Let us now consider in particular a system of ODEs, which we write in the form

$$\dot{x}^i \ - \ f^i(t;\mathbf{x}) \ = \ 0 \ . \tag{1.9}$$

If we act on $(t;\mathbf{x})$ by $X_0 = \tau(t;\mathbf{x})(\partial/\partial_t) + \xi^i(t;\mathbf{x})(\partial/\partial x^i)$ we are operating an infinitesimal transformation

$$\begin{cases} x^i \to \widetilde{x}^i = x^i + \varepsilon \xi^i(t;\mathbf{x}) \ , \\ t \to \widetilde{t} = t + \varepsilon \tau(t;\mathbf{x}) \ ; \end{cases} \tag{1.10}$$

obviously knowing how x and t change enables to determine how the $\dot{x}$ change as well: with some computation – or applying the prolongation formula [1-4] – we get that

$$\dot{x}^i \equiv v^i \to \widetilde{v}^i = v^i + \varepsilon \chi^i(t;\mathbf{x},v) \ , \tag{1.11}$$

where explicitly

$$\chi^i \ = \ \partial_t \xi^i + \dot{x}^j \partial_j \xi^i - \dot{x}^i \partial_t \tau - \dot{x}^i \dot{x}^j \partial_j \tau \ ; \tag{1.12}$$

the last term disappears if we consider projectable vector fields. By applying the prolonged vector field

$$X_0^{(1)} = X_0 + \chi^i(\partial/\partial \dot{x}^i) \ , \tag{1.13}$$

on (1.9) we get $\tau \partial_t f^i + \xi^j \partial_j f^i - \chi^i = 0$; using the expression for χ^i given above, substituting for $\dot{x}^i$ according to (1.9) itself – which ensures we are on S_Δ – and requiring this to vanish, we obtain the *determining equations* for the generators of Lie-point symmetries of the dynamical system (1.9), i.e.

Lemma 1. *The symmetry generators for the dynamical systems (1.9) are vector fields* $X_0 = \tau(\partial/\partial_t) + \xi^i(\partial/\partial x^i)$ *with the functions* (τ,ξ) *satisfying*

$$\left(\tau \partial_t + \xi^j \cdot \partial_j\right) \ f^i \ - \ \left(\partial_t + f^j \cdot \partial_j\right) \ \xi^i \ + \ f^i \ \left(\partial_t + f^j \cdot \partial_j\right) \ \tau \ = \ 0 \ . \tag{1.14}$$

Introducing the notation

$$\{f,\xi\}^i \ := \ (f^j \cdot \partial_j)\xi^i - (\xi^j \cdot \partial_j) f^i \ , \tag{1.15}$$

we notice that for projectable symmetries, i.e. for $\tau = \tau(t)$, we get the

Corollary 1. *The projectable symmetries of (1.9) are given by vector fields* X_0 *as above with* (τ,ξ) *satisfying*

$$\partial_t(\xi^i - \tau f^i) + \{f,\xi\}^i \ = \ 0 \ . \tag{1.16}$$

For given functions $f^i(t;\mathbf{x})$ these partial differential equations always [3] have non-trivial solutions $\tau(t)$ and $\xi^i(t;\mathbf{x})$, for example we can fix $\tau(t)$ or $\xi^i(t;\mathbf{x})$ and compute $\xi^i(t;\mathbf{x})$ or $\tau(t)$. Although an infinite number of symmetries of (1.16) exist, there is no constructive way to find them.

For further detail on symmetries of (deterministic) ODEs and PDEs, and applications, the reader is referred to [1-8].

2. Symmetries of stochastic ODEs I.

Now, let us turn to stochastic equations [9-13], and let us consider a Ito equation

$$dx^i \ = \ f^i(t;\mathbf{x})\,dt \ + \ \sigma^i_k(t;\mathbf{x})\,dw^k \ , \tag{2.1}$$

where f and σ are smooth functions, $\sigma(t;\mathbf{x})$ is a nonzero matrix and the w^k are independent homogeneous standard Wiener processes, so that

$$\langle |w^i(t) - w^j(s)|^2 \rangle = \delta^{ij}(t-s) \ . \tag{2.2}$$

The equation (2.1) should be seen as a map from the vector Wiener process $\mathbf{w}(t) = \{w^1(t), ..., w^n(t)\}$ to the stochastic process undergone by $\{x^1(t),...,x^n(t)\}$, and its meaning is precisely that of defining the vector stochastic process $\mathbf{x}(t)$.

We recall that the Ito equation (2.1) is equivalent to the Stratonovich equation

$$dx^i \ = \ b^i(t;\mathbf{x})\,dt \ + \ \sigma^i_k(t;\mathbf{x}) \circ dw^k \ , \tag{2.3}$$

with (we write $\sigma_{ik} \equiv \sigma^i_k$ for ease of notation)

$$b^i(t;\mathbf{x}) = f^i(t;\mathbf{x}) - \frac{1}{2}\left[\sigma_{kj}(t;\mathbf{x})\frac{\partial \sigma^T_{ij}(t;\mathbf{x})}{\partial x^k}\right] \ . \tag{2.4}$$

Considering Stratonovich equations can be more convenient from the point of view of keeping track of the action of changes of coordinates (as it was done e.g. in Misawa's work [14]), since these transform according to the familiar chain rule; on the other side, if we consider a function $y = \Phi(x)$, its evolution in terms of Ito equations is described by the *Ito formula*:

$$\begin{aligned} dy^i =& \frac{\partial \Phi^i}{\partial x^j}\,dx^j + \frac{1}{2}\frac{\partial^2 \Phi^i}{\partial x^j \partial x^k}\,dx^j\,dx^k \\ =& \left[f^j \partial_j \Phi^i + (1/2)\partial^2_{jm}\Phi^i(\sigma\sigma^T)^{jm}\right]\,dt \ + \ \left[(\partial_j \phi^i)\sigma^{jk}\right]dw^k \ . \end{aligned} \tag{2.5}$$

We also recall that to the Ito equation (2.1) (or the equivalent Stratonovich equation) is associated the corresponding Fokker-Planck equation

$$\partial_t \rho \ = \ -\,\partial_i\,(f^i \rho) \ + \ \frac{1}{2}\,\partial^2_{ij}\,\left[(\sigma\sigma^T)^{ij}\,\rho\right] \ , \tag{2.6}$$

describing the evolution of the probability measure $\rho(t;\mathbf{x})$ for the stochastic process described by (2.1). Equations (2.1) and (2.6) contains the same *statistical* information [9-13], provided σ satisfies the **non-degeneracy condition** $\sigma\sigma^T \neq 0$, which we will assume throughout the paper.

`Remark 1.` Equation (2.6) describes then the time evolution of the probability measure $\rho(\mathbf{x},t)$ under the stochastic process (2.1). It is obvious that for this interpretation $\rho(\mathbf{x},t)$ should be subject to the condition

$$\int_{-\infty}^{+\infty} \rho(\mathbf{x},t)\, dx^1 ... dx^n \ = \ 1 \tag{2.7}$$

(it is enough to impose this at $t = 0$); this is relevant in connection with the allowed transformations of $(\mathbf{x},t;\rho)$: only transformation preserving this normalization do represent symmetries of the Fokker-Planck compatible with its probabilistic interpretation, and one should expect a correspondence between symmetries of the Ito equation and these – rather than all the symmetries of the Fokker-Planck – as we discuss below. $\odot$

`Remark 2.` As stressed above, the equivalence between the Ito equation (2.1) and the associated Fokker-Planck equation (2.6) [with the additional constraint (2.7)] is only statistical. It is important to stress that different Ito equations which have the same f and different matrices σ can give the same term $\sigma\sigma^T$ and thus the same Fokker-Planck equation; a simple example is provided e.g. by σ orthogonal $[\sigma \in O(n)]$: in this case we have by definition $\sigma\sigma^T = I$, so that all the Ito equations with the same f and any orthogonal matrix σ give the same Fokker-Planck equation. Similarly, σ and $\widetilde{\sigma} = \sigma B$, with B any orthogonal matrix, will give the same Fokker-Planck equation (and conversely σ and $\widetilde{\sigma}$ give the same Fokker-Planck – with the same f – only if there is an orthogonal matrix B such that the above relation is satisfied [12]).

If we consider a continuous variation of σ, say $\sigma + \varepsilon\gamma$, in the Ito equation, the associated Fokker-Planck equations remains unchanged provided $(\sigma+\varepsilon\gamma)(\sigma+\varepsilon\gamma)^T = \sigma\sigma^T$, which at order ε is simply $\sigma\gamma^T + \gamma\sigma^T = 0$ (which does *not* imply $\gamma = 0$); this simple observation will be of use in section 5.

The stochastic processes described by two different Ito equation having the same associated Fokker-Planck equation have the same statistical properties (the probability measures evolve in the same way), but *are* different: the same realization of the Wiener process $\mathbf{w}(t)$ leads to different sample paths.

It is maybe worth stressing that if we have Fokker-Planck equations which are equivalent modulo a simple transformation, e.g. are mapped one into the other by a rescaling of the time variable, but have however different σ and/or f, these will *not* be considered equivalent. $\odot$

Let us now consider a near-identity change of coordinates, passing from $\mathbf{x}$ to $\mathbf{y}$ via

$$x^i \to y^i = x^i + \varepsilon \xi^i(t;\mathbf{x}) \ . \tag{2.8}$$

Using the Ito formula, we have

$$
\begin{aligned}
dy^i &= dx^i + \varepsilon d\xi^i \\
&= f^i(t;\mathbf{x})dt + \sigma^i_k(t;\mathbf{x})dw^k + \\
&\quad + \varepsilon\{[\partial_t\xi^i + f^j\partial_j\xi^i + (1/2)(\sigma\sigma^T)^{jk}\partial^2_{jk}\xi^i]dt + (\partial_j\xi^i)\sigma^j_k dw^k\} \ ;
\end{aligned}
\tag{2.9}
$$

at first order in ε,

$$
f^i(t;\mathbf{x}) = f^i(t;\mathbf{y}) - \varepsilon\xi^j(t;\mathbf{y})\frac{\partial f^i}{\partial y^j} \ ; \ \sigma^i_k(t;\mathbf{x}) = \sigma^i_k(t;\mathbf{y}) - \varepsilon\xi^j\frac{\partial\sigma^i_k}{\partial y^j} \ . \tag{2.10}
$$

In other words, the transformation (2.8) maps the Ito equation (2.1) into a new Ito equation

$$
dy^i \ = \ \widetilde{f}^i(t;\mathbf{y})\, dt \ + \ \widetilde{\sigma}^i_k(t;\mathbf{y})dw^k \ , \tag{2.11}
$$

where we have

$$
\begin{aligned}
\widetilde{f}^i &= f^i \ + \ \varepsilon\,[\partial_t\xi^i + f^j\partial_j\xi^i - \xi^j\partial_j f^i + (1/2)(\sigma\sigma^T)^{jk}\partial^2_{jk}\xi^i] \\
\widetilde{\sigma}^i_k &= \sigma^i_k \ + \ \varepsilon\,[\sigma^j_k\partial_j\xi^i - \xi^j\partial_j\sigma^i_k] \ .
\end{aligned}
\tag{2.12}
$$

When the transformation (2.8) maps (2.1) into itself, i.e. when (2.11) coincides with (2.1) [up to terms $o(\varepsilon)$], we say that (2.8) is a (Lie-point) spatial **symmetry** of (2.1). Thus, Lie-point symmetries of (2.1) are identified by the vanishing of terms $O(\varepsilon)$ in (2.12):

Lemma 2. *The Lie-point spatial symmetries of the Ito equation (2.1) are given by* $X_0 = \xi^i(t;\mathbf{x})(\partial/\partial x^i)$ *with the* $\xi^i(t;\mathbf{x})$ *satisfying the determining equations for spatial symmetries:*

$$
\begin{cases}
\partial_t\xi^i + (f^j\cdot\partial_j)\xi^i - (\xi^j\cdot\partial_j)f^i + \frac{1}{2}\left(\sigma\sigma^T\right)^{jk}\partial^2_{jk}\xi^i = 0 \\
(\sigma^j_k\cdot\partial_j)\xi^i - (\xi^j\cdot\partial_j)\sigma^i_k = 0 \ .
\end{cases}
\tag{2.13}
$$

We could of course consider transformations acting also on t: in this case we would be modifying the processes $\mathbf{w}(t)$ as well, and should pay some extra care; we will consider this case in the next section.

`Remark 3.` Notice that for $\sigma = 0$ the (2.13) reduce to the familiar determining equations for Lie-point time-independent symmetries of a dynamical system $\dot{x}^i = f(t;\mathbf{x})$ [4]. $\odot$

`Remark 4.` If $\sigma = \sigma(t)$ does not depend on the spatial variables, the second of (2.13) reduces to $\sigma^j_k\partial_j\xi^i = 0$, which in turns imply the vanishing of the term $(\sigma\sigma^T)^{jk}\partial^2_{jk}\xi^i$, which in this case can be rewritten as $\sigma^k_p\partial_k(\sigma^j_p\partial_j\xi^i)$. Thus, (2.13) are in this case equivalent to the determining equations for symmetries of the deterministic part of the Ito equation [see (1.16) and recall now we are assuming $\tau = 0$] with the additional condition $\sigma^j_k\partial_j\xi^i = 0$. $\odot$

`Remark 5`. In studying the symmetry of deterministic dynamical systems, one introduces the bracket $\{f,g\} := (f\cdot\nabla)g - (g\cdot\nabla)f$; with this notation, (2.13) reads

$$\begin{cases} \partial_t \xi^i + \{f,\xi\}^i + (1/2)(\sigma\sigma^T)^{jk}\partial^2_{jk}\xi^i = 0 \\ \{\sigma_k,\xi\}^i = 0 \ . \end{cases} \tag{2.14}$$

$\odot$

3. Symmetry of stochastic ODEs – II.

We will consider again the Ito equation (2.1) and how this is transformed under a change of coordinates, but now we will consider (projectable) transformations which involve the time as well, i.e. we will not impose $\tau = 0$ in (1.8), and thus (2.8) should be supplemented with

$$t \to s = t + \varepsilon\tau(t) \ . \tag{3.1}$$

Then, the equation (2.9) applies, with

$$\begin{aligned} f^i(t;\mathbf{x}) &= f^i - \varepsilon\xi^j \frac{\partial f^i}{\partial y^j} - \varepsilon\tau(s)\partial_s f^i \\ \sigma^i_k(t;\mathbf{x}) &= \sigma^i_k - \varepsilon\xi^j \frac{\partial \sigma^i_k}{\partial y^j} - \varepsilon\tau(s)\partial_s \sigma^i_k \\ dw^k &= (1-\varepsilon\partial_s\tau/2)\, d\widetilde{w}^k \\ dt &= (1-\varepsilon\partial_s\tau) ds \ . \end{aligned} \tag{3.2}$$

(The transformations of the Wiener process $\mathbf{w}(t)$ into $\widetilde{\mathbf{w}}(t)$ is discussed in appendix A). Obviously in the r.h.s. of the above equation we have omitted the dependence of f,σ,ξ for ease of notation, but these should always be seen as functions of s and $\mathbf{y}$.

Substituting (2.1) in (2.9) and using (3.2), we see that the Ito equation (2.1) is now mapped into a new Ito equation (2.11), again with

$$\widetilde{f}^i(s;\mathbf{y}) \ = \ f^i(s;\mathbf{y}) \ + \varepsilon(\delta f)^i \ , \ \widetilde{\sigma}^i_k(s;\mathbf{y}) \ = \ \sigma^i_k(s;\mathbf{y}) \ + \ \varepsilon(\delta\sigma)^i_k \ ; \tag{3.3}$$

and is mapped to itself if the terms of order ε in this vanish; again from (2.1) and (2.9), with (3.2), and with a slight change of notation to compare easily with the previous Lemma 2, we have proven that:

Lemma 3. *The projectable vector field $X_0 = \tau(t)(\partial/\partial t) + \xi^i(t;\mathbf{x})(\partial/\partial x^i)$ is a symmetry generator for the Ito equation (2.1) if and only if $\tau(t)$ and the $\xi^i(t;\mathbf{x})$ satisfy the* **full determining equation** *for projectable symmetries of a Ito equation:*

$$\begin{cases} \partial_t \xi^i + (f^j\cdot\partial_j)\xi^i - (\xi^j\cdot\partial_j)f^i - \partial_t(f^i\tau) + \frac{1}{2}(\sigma\sigma^T)^{jk}\partial^2_{jk}\xi^i \ = \ 0 \\ (\sigma^j_k\cdot\partial_j)\xi^i - (\xi^j\cdot\partial_j)\sigma^i_k - \tau\partial_t\sigma^i_k - \frac{1}{2}\sigma^i_k\partial_t\tau \ = \ 0 \ . \end{cases} \tag{3.4}$$

These are $n + n^2$ equations for the $n + 1$ functions ξ^i, τ; thus for $n > 1$ they can have solution only in very exceptional cases. This should not be surprising, as symmetry is a non-generic property.

`Remark 6`. With the notation introduced in remark 5, and writing also $-(1/2)\sigma\sigma^T = A$, these read

$$\begin{cases} \partial_t \left(\xi^i - \tau f^i\right) + \{f, \xi\}^i - A^{jk} \partial^2_{jk} \xi^i \ = \ 0 \\ \{\sigma_k, \xi\}^i - \tau \partial_t \sigma^i_k - (1/2) \sigma^i_k \partial_t \tau \ = \ 0 \ . \end{cases} \tag{3.5}$$

$\odot$

`Remark 7`. The symmetries which are linear in x, i.e. such that $\xi^i(t; \mathbf{x}) = M^i_j(t) x^j$, are given by the same equations as for symmetries of the deterministic part of the Ito equation (2.1), which is nothing else than (1.16), plus the additional condition

$$M^i_j \sigma^j_k = M^j_p x^p \partial_j \sigma^i_k + \tau \partial_t \sigma^i_k + (1/2) \sigma^i_k \partial_t \tau \ ; \tag{3.6}$$

for $\tau = 0$ these further simplify. In particular, if $\tau = 0$ and σ is at most linear in x, $\sigma^i_j = S^i_j(t) + R^i_{jk}(t) x^k$, then (3.6) read

$$M^i_j S^j_k = 0 \quad , \quad M^i_j R^j_{kp} = R^i_{kj} M^j_p \ . \tag{3.7}$$

$\odot$

`Remark 8`. A relevant case in applications is the one where $f^i = f^i(\mathbf{x})$ and $\sigma^i_k = \sigma^i_k(t)$ or even $\sigma^i_k = \mathrm{const} = S^i_k$; this correspond to an autonomous dynamical system subject to a noise which depends only on t or even a constant noise. In this case (3.4), (3.5) can be discussed quite completely. Indeed, with σ independent of the spatial coordinates $\mathbf{x}$ the second of these reads

$$2 \sigma^j_k \partial_j \xi^i = 2 \tau \partial_t \sigma^i_k + \sigma^i_k \partial_t \tau \ ; \tag{3.8}$$

as the r.h.s. only depends on t, by differentiating with respect to x^m we get the equation $\sigma^j_k (\partial^2 \xi^i / \partial x^j \partial x^m) = 0$: i.e., the (symmetric) matrix of second derivatives of ξ^i, H^i_{jm} must be such that $\sigma H^i = H^i \sigma = 0$. Notice that in particular, if $\sigma^i_k(t) = \lambda_{(i)}(t) \delta^i_k$ with all $\lambda_{(i)} \not\equiv 0$ (or however if $\sigma{-1}$ exists), this means that ξ can be at most linear in the $\mathbf{x}$. $\odot$

4. Symmetries of the Fokker-Planck equation.

We will now derive the determining equations for *projectable* symmetry generators of the Fokker-Planck equation (2.6) in arbitrary spatial dimensions; as already mentioned in the introduction, general symmetries of the Fokker-Planck equations (with some limitations on σ) in one and two space dimensions have been completely classified [16-18].

It will be convenient to rewrite the Fokker-Planck equation as

$$u_t \ + \ A^{ij} \, \partial^2_{ij} u \ + \ B^i \, \partial_i u \ + \ C u \ = \ 0 \ , \tag{4.1}$$

where $\partial_i u$ denotes the partial derivative with respect to x^i and so on; and the coefficients A, B, C depend on t and $\mathbf{x}$ only – i.e. do not depend on u – and are given explicitely by

$$\begin{aligned} A^{ij}(t;\mathbf{x}) \ &= \ - \ \frac{1}{2} (\sigma \sigma^T)^{ij} \\ B^i(t;\mathbf{x}) \ &= f^i \ - \ \partial_j (\sigma \sigma^T)^{ij} \\ C(t;\mathbf{x}) \ &= (\partial_i \cdot f^i) \ - \ \frac{1}{2} \partial^2_{ij} (\sigma \sigma^T)^{ij} \ . \end{aligned} \tag{4.2}$$

We need to derive the second prolongation $X_0^{(2)}$ of the projectable vector field (1.8), and more precisely we need to compute the coefficients $\Phi_t, \Phi_i, \Phi_{ik}$ of respectively $\partial/\partial u_t$, $\partial/\partial u_i$ and $\partial/\partial u_{ik}$ in $X_0^{(2)}$.

Using the general prolongation formula and the fact that for X_0 projectable one has $\tau_u = \xi^i_u = \partial_i \tau = 0$, we get with standard algebra

$$\begin{aligned} \Phi_t &= \phi_t - \xi^j_t \partial_j u + (\phi_u - \tau_t) u_t \\ \Phi_i &= \partial_i \phi + \phi_u \partial_i u - \partial_i \xi^j \partial_j u \\ \Phi_{ik} &= \partial^2_{ik} \phi + \partial_k \phi_u \partial_i u + \partial_i \phi_u \partial_k u - \partial^2_{ik} \xi^j \partial_j u + \phi_u \partial^2_{ik} u - \\ & \quad - \partial_i \xi^j \partial^2_{jk} u - \partial_k \xi^j \partial^2_{ij} u + \phi_{uu} \partial_i u \partial_k u \ . \end{aligned} \tag{4.3}$$

Applying $X_0^{(2)}$ on (4.1) yields

$$\Phi_t + A^{ik} \Phi_{ik} + B^i \Phi_i + C \phi + \Theta \ = 0 \ , \tag{4.4}$$

with

$$\Theta = (\xi^j \partial_j A^{ik} + \tau A^{ik}_t) \partial^2_{ik} u + (\xi^j \partial_j B^i + \tau B^i_t) \partial_i u + (\xi^j \partial_j C + \tau C_t) u \ . \tag{4.5}$$

Using the above explicit expressions for the Φ's, and substituting for u_t according to (4.1), we get an expression of the form (see below for the explicit expressions of the coefficients)

$$\eta^{ik}(t;\mathbf{x},u) \partial_i u \partial_k u + \gamma^{ik}(t;\mathbf{x},u) \partial^2_{ik} u + \mu^i(t;\mathbf{x},u) \partial_i u + \nu(t;\mathbf{x},u) \ = 0 \ , \tag{4.6}$$

which must be zero for X_0 to be a symmetry of the FP equation. This means, of course, that the η, γ, μ, ν must vanish separately: we obtain in this way the determining equations

$$\begin{aligned} \eta^{ik} &\equiv A^{ik} \phi_{uu} = 0 \ , \\ \gamma^{ik} &\equiv \tau A^{ik}_t + \tau_t A^{ik} + \xi^m \partial_m A^{ik} - A^{im} \partial_m \xi^k - A^{mk} \partial_m \xi^i = 0 \\ \mu^i &\equiv \tau B^i_t + \tau_t B^i + \xi^m \partial_m B^i - B^m \partial_m \xi^i - \xi^i_t + A^{ik} \partial_k \phi_u + A^{mi} \partial_m \phi_u - A^{mk} \partial^2_{mk} \xi^i = 0 \\ \nu &\equiv \phi_t - (\phi_u - \tau_t) C u + A^{ik} \partial^2_{ik} \phi + B^i \partial_i \phi + C \phi + (\xi^m \partial_m C + \tau C_t) u = 0 \ . \end{aligned} \tag{4.7}$$

Let us recall that we assumed that σ is not degenerate, so that the A^{ik} are not all zero. The first of these means then $\phi_{uu} = 0$, and we can write

$$\phi(t;\mathbf{x},u) \ = \ \alpha(t;\mathbf{x}) + \beta(t;\mathbf{x})\, u \ ; \tag{4.8}$$

notice that now, since τ and the ξ^i's do not depend on u, all the u dependencies are explicit.

The second of (4.7) is not changed, while the third one can now be written as

$$\tau B^i_t + \tau_t B^i + \xi^m \partial_m B^i - B^m \partial_m \xi^i - \xi^i_t + A^{ik}\partial_k\beta + A^{mi}\partial_m\beta - A^{mk}\partial^2_{mk}\xi^i = 0 \ ; \tag{4.9}$$

as for the last one, this decouples into two separate equations (since in it the coefficient of u and the term not containing u must vanish separately): the one involving the coefficient of u yields

$$\beta_t + \partial_t(\tau C) + \xi^m \partial_m C + A^{ik}\partial^2_{ik}\beta + B^i \partial_i \beta = 0 \ , \tag{4.10}$$

while terms not containing u give

$$\alpha_t + A^{ik}\partial^2_{ik}\alpha + B^i\partial_i\alpha + C\alpha = 0 \ . \tag{4.11}$$

This is nothing else than the FP equation itself for α, and its appearance is a consequence of the linearity of the FP equation, as it expresses the linear superposition principle [1-4]; we will thus not consider it any more in our discussion.

Summarizing, we have proven that:

Lemma 4. *The projectable symmetries of the Fokker-Planck equation (4.1) are given by vector fields in the form (1.8) with ϕ satisfying (4.8); apart from the trivial symmetries $X_\alpha = \alpha(t;\mathbf{x})\partial_u$ with $\alpha(t;\mathbf{x})$ satisfying (4.1), corresponding to the linear superposition principle, the other symmetries are given by (τ,ξ,β) satisfying the determining equations for projectable symmetries of the Fokker-Planck equation:*

$$\begin{cases} \partial_t(\tau A^{ik}) + (\xi^m\partial_m A^{ik} - A^{im}\partial_m\xi^k - A^{mk}\partial_m\xi^i) = 0 \\ \partial_t(\tau B^i) - \left[\xi^i_t + B^m\partial_m\xi^i - \xi^m\partial_m B^i\right] + (A^{ik}\partial_k\beta + A^{mi}\partial_m\beta) - A^{mk}\partial^2_{mk}\xi^i = 0 \\ \partial_t(\tau C) + \beta_t + A^{ik}\partial^2_{ik}\beta + B^i\partial_i\beta + \xi^m\partial_m C = 0 \ . \end{cases} \tag{4.12}$$

5. Symmetries of the Ito equation versus symmetries of the associated Fokker-Planck equation.

We are specially interested in discussing how the symmetries of the partial differential equation (4.1) and those of the symmetries of the system of stochastic ODEs (2.1) are related.

We notice that in the above (4.12), we have coefficients A, B, C; however, these are not independent functions: first of all, $A^{ik} = A^{ki}$, and moreover

$$B^i = f^i + 2\partial_k A^{ik} \ , \quad C = \partial_i f^i + \partial^2_{ik} A^{ik} \ . \tag{5.1}$$

Using this, (4.12) reads

$$\begin{cases} \partial_t(\tau A^{ik}) + \left(\xi^m \partial_m A^{ik} - A^{im}\partial_m \xi^k - A^{km}\partial_m \xi^i\right) \ = \ 0 \\ \left[\partial_t(\xi^i - \tau f^i) + \{f,\xi\}^i - A^{mk}\partial^2_{mk}\xi^i\right] - \\ \quad -2\left[\partial_t(\tau\partial_k A^{ik}) + A^{ik}\partial_k\beta - \ A^{mk}\partial^2_{mk}\xi^i - \partial_k A^{mk}\partial_m\xi^i + \xi^m\partial^2_{km}A^{ik}\right] \ = \ 0 \\ \partial_t\left[\beta + \tau(\partial_i f^i + \partial^2_{ik}A^{ik})\right] + f^i\partial_i\beta + \\ \quad +A^{ik}\partial^2_{ik}\beta + 2\partial_k A^{ik}\partial_i\beta + \xi^m\partial^2_{im}f^i + \xi^m\partial^3_{ikm}A^{ik} \ = \ 0 \ . \end{cases} \tag{5.2}$$

The equations of the above system can be simplified if one eliminates the term $\partial_t(\tau\partial_k A^{ik})$ in the second equation and the term $\tau(\partial_i f^i + \partial^2_{ik}A^{ik})$ in the third one. To eliminate $\partial_t(\tau\partial_k A^{ik})$ we multiply the first equation in (5.2) by 2, derive it with respect to x^k and take the sum over all k; then we add the resulting equation to the second equation in (5.2). This yields

$$\left[\partial_t(\xi^i - \tau f^i) + \{f,\xi\}^i - A^{mk}\partial^2_{mk}\xi^i\right] - 2\left[A^{ik}\partial_k\beta + \ A^{im}\partial^2_{mk}\xi^k\right] \ = \ 0 \ . \tag{5.3}$$

Next, we eliminate $\tau(\partial_i f^i + \partial^2_{ik}A^{ik})$ in the third of (5.2). First of all, we derive the second equation with respect to x^i and sum it on all i. Next, we derive the first equation with respect to x^k and x^i, take the sum over the indices i and k, and multiply by -1 the resulting equation. Finally, we add to the third equation the two equations obtained in this way, getting the following equivalent form of the system (5.2)

$$\begin{cases} \partial_t(\tau A^{ik}) + \left(\xi^m\partial_m A^{ik} - A^{im}\partial_m\xi^k - A^{km}\partial_m\xi^i\right) \ = \ 0 \\ \left[\partial_t(\xi^i - \tau f^i) + \{f,\xi\}^i - A^{mk}\partial^2_{mk}\xi^i\right] - 2\left[A^{ik}\partial_k\beta + \ A^{im}\partial^2_{mk}\xi^k\right] \ = \ 0 \\ \left[\partial_t + f^i\partial_i - A^{ik}\partial^2_{ik}\right]\left[\beta + \partial_m\xi^m\right] \ = \ 0 \ . \end{cases} \tag{5.4}$$

We remind that solving these one obtains the functions $\tau(t)$, $\xi^i(t;\mathbf{x})$ and $\beta(t;\mathbf{x})$ which determine the symmetries (apart from the trivial X_α ones, see above) of the Fokker-Planck equation (4.1) via (1.8) and (4.8).

Let us consider the first of (5.4): recalling that $A^{ik} = -(1/2)\sigma^i_j\sigma^k_m\delta^{jm}$, and taking away the common factor (1/2), this is rewritten as

$$\begin{array}{l} \sigma^i_j\left[\sigma^m_j\partial_m\xi^k - \xi^m\partial_m\sigma^k_j - \tau\partial_t\sigma^k_j - (1/2)\sigma^k_j\partial_t\tau\right] \\ + \ \sigma^k_j\left[\sigma^m_j\partial_m\xi^i - \xi^m\partial_m\sigma^i_j - \tau\partial_t\sigma^i_j - (1/2)\sigma^i_j\partial_t\tau\right] \ = \ 0 \ ; \end{array} \tag{5.5}$$

notice the two terms are obtained from each other under the exchange $i \leftrightarrow k$. For later discussion, we denote the term in square brackets as Γ: with this (5.5) reads simply $\sigma^i_j\Gamma^k_s\delta^{js} + \sigma^k_j\Gamma^i_s\delta^{js} = 0$.

If now we look back at the second of (3.5), this is just the same as the terms into square brackets: this means that if the second of (3.5) is satisfied, then necessarily the first of (5.4) is also satisfied.

Let us now focus on the second of (5.4): obviously the terms in the first square bracket are just the l.h.s. of the first of (3.5): if the latter holds, then the former reduces to

$$A^{ik}\partial_k\beta = -A^{im}\partial^2_{mk}\xi^k \ . \tag{5.6}$$

Obviously β is not present in symmetries of the Ito equation, so we can choose it as to satisfy the third of (5.4) and (5.6): for this it suffices to choose

$$\beta \ = \ -\partial_m\xi^m + c_0 \ = \ -\mathrm{div}(\xi) + c_0 \ . \tag{5.7}$$

We have thus proved that:

Lemma 5. *Let $X_0 = \tau(\partial/\partial t) + \xi^i(\partial/\partial x^i)$ be a symmetry of the Ito equation (2.1). Then X_0 can be extended to a symmetry $X_1 = X_0 + \phi(\partial/\partial u)$ of the associated Fokker-Planck equation (4.1); the extension is given by $\phi = \alpha(t,\mathbf{x}) + \beta(t;\mathbf{x})u$ with α a solution of the (4.1) itself and β satisfying (5.7).*

We would now like to consider the opposite question, i.e. if and when a symmetry of the Fokker-Planck equation associated to a Ito equation can be projected to a symmetry of the Ito equation itself. For this is obviously necessary that the (5.6) holds, but (5.7) is a sufficient (not necessary) condition for (5.4), and the same is true of $\Gamma = 0$.

As mentioned above, we expect a complete *statistical* equivalence of the description of a stochastic process in terms of its Ito equation (2.1) or of the associated Fokker-Planck equation (2.6); however as we also mentioned (see remark 1) the probabilistic interpretation of (2.6) requires to restrict it to functions satisfying the normalization condition (2.7); moreover the equivalence is only statistical: we could have a transformation which maps a Ito equation into a different one with the same statistical properties (see remark 2) and thus the same associated Fokker-Planck equation: this would be a symmetry of the Fokker-Planck but not of the Ito equation.

Thus we expect that there is a (partial) correspondence between symmetries of (2.1) and *those symmetries of (2.6) which preserve the normalization condition (2.7)*, rather than all symmetries of (2.6).

The condition to be satisfied for a vector field of the form (4.8) to preserve (2.7) are discussed in appendix B, and are simply that the integral of $\alpha(\mathbf{x},t)$ in $d\mathbf{x}$ vanishes (here we are considering $\alpha \equiv 0$), and that

$$\beta = -\mathrm{div}(\xi) \ . \tag{5.8}$$

The latter condition guarantees automatically that the third of (5.4) holds, and also that the second square bracket in the second of (5.4) vanishes, so that this equation coincides with the first of (3.5); it remains to discuss the relation between the first of (5.4) and the second of (3.5).

As remarked above, these two equations can be rewritten as $\sigma\Gamma^T + \Gamma\sigma^T = 0$ and $\Gamma = 0$. The second of these imply the first, but the converse is not true: there is *not* a complete equivalence. This should not be too surprising: it just corresponds to the possibility that the same Fokker-Planck equation can correspond to different Ito equations, so the transformations with $\Gamma \neq 0$ but $\sigma\Gamma^T + \Gamma\sigma^T = 0$ will be those which map a Ito equation E_0 into a different Ito equation E_1 which has the *same* Fokker-Planck associated equation, this transformation will be a symmetry of the Fokker-Planck equation without being a symmetry of the Ito equation. Indeed, in remark 2 it was noticed that $\sigma\Gamma^T + \Gamma\sigma^T = 0$ (with $\Gamma \neq 0$) is precisely the condition for the occurrence of this situation.

Lemma 6. *Let $X_1 = X_0 + \phi(\partial/\partial u)$, with $X_0 = \tau(t)(\partial/\partial t) + \xi^i(\mathbf{x},t)(\partial/\partial x^i)$, be a symmetry of the Fokker-Planck equation (2.6), (4.1) associated to the Ito equation (2.1); then this preserves the normalization condition (2.7) if and only if $\phi(\mathbf{x},t;u) = \alpha(\mathbf{x},t) + \beta(\mathbf{x},t)u$ with $\int \alpha d\mathbf{x} = 0$ and $\beta = -\mathrm{div}(\xi)$. In this case X_0 transform the Ito equation into a (generally, different) Ito equation with the same statistical properties; if moreover Γ defined in (5.5) satisfy $\Gamma^i_k = 0$, then X_0 is a symmetry of the Ito equation (2.1).*

Notice that if we are analyzing a given Fokker-Planck equation, we can consider at once all the Ito equations compatible with it; the last part of the above Lemma can be reformulated by saying that X_0 is a symmetry only for those Ito equations (among those having the considered associated Fokker-Planck equation) whose σ is such to satisfy $\Gamma = 0$.

`Remark 9`. It is easy to check that the trivial symmetries $\xi^i = 0$, $\tau = 0$ and $\beta = c_0$, which are always symmetries of both the Ito (2.1) and the Fokker-Planck (4.1) equations, are solutions of (3.5) and (5.4), as they should. $\odot$

`Remark 10`. From the above system (5.4), restricting to the case where $f(t,x) = f(x)$ and $\sigma(t,x) = \sigma(x)$, we recover the results of Cicogna and Vitali [16] for the one-dimensional setting. From (5.2) it is also possible to recover the results of Shtelen and Stogny [17] for the two-dimensional Kramers equation, as well as the recent results by Finkel [18] for the two-dimensional Fokker-Planck equation

$$u_t - \frac{u_{xx}}{2} - \frac{u_{yy}}{2} + \frac{a_1}{x} u_x + a_2 u_y - \frac{a_1}{x^2} u = 0 \ ,$$

where a_1, a_2 are constants and $a_1 > 0$. $\odot$

`Remark 11`. The discussion of this section shows that if we are looking for the symmetries of a Fokker-Planck equation compatible with its probabilistic interpretation as the diffusion equation associated to a Ito equation, we do not have to deal with the general form (1.6) of the symmetry vector field, but we can use instead the ansatz

$$X_0 \ = \ \tau(t)\frac{\partial}{\partial_t} \ + \ \xi^i(t,\mathbf{x})\frac{\partial}{\partial x^i} \ - \ \mathrm{div}(\xi) u \frac{\partial}{\partial u} \ ; \tag{5.9}$$

needless to say, this ansatz does substantially simplify the analysis of symmetries, which is quite involved in the general case (see e.g. [18]). $\odot$

6. Examples.

In this section we treat some simple cases to illustrate our results and to check the correspondence between symmetries of a stochastic ODEs and of the associated Fokker-Planck equation. Symmetries of Fokker-Planck equations in one and two spatial dimensions were studied in [16,17]; we will use these works to check many of our results.

In the following, $\alpha(\mathbf{x},t)$ will denote an arbitrary solution of the Fokker-Planck equation. In all cases, the Fokker-Planck equation admits the symmetries $X_\alpha = \alpha(\mathbf{x},t)\partial_u$ and the scaling symmetry $X_0 = u\partial_u$, both implied by the linearity of the equation; notice that the latter changes the normalization of solutions (and should therefore be discarded in view of the probabilistic interpretation of the Fokker-Planck equation), while the former leaves normalization unchanged (and is thus acceptable in the present context) only for $\int \alpha(\mathbf{x},t)dx^1...dx^n = 0$; we will not discuss this point any more in the following example.

`Example 1.` As the first one-dimensional example, we consider the case $f(t,x) = 0$, $\sigma(t,x) = \sigma_0 = \text{const} \neq 0$, i.e. the equation

$$dx = \sigma_0 \, dw(t) \tag{6.1}$$

which represents a free particle subject to constant noise. The corresponding Fokker-Planck equation is simply the heat equation $u_t = (\sigma_0^2/2)u_{xx}$. The symmetries of the heat equation are well known to be [1]

$$\begin{aligned} v_1 &= \partial_t \\ v_2 &= \partial_x \\ v_3 &= u\partial_u \\ v_4 &= \sigma_0^2 t\partial_x - \sigma_0 x u\partial_u \\ v_5 &= 2t\partial_t + x\partial_x \\ v_6 &= t^2\partial_t + xt\partial_x - (1/2)(t + x^2/\sigma_0^2)u\partial_u \\ v_\alpha &= \alpha(x,t)\partial_u \end{aligned} \tag{6.2}$$

Of these, v_1, v_2 and v_5 (which do not act on u) are also symmetries of the Ito equation (6.1), as is easily checked using (3.4). Notice that (5.8) is satisfied for these, and is not satisfied for v_3, v_4 and v_6.

Let us check that v_1, v_2 and v_5 do actually span the symmetry algebra of (6.1): in this case (3.4) reads simply $\xi_t + (1/2)\sigma_0^2\xi_{xx} = 0$ and $\xi_x - (1/2)\tau_t = 0$; differentiating the latter in x we get $\xi_{xx} = 0$ (since τ does not depend on x), and therefore $\xi = a(t)x + b(t)$; plugging this into the first equation we easily get that $a(t)$ and $b(t)$ are actually constants, and thus

$$\xi(x,t) = c_1 x + c_2 \ , \ \ \tau(t) = 2c_1 t + c_3 \ ; \tag{6.3}$$

therefore v_1, v_2, v_5 do indeed span the symmetry algebra of (6.1). $\odot$

`Example 2.` Let us now consider the case $f(t,x) = 1$, $\sigma(t,x) = x$, i.e. the Ito equation

$$dx = dt + x\, dw(t) \ . \tag{6.4}$$

Inserting $f = 1$ and $\sigma = x$ into (3.5), these read $\xi_t - \tau_t + \xi_x + (x^2/2)\xi_{xx} = 0$ and $x\xi_x - \xi - (x/2)\tau_t = 0$. The latter has general solution (as easily seen e.g. by differentiating twice in x) $\xi(x,t) = (x/2)\ln(x)\tau_t + xa(t)$; inserting this into the former one and requiring the vanishing of the coefficients of $x\ln(x)$, $\ln(x)$, x and of the x-independent terms we have that $a(t) = 0$ and $\tau(t) = b_0$, where b_0 is a constant. This means $\xi = 0$.

Thus, for this choice of functions $f(t,x)$ and $\sigma(t,x)$ the equations (3.5) and (5.4) have only the solutions $\xi = 0$, $\beta = c_0$ and $\tau = b_0$, which correspond to the trivial symmetries. These symmetries (to which one should add $v_\alpha = \alpha(x,t)\partial_u$ with acceptable α) are all the symmetries of the Fokker-Planck equation [16] and of the Ito equation. Notice that (5.8) enforces $\beta = c_0 = 0$. $\odot$

`Example 3.` We next consider the case $f(t,x) = x$, $\sigma(t,x) = 1$, i.e. the Ito equation

$$dx = xdt + dw(t) \ ; \tag{6.5}$$

the corresponding Fokker-Planck equation is

$$u_t - (1/2)u_{xx} + xu_x + u = 0 \ . \tag{6.6}$$

The symmetries of (6.6) are (see [16])

$$\begin{aligned}
v_1 &= \partial_t \\
v_2 &= u\partial_u \\
v_3 &= e^t \partial_x \\
v_4 &= e^{-t}\left[\partial_x + 2xu\partial_u\right] \\
v_5 &= e^{2t}\left[\partial_t + x\partial_x - u\partial_u\right] \\
v_6 &= e^{-2t}\left[-\partial_t + x\partial_x + 2x^2u\partial_u\right] \\
v_\alpha &= \alpha(x,t)\partial_u \ .
\end{aligned} \tag{6.7}$$

Notice that v_1, v_3 and v_5 satisfy (5.8), which is not satisfied by v_2, v_4 and v_6.

As for symmetries of the Ito equation (6.5), we write eqs. (3.4) in this case, and again differentiate the second one in x, obtaining $\xi = a(t)x + b(t)$, $\tau_t = 2a(t)$; from the first equation we get then $\xi(x,t) = e^{2t}c_1 x + e^t c_2$, $\tau(t) = e^{2t}c_1 + c_3$. This shows that the symmetries of (6.5) are v_1, v_3 and

$$\widetilde{v}_5 = e^{2t}\left[\partial_t + x\partial_x\right] \ , \tag{6.8}$$

which is the projection of v_5 to the (x,t) space. $\odot$

`Example 4.` As a first example in two space dimensions [with coordinates $(x_1,x_2)=(x,y)$], we choose

$$f=\begin{pmatrix} y \\ -k^2 y\end{pmatrix} \ \ ; \ \ \sigma=\begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2k^2}\end{pmatrix} \ . \tag{6.9}$$

Thus, we are considering the equations (see also [17])

$$\begin{aligned} dx \ &= y\,dt \\ dy \ &= \ - \ k^2 y\,dt \ + \ \sqrt{2k^2}\,dw(t) \end{aligned} \tag{6.10}$$

with k^2 a positive constant.

The corresponding Fokker-Planck equation is the Kramers equation

$$u_t \ = \ k^2 u_{yy} - y u_x + k^2 y u_y + k^2 u \ ; \tag{6.11}$$

the symmetries of this were studied in [17] and are

$$\begin{aligned} v_1 \ &= \partial_t \\ v_2 \ &= u\partial_u \\ v_3 \ &= e^{-k^2 t}\,[k^{-2}\partial_x - \partial_y] \\ v_4 \ &= t\partial_x + \partial_y - (1/2)(y+k^2 x)u\partial_u \\ v_5 \ &= \partial_x \\ v_6 \ &= e^{k^2 t}\,[k^{-2}\partial_x + \partial_y - yu\partial_u] \\ v_\alpha \ &= \alpha(x,y,t)\,\partial_u \ . \end{aligned} \tag{6.12}$$

Here v_1,v_3 and v_5 satisfy (5.8), while for v_2,v_4 and v_6 this is violated.

According to our definition, the symmetries of the equations (6.10) are again v_1, v_3 and v_5. Notice that, in this case, from the second equation of (3.4) if we take $i=k=2$ we can integrate the obtained equation on $x_2=y$ to get $\xi^y=(1/2)\tau_t y+g(x,t)$. For $i=1$ and $k=2$ we obtain that ξ^1 does not depend on x_2, so $\xi^1=a(x_1,t)$. Substituting the functions ξ^1 and ξ^2 in the first equation of (3.4) and taking $i=1$ and $i=2$ one gets two linear equations in powers of x_2. Equating to zero the coefficients of powers of x_2 and the x_2-independent terms in both equations we obtain

$$\tau=c_1 \ , \ \xi^1=c_3 k^{-2} e^{-k^2 t}+c_5 \ , \ \xi^2=-c_3 e^{-k^2 t} \ . \tag{6.13}$$

Since $\mathrm{div}\xi=0$, we also have $\beta=0$ due to (5.8). $\odot$

`Example 5.` We consider next the case

$$f=\begin{pmatrix} a_1/x \\ a_2\end{pmatrix} \ \ , \ \ \sigma=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \tag{6.14}$$

i.e. the equation

$$\begin{aligned} dx &= (a_1/x)dt + dw_1(t) \\ dy &= a_2 dt + dw_2(t) \ . \end{aligned} \tag{6.15}$$

The associated Fokker-Planck equation is

$$u_t \ = \ \frac{1}{2}\,(u_{xx} + u_{yy}) \ + \frac{a_1}{x^2} u - \frac{a_1}{x} u_x - a_2 u_y \ ; \tag{6.16}$$

this has been studied by Finkel in [18], and its symmetries are

$$\begin{aligned} v_1 &= \partial_t \\ v_2 &= u\partial_u \\ v_3 &= \partial_y \\ v_4 &= 2t\partial_t + x\partial_x + (y + a_2 t)\partial_y - 2u\partial_u \\ v_5 &= \ - t\partial_y + (y - a_2 t) u\partial_u \\ v_6 &= t\,[t\partial_t + x\partial_x + y\partial_y] \ + \ [t(a_1 + a_2 y - 1) - (1/2)(x^2 + y^2 + a_2^2 t^2)]\, u\partial_u \\ v_\alpha &= \alpha(x,y,t)\,\partial_u \ . \end{aligned} \tag{6.17}$$

The symmetries of (6.15) are spanned by v_1, v_3 and v_4, and one can check that these satisfy (5.8), and that v_2, v_5 and v_6 do not comply with it. $\odot$

`Example 6.` As discussed in section 5, the correspondence between normalization-preserving symmetries of the Fokker-Planck equation and symmetries of the Ito equation is not complete; as in previous examples there was always correspondence between symmetries of a Ito equation and all the normalization-preserving symmetries of the associated Fokker-Planck equation, we are now going to give a very simple example of a case in which this does not happen.

Consider the two-dimensional Ito system (with zero drift)

$$\begin{aligned} dx^1 &= \cos(t) dw^1 \ - \ \sin(t) dw^2 \\ dx^2 &= \sin(t) dw^1 \ + \ \cos(t) dw^2 \ ; \end{aligned} \tag{6.18}$$

the corresponding Fokker-Planck equation is now just the two dimensional heat equation $u_t = (1/2)\triangle u$.

It is now immediate to check that the vector field $X_0 = \partial_t$ is a solution of (5.4) and thus a symmetry of the Fokker-Planck equation; on the other hand, the second equation of (3.5) is not satisfied and thus X_0 is *not* a symmetry of (6.18). Obviously, the case of any orthogonal σ with $\partial_t \sigma^i_k \not\equiv 0$ will be exactly the same. $\odot$

In the previous examples, we have always considered one or two dimensional cases for which the symmetries of the Fokker-Planck equation were known, and checked that the symmetries of the Ito equations we were considering did indeed correspond to the symmetries of the associated Fokker-Planck equations.

In the following examples we will consider n-dimensional stochastic equations, and will discuss the symmetries of the Ito equation and the normalization preserving symmetries of the associated Fokker-Planck equation, without attempting to determine other symmetries of the latter, not interesting in the present context. These n-dimensional cases were oviously not considered in the Cicogna-Vitali and the Finkel symmetry classifications [16,18].

`Example 7.` We will start by considering n uncoupled equations for equal "Langevin harmonic oscillators" subject to independent stochastic noises [9,22]; this system has $f^i = -x^i$ and $\sigma^i_j = s_i \delta_{ij}$, and thus is described by the Ito system

$$dx^i \ = \ -x^i \, dt \ + \ \sqrt{2 s_i} \, dw^i \ , \quad i = 1, 2, ..., n \tag{6.19}$$

(no sum on i) where we assume all the s_i are strictly positive; the corresponding Fokker-Planck equation is

$$\partial_t u \ = \ \sum_{i=1}^n \left[s_i \partial^2_{ii} u + x^i \partial_i u + u \right] \ . \tag{6.20}$$

From the second of (3.5) we get $\xi^i(\mathbf{x}, t) = \gamma^i(t) + (1/2) \tau_t x^i$, and the first of (3.5) shows then easily that $\gamma^i(t) = d_i e^{-t}$ while $\tau = c_1 e^{-2t} + c_2$, so that in particular $\tau_t = -2 c_1 e^{-2t}$. Therefore (5.7) yields $\beta = n c_1 e^{-2t} + d_0$, and (5.8) forces $d_0 = 0$.

Thus the symmetries of (6.20) compatible with its probabilistic meaning are spanned by (here and in the following $\partial_i \equiv \partial / \partial x^i$)

$$\begin{aligned} v_1 &= \partial_t \\ v_2 &= e^{-2t} \left[\partial_t - \sum_{i=1}^n x^i \partial_i + n u \partial_u \right] \\ v_\alpha &= \alpha(\mathbf{x}, t) \partial_u \\ v_{q_i} &= e^{-t} \partial_i \qquad (i = 1, 2, ..., n) \ . \end{aligned} \tag{6.21}$$

$\odot$

`Example 8.` We will now consider the class of Ito systems in which the drift is a linear function, $f^i(t; \mathbf{x}) = M^i_k x^k$ (we assume $M \neq 0$), and the matrix σ^i_j is constant; we will suppose this is invertible. Now the Ito equation (2.1) reads

$$dx^i = M^i_k x^k dt + \sigma^i_k dw^k \ . \tag{6.22}$$

This system represents an Ornstein-Uhlenbeck process [22] and its corresponding Fokker-Planck equation is [with $A = -(1/2) \sigma \sigma^T$]

$$u_t - A^{ij} \partial^2_{ij} u + M^i_k x^k \partial_i u + M^k_k = 0 \ . \tag{6.23}$$

In this case (3.5) reads

$$\begin{cases} \partial_t \xi^i + M^j_k x_k \partial_j \xi^i - \xi^j M^i_j - M^i_k x_k \partial_t \tau + A^{jk} \partial^2_{jk} \xi^i \ = \ 0 \\ \sigma^j_k \partial_j \xi^i - (1/2) \sigma^i_k \partial_t \tau \ = \ 0 \ . \end{cases} \tag{6.24}$$

Taking the derivatives with respect to x_m in the second of these, one has $\sigma^j_k \partial^2_{jm} \xi^i = 0$; thanks to the invertibility of σ, this implies $\partial^2_{jm} \xi^i = 0$ and we can write $\xi^i(t; \mathbf{x}) = L^i{}_j(t) x^j + P^i(t)$. With this, (6.24) reads

$$\begin{cases} [\partial_t L^i{}_k + L^i{}_j M^j{}_k - M^i{}_j L^j{}_k - \tau_t M^i{}_k] \, x^j \ + \ [\partial_t P^i - M^i{}_j P^j] = 0 \\ L^i{}_j \sigma^j{}_k - \frac{1}{2} \tau_t \sigma^i{}_k = 0 \ , \end{cases} \tag{6.25}$$

and in the first of these the two terms in square brackets have to vanish separately. Thus, we obtain in matrix notation the system [here $\theta(t) = \tau_t$]

$$\begin{cases} \partial_t L \ + \ [L, M] \ = \ \theta \, M \\ \dot{P} \ = \ M P \\ L \, \sigma \ = \ (\theta/2) \, \sigma \ ; \end{cases} \tag{6.26}$$

Multiplying the last of these by σ^{-1} from the right, we get $L = (\theta/2) I$, which guarantees $[L, M] = 0$ and hence reduces the first one to

$$\theta_t \, I = \theta M \ ; \tag{6.27}$$

for $M \neq I$ this entails $\theta = 0$ and thus $L = 0$, $\tau = c_0$; P will be a solution to $\dot{P} = MP$, i.e. $P(t) = \exp[Mt] P(0)$; this depends on the n arbitrary constants identifying the initial condition $P(0)$.

For $M = I$, (6.27) yields $\theta = c_1 e^t$ and hence $\tau = c_1 e^t + c_2$; in this case $\dot{P} = P$ and thus $P(t) = e^t P(0)$, which again depends on the n arbitrary constants identifying the initial condition $P(0)$.

As for β, we notice that $\mathrm{div}(\xi^i) = \mathrm{Tr}(L)$; therefore for $M \neq I$ we have $\beta = 0$, while for $M = I$ we have $\beta = (n\theta/2)$. $\odot$

`Example 9.` In many cases, relevant for applications, one considers Ito equations with

$$\sigma^i_k = b \delta^i_k . \tag{6.28}$$

In this case, the determining equations (3.5) reduce to

$$\begin{cases} \partial_t (\xi^i - \tau f^i) + \{f, \xi\}^i = 0 \\ \xi^i_k = (\tau_t/2) \delta^i_k \ , \end{cases} \tag{6.29}$$

implying in particular $\xi^i(t;\mathbf{x}) = h^i(t) + (\tau_t/2)x^i$ and therefore $\mathrm{div}(\xi) = n\tau_t/2 = -\beta$. $\odot$

`Example 10.` Finally, we consider now the n-dimensional nonlinear case

$$dx^i = -(1-\lambda\|x\|^2)x^i\,dt + dw^i\ , \tag{6.30}$$

where $\|x\|$ is the norm of the vector $\mathbf{x}$. By inserting $f^i = -x^i(1-\lambda\|x\|^2)$ and $\sigma^i_j = \delta_{ij}$ in the determining equations (3.5) we obtain, from the second of these, $\xi^i(t;\mathbf{x}) = h^i(t) + (\tau_t/2)x^i$. Inserting this into the first of (3.5) and isolating the coefficients of different powers of x, we get that for $\lambda \neq 0$ the only symmetry is given by $v_1 = \partial_t$, while for $\lambda = 0$ we are in the situation discussed in example 7 and thus we also get the symmetries described there.

Notice that at first sight one could have thought that (6.30) had a rotation symmetry; this is not the case because within the class of transformations we are considering we can rotate the vector $\mathbf{x}$, but not the vector Wiener process $\mathbf{w}(t)$; transformations allowing to rotate $\mathbf{w}(t)$ as well will be considered elsewhere. $\odot$

Appendix A.

In this appendix we derive the formula used in (3.2) for the transformation induced on a Wiener process by a near-identity change of the time coordinate,

$$t \to s = t + \varepsilon \tau(t) \ . \tag{A.1}$$

Notice that physically we want the time transformation to be invertible; this requires $\tau'(t) > -(1/\varepsilon)$.

Let us consider a Wiener process $w(t)$: the probability that it undergoes a change $z = dw$ in the time interval $\theta = dt$ has a density

$$dp(z;\theta) \ = \ \frac{1}{\sqrt{2\pi\theta}} \ e^{-z^2/\theta} \ dz \ ; \tag{A.2}$$

however once we pass to s as time coordinate,

$$\theta \ = \ dt \ = \ \frac{1}{1+\varepsilon\tau'} \, ds \tag{A.3}$$

and (A.2) should be changed accordingly for $w(t)$ expressed as $w(s) \equiv w[t(s)]$. If we consider, however,

$$\zeta \ = \ \sqrt{1+\varepsilon\tau'} \ z \tag{A.4}$$

and the stochastic process

$$\widetilde{w}(s) \ = \ \sqrt{1+\varepsilon\tau'} \ w(s) \ , \tag{A.5}$$

it is immediate to check that the probability that $\widetilde{w}(s)$ undergoes a change $\zeta = d\widetilde{w}$ in the time interval $\theta = ds$ has a density which is just

$$d\widetilde{p}(\zeta;\theta) \ = \ \frac{1}{\sqrt{2\pi\theta}} \ e^{-\zeta^2/\theta} \ d\zeta \ . \tag{A.6}$$

We interpret this by saying that under (A.1) the Wiener process $w(t)$ is changed according to (A.5). Notice that therefore, at first order in ε,

$$d\widetilde{w} \ = \ (1+\varepsilon\tau'/2) \, dw \ ; \tag{A.7}$$

this is precisely the formula used in section 3.

Appendix B. Normalization of solutions.

Let $u = \rho(\mathbf{x},t) \geq 0$ be a solution to the Fokker-Planck equation satisfying the normalization condition (2.7) $I = 1$, where

$$I \ := \ \int_{-\infty}^{+\infty} \rho(\mathbf{x},t)\, dx^1 ... dx^n \ ; \tag{B.1}$$

we want to discuss how I changes under a symmetry transformation.

First of all we recall that positivity of ρ together with $I = 1$ imply that $\rho(\mathbf{x},t)$ and $\partial_i \rho(\mathbf{x},t)$ go to zero sufficiently fast for $|\mathbf{x}| \to \infty$, and this in turn guarantees that $dI/dt = 0$, i.e. the normalization is preserved under the Fokker-Planck flow.

If now we operate an arbitrary transformation in which

$$t \to \widetilde{t} = t + \varepsilon \tau(t) \ , \ x^i \to \widetilde{x}^i = x^i + \varepsilon \xi^i(t,\mathbf{x}) \ , \ u \to \widetilde{u} = u + \varepsilon \phi(t,\mathbf{x};u) \ , \tag{B.2}$$

we have that $\rho(\mathbf{x},t) \to \widetilde{\rho}(\mathbf{x},t) + \varepsilon \phi(t,\mathbf{x};u)$; for $\phi = \beta u$ (see section 4), this reads

$$\rho(\mathbf{x},t) \to \widetilde{\rho}(\mathbf{x},t) + \varepsilon [\alpha(\mathbf{x},t) + \beta(\mathbf{x},t)\rho(\mathbf{x},t)] \ . \tag{B.3}$$

As for the volume element $dx^1 ... dx^n$, under (B.2) this is changed into $d\widetilde{x}^1 ... d\widetilde{x}^n$; using $d\widetilde{x}^i = dx^i - \varepsilon \partial \xi^i / \partial x^j dx^j + o(\varepsilon)$, we have that at first order in ε,

$$d\widetilde{x}^1 ... d\widetilde{x}^n \ = \ [1 + \mathrm{div}(\xi)]\, dx^1 ... dx^n \ . \tag{B.4}$$

Thus, again at first order in ε, I is changed to $\widetilde{I} = I + \varepsilon J + O(\varepsilon^2)$, where

$$J \ = \ \int_{-\infty}^{+\infty} \{ [\beta + \mathrm{div}(\xi)]\, \rho \ + \ \alpha \} \ dx^1 ... dx^n \ . \tag{B.5}$$

Obviously this has to vanish for the normalization to be preserved, i.e. we require $J = 0$; notice that this must hold for *any* solution of the Fokker-Planck if (B.2) is a symmetry of (2.6), (2.7). This yields at once that we must have

$$\int_{-\infty}^{+\infty} \alpha(\mathbf{x},t)\, dx^1 ... dx^n \ = \ 0 \tag{B.6}$$

and

$$\beta(\mathbf{x},t) \ = \ - \mathrm{div}(\xi) \ ; \tag{B.7}$$

this is precisely the condition (5.8) [i.e. (5.7) with with $c_0 = 0$] given in section 5.

Notice also that if we consider the Fokker-Planck with the normalization condition (2.7), an $\alpha(\mathbf{x},t)$ satisfying (B.6) is *not* an allowed solution, and the vector field X_α (see the examples) is not corresponding to the linear superposition principle; this corresponds to the fact that the function space [identified by (2.7)] in which we set our problem is *not* a linear space.

References

1. P.J. Olver, *Applications of Lie groups to differential equations*, Springer (Berlin) 1986.
2. G.W. Bluman and S. Kumei, *Symmetries and differential equations*, Springer (New York) 1989.
3. H. Stephani, *Differential equations. Their solution using symmetries*, Cambridge University Press (Cambridge) 1989.
4. G. Gaeta, *Nonlinear Symmetries and Nonlinear Equations*, Kluwer (Dordrecht) 1994.
5. P. Winternitz, "Lie groups and solutions of nonlinear partial differential equations", in: *Integrable systems, quantum groups, and quantum field theories* (NATO-ASI C 409), L.A. Ibort and M.A. Rodríguez editors, Kluwer (Dordrecht) 1993.
6. N.H. Ibragimov, *Russ. Math. Surv.* **47** (1992), 89.
7. P.J. Olver, *Equivalence, Invariants, and Symmetry*, Cambridge University Press (Cambridge) 1995.
8. V. Arnold, *Ordinary Differential Equations*, Springer (Berlin), 1993^2.
9. N.G. van Kampen, *Stochastic processes in Physics and Chemistry*, North Holland (Amsterdam) 1992^2.
10. C. DeWitt-Morette and K.D. Elworthy, *Phys. Rep.* **77** (1981), 125-167; F. Guerra, *Phys. Rep.* **77** (1981), 263-312.
11. B. Oksendal *Stochastic Differential Equations*, Springer (Berlin) 1985.
12. C. W. Gardiner *Handbook of Stochastic Methods*, Springer 1989.
13. L. Arnold, *Stochastic dynamical systems*, Springer (Berlin) 1998.
14. T. Misawa, *J. Math. Phys.* **29** (1988), 2178; *Phys. Lett. A* **195** (1994), 185; *J. Phys. A* **27** (1994), L777; *Nuovo Cimento B* **113** (1998), 421.
15. L. Arnold and P. Imkeller, *Probability Theory and Related Fields* **110** (1998), 559-588.
16. G. Cicogna and D. Vitali, *J. Phys. A: Math. Gen.* **22** (1989), L453; *J. Phys. A: Math. Gen.* **23** (1990), L85.
17. W.M. Shtelen and V.I. Stogny, *J. Phys. A: Math. Gen.* **22** (1989), L539-L543.
18. F. Finkel, *J. Phys. A: Math. Gen.* **32** (1999), 2671-2684.
19. M. Kardar, G. Parisi and Y.C. Zhang, *Phys. Rev. Lett.* **56** (1986), 889.
20. T. Halpin-Healy and Y.C. Zhang *Phys. Rep.* **254** (1995), 215.
21. A.L. Barabási and H.E. Stanley, *Fractals concepts in surface growth*, Cambridge University Press (Cambridge) 1995.
22. H. Risken *The Fokker-Planck Equation*, Springer (Berlin) 1989.