Nonlinear S-transform and Critical Point Convergence for a Quantum Hierarchical Model

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Abstract

A sequence of measures $\{\nu_n\}$ on a separable Hilbert space $\mathcal{H}$ generated by a nonlinear map is considered. For a special choice of $\mathcal{H}$ and

ν_0, such a sequence describes the Euclidean Gibbs states of a chain of interacting quantum anharmonic oscillators. Each ν_n has a Laplace transform F_n, which is an entire function on $\mathcal{H}$. The sequence $\{F_n\}$ can be generated by a nonlinear generalization of the S-transform known in Gaussian Analysis, defined as a holomorphic map on certain spaces of entire functions. A family of fixed points for this map is found and analyzed. In the case where $\{F_n\}$ describes the mentioned oscillators, it is proven that this sequence converges to both stable and unstable fixed point. The convergence to the unstable fixed point corresponds to the appearance of the strong dependence between the oscillators peculiar to the critical point.

Contents

1 Introduction

Methods of infinite dimensional analysis created during the last decades have found important applications in several domains of modern mathematics. Among them is a rigorous construction of models of quantum statistical physics [1]. Conversely, the problems posed by quantum statistical physics stimulate a further development of methods of infinite dimensional analysis. In particular, the important task here is to generalize to the infinite dimensional case appropriate tools of finite dimensional nonlinear analysis, especially those used to study such essentially nonlinear phenomena as phase transitions. This paper is intended as a step in such a direction.

The role of the S-transform in Gaussian Analysis is well known (see e.g. [7] and [9]). In Section 2, we consider a sequence of measures $\{\nu_n\}$ on a real separable Hilbert space $\mathcal{H}$ generated by a nonlinear map. For a special choice of $\mathcal{H}$ and ν_0, such a sequence describes the Euclidean Gibbs states of a chain of interacting quantum anharmonic oscillators. Each measure ν_n has a Laplace transform F_n, which is an entire function on $\mathcal{H}$. The sequence $\{F_n\}$ may be generated by a nonlinear generalization of the S-transform (NLST). Thus the weak convergence of the sequence $\{\nu_n\}$ may be shown by proving the convergence of $\{F_n\}$ to the NLST fixed points. We study these objects in Section 3 where appropriate locally convex spaces of entire functions of order of growth at most two , defined on infinite dimensional Hilbert spaces, are introduced. Then we define the NLST, prove that it is a holomorphic map, describe a family of fixed points for it, and study their stability by means of the NLST Fréchet derivative. This information is then used to prove the convergence of the sequences $\{F_n\}$ to a stable and to an unstable fixed point. The convergence to the latter occurs at a certain threshold value of the parameter which describes the interaction between the oscillators. It corresponds to the appearance of a strong dependence between them, which is peculiar to the critical point. The necessary information about the model and its description in terms of measures on infinite dimensional spaces (i.e. in terms of Euclidean Gibbs states) is given in Section 4. Section 5 contains the proof of the theorem stating the mentioned convergence. The proof is based upon a number of lemmas, which in turn are proven in Section 6.

This paper may be considered as a general formulation and extension of methods discussed in a particular case in [2]–[5] (to which we refer for further motivations and some details).

2 General Setup

Consider a real separable Hilbert space $\mathcal{H}$ with inner product $(.,.)$ and norm $|.|$. Let $\mathcal{B}(\mathcal{H})$ stand for the set of Borel subsets of $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$ and $\omega \in \mathcal{H}$, we write $A - \omega = \{\xi \mid \xi + \omega \in A\}$ and $-A = 0 - A$. Given two probability measures on $\mathcal{H}$, their convolution $\mu \star \nu$ is defined by

$$(\mu \star \nu)(A) = \int_{\mathcal{H}} \mu(A - \omega)\nu(d\omega).$$

Let $\mathcal{M}(\mathcal{H})$ be the space of all probability measures on $\mathcal{H}$ and let $\mathcal{M}_0(\mathcal{H})$ be the subset of $\mathcal{M}(\mathcal{H})$ containing even probability measures (i.e. such that $\mu(-A) = \mu(A)$) possessing the property

$$\int_{\mathcal{H}} \exp(a|\omega|^2)\mu(d\omega) < \infty, \quad \forall a > 0. \tag{2.1}$$

For $M \in I\!N$, let $\nu^{\star M}$ stand for the convolution of M copies of the probability measure ν. Define for $\nu \in \mathcal{M}(\mathcal{H})$, $\theta \geq 0$, and $\delta \geq 0$:

$$\begin{aligned} \mathbf{N}_M^{\theta,\delta}(\nu)(d\omega) &= \frac{1}{Z_\nu} \exp\left(\frac{1}{2}\theta^2 M^{-1-\delta}|\omega|^2\right) \nu^{\star M}(M^{-(1+\delta)/2} d\omega), \qquad (2.2) \\ Z_\nu &= \int_{\mathcal{H}} \exp\left(\frac{\theta}{2}|\omega|^2\right) \nu^{\star M}(d\omega). \end{aligned}$$

Clearly, $\mathbf{N}_M^{\theta,\delta}$ maps $\mathcal{M}_0(\mathcal{H})$ into itself. Let us consider the sequence $\{\nu_n\} \subset \mathcal{M}_0(\mathcal{H})$ generated by (2.2) according to

$$\nu_{n+1} = \mathbf{N}_M^{\theta,\delta}(\nu_n), \quad \nu_0 = \nu \in \mathcal{M}_0(\mathcal{H}), \quad n \in I\!N_0 \stackrel{\text{def}}{=} I\!N \cup \{0\}. \tag{2.3}$$

For $\theta = \delta = 0$, one may expect that $\{\nu_n\}$ converges weakly to some Gaussian measure with the covariance operator depending on the choice of ν_0. It is also naturally to expect that this asymptotic behaviour of $\{\nu_n\}$ is stable, i.e. the same convergence holds for small positive values of θ and $\delta = 0$. This means that for θ zero or small enough but δ positive, the corresponding sequence $\{\nu_n\}$ will converge for $n \to \infty$ to the δ–measure concentrated at zero. Does there exist the threshold value of θ which bounds this stability region and for which the corresponding sequence is no longer degenerate? If yes, it would ensure that a strong dependence, in contrast to the weak dependence corresponding to the case of "small" θ, appears for such θ. A similar phenomenon is known in statistical physics under the name "critical point behaviour". It occurs at a critical (threshold) value of the temperature. In this paper, a positive answer has been found for δ restricted to the interval $\delta \in (0, 1/2)$ and for the choice of $\mathcal{H}$ and ν_0 which we present below. For this choice, the sequence of measures $\{\nu_n\}$ describes the Euclidean Gibbs states of the model of hierarchically interacting quantum anharmonic oscillators. In this case the mentioned convergence ensures the appearance of the strong dependence between their oscillations, peculiar to the critical point. The space $\mathcal{H}$ is infinite dimensional, a fact which corresponds to the quantum character of the model. Its hierarchical nature appears in the fact that the measures ν_n are arranged in the sequence described above where $n \in I\!N_0$ numbers the hierarchy levels. For a detailed information regarding such models and their description by means of measures on infinite dimensional spaces, we refer to our previous papers [2]–[4].

In what follows, as $\mathcal{H}$ we choose the real Hilbert space $L^2(I_\beta)$, $I_\beta \stackrel{\rm def}{=} [0, \beta]$, where $\beta > 0$ stands for the inverse temperature T^{-1}. From now on we fix M and θ in (2.2) by setting $M = 2$ and $\theta^2 = (2^\delta - 1)2^{-1-\delta}$. Such a choice of θ has a physical motivation (see [2], [3]). Doing so we are going to seek the threshold value mentioned above not for θ but for the inverse temperature β (which is a parameter more suitable for the physical interpretation of the model). Consider the following strictly positive trace class operator on $\mathcal{H} = L^2(I_\beta)$

$$S = (-m\Delta_\beta + 1)^{-1}, \tag{2.4}$$

where Δ_β stands for the Laplace operator in $L^2(I_\beta)$ and $m > 0$ is the reduced physical mass of the oscillator. For this operator, one can define an even Gaussian measure γ_S on $\mathcal{H}$ having S as a covariance operator. This measure is uniquely determined by its Fourier transform

$$\int_{\mathcal{H}} \exp(i(\varphi, \omega))\gamma_S(d\omega) = \exp\left(-\frac{1}{2}(S\varphi, \varphi)\right), \quad \varphi \in \mathcal{H}. \tag{2.5}$$

Let

$$\Omega_\beta = \{\omega \in C([0, \beta] \to I\!R) \mid \omega(0) = \omega(\beta)\}.$$

One may show that $\gamma_S(\Omega_\beta) = 1$ and the measure γ_S corresponds to the oscillator bridge process of length β [15]. The starting element ν_0 of the sequence $\{\nu_n\}$ is intended to describe the anharmonic oscillator and will be obtained as a perturbation of γ_S in a way we are going to present below. For $x \in I\!R$, we set

$$V(x) = a_1 x^2 + a_2 x^4, \quad a_1 \in I\!R, \quad a_2 > 0. \tag{2.6}$$

Then we define a measure $\nu_0 = \nu \in \mathcal{M}(\mathcal{H})$ in such a way that for $\omega \in \Omega_\beta$

$$\nu(d\omega) = \frac{1}{Y} \exp\left(\int_{I_\beta} V(\omega(\tau))d\tau\right) \gamma_S(d\omega), \quad \omega \in \Omega_\beta. \tag{2.7}$$

where $Y > 0$ is the normalization constant (for more details see e.g. [1] and [8]). Then $\nu(\Omega_\beta) = 1$ and the integrals

$$F_{2l}(\tau_1, \ldots, \tau_{2l}) \stackrel{\rm def}{=} \int_{\mathcal{H}} \omega(\tau_1) \ldots \omega(\tau_{2l})\nu(d\omega)$$

$$= \int_{\Omega_\beta} \omega(\tau_1)\ldots\omega(\tau_{2l})\nu(d\omega), \ l \in I\!N, \tag{2.8}$$

are continuous as functions of $(\tau_1, \ldots, \tau_{2l}) \in I_\beta^{2l}$. They are invariant with respect to the shifts

$$F_{2l}(\tau_1, \ldots, \tau_{2l}) = F_{2l}(\tau_1 + \vartheta, \ldots, \tau_{2l} + \vartheta), \quad \vartheta \in I_\beta, \tag{2.9}$$

where addition is modulo β. Set $U_2(\tau_1, \tau_2) \stackrel{\mathrm{def}}{=} F_2(\tau_1, \tau_2)$ and

$$\begin{aligned} U_4(\tau_1, \ldots, \tau_4) &\stackrel{\mathrm{def}}{=} F_4(\tau_1, \ldots, \tau_4) - F_2(\tau_1, \tau_2)F_2(\tau_3, \tau_4) \\ - &\ F_2(\tau_1, \tau_3)F_2(\tau_2, \tau_4) - F_2(\tau_1, \tau_4)F_2(\tau_2, \tau_3). \end{aligned} \tag{2.10}$$

Then (2.9) implies that the following integrals do not depend on τ

$$\hat{U}(0) \stackrel{\mathrm{def}}{=} \int_{I_\beta} U_2(\tau, \tau')d\tau', \tag{2.11}$$

$$X \stackrel{\mathrm{def}}{=} \int_{I_\beta^2} U_4(\tau, \tau, \tau', \tau'')d\tau' d\tau''. \tag{2.12}$$

For $\delta < 1/2$, we introduce

$$\varepsilon = \frac{1 - 2\delta}{4}, \quad u(\delta) \ = \frac{2^\delta - 2^{-\varepsilon}}{2^\delta - 1}, \tag{2.13}$$

$$\kappa(u) = \frac{2^{-\delta}}{1 - (1 - 2^{-\delta})u}, \quad u \in (0, (1 - 2^{-\delta})^{-1}), \tag{2.14}$$

$$w(\delta) = 2^{2(1-\varepsilon)-\delta} u(\delta) \frac{1 - 2^{-\varepsilon}}{2^\delta - 1}. \tag{2.15}$$

Then for $u < u(\delta)$

$$\kappa(u) < \kappa(u(\delta)) = 2^\varepsilon. \tag{2.16}$$

It is not difficult to show that the integrals (2.8) depend continuously on β, which implies via (2.11) that $\hat{U}(0)$ is a continuous function of β. Thus we define $\beta_0^\pm$ as the smallest values of β obeying the equations

$$\begin{aligned} \hat{U}(0) &= 1, \quad \text{for} \quad \beta = \beta_0^-; \\ \hat{U}(0) &= u(\delta), \quad \text{for} \quad \beta = \beta_0^+. \end{aligned} \tag{2.17}$$

Definition 2.1 *Given $\delta \in (0, 1/2)$, the set $\mathcal{M}_\delta$ consists of measures having the form (2.7) such that:*

(i) $\beta_0^\pm$ *defined by (2.17) exist;*

(ii) *for all β from the closed interval with the endpoints $\beta_0^\pm$, one has*

$$|X| < w(\delta). \tag{2.18}$$

Below we discuss the question whether $\mathcal{M}_\delta$ is actually nonempty. Now we are in a position to state our main result. To this end we define the following rank-one operator on $L^2(I_\beta)$:

$$(B_\beta \varphi)(\tau) \ = \frac{1}{\beta} \int_{I_\beta} \varphi(\tau) d\tau. \tag{2.19}$$

Theorem 2.1 *For every $\nu \in \mathcal{M}_\delta$, there exists β_* such that for $\beta = \beta_*$, the sequence of measures $\{\nu_n \mid \nu_0 = \nu\}$ defined by (2.2), (2.3) converges weakly to the zero mean Gaussian measure which has B_{β_*} as covariance operator. For $\beta < \beta_*$, this sequence converges weakly to the measure degenerate at zero.*

The idea of the proof may be outlined as follows. For every measure ν_n, we define its Laplace transform

$$F_n(\xi) = \int_{\mathcal{H}} \exp[(\xi, \omega)] \nu_n(d\omega), \quad \xi \in \mathcal{H}. \tag{2.20}$$

Since ν_n possesses the property (2.1), F_n is defined on the whole space $\mathcal{H}$ and one can introduce a topological vector space containing such functions. Then we construct the operator which maps F_{n-1} into F_n. It appears that this operator is a nonlinear version of the generalized S–transform known in White Noise Analysis (see e.g. [7] and [9]), thus it is called here a nonlinear S–transform (NLST). The stated weak convergence of $\{\nu_n\}$ is then obtained by showing the conditional compactness of this sequence and by proving the pontwise convergence of the sequence $\{F_n\}$ to the Laplace transform of the corresponding limit measure.

3 Nonlinear S–Transform

Let the space $\mathcal{H}$ introduced above be embedded densely into a real Hilbert space $\mathcal{H}_-$ with inner product $(. \, , \, .)_-$ and norm $| \, . \, |_-$. Assume also that the embedding operator belongs to the Hilbert–Schmidt class. The space $\mathcal{H}_-$

is chosen in such a way that F_0, defined by (2.20) with $\nu_0 \in \mathcal{M}_\delta$, may be continuously extended to $\mathcal{H}_-$. This possibility follows from Theorem 1.7 of [7] p.109. Below we use some methods of the theory of analytic functions of a single complex variable. To this end we introduce the complexifications $\mathcal{H}^{(c)}$, $\mathcal{H}_-^{(c)}$ of the above spaces. For ξ, $\eta \in \mathcal{H}$, we write $\xi + i\eta = \zeta \in \mathcal{H}^{(c)}$, $|\zeta|^2 = |\xi|^2 + |\eta|^2$, and $A\zeta = A\xi + iA\eta$, $(\zeta, \zeta') = (\xi, \xi') - (\eta, \eta') + i(\xi, \eta') + i(\xi', \eta)$, where $A : \mathcal{H} \to \mathcal{H}$ is a linear operator.

For an entire function $F : \mathcal{H}_-^{(c)} \to \mathbb{C}$ and some $\alpha > 0$, we set

$$\|F\|_\alpha = \sup \left\{ |F(\zeta)| \exp\left(-\frac{1}{2}\alpha |\zeta|_-^2\right) \mid \zeta \in \mathcal{H}_-^{(c)} \right\}, \tag{3.1}$$

$$\mathcal{F}_\alpha = \{F \in \mathcal{F} | \|F\|_{\alpha'} < \infty, \forall \alpha' > \alpha\}, \quad \alpha \geq 0. \tag{3.2}$$

The latter set endowed with the topology generated by the family $\{\| \,.\, \|_{\alpha'}, \alpha' > \alpha\}$ becomes a Fréchet space. Now let us consider the transformation (2.20) with $\zeta = \xi + i\eta$, instead of real ξ, and $\nu_0 \in \mathcal{M}_\delta$. It defines F_0 as a function on $\mathcal{H}^{(c)}$, which can be extended to $\mathcal{H}_-^{(c)}$. One can easily show that this extension (which is also written as F_0) belongs to $\mathcal{F}_0$.

Let $O : \mathcal{H} \to \mathcal{H}_-$ stand for the embedding operator and let $O^* : \mathcal{H}_- \to \mathcal{H}$ be its adjoint. Then

$$T \stackrel{\text{def}}{=} OO^* : \mathcal{H}_- \to \mathcal{H}_-, \tag{3.3}$$

is a strictly positive trace class operator (since O belongs to the Hilbert–Schmidt class, more details can be found e.g. in [7] pp.3-13). T defines on $\mathcal{H}_-$ a zero mean Gaussian measure γ_T having such T as covariance operator. Then $(\mathcal{H}_-, \mathcal{B}(\mathcal{H}_-), \gamma_T)$ is *the Gaussian space associated with* $(\mathcal{H}, | \,.\, |)$ (see [9], p.3).

For positive σ, θ and for appropriate F, we define

$$F \mapsto \mathbf{G}^{\sigma,\theta}(F), \quad \mathbf{G}^{\sigma,\theta}(F)(\zeta) = \int_{\mathcal{H}_-} F^2(\sigma\zeta + \theta\omega)\gamma_T(d\omega), \quad \zeta \in \mathcal{H}_-^{(c)}, \tag{3.4}$$

which can be expressed in terms of the generalized S–transform $\mathbf{S}_{\sigma,\theta}$ used in White Noise Analysis

$$\mathbf{G}^{\sigma,\theta}(F) = \mathbf{S}_{\sigma,\theta}(F^2).$$

(see [9], p.13, the operator $\mathcal{S}$ introduced in [9] is in our present notations $\mathbf{S}_{1,1}$). To define $\mathbf{G}^{\sigma,\theta}$ as an operator on the spaces $\mathcal{F}_\alpha$ we will need some technical background, which we are going to provide. By means of Theorem 1.6 [7], p.106 one can prove the following version of Fernique's theorem.

Proposition 3.1 *Let T and γ_T be as above and let A be a symmetric bounded linear operator on $\mathcal{H}_-$ such that*

$$\operatorname{trace}(TA) < 1. \tag{3.5}$$

Then the expression

$$\exp\left(\frac{1}{2}(A\omega, \omega)_-\right)\gamma_T(d\omega) = K_D \gamma_D(d\omega), \tag{3.6}$$

defines on $\mathcal{H}_-$ an even Gaussian measure γ_D with the covariance operator

$$D = (1 - TA)^{-1}T, \tag{3.7}$$

where

$$K_D = \int_{\mathcal{H}_-} \exp\left(\frac{1}{2}(A\omega, \omega)_-\right)\gamma_T(d\omega) = [\det(1 - TA)]^{-1/2}.$$

Let A be as above and G be such that the function

$$F(\zeta) = \exp\left(\frac{1}{2}(A\zeta, \zeta)_-\right)G(\zeta), \tag{3.8}$$

is in the domain of $\mathbf{G}^{\sigma,\theta}$, which is described below.

Lemma 3.1 *Let*

$$\operatorname{trace}(TA) < \frac{1}{2\theta^2}. \tag{3.9}$$

Then for F given by (3.8):

$$\begin{aligned} \mathbf{G}^{\sigma,\theta}(F)(\zeta) &= K_C \exp\left(\sigma^2(A\zeta, \zeta)_- + 2\sigma^2\theta^2(A\zeta, CA\zeta)_-\right) \\ &\quad \mathbf{G}_C^{\sigma,\theta}(G)((1 + 2\theta^2 CA)\zeta), \end{aligned} \tag{3.10}$$

where

$$C = (1 - 2\theta^2 TA)^{-1}T, \tag{3.11}$$

$$K_C = [\det(1 - 2\theta^2 TA)]^{-1/2}. \tag{3.12}$$

Proof. Inserting F as given by (3.8) into (3.4) and applying Proposition 3.1 we obtain

$$\begin{aligned} \mathbf{G}^{\sigma,\theta}(F)(\zeta) &= \exp\left(\sigma^2(A\zeta, \zeta)_-\right)K_C \\ &\quad \int_{\mathcal{H}_-} G^2(\sigma\zeta + \theta\omega)\exp\left(2\sigma\theta(A\zeta, \omega)_-\right)\gamma_C(d\omega), \end{aligned} \tag{3.13}$$

with C given by (3.11). The next step is to shift γ_C (see e.g. [7], p.151) in order to include exp(.) under the integral in (3.13) into the shifted measure. Let $y \in \mathcal{H}_-$ be chosen such that $C^{-1}y \in \mathcal{H}_-$. Then we set

$$\gamma_{C,y}(d\omega) = \exp\left(-(C^{-1}y,\omega)_- - \frac{1}{2}(C^{-1}y,y)_-\right)\gamma_C(d\omega).$$

The shifted measure has the property

$$\int \Psi(\omega)\gamma_{C,y}(d\omega) = \int \Psi(\omega - y)\gamma_C(d\omega), \tag{3.14}$$

that yields:

$$\mathbf{G}^{\sigma,\theta}(F)(\zeta) = \exp\left(\sigma^2(A\zeta,\zeta)_-\right) K_C$$
$$\int_{\mathcal{H}_-} G^2(\sigma\zeta+\theta\omega)\exp\{2\sigma\theta(A\zeta,\omega)_- + (C^{-1}y,\omega)_- + \frac{1}{2}(C^{-1}y,y)_-\}\gamma_{C,y}(d\omega).$$

Setting here

$$y = -2\sigma\theta CA\zeta, \tag{3.15}$$

one arrives at

$$\begin{aligned}\mathbf{G}^{\sigma,\theta}(F)(\zeta) &= \exp\left(\sigma^2(A\zeta,\zeta)_- + 2\sigma^2\theta^2(A\zeta,CA\zeta)_-\right)\\ &\quad K_C\int_{\mathcal{H}_-} G^2(\sigma\zeta+\theta\omega)\gamma_{C,y}(d\omega).\end{aligned} \tag{3.16}$$

Then by means of (3.14)

$$\begin{aligned}\mathbf{G}^{\sigma,\theta}(F)(\zeta) &= \exp\left(\sigma^2(A\zeta,\zeta)_- + 2\sigma^2\theta^2(A\zeta,CA\zeta)_-\right)\\ &\quad K_C\int_{\mathcal{H}_-} G^2(\sigma\zeta+\theta(\omega+2\sigma\theta CA\zeta))\gamma_C(d\omega),\end{aligned}$$

which gives (3.10). ■

Now we may define $\mathbf{G}^{\sigma,\theta}$ as a holomorphic map on appropriate spaces of entire functions. Let $\|T\|$ stand for the operator norm of T as an operator in $\mathcal{H}_-$ and suppose $\sigma < 1/\sqrt{2}$. Let also

$$\alpha_{\max} \stackrel{\text{def}}{=} \min\left\{\frac{1-2\sigma^2}{2\theta^2\|T\|}\ ;\ \frac{1}{2\theta^2\mathrm{trace}T}\right\}. \tag{3.17}$$

Lemma 3.2 *For $\alpha < \alpha_{\max}$, the map $\mathbf{G}^{\sigma,\theta}$ (3.4) is holomorphic, i.e. Fréchet differentiable, on $\mathcal{F}_\alpha$.*

Proof. For $\alpha' \in (\alpha, \alpha_{\max})$ and two functions $F_1, F_2 \in \mathcal{F}_\alpha$, we estimate

$$\Phi(\zeta) \stackrel{\text{def}}{=} \int_{\mathcal{H}_-} F_1(\sigma\zeta + \theta\omega) F_2(\sigma\zeta + \theta\omega)\gamma_T(d\omega).$$

The definitions (3.1), (3.2) yield

$$|F_i(\zeta)| \le \|F_i\|_{\alpha'} \exp\left(\frac{1}{2}\alpha'|\zeta|_-^2\right),\ \zeta = \xi + i\eta.$$

Therefore,

$$|\Phi(\zeta)| \le \int_{\mathcal{H}_-} |F_1(\sigma\zeta + \theta\omega)|\ |F_2(\sigma\zeta + \theta\omega)|\gamma_T(d\omega) \le \|F_1\|_{\alpha'}\|F_2\|_{\alpha'}$$
$$\int_{\mathcal{H}_-} \exp\left\{\alpha'(\sigma^2|\xi|_-^2 + 2\sigma\theta(\xi,\omega)_- + \sigma^2|\eta|_-^2 + \theta^2|\omega|_-^2)\right\}\gamma_T(d\omega).$$

Integrating over $\mathcal{H}_-$, one obtains

$$\begin{aligned}|\Phi(\zeta)| &\le \|F_1\|_{a'}\|F_2\|_{\alpha'} K \exp(\alpha'\sigma^2|\eta|_-^2 + \alpha'\sigma^2((1-2\alpha'\theta^2 T)^{-1}\xi,\xi)_-)\\ &\le K\|F_1\|_{\alpha'}\|F_2\|_{\alpha'} \exp\left\{\frac{\alpha'\sigma^2}{1-2\alpha'\theta^2\|T\|}|\zeta|_-^2\right\},\end{aligned}$$

where

$$K = [\det(1-2\alpha'\theta^2 T)]^{-1/2} \le [1-2\alpha'\theta^2 \mathrm{trace} T]^{-1/2}.$$

This yields for $\alpha' < \alpha_{\max}$,

$$\|\int_{\mathcal{H}_-} F_1(\sigma\zeta + \theta\omega)F_2(\sigma\zeta + \theta\omega)\gamma_T(d\omega)\|_{\alpha'} \le K\|F_1\|_{\alpha'}\|F_2\|_{\alpha'}. \tag{3.18}$$

Putting here $F_1 = F_2 = F$, one gets

$$\|\mathbf{G}^{\sigma,\theta}(F)\|_{\alpha'} \le K\|F\|_{\alpha'}^2. \tag{3.19}$$

For $F \in \mathcal{F}_\alpha$, $G \in \mathcal{F}_\alpha$, we have

$$\begin{aligned}&\mathbf{G}^{\sigma,\theta}(F+G)(\zeta) - \mathbf{G}^{\sigma,\theta}(F)(\zeta) = \\ &2\int_{\mathcal{H}_-} F(\sigma\zeta+\theta\omega)G(\sigma\zeta+\theta\omega)\gamma_T(d\omega) + \mathbf{G}^{\sigma,\theta}(G)(\zeta).\end{aligned} \tag{3.20}$$

The latter term can be estimated by (3.19). Therefore, the Fréchet derivative of $\mathbf{G}^{\sigma,\theta}$ at $F \in \mathcal{F}_\alpha$ is the following linear operator on $\mathcal{F}_\alpha$

$$\{(\mathbf{G}^{\sigma,\theta}(F))'G\}(\zeta) = 2\int_{\mathcal{H}_-} F(\sigma\zeta + \theta\omega)G(\sigma\zeta + \theta\omega)\gamma_T(d\omega). \tag{3.21}$$

That $(\mathbf{G}^{\sigma,\theta}(F))'$ is bounded can be shown by taking $F_1 = F$, $F_2 = G$ in (3.18). $\blacksquare$

Remark 3.1 *We can extend $\mathbf{G}^{\sigma,\theta}$ preserving all its properties stated above to $\mathcal{F}_{\alpha_{\max}}$ provided $\alpha_{\max}$ is strictly less than $(2\theta^2 \mathrm{trace} T)^{-1}$, or to other spaces, e.g. the space consisting of the functions given by (3.8) with $G \in \mathcal{F}_0$ and such A that the constant (3.12) exists.*

Definition 3.1 *We denote by $\mathcal{F}_S$ the maximal space on which the map $\mathbf{G}^{\sigma,\theta}$ can be defined maintaining the properties stated by Lemma 3.2.*

The problem we are concerned now is the existence of fixed points of the operator $\mathbf{G}^{\sigma,\theta}$ and their stability. Among the possible fixed points one may distinguish a family, which can be described in an evident way. It consists of Gaussian fixed points and is defined as follows. Let $\mathcal{Q}_{\mathrm{fin}} = \{Q \subset I\!N \mid |Q| < \infty\}$, where $|Q|$ means the cardinality. Let also $\{\omega_j, j \in I\!N\}$ be the base of $\mathcal{H}_-$ formed by the eigenvectors of the trace class operator T (see (3.3)). For a nonempty $Q \in \mathcal{Q}_{\mathrm{fin}}$, let $\mathcal{H}_Q$ be the linear span of $\{\omega_j, j \in Q\}$. Let W_Q be the orthogonal projector onto $\mathcal{H}_Q$ and T_Q be the restriction of T onto $\mathcal{H}_Q$. For a nonempty $Q \in \mathcal{Q}_{\mathrm{fin}}$, we put

$$A_Q = \frac{1 - 2\sigma^2}{2\theta^2} T_Q^{-1} W_Q, \tag{3.22}$$

and $A_Q = 0$ for $Q = \emptyset$. Note that T_Q^{-1} exists for each nonempty $Q \in \mathcal{Q}_{\mathrm{fin}}$ – a fact, which follows from the strict positivity of T.

Proposition 3.2 *The elements of the family*

$$\mathcal{A} = \{F_Q(\zeta) = (2\sigma^2)^{\frac{|Q|}{2}} \exp\left(\frac{1}{2}(A_Q\zeta, \zeta)_-\right) \mid Q \in \mathcal{Q}_{\mathrm{fin}}\} \tag{3.23}$$

are fixed points of $\mathbf{G}^{\sigma,\theta}$.

Proof. We have $F_\emptyset \equiv 1$, which is a fixed point. For a nonempty $Q \in \mathcal{Q}_{\rm fin}$, using (3.10) with $G \equiv 1$ and imposing corresponding conditions on A, we obtain (3.22). ■

The stability of the above fixed points can be studied by means of the eigenvalues of the Fréchet derivative of $\mathbf{G}^{\sigma,\theta}$ at the points F_Q. Thus let us consider the eigenvalue problem

$$(\mathbf{G}^{\sigma,\theta}(F_Q))' G = \lambda G. \tag{3.24}$$

Let $G(\zeta) = F_Q(\zeta) P(\zeta)$ with P being an even polynomial – we restrict ourselves to this case since only even measures, and hence even functions F_n, are to be considered here. Inserting such G into (3.21) we obtain

$$2(2\sigma^2)^{|Q|} \int_{\mathcal{H}_-} \exp\left[(A_Q(\sigma\zeta + \theta\omega), \sigma\zeta + \theta\omega)_-\right] P(\sigma\zeta + \theta\omega)\gamma_T(d\omega)$$
$$= \lambda(2\sigma)^{\frac{|Q|}{2}} \exp\left(\frac{1}{2}(A_Q\zeta, \zeta)_-\right) P(\zeta).$$

By means of Lemma 3.1 the latter equality can be transformed into the equation:

$$2\int_{\mathcal{H}_-} P(\sigma L_Q \zeta + \theta\omega)\gamma_{C_Q}(d\omega) = \lambda P(\zeta), \tag{3.25}$$

where

$$\sigma L_Q = \sigma(1 - W_Q) + \frac{1}{2\sigma} W_Q, \tag{3.26}$$

$$C_Q = (1 - W_Q)T + \frac{1}{2\sigma^2} W_Q T. \tag{3.27}$$

Let $\deg P = 2k$, then one may put

$$P = P_k + \delta P, \quad \deg P_k = 2k, \ P_k(\alpha\zeta) = \alpha^{2k} P_k(\zeta), \ \deg \delta P < 2k, \tag{3.28}$$

and obtain from (3.25)

$$2P_k(\sigma L_Q \zeta) + P'(\zeta) = \lambda P_k(\zeta) + \lambda \delta P(\zeta), \tag{3.29}$$

with P' such that $\deg P' < 2k$. Let $\mathcal{H}^{(1)} = \{\zeta = \xi + i\eta \mid \xi \in \mathcal{H}_Q \text{ or } \eta \in \mathcal{H}_Q\}$ and $\mathcal{H}^{(2)} = \{\zeta = \xi + i\eta \mid \xi \in \mathcal{H}_- \setminus \mathcal{H}_Q \text{ or } \eta \in \mathcal{H}_- \setminus \mathcal{H}_Q\}$. We set

$$\begin{aligned} \mathcal{P}_Q &= \{P \mid P_k(\zeta) = 0, \ \forall \zeta \in \mathcal{H}^{(1)}\}, \\ \tilde{\mathcal{P}}_Q &= \{P \mid P_k(\zeta) = 0, \ \forall \zeta \in \mathcal{H}^{(2)}\}, \end{aligned} \tag{3.30}$$

and obtain

$$
\begin{aligned}
P_k(\sigma L_Q \zeta) &= P_k(\sigma\zeta), \quad \text{for} \quad P \in \mathcal{P}_Q, \\
P_k(\sigma L_Q \zeta) &= P_k(\frac{1}{2\sigma}\zeta), \ \text{for} \ \ P \in \tilde{\mathcal{P}}_Q.
\end{aligned} \tag{3.31}
$$

Inserting such P into (3.29) and taking into account the homogeneity of P_k (3.28), as well as (3.31), we arrive at

$$
\lambda_k = 2\sigma^{2k}; \quad P \in \mathcal{P}_Q, \ 2k = \deg P, \ k \in \mathbb{Z}_+, \tag{3.32}
$$

$$
\tilde{\lambda}_k = \frac{2}{(2\sigma)^{2k}}; \quad P \in \tilde{\mathcal{P}}_Q, \ 2k = \deg P, \ k \in \mathbb{N}. \tag{3.33}
$$

We recall that a fixed point F_Q is stable if all the eigenvalues of $(\mathbf{G}^{\sigma,\theta}(F_Q))'$ do not exceed one. F_Q is l-unstable ($l \in \mathbb{N}$) if there exist l eigenvalues which are greater than one. For every unstable fixed point, there exist "directions" such that arbitrary small perturbations δF of F in these directions produce the effect that the sequence $\{F_n, \ n \in \mathbb{Z}_+\}$, defined recursively

$$
F_n = \mathbf{G}^{\sigma,\theta}(F_{n-1}), \quad F_0 = F_Q + \delta F, \tag{3.34}
$$

does not approach F_Q as $n \to \infty$. Such directions correspond to the eigenvalues λ of $(\mathbf{G}^{\sigma,\theta}(F_Q))'$ which are greater than one. In our case, $\lambda_0 = 2 > 1$, thus the sequence $\{F_n, \ n \in \mathbb{Z}_+\}$ with $F_0(\zeta) = K_0 \exp(\frac{1}{2}(A_Q\zeta,\zeta)_-)$, $K_0 \neq (2\sigma^2)^{|Q|/2}$ does not converge to the fixed point F_Q. To exclude this trivial instability and having in mind that each F_n ought to be the Laplace transform of a probability measure, we redefine the NLST by putting instead of (3.4)

$$
\hat{\mathbf{G}}^{\sigma,\theta}(F)(\zeta) = \frac{\int_{\mathcal{H}_-} F^2(\sigma\zeta + \theta\omega)\gamma_T(d\omega)}{\int_{\mathcal{H}_-} F^2(\theta\omega)\gamma_T(d\omega)}. \tag{3.35}
$$

Accordingly, the family $\mathcal{A}$ (3.23) transforms into

$$
\hat{\mathcal{A}} = \{F_Q(\zeta) = \exp\left(\frac{1}{2}(A_Q\zeta,\zeta)_-\right) \ | \ Q \in \mathfrak{Q}_{\text{fin}}\}. \tag{3.36}
$$

Let us set

$$
\hat{\mathcal{F}} = \{F \in \mathcal{F}_S \ | \ F(\zeta) = F(-\zeta), \ F(0) = 1\}. \tag{3.37}
$$

Then $\hat{\mathcal{A}} \subset \hat{\mathcal{F}}$, and $\hat{\mathbf{G}}^{\sigma,\theta}$ maps $\hat{\mathcal{F}}$ into itself. The stability of each $F_Q \in \hat{\mathcal{A}}$ under the perturbations within $\hat{\mathcal{F}}$ is defined by the eigenvalues (3.32), (3.33) but now with $k \in I\!N$, i.e. $k \neq 0$. Since $\sigma < 1/\sqrt{2}$, all $\lambda_k < 1$, $k \in I\!N$. At the same time we have $\tilde{\lambda}_1 > 1$, which means that F_Q is unstable whenever $\tilde{\mathcal{P}}_Q$ is nonempty. The only possibility for $\tilde{\mathcal{P}}_Q$ to be empty is $A_Q = 0$, which occurs for $Q = \emptyset$. Thus the only stable fixed point in $\hat{\mathcal{A}}$ is $F_\emptyset \equiv 1$. In what follows, the sequence $\{F_n \in \hat{\mathcal{F}},\ n \in Z\!\!Z_+\}$ defined recursively

$$F_n = \hat{\mathbf{G}}^{\sigma,\theta}(F_{n-1}),\ n \in I\!N, \tag{3.38}$$

with $\sigma \in (0, 1/\sqrt{2})$, may converge as $n \to \infty$ to the stable fixed point $F_\emptyset \equiv 1$, or to some unstable fixed point $F_Q \in \hat{\mathcal{A}}$. An example can be constructed by choosing $F_0(z) = \exp(\frac{1}{2}(A_0\zeta, \zeta)_-)$ with a suitable operator A_0. In this case we have $F_n(\zeta) = \exp(\frac{1}{2}(A_n\zeta, \zeta)_-)$, and the sequence $\{A_n\}$ may be described explicitly. We do not consider this simple example (which describes the harmonic oscillators) and will be concerned with the case of a non-Gaussian F_0, where all F_n are also non–Gaussian and describe the anharmonic oscillators. For every ν_n, the Laplace transform (2.20) belongs to $\hat{\mathcal{F}}$ and one can easily prove the following statement.

Proposition 3.3 *Let the sequence of measures $\{\nu_n\}$ be defined by (2.2), (2.3) with the operator $\mathbf{N}_2^{\theta,\delta}$. Then the sequence of the Laplace transforms $\{F_n\}$ defined by (2.20) obeys the recursion (3.38) with $\sigma = 2^{-(1+\delta)/2}$.*

It should be stressed here that the explicit form of the operator T, as well as of the space $\mathcal{H}_-$, has no relevance – for any pair $(\mathcal{H}_-, T)$ obeying the above conditions, Proposition 3.3 holds. An example, which corresponds to our choice $\mathcal{H} = L^2(I_\beta)$, is $\mathcal{H}_- = W_2^{-1}(I_\beta)$, where the latter is the Sobolev space of square integrable generalized functions of index -1.

4 Quantum Hierarchical Model

As it was pointed out above, the non–Gaussian sequence of functions generated by the map (3.35) with the starting element F_0 being the Laplace transform of the measure (2.7) describes the Euclidean Gibbs states of the model of hierarchically interacting quantum anharmonic oscillators. It is characterized by a number of parameters. A part of them describes the individual properties of the oscillators, they define the starting element F_0 (see (2.6), (2.7)). The other parameters describe the interaction between the oscillators and define in turn the properties of the map $\hat{\mathbf{G}}^{\sigma,\theta}$. This interaction grows proportionally to the inverse temperature β. When the inverse temperature reaches its critical value $\beta_* > 0$, the oscillations become strongly dependent, which is known as the critical point behaviour of the model. Having in mind Proposition 3.3 we fix σ and θ

$$\theta^2 = (2^\delta - 1)2^{-1-\delta} = \frac{1}{2} - \sigma^2, \quad \sigma = 2^{-(1+\delta)/2}, \quad \delta \in (0, 1/2), \tag{4.1}$$

and set for this choice, $\mathbf{G}_\beta \stackrel{\text{def}}{=} \hat{\mathbf{G}}^{\sigma,\theta}$. Then the sequence $\{F_n\}$ is given as follows

$$\begin{aligned} F_n(\zeta) &= \mathbf{G}_\beta(F_{n-1})(\zeta) \qquad\qquad (4.2) \\ &= \frac{\int_{\mathcal{H}_-} F_{n-1}^2(2^{-(1+\delta)/2}\zeta + 2^{-(1+\delta)/2}\sqrt{2^\delta - 1}\omega)\gamma_T(d\omega)}{\int_{\mathcal{H}_-} F^2(2^{-(1+\delta)/2}\sqrt{2^\delta - 1}\omega)\gamma_T(d\omega)}, \ \zeta \in \mathcal{H}_-^{(c)}, \\ F_0(\varphi) &= \int_{\mathcal{H}} \exp((\varphi, \omega))\nu_0(d\omega), \ \varphi \in \mathcal{H}^{(c)}, \quad \nu_0 \in \mathcal{M}_\delta. \end{aligned}$$

As an even entire function F_n can be written in the form of the expansion

$$F_n = \sum_{l=0}^{\infty} \frac{1}{(2l)!} F_{2l}^{(n)}, \tag{4.3}$$

which is absolutely convergent on the whole $\mathcal{H}_-^{(c)}$. Since $F_n(0) = 1$, the function

$$U^{(n)} \stackrel{\text{def}}{=} \log F_n, \tag{4.4}$$

is holomorphic in a neighborhood of zero in $\mathcal{H}_-^{(c)}$, where the expansion

$$U^{(n)} = \sum_{l=1}^{\infty} \frac{1}{(2l)!} U_{2l}^{(n)} \tag{4.5}$$

converges absolutely. Both $F_{2l}^{(n)}$ and $U_{2l}^{(n)}$ are homogeneous $2l$-linear continuous functions on the corresponding tensor products of $\mathcal{H}_-^{(c)}$. The embedding operator $O : \mathcal{H} \to \mathcal{H}_-$ belongs to the Hilbert-Schmidt class, thus the kernel theorem [7] implies that for $\varphi \in \mathcal{H}^{(c)}$, these functions possess the integral representations:

$$F_{2l}^{(n)} = \int_{I_\beta^{2l}} F_{2l}^{(n)}(\tau_1, \tau_2, \ldots, \tau_{2l})\varphi(\tau_1)\ldots\varphi(\tau_{2l})d\tau_1 d\tau_2 \ldots d\tau_{2l}, \tag{4.6}$$

$$U_{2l}^{(n)} = \int_{I_\beta^{2l}} U_{2l}^{(n)}(\tau_1, \tau_2, \ldots, \tau_{2l}) \varphi(\tau_1) \ldots \varphi(\tau_{2l}) d\tau_1 d\tau_2 \ldots d\tau_{2l}. \tag{4.7}$$

A particular case of (4.6) has appeared as (2.8). Just as in that particular case the above kernels belong to $C(I_\beta^{2l})$ and are invariant with respect to the shifts (2.9). Therefore, as in (2.11), (2.12) the following integrals do not depend on τ:

$$\hat{U}_n(0) \stackrel{\text{def}}{=} \int_{I_\beta} U_2^{(n)}(\tau, \tau') d\tau', \tag{4.8}$$

$$X_n \stackrel{\text{def}}{=} \int_{I_\beta} U_4^{(n)}(\tau, \tau, \tau', \tau'') d\tau' d\tau''. \tag{4.9}$$

Now let us describe the fixed points of $\mathbf{G}_\beta$ which are the Laplace transforms of the limiting Gaussian measures mentioned in Theorem 2.1. For the choice of θ and σ (4.1), the operator A_Q is $T_Q^{-1} W_Q$ (see (3.22)). The stable fixed point $F_\emptyset$ corresponds to $A_Q = 0$. It is the Laplace transform of the δ-measure concentrated at zero, which is the limiting Gaussian measure for $\beta < \beta_*$. In the case $\beta = \beta_*$ one should have

$$(A_Q \varphi, \varphi)_- = (B_{\beta_*} \varphi, \varphi),$$

where the rank–one operator B_β was introduced in (2.19). This means that $Q = \{1\}$ and W_Q is the projector on the one–dimensional subspace $\mathcal{H}_Q \subset L^2(I_\beta) \subset W_2^{-1}(I_\beta)$ consisting of constant functions and spanned by $\omega_1 \equiv 1/\sqrt{\beta}$. The subspace $\mathcal{H}_Q$ corresponds to the eigenvalue $s_1 = 1$, thus $|\omega_1|_- = |\omega_1| = 1$, $\|T\|^2 = s_1 = 1$, and $\alpha_{\max} = 1$ (see Remark 3.1). Then

$$\begin{aligned} \exp\left(\frac{1}{2}(A_Q \varphi, \varphi)_-\right) &= \exp\left(\frac{1}{2}(\varphi, \omega_1)^2\right) \\ &= \exp\left(\frac{1}{2}(B_\beta \varphi, \varphi)\right) \in \hat{\mathcal{A}}. \end{aligned} \tag{4.10}$$

Since A_Q has rank one, the fixed point (4.10) is one-dimensional. This means in turn that the measure which has it as a Laplace transform is supported on the one-dimensional subspace of $L^2(I_\beta)$ consisting of constants just like in the classical (nonquantum) case (see [5]). The classical character of the critical point in the hierarchical quantum spin model was shown in [12].

In the sequel we use also the operator description of the model under consideration. The starting measure (2.7) is connected with a one-particle Hamiltonian. It is the following operator

$$H = -\frac{1}{2m}\left(\frac{d}{dx}\right)^2 + \frac{1}{2}x^2 + V(x), \quad x \in I\!R, \tag{4.11}$$

densely defined in the Hilbert space $\mathcal{H}^{\rm osc} = L^2(I\!R, dx)$, which describes a quantum particle performing one-dimensional oscillations around the equilibrium position $x = 0$. Here m is the same as in (2.4) and the function V is given by (2.6). Let q denote the position operator which describes the oscillator displacements from the equilibrium and acts in $\mathcal{H}^{\rm osc}$ as follows: $(q\psi)(x) = x\psi(x)$. For all $\beta > 0$, the operators $q^l \exp(-\beta H)$, $l \in Z\!\!\!Z_+$ belong to the trace class. Let us consider a countable chain of copies of the described oscillator, numbered by $j \in I\!N$. Introduce

$$\Lambda_{n,j} = \{s \in I\!N|\ 2^n(j-1)+1 \leq s \leq 2^n j\},\ j \in I\!N,\ n \in Z\!\!\!Z_+. \tag{4.12}$$

Then

$$\Lambda_{n,j} = \bigcup_{s\in\Lambda_{l,j}} \Lambda_{n-l,s},\ l = 1, 2, \ldots, n. \tag{4.13}$$

For each $\Lambda_{n,j}$, a Hamiltonian $H_{\Lambda_{n,j}}$, which describes the particles located in $\Lambda_{n,j}$, is defined recursively on the base of (4.13):

$$H_{\Lambda_{n,j}} = -\frac{1}{2}(2^\delta - 1)2^{-n(1+\delta)}(\sum_{s\in\Lambda_{n,j}} q_s)^2 + \sum_{s\in\Lambda_{1,j}} H_{\Lambda_{n-1,s}}, \tag{4.14}$$

where δ was introduced in (2.2), and the starting element $H_{\Lambda_{0,j}}$ is the one–particle Hamiltonian given by (4.11). Each $H_{\Lambda_{n,j}}$ is an essentially self–adjoint operator in the space $\mathcal{H}^{\rm osc}_{\Lambda_{n,j}}$, which is the tensor product of the $\mathcal{H}^{\rm osc}$ copies indexed by $s \in \Lambda_{n,j}$. The Gibbs state of the particles located in $\Lambda_{n,j}$ may be described by means of temperature Green functions, which are defined as follows. For $0 \leq \tau_1 \leq \tau_2 \leq \ldots \leq \tau_l$, $l \in I\!N$, we set

$$\Gamma^{\beta,\Lambda_{n,j}}_{A_1,\ldots,A_l}(\tau_1, \ldots, \tau_l) = \frac{1}{Z_{\Lambda_{n,j}}} \tag{4.15}$$
$$\mathrm{trace}\left\{A_1 \exp(-(\tau_2 - \tau_1)H_{\Lambda_{n,j}}) \ldots A_l \exp(-(\beta - \tau_l + \tau_1)H_{\Lambda_{n,j}})\right\},$$

where

$$Z_{\Lambda_{n,j}} = \mathrm{trace}\left\{\exp(-\beta H_{\Lambda_{n,j}})\right\}.$$

A_i, $i = 1, \ldots, l$ are densely defined operators in $\mathcal{H}^{\rm osc}_{\Lambda_{n,j}}$, and such that the trace in (4.15) exists. Put

$$A_i = A_{\Lambda_{n,j}} = 2^{-\frac{1}{2}n(1+\delta)} \sum_{s\in\Lambda_{n,j}} q_s. \tag{4.16}$$

Note that $\Lambda_{n,j}$ consists of 2^n points, which means that the normalization of the above sum, due to the presence of the factor $2^{-n\delta/2}$ with $\delta > 0$, is abnormal (comparing to that used in the case of the weak dependence described by the central limit theorem). Therefore, if the dependence between the displacements q_s is weak, the sequence $\{A_{\Lambda_{n,j}}, n \in I\!N\}$ ought to degenerate at zero. The latter means that the sequences of Green functions defined by (4.15) with such A_i converge to zero for all values of their variables and for all $l \in I\!N$. To simplify notations we set

$$\Gamma_{2l}^{(n)}(\tau_1, \ldots, \tau_{2l}) = \Gamma_{A_1,\ldots,A_{2l}}^{\beta,\Lambda_{n,j}}(\tau_1, \ldots, \tau_{2l}), \tag{4.17}$$

where the A_i are given by (4.16). The next assertion establishes the relationship between the operator and functional integral approaches in quantum statistical mechanics. It directly follows from [1] and was used in our works [2], [3] devoted to this model, to which we refer for a more detailed description of the model and technical background.

Proposition 4.1 *For all $\varphi \in L^2(I_\beta)$, one has*

$$F_n(\varphi) = \sum_{l=1}^{\infty} \frac{1}{(2l)!} \int_{I_\beta^{2l}} \Gamma_{2l}^{(n)}(\tau_1, \ldots, \tau_{2l})\varphi(\tau_1)\ldots\varphi(\tau_{2l})d\tau_1 \ldots d\tau_{2l}.$$

In other words, the functions introduced in (4.6) coincide with those defined by (4.15) – (4.17).

At the end of this section let us discuss the question whether the set of measures $\mathcal{M}_\delta$ introduced by Definition 2.1 is actually nonempty. First we remark that the statement analogous to Theorem 2.1 which describes the classical hierarchical models was proved in [10], where it was shown that the corresponding set of measures is nonempty if the parameters of V (2.6) satisfy certain conditions. Second, each measure of the type of (2.7) converges weakly, when $m \to +\infty$, to the measure describing the classical oscillator – an element of the mentioned classical hierarchical model. This was proved in [5]. Therefore, the family $\mathcal{M}_\delta$ is nonempty for an appropriate choice of V and large values of the reduced physical mass m. On the other hand, as it was proved in [3], there exists $m_* > 0$ such that, for $m < m_*$, one has $\hat{U}(0) < 1$ for all β. This means that no threshold β_* exists, and hence no strong dependence like that described by Theorem 2.1 may occur, in this case. In physical terms this means that small values of m make critical phenomena impossible in such models.

5 Proof of the Theorem

Introduce

$$\mathcal{K} = \{k \mid k = \frac{2\pi}{\beta}\kappa, \ \kappa \in \mathbb{Z}\}, \tag{5.1}$$

and

$$\hat{U}_n(k) = \int_{I_\beta} U_2^{(n)}(0,\tau)\cos(k\tau)d\tau, \quad k \in \mathcal{K}. \tag{5.2}$$

Then the value of $\hat{U}_n(k)$ for $k = 0$ coincides with the one given by (4.8) and

$$U_2^{(n)}(\tau_1, \tau_2) = \frac{1}{\beta}\sum_{k\in\mathcal{K}} \hat{U}_n(k)\cos[k(\tau_1 - \tau_2)]. \tag{5.3}$$

We also set

$$\begin{aligned} Y_n &= \int_{I_\beta^3} U_4^{(n)}(\tau_1, \tau_2, \tau_3, \tau_4)d\tau_2 d\tau_3 d\tau_4 \\ &= \frac{1}{\beta}\int_{I_\beta^4} U_4^{(n)}(\tau_1, \tau_2, \tau_3, \tau_4)d\tau_1 d\tau_2 d\tau_3 d\tau_4. \end{aligned} \tag{5.4}$$

The proof of our theorem relies upon the lemmas stated just below, which hold for any choice of the initial element $\nu_0 \in \mathcal{M}_\delta$.

Lemma 5.1 *For all $n \in \mathbb{Z}_+$ and $l \in \mathbb{N}$, the following estimates hold true everywhere*

$$\begin{aligned} &\int_{I_\beta^2} U_4^{(n)}(\tau, \tau, \tau_1, \tau_2)d\tau_1 d\tau_2 \\ \leq \ &\int_{I_\beta^2} U_4^{(n)}(\tau, \tau', \tau_1, \tau_2)d\tau_1 d\tau_2; \end{aligned} \tag{5.5}$$

$$(-1)^{l-1}U_{2l}^{(n)}(\tau_1, \tau_2, \ldots, \tau_{2l}) \geq 0; \tag{5.6}$$

$$F_{2l}^{(n)}(\tau_1, \tau_2, \ldots, \tau_{2l}) \geq 0. \tag{5.7}$$

Lemma 5.2 *There exists β_* such that for $\beta = \beta_*$, the sequences $\{X_n, n \in \mathbb{Z}_+\}$, $\{Y_n, n \in \mathbb{Z}_+\}$ converge to zero, and the sequence $\{\hat{U}_n(0), n \in \mathbb{Z}_+\}$ converges to one, as $n \to \infty$. For $\beta < \beta_*$, all these sequences converge to zero.*

Lemma 5.3 *For every $\beta > 0$, $n \in I\!N$, and arbitrary nonzero $k \in \mathcal{K}$,*

$$0 \le \hat{U}_n(k) \le \frac{1}{mk^2}|\Lambda_n|^{-\delta} = \frac{2^{-n\delta}}{mk^2}. \tag{5.8}$$

The latter statement has two evident and important corollaries. Set

$$\mathcal{K}_N = \{k \in \mathcal{K} \mid k = \frac{2\pi}{\beta}\kappa, \ |\kappa| \le N, \ N \in I\!N\}, \quad \mathcal{K}_N^c = \mathcal{K} \setminus \mathcal{K}_N. \tag{5.9}$$

Corollary 5.1 *For every $\beta > 0$,*

$$\sup_{n \in \mathbb{Z}_+} \left\{ \sum_{k \in \mathcal{K}_N^c} \hat{U}_n(k) \right\} \longrightarrow 0, \quad N \to +\infty,$$

hence the sequence $\{\nu_n\}$ is conditionally compact (see [13], p.154).

Corollary 5.2 *For every β,*

$$\sum_{k \in \mathcal{K} \setminus \{0\}} \hat{U}_n(k) \longrightarrow 0, \quad n \to +\infty.$$

This fact follows from the positivity of δ, which defines the abnormal normalization corresponding to the critical point convergence. Therefore, the general conclusion following from the above statement is that for β finite, the one-dimensional (i.e. classical) fixed points characterize the critical point convergence in such models.

Let $\xi_1 \in L^2(I_\beta)$ be such that $\xi_1(\tau) \equiv 1$.

Lemma 5.4 *For every $n \in I\!N$, the following representation holds*

$$F_n(z\xi_1) = \prod_{j=1}^{\infty}(1 + c_j^{(n)} z^2), \quad z \in \mathbb{C}, \tag{5.10}$$

where

$$c_j^{(n)} > 0, \ \sum_j c_j^{(n)} < \infty, \ c_1^{(n)} \ge c_j^{(n)}.$$

For $\beta \le \beta_*$, there exists $c > 0$ such that, for all $n \in I\!N$,

$$c_1^{(n)} \le c. \tag{5.11}$$

Corollary 5.3 *Let c be as above. Then for every $n \in \mathbb{N}$, the following estimates hold*

$$\begin{aligned} |Y_n| &< 6c\hat{U}_n(0), & (5.12)\\ \int_{I_\beta^{2l}} \left|U_{2l}^{(n)}(\tau_1,\ldots,\tau_{2l})\right| d\tau_1\ldots d\tau_{2l} &< \frac{(2l-1)!}{6}c^{l-2}\beta|Y_n|,\ l>2. & (5.13)\end{aligned}$$

Proof. From the representations (4.4)–(4.7) and (5.10), one has ($l \geq 1$):

$$\int_{I_\beta^{2l}} U_{2l}^{(n)}(\tau_1,\ldots,\tau_{2l})d\tau_1\ldots d\tau_{2l} = 2(2l-1)!(-1)^{l-1}\sum_{j=1}^{\infty}\left(c_j^{(n)}\right)^l. \qquad (5.14)$$

Taking into account (4.8), (5.4), the estimates (5.11), and $c_1^{(n)} \geq c_j^{(n)}$, one obtains

$$\begin{aligned} \beta\hat{U}_n(0) &= 2\sum_j c_j^{(n)} > 2c_1^{(n)}, & (5.15)\\ \beta\,|Y_n| &= 12\sum_j\left(c_j^{(n)}\right)^2 < 6c_1^{(n)}\beta\hat{U}_n(0) \leq 6c\beta\hat{U}_n(0). \end{aligned}$$

The latter yields (5.12), the estimate (5.13) follows from (5.14) and (5.6). ■

For a continuous complex valued function $\varphi : I_\beta \to \mathbb{C}$, i.e. for $\varphi \in C(I_\beta)$, we set

$$\|\varphi\|_{C(I_\beta)} = \sup\{|\varphi(\tau)| \mid \tau \in I_\beta\}.$$

Corollary 5.4 *For every $\varphi \in C(I_\beta)$ and $n \in \mathbb{Z}_+$, $F_n(z\varphi)$ is an entire function of $z \in \mathbb{C}$, which obeys the estimate*

$$|F_n(z\varphi)| \leq \exp\left(\frac{1}{2}\beta\hat{U}_n(0)|z|^2\|\varphi\|^2_{C(I_\beta)}\right). \qquad (5.16)$$

Proof. To estimate F_n on $C(I_\beta)$ we exploit its representation (4.3). Having in mind that $C(I_\beta) \subset \mathcal{H}^{(c)}$ we may use the representation (4.6), the positivity property (5.7), (5.10), (5.15), and obtain

$$\begin{aligned} |F_n(z\varphi)| &\leq \sum_{l=0}^{\infty}\frac{1}{(2l)!}\,|z|^{2l}\,\|\varphi\|^{2l}_{C(I_\beta)}\int_{I_\beta^{2l}}|F_{2l}^{(n)}(\tau_1,\tau_2,\ldots,\tau_{2l})|d\tau_1 d\tau_2\ldots d\tau_{2l}\\ &= F_n(|z|\,\|\varphi\|_{C(I_\beta)}\xi_1) = \prod_{j=1}^{\infty}(1+c_j^{(n)}\,|z|^2\,\|\varphi\|^2_{C(I_\beta)})\\ &\leq \prod_{j=1}^{\infty}\exp(c_j^{(n)}|z|^2\|\varphi\|^2_{C(I_\beta)}) = \exp\left(\frac{1}{2}\beta\hat{U}_n(0)|z|^2\|\varphi\|^2_{C(I_\beta)}\right). \end{aligned}$$

For every $z \in \mathbb{C}$, $F_n(z\varphi)$ is an entire function of $z\varphi \in \mathcal{H}_-^{(c)}$, thus it is an entire function of z for every fixed $\varphi \in C(I_\beta) \subset \mathcal{H}_-^{(c)}$. ■

Proof of Theorem 2.1. If we prove the pointwise convergence of the sequences $\{F_n\}$ on the dense subset $C(I_\beta) \setminus \mathcal{H}^{(c)}$ to the Laplace transforms of the corresponding measures (for $\beta < \beta_*$ and for $\beta = \beta_*$), then the weak convergence of $\{\nu_n\}$ will follow from Corollary 5.1 by Lemma 2.1 [13], p.153. For $\beta \leq \beta_*$, let $b(\beta) = 0$ if $\beta < \beta_*$, and $b(\beta_*) = 1$. Then we should show that, for all $\varphi \in C(I_\beta)$,

$$F_n(\varphi) \to \exp\left\{\frac{b(\beta)}{2\beta}\left(\int_{I_\beta}\varphi(\tau)d\tau\right)^2\right\}, \quad n \to \infty, \quad \beta \leq \beta_*. \tag{5.17}$$

First we prove that, for $\|\varphi\|_{C(I_\beta)} < 1/\sqrt{c}$ (c was introduced in (5.11)),

$$\log F_n(\varphi) = U^{(n)}(\varphi) \to \frac{b(\beta)}{2\beta}\left(\int_{I_\beta}\varphi(\tau)d\tau\right)^2.$$

Applying (4.5), (4.7) one has

$$\begin{aligned}
&\left|U^{(n)}(\varphi) - \frac{b(\beta)}{2\beta}\left(\int_{I_{\beta_*}}\varphi(\tau)d\tau\right)^2\right| \\
\leq\ & \frac{1}{2}\|\varphi\|^2_{C(I_\beta)}\int_{I_\beta^2}\left|U_2^{(n)}(\tau_1,\tau_2) - \frac{b(\beta)}{\beta}\right| d\tau_1 d\tau_2 \\
+\ & \sum_{l=2}^{\infty}\frac{1}{(2l)!}\|\varphi\|^{2l}_{C(I_\beta)}\int_{I_\beta^{2l}}|U_{2l}^{(n)}(\tau_1,\tau_2,\ldots,\tau_{2l})|d\tau_1 d\tau_2\ldots\tau_{2l}.
\end{aligned} \tag{5.18}$$

The second term can be estimated by means of (5.13), which gives its upper bound

$$-\frac{\beta}{12c^2}|Y_n|\log\left(1 - c\|\varphi\|^2_{C(I_\beta)}\right) \longrightarrow 0, \quad n \to \infty,$$

since $Y_n \to 0$ for $\beta \leq \beta_*$ by Lemma 5.2. To estimate the first term on the right-hand side of (5.18) we use (5.3), which gives

$$\begin{aligned}
&\int_{I_\beta^2}\left|U_2^{(n)}(\tau_1,\tau_2) - \frac{b(\beta)}{\beta}\right| d\tau_1 d\tau_2 \\
=\ & \frac{1}{\beta}\int_{I_\beta^2}\left|\hat{U}_n(0) - b(\beta) + \sum_{k\in\mathcal{K}\setminus\{0\}}\hat{U}_n(k)\cos[k(\tau_1-\tau_2)]\right| d\tau_1 d\tau_2 \\
\leq\ & \beta\left|\hat{U}_n(0) - b(\beta)\right| + \beta\sum_{k\in\mathcal{K}\setminus\{0\}}\hat{U}_n(k).
\end{aligned}$$

Here the first term tends to zero by Lemma 5.2 whereas the second one tends to zero by Corollary 5.2. Thus we have proved the pointwise convergence in the ball $\|\varphi\|_{C(I_\beta)} < 1/\sqrt{c}$. This convergence implies that, for such φ, the sequence $\{F_n(z\varphi)\}$ of entire functions of a single complex variable converges for $|z| < 1$. Since the sequence $\{\hat{U}_n(0)\}$ is bounded for $\beta \leq \beta_*$ (see Lemma 5.2), the sequence $\{F_n(z\varphi)\}$ is bounded in view of (5.16). Therefore, by Vitali's theorem it converges uniformly on compact subsets of $\mathbb{C}$, and hence pointwise on $\mathbb{C}$. For every $\tilde{\varphi} \in C(I_\beta)$, one finds $z \in \mathbb{C}$ and φ in the mentioned ball such that $\tilde{\varphi} = z\varphi$, which completes the proof. ■

6 Proof of Lemmas

We start with the following assertion.

Proposition 6.1 *For arbitrary $n \in \mathbb{N}$, the following inequalities hold:*

$$\hat{U}_n(0) \leq \kappa(\hat{U}_{n-1}(0))\hat{U}_{n-1}(0); \tag{6.1}$$

$$\hat{U}_n(0) \geq \kappa(\hat{U}_{n-1}(0))\hat{U}_{n-1}(0) + \frac{1}{2}(1 - 2^{-\delta}) \left[\kappa(\hat{U}_{n-1}(0))\right]^3 2^{2\delta-1} X_{n-1}; \tag{6.2}$$

$$2^{2\delta-1} \left[\kappa(\hat{U}_{n-1}(0))\right]^4 X_{n-1} \leq X_n < 0; \tag{6.3}$$

$$0 \leq \hat{U}_n(k) \leq \hat{U}_n(0), \tag{6.4}$$

where $\kappa(u)$ and X_n are defined by (2.14) and (4.9) respectively.

The proof of this Proposition, as well as of Lemma 5.1, may be given in the framework of a lattice approximation technique similar to that developed in constructive quantum field theory (see [14], [15]). This technique was applied in [3] where some of the mentioned estimates were proved (e.g. (6.1) in Lemma 2.2). The full exposition of it will be done in our forthcoming paper [6].

The following assertion is the main tool in proving Lemma 5.2.

Lemma 6.1 *Let $\mathcal{I}_n$, $n \in \mathbb{Z}_+$, be the triple of statements (i_n^1, i_n^2, i_n^3) where*

$$\begin{aligned} i_n^1 &= \{\exists \beta_n^+ \in [\beta_0^-, \beta_0^+] : \ \hat{U}_n(0) = u(\delta), \ \beta = \beta_n^+; \ \hat{U}_n(0) < u(\delta), \ \forall \beta < \beta_n^+\}; \\ i_n^2 &= \{\exists \beta_n^- \in [\beta_0^-, \beta_0^+] : \ \hat{U}_n(0) = 1, \ \beta = \beta_n^-; \ \hat{U}_n(0) < 1, \ \forall \beta < \beta_n^-\}; \\ i_n^3 &= \{\forall \beta \in (0, \beta_n^+) : \ |X_n| < w(\delta)\}. \end{aligned}$$

Then (i) *$\mathcal{I}_0$ is true;* (ii) *$\mathcal{I}_{n-1}$ implies $\mathcal{I}_n$.*

Proof. $\mathcal{I}_0$ is true in view of Definition 2.1. Let us prove the implication. For $\beta = \beta_n^+$, $\kappa(\hat{U}_n(0)) = 2^\varepsilon$ and $\kappa(\hat{U}_n(0)) < 2^\varepsilon$ for $\beta < \beta_n^+$ (see (2.13) – (2.16)). Let $\beta = \beta_{n-1}^+$, then (2.15), (6.2), and i_{n-1}^3 yield

$$\begin{aligned}\hat{U}_n(0) &\geq 2^\varepsilon u(\delta) + \frac{1}{2}(1-2^{-\delta})2^{3\varepsilon}2^{2\delta-1}X_{n-1} \\ &> 2^\varepsilon u(\delta)\left[1 - 2^{2(\varepsilon-1)+\delta}(2^\delta-1)\frac{w(\delta)}{u(\delta)}\right] = u(\delta).\end{aligned} \tag{6.5}$$

For $\beta = \beta_{n-1}^-$, the estimate (6.1) gives

$$\hat{U}_n(0) \leq 1. \tag{6.6}$$

Taking into account the β-continuity of $\hat{U}_n(0)$ (which one may prove recursively having this property for $n = 0$) and the estimates (6.5) and (6.6), one concludes that there exists at least one value $\tilde{\beta}_n^+ \in (\beta_{n-1}^-, \beta_{n-1}^+)$ such that $\hat{U}_n(0) = u(\delta)$. Then we put $\beta_n^+ \overset{\text{def}}{=} \min \tilde{\beta}_n^+$. The mentioned continuity of $\hat{U}_n(0)$ yields also $\hat{U}_n(0) < u(\delta)$ for $\beta < \beta_n^+$. Thus i_n^1 is true. The existence of $\beta_n^- \in [\beta_{n-1}^-, \beta_{n-1}^+)$ can be proven similarly with $\beta_n^- = \min\{\beta \in [\beta_{n-1}^-, \beta_{n-1}^+) \mid \hat{U}_n(0) = 1\}$. Thus i_n^2 is also true. For $\beta < \beta_n^+ < \beta_{n-1}^+$, we have $\kappa(\hat{U}_{n-1}(0)) < 2^\varepsilon$, which yields

$$|X_n| < 2^{2\delta-1}2^{4\varepsilon}\,|X_{n-1}| = |X_{n-1}| < w(\delta), \tag{6.7}$$

hence i_n^3 is true as well. The proof is concluded by remarking that

$$[\beta_n^-, \beta_n^+] \subset [\beta_{n-1}^-, \beta_{n-1}^+) \subset [\beta_0^-, \beta_0^+]. \tag{6.8}$$

■

Lemma 6.2 *There exists $\beta_* \in [\beta_0^-, \beta_0^+]$ such that for $\beta = \beta_*$, the following inequalities hold for all $n \in \mathbb{Z}_+$:*

$$1 \leq \hat{U}_n(0) < u(\delta). \tag{6.9}$$

For $\beta < \beta_*$, the above upper estimate also holds, moreover, there exists $K(\beta) > 0$ such that, for all $n \in \mathbb{Z}_+$

$$\hat{U}_n(0) \leq K(\beta)2^{-n\delta}. \tag{6.10}$$

Proof. Consider the set $\Delta_n \stackrel{\text{def}}{=} \{\beta \in (0, \beta_n^+) | 1 \le \hat{U}_n(0) < u(\delta)\}$. Just above we have shown that it is nonempty and $\Delta_n \subseteq [\beta_n^-, \beta_n^+)$. Let us prove that $\Delta_n \subseteq \Delta_{n-1}$. Suppose there exists some $\beta \in \Delta_n$, which does not belong to Δ_{n-1}. For this β, either $\hat{U}_{n-1}(0) < 1$ or $\hat{U}_{n-1}(0) \ge u(\delta)$. Hence either $\hat{U}_n(0) < 1$ or $\hat{U}_n(0) > u(\delta)$ (it can be proved as above). This is in conflict with the assumption $\beta \in \Delta_n$. Now let D_n be the closure of Δ_n, then

$$D_n = \{\beta \in [\beta_n^-, \beta_n^+] | 1 \le \hat{U}_n(0) \le u(\delta)\}, \tag{6.11}$$

is nonempty, closed, and $D_n \subseteq D_{n-1} \subseteq \ldots \subset [\beta_0^-, \beta_0^+]$. Set $D_* = \bigcap_n D_n$, then $D_* \subset [\beta_0^-, \beta_0^+]$ is also nonempty and closed. Now let us show that, for every $\beta \in D_*$, the sharp upper bound in (6.9) holds for all $n \in I\!N$. Suppose $\hat{U}_n(0) = u(\delta)$ for some $n \in I\!N$. Then (6.5) yields $\hat{U}_m(0) > u(\delta)$ for all $m > n$, which means that this β does not belong to all D_m, and hence to D_*. Set $\beta_* = \min D_*$. We have just proved that, for $\beta = \beta_*$, (6.9) holds, thus it remains to prove the second part of the Lemma. Take $\beta < \beta_*$. If $\hat{U}_n(0) \ge 1$ for all $n \in I\!N$, then either (6.9) holds or there exists such n_0 that $\hat{U}_{n_0}(0) \ge u(\delta)$. This means either $\beta \in D_*$ or $\beta > \inf \beta_n^+$. Both these cases contradict the assumption $\beta < \beta_*$. Hence there exists n_0 such that $\hat{U}_n(0) < 1$ for all $n \ge n_0$. In what follows, the definition (2.14) and the estimate (6.1) imply that the sequences $\{\hat{U}_n(0),\ n \ge n_0\}$ and $\{\kappa(\hat{U}_n(0)),\ n \ge n_0\}$ are strictly decreasing. Then for all $n > n_0$, one has (see (6.1))

$$\begin{aligned} \hat{U}_n(0) &\le \kappa(\hat{U}_{n-1}(0))\hat{U}_{n-1}(0) \le \ldots \\ &\le \kappa(\hat{U}_{n-1}(0)\kappa(\hat{U}_{n-2}(0)) \ldots \kappa(\hat{U}_{n_0}(0))\hat{U}_{n_0}(0) < (\kappa(\hat{U}_{n_0}(0))^{n-n_0}. \end{aligned}$$

Since $\kappa(\hat{U}_{n_0}(0)) < 1$, one gets $\sum_{n=0}^{\infty} \hat{U}_n(0) < \infty$. Thus

$$\prod_{n=1}^{\infty} \left[1 - (1 - 2^{-\delta})\hat{U}_{n-1}(0)\right]^{-1} \stackrel{\text{def}}{=} K_0 < \infty. \tag{6.12}$$

Finally, we apply (6.1) once again and obtain

$$\begin{aligned} \hat{U}_n(0) &\le 2^{-n\delta} \left[1 - (1 - 2^{-\delta})\hat{U}_{n-1}(0)\right]^{-1} \ldots \left[1 - (1 - 2^{-\delta})\hat{U}_0(0)\right]^{-1} \hat{U}_0(0) \\ &< 2^{-n\delta} K_0 u(\delta) \stackrel{\text{def}}{=} K(\beta) 2^{-n\delta}. \end{aligned}$$

■

Proof of Lemma 5.2. Consider the case $\beta = \beta_*$ where the estimate (6.9) hold. First we show that $X_n \to 0$. Making use of (6.3) we obtain

$$0 > X_n \ge 2^{2\delta - 1} \left[\kappa(\hat{U}_{n-1}(0))\right]^4 X_{n-1} > X_{n-1} > X_{n-2} > \ldots > -w(\delta).$$

Therefore, $\{X_n\}$ is strictly increasing and bounded, hence it converges and its limit, say X_*, obeys the condition $X_* > X_0 > -w(\delta)$. Assume that $X_* < 0$. Then (6.3) yields $\kappa(\hat{U}_n(0)) \to 2^{\varepsilon}$ hence $\hat{U}_n(0) \to u(\delta)$. Passing to the limit $n \to \infty$ in (6.2) one obtains $X_* = -w(\delta)$ which contradicts the above condition. Thus $X_* = 0$. To show $\hat{U}_n(0) \to 1$ we set

$$\Xi_n = \frac{1}{2}(1 - 2^{-\delta})\left[\kappa(\hat{U}_{n-1}(0))\right]^3 2^{2\delta-1} X_{n-1}. \tag{6.13}$$

Combining (6.1) and (6.2) one obtains

$$0 \geq \hat{U}_n(0) - \kappa(\hat{U}_{n-1}(0))\hat{U}_{n-1}(0) \geq \Xi_n \to 0. \tag{6.14}$$

For $\beta = \beta_*$, the sequence $\{\hat{U}_n(0)\} \subset [1, u(\delta))$ in view of Lemma 6.2. By (6.14) all its accumulation points in the interval $\in [1, u(\delta)]$ ought to solve the equation

$$U - \kappa(U)U = 0$$

There is only one such $U_* = 1$, which is thus the limit of the whole sequence $\{\hat{U}_n(0)\}$. For $\beta < \beta_*$, $\hat{U}_n(0) \to 0$ in view of (6.10), which yields in turn $\kappa(\hat{U}_n(0)) \to 2^{-\delta}$. The latter and (6.3) imply $X_n \to 0$. To prove the convergence of $\{Y_n\}$ we use (4.9), (5.4), and Lemma 5.1, which yields $|Y_n| \leq |X_n|$ and hence $Y_n \to 0$ for $\beta \leq \beta_*$. ∎

Proof of Lemma 5.3. The lower bound in (5.8) follows from (6.4). By the definitions (4.4) – (4.7) and (4.15) – (4.17), and by means of Proposition 4.1 one obtains

$$\begin{aligned} U_2^{(n)}(0,\tau) &= \Gamma_2^{(n)}(0,\tau) \qquad\qquad (6.15)\\ &= \frac{1}{Z_{\Lambda_{n,j}}} \mathrm{trace}\left\{A_{\Lambda_{n,j}} \exp\left[-\tau H_{\Lambda_{n,j}}\right] A_{\Lambda_{n,j}} \exp\left[-(\beta-\tau)H_{\Lambda_{n,j}}\right]\right\}. \end{aligned}$$

It may be shown that every $H_{\Lambda_{n,j}}$ is invertible on its domain and its inverse is a compact positive operator on $\mathcal{H}^{\mathrm{osc}}_{\Lambda_{n,j}}$. Let $\{E_p^{(n)}, p \in I\!N\}$ be the set of the eigenvalues of $H_{\Lambda_{n,j}}$. We denote the corresponding eigenfunctions by $\Psi_p^{(n)}$ and set

$$A_{pp'}^{(n)} \stackrel{\mathrm{def}}{=} (A_{\Lambda_{n,j}}\Psi_p^{(n)}, \Psi_{p'}^{(n)})_{\mathcal{H}^{\mathrm{osc}}_{\Lambda_{n,j}}}.$$

Then one gets from (6.15)

$$U_2^{(n)}(0,\tau) = \frac{1}{Z_{\Lambda_{n,j}}} \sum_{p,p'} \left(A_{pp'}^{(n)}\right)^2 \exp\left[-\beta E_{p'}^{(n)} + \tau(E_p^{(n)} - E_{p'}^{(n)})\right],$$

which can be used together with the definition (5.2) to obtain

$$\begin{aligned}
\hat{U}_n(k) &= \frac{1}{Z_{\Lambda_{n,j}}} \sum_{p,p'} \left(A_{pp'}^{(n)}\right)^2 \frac{E_p^{(n)} - E_{p'}^{(n)}}{k^2 + (E_p^{(n)} - E_{p'}^{(n)})^2} \left(\exp[-\beta E_{p'}^{(n)}] - \exp[-\beta E_p^{(n)}]\right) \\
&\leq \frac{1}{k^2} \frac{1}{Z_{\Lambda_{n,j}}} \sum_{p,p'} \left(A_{pp'}^{(n)}\right)^2 (E_p^{(n)} - E_{p'}^{(n)}) \left(\exp[-\beta E_{p'}^{(n)}] - \exp[-\beta E_p^{(n)}]\right) \\
&= \frac{1}{k^2} \frac{1}{Z_{\Lambda_{n,j}}} \mathrm{trace} \left\{ \left[A_{\Lambda_{n,j}}, \left[H_{\Lambda_{n,j}}, A_{\Lambda_{n,j}}\right]\right] \exp\left(-\beta H_{\Lambda_{n,j}}\right)\right\}, \qquad (6.16)
\end{aligned}$$

where [. , .] stands for the commutator. The operator $H_{\Lambda_{n,j}}$ can by written in the following form (see (4.14) and (4.11))

$$H_{\Lambda_{n,j}} = -\frac{1}{2m} \sum_{s \in \Lambda_{n,j}} \left(\frac{\partial}{\partial x_s}\right)^2 + V_{\Lambda_{n,j}}(q),$$

where $V_{\Lambda_{n,j}}$ is a polynomial in $q_s, s \in \Lambda_{n,j}$. Thus the double commutator in (6.16) may be computed explicitly. It equals $|\Lambda_{n,j}|^{-\delta}/m$, which yields (5.8). ■

Proof of Lemma 5.4 . Within the lattice approximation approach on the base of the Lieb - Sokal theorem [11], one can show that for all $n \in \mathbb{Z}_+$, the function $f_n(z) = F_n(z\xi_1)$ is different from zero on $\{z \in \mathbb{C} \; |\mathrm{Re} z \neq 0\}$. This function is entire and even, its growth is of order less then two. Then it should possess purely imaginary zeros, hence its Hadamard's representation ought to be (5.14). The bound c used in (5.11) one may compute on the base (5.15), that yields $\beta_* u(\delta)/2$, where (6.9) was used. ■

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