Construction of the renormalized $\mathrm{GN}_{2-\epsilon}$ trajectory

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Abstract

We construct the renormalized Gross-Neveu trajectory in $2-\epsilon$ dimensions. Our construction uses a contraction mapping for an extended renormalization group. The extension is a running coupling with linear step β-function. The contraction mapping relies on norm estimates for a fermionic momentum space renormalization group.

1 Introduction

In this paper, we construct the renormalized trajectory of the (chiral) Gross–Neveu model in $2-\epsilon$ dimensions. The two dimensional model was introduced by Gross and Neveu [1] and by Mitter and Weisz [2], both as a model for asymptotic freedom and for dynamical mass generation. In this paper, we consider a super-renormalizable deformation of its renormalization flow. The deformation mimicks a dimensional continuation, without being a regularization. We use it to illustrate how rigorous control of a fermionic ultraviolet limit can be gained by general norm bounds on fermionic renormalization groups. Our construction relies on a cumulant bound, which was proved first by Gawedzki and Kupiainen in [3]. A simplified proof by Lesniewski appeared later in [4]. In an accompanying paper [5], we will give a fresh proof of the cumulant bound and its implications on norm estimates for fermionic renormalization groups.

Our construction is furthermore based on a non-perturbative implementation of the beta function method of [6, 7]. In this approach, one computes renormalized field theories directly as invariant curves emerging from a renormalization fixed point. The fundamental dynamical equations are the condition of renormalization invariance and a tangent (or first order) condition, which selects a particular curve.

The two dimensional Gross-Neveu model has attracted a lot of attention by rigorous renormalization theorists, both because of its simplicity and its interesting non-perturbative features. We mention the work of Gawedzki and Kupiainen [3]; Feldman, Magnen, Rivasseau, and Seneor [8]; Iagolnitzer and Magnen [9, 10]; Kopper, Magnen and Rivasseau [11]. Recent work of Disertori and Rivasseau [12] simplifies earlier constructions by avoiding the use of phase space expansion technology. Our work has the same intention although it proceeds along a different route. Another simple, and conceptually rather different, approach, which organizes perturbation theory in a *ring expansion*, was developed in [13]. Although it has not been applied to a construction of the Gross-Neveu model, it is an alternative to our bounds.

In the following, we briefly describe the model and our main result, leaving detailed definitions for later sections. The ϵ comes from a modification of the massless free propagator, which reads

$$\widehat{C}(p) = \frac{\zeta \not{p}}{|p|^{2+\epsilon}} \tag{1}$$

in momentum space. Here $\zeta \in \mathbb{C}$; $\not{p} = p_1\gamma_1 + p_2\gamma_2$, where γ_1 and γ_2 are two–dimensional hermitean Dirac matrices, with $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}$; and finally $0 < \epsilon < \frac{2}{3}$. As an interaction, we can take any chirally invariant four-fermion interaction. For instance, the two choices

$$\mathcal{O}_{GN}(\psi) = \int \mathrm{d}x \left(\bar{\psi}\psi(x)\right)^2 \tag{2}$$

and

$$\mathcal{O}_{GN}(\psi) = \int \mathrm{d}x \left\{ \left(\bar{\psi}\psi(x)\right)^2 + \left(\bar{\psi}\mathrm{i}\gamma_5\psi(x)\right)^2 \right\}, \tag{3}$$

which correspond to the Gross–Neveu model with discrete or continuous chiral symmetry, are allowed. Here $\bar{\psi}\psi(x) = \sum_{a,\sigma} \bar{\psi}_{\sigma,a}(x)\psi_{\sigma,a}(x)$, where $\sigma \in \{1, 2\}$ is the spin and $a \in \{1, \ldots, N\}$ is the colour index. A function of $F(\psi, \bar{\psi})$ has a continuous chiral invariance if

$$F(\psi, \bar{\psi}) = F\big(\mathrm{e}^{\mathrm{i}\alpha\gamma_5}\psi, \bar{\psi}\mathrm{e}^{\mathrm{i}\alpha\gamma_5}\big) \tag{4}$$

for all $\alpha \in \mathbb{R}$, regarding ψ as a column vector and $\bar{\psi}$ as a row vector with respect to the spin indices. γ_5 is a hermitian matrix that anticommutes with the matrices γ_1 and γ_2 and whose square is $(\gamma_5)^2 = 1$. The interaction (2) is only invariant under the above transformation if $\alpha = \pi$. But even this discrete symmetry forbids a mass term $m \int \mathrm{d}x\ \bar{\psi}(x)\psi(x)$.

A construction of the model can be obtained by iteration of a renormalization group transformation R_L, which combines an integration over fluctuations with a rescaling step. We use the scaled momentum space renormalization group as in [3], given by

$$R_L(V)(\Psi) = \log \int \mathrm{d}\mu_{C_L}(\Phi)\ \exp\Big(V(S_L\Psi + \Phi)\Big) - \text{const.}, \tag{5}$$

Here C_L is a two sided regularization of (1), with unit ultraviolet cutoff and infrared cutoff L^{-1} in units of mass, $\mathrm{d}\mu_{C_L}(\Phi)$ is the corresponding fermionic Gaussian measure (10), S_L is a dilatation by a scale factor of L, and $V(\Psi)$ is a fermionic potential, which perturbs the free model (details follow below). The constant is subtracted to make $R_L(V)(0) = 0$. Eq. (5) satisfies the semi-group law $R_L R_{L'} = R_{L\,L'}$. Consequently, an n-fold iteration of (5) is equivalent to a single step with scale L^n. For technical reasons, we prefer a discrete renormalization group with a rather large scale L. The coupling constant g in front of the interaction will have to be small, its maximal value γ depending on L. It may be complex, but since our bounds involve only $|g|$ and the beta function will only amount to multiplication by a real scale factor, we may take $g > 0$ without loss of generality.

Theorem. *There are $L > 1$ and $\gamma \leq 1$ such that, for all $0 \leq g \leq \gamma$, the following holds.*

1. *Let $g_N = L^{-2\epsilon N} g$, and $V_0^{(N)} = g_N\, \mathcal{O}_{GN}(\psi)$, with $\mathcal{O}_{GN}(\psi)$ given by (2) or (3). Then the limit*

$$V(\psi, g) = \lim_{N \to \infty} (R_L)^N (V_0^{(N)})(\psi) \tag{6}$$

   exists.

2. *Let $\beta_L(g) = L^{-2\epsilon} g$. Then the composition of R_L with the application of the step beta function β_L has a fixed point. For all $g < L^{-2\epsilon}\gamma$,*

$$R_L\big(V(\psi, g)\big) = V(\psi, \beta_{L^{-1}}(g)). \tag{7}$$

3.

$$V(\psi, g) = g\, \mathcal{O}_{GN}(\psi) + g^{\frac{7}{4}} \mathcal{V}(\psi, g), \tag{8}$$

   where $\mathcal{V}$ is small in a norm that depends on L (the details will be given in Section 4).

We prove the Theorem by showing that, in an appropriate Banach space of coupling constants g and interactions, the extended renormalization group T_L, defined by

$$V(\psi, g) \mapsto R_L(V)\big(\psi, \beta_L(g)\big), \tag{9}$$

is a contraction mapping on a cone emerging from the free field fixed point, which corresponds to a ball of second order perturbations $\mathcal{V}$. The interactions in this Banach space are analytic in the fields, chirally invariant, and have exponential spatial decay. Their decay length is determined by that of the fluctuation covariance C_L, and is of the order $O(L)$. [1]

1.1 Setup

We consider continuum functional integrals with ultraviolet and infrared cutoff. [2] Our cutoffs will be built into the propagator. The model is then defined by a regularized propagator together with an effective (inter-) action. The details of this standard setup are, for example, given in [14]. It is also possible to regard the Grassmann variables merely as a convenient way of organizing infinite systems of equations for antisymmetric functions.

Our fermionic fields Ψ are indexed by $\mathbb{X} = \mathbb{R}^2 \times \Lambda$, where Λ is a discrete set, in our case $\Lambda = \{1, -1\} \times \{1, 2, \ldots, N\} \times \{1, -1\}$, where the first index is the spin index, the second a colour index, and the third distinguishes between ψ and $\bar{\psi}$ according to $\Psi(x, \sigma, a, 1) = \bar{\psi}_{\sigma,a}(x)$ and $\Psi(x, \sigma, a, -1) = \psi_{\sigma,a}(x)$. The Grassmann Gaussian integral corresponding to a free theory with a propagator C is determined by

$$\int \mathrm{d}\mu_C(\Psi)\ \mathrm{e}^{(\eta,\Psi)} = \mathrm{e}^{\frac{1}{2}(\eta, C\eta)}. \tag{10}$$

Here the $\eta(X)$ are Grassmann source fields labelled by $X \in \mathbb{X}$, and (η, Ψ) is an abbreviation for $\int_{\mathbb{X}} \mathrm{d}X\, \eta(X)\, \Psi(X)$, the integral over $\mathbb{X}$ meaning $\int \mathrm{d}X F(X) = \int \mathrm{d}^2x \sum_\lambda F(x, \lambda)$.

The fluctuation integral in (5) is well–defined if the covariance $C(X, X')$ is a bounded function of X and X'. The inverse Fourier transform of (1) is not bounded; the construction proceeds by first replacing it by an ultraviolet cutoff covariance which is a finite sum of bounded covariances. The ultraviolet cutoff is removed by taking the number of terms in the sum to infinity, and at the same time letting the coupling constant flow in the way described in the Theorem. The terms in the sum are given by the single–scale covariance of our model,

$$C_L((x, \sigma, a, -1), (x', \sigma', a', 1)) = \mathbf{C}_{1,L}(x, \sigma, a; x', \sigma', a') = -C_L((x', \sigma', a', 1), (x, \sigma, a, -1)), \tag{11}$$

[1] In the scaled renormalization group, one iterates the same transformation, and localization properties depend on this iterated transformation rather than on a flowing scale. Translated to a non-scaled renormalization group, where fluctuation propagators come on different scales, the localization scale becomes proportional to the ultraviolet cutoff.

[2] The continuum regularized functional integral again can be defined by discretizing the regularized field theory to a finite lattice. One then performs both its infinite-volume and zero lattice spacing limit in the presence of continuum cutoffs. We remark that the bounds given in Section 2 imply that the effective action converges as the lattice cutoff is removed.

and zero when the charge indices coincide: $C_L((\cdot,j),(\cdot,j)) = 0$. It is given by the following Dirac propagator

$$\mathbf{C}_{L,L'}(x,\sigma,a;x',\sigma',a') = \delta_{aa'} \int đp \ \mathrm{e}^{\mathrm{i}p(x-x')} \frac{\not{p}_{\sigma,\sigma'}}{|p|^{2+\epsilon}} \Big(\hat{\chi}(Lp) - \hat{\chi}(L'p)\Big), \tag{12}$$

which is two-sided regularized in momentum space with the help of the cutoff function

$$\hat{\chi}(p) = \frac{1}{\Gamma(1+\frac{\epsilon}{2})} \int_{p^2}^{\infty} \mathrm{d}t \ \mathrm{e}^{-t} t^{\frac{\epsilon}{2}}. \tag{13}$$

(This particular regulator has the advantage that the cutoff propagator (12) becomes analytic in momentum space.)

The covariance with unit infrared cutoff and ultraviolet cutoff L^N can be written as a telescope sum

$$\mathbf{C}_{L^{-N},1} = \sum_{m=1}^{N} \mathbf{C}_{L^{-m},L^{-m+1}}; \tag{14}$$

in terms of self-similar $\mathbf{C}$s, which are supported on narrow momentum slices,

$$\mathbf{C}_{L^{-m},L^{-m+1}}(x,\sigma,a;x',\sigma',a') = L^{2m\sigma} \mathbf{C}_{1,L}(L^m x,\sigma,a;L^m x',\sigma',a'). \tag{15}$$

1.2 The RG transformation

The exponent σ denotes the scaling dimension of the massless free fermionic field. In our model, $\sigma = \frac{1}{2}(1-\epsilon)$. The associated scale transformation of fields reads

$$S_L(\Psi)(x,\lambda) = L^{-\sigma}\, \Psi\left(L^{-1}\, x, \lambda\right). \tag{16}$$

With its help, the self-similarity property of the telescoped covariances (15) becomes

$$\mathbf{C}_{L^{-m},L^{-m+1}} = S_{L^{-m}} \mathbf{C}_{1,L} (S_{L^{-m}})^T \tag{17}$$

in operator notation, where T denotes the transposition. The additive decomposition (14) of the covariance implies that the exponential of the effective action at scale 1 is

$$\int \mathrm{d}\mu_{C_{L^{-N},1}}(\Psi) \mathrm{e}^{\mathbf{V}(\Psi+\Phi)} = \int \prod_{m=1}^{N} \mathrm{d}\mu_{C_{1,L}}(\Psi_m) \mathrm{e}^{\mathbf{V}\left(S_{L^{-1}}\Psi_1 + \ldots + S_{L^{-m}}\Psi_m + \Phi\right)} \tag{18}$$

and is thus equal to $\mathrm{e}^{(R_L \circ \cdots \circ R_L)(V)(\Phi)}$, provided that V is a scaled version of the bare potential, namely $V(\Psi) = \mathbf{V}(S_{L^{-m}}\Psi)$. Notice that because of this rescaling, the infrared cutoff of the theory, defined by the left hand side of (18), is one and not L^{-N}, and is not changed by the scaling of the bare potential. More generally, one obtains the scaled renormalization group by a multi-scale transformation, where each multi-scale component is rescaled to a unit scale.

Conversely, one reconstructs the non-scaled renormalization flow by the introduction of a (physical) renormalization scale, often together with a renormalization condition on a coupling parameter. We emphasize that the converse step thus requires an additional datum. [3]

We may decompose R_L into two parts, $R_L = S_L \circ F_L$, with F_L an integration over the fluctuations

$$F_L(V)(\Phi) = \log \int \mathrm{d}\mu_{C_{1,L}}(\Psi) \mathrm{e}^{V(\Psi+\Phi)} - \text{const.} \tag{19}$$

We will derive an estimate on the renormalization group in terms of estimates on these two parts. A field independent constant, which is proportional to the volume, is subtracted in order to preserve the condition $V(0) = 0$ in the RG flow. In statement 1 of the Theorem, the rescaling of the initial coupling constant as a function of the renormalized coupling constant g is given. In the next section, we specify a set of potentials V to which the RG transformation can be applied.

An important property is that the cutoff covariances are of the form $\not{d}$, so that

$$\mathrm{e}^{\mathrm{i}\alpha\gamma_5}\, \mathbf{C}_{L^{-m},L^{-m+1}}\, \mathrm{e}^{\mathrm{i}\alpha\gamma_5} = \mathbf{C}_{L^{-m},L^{-m+1}}. \tag{20}$$

Thus, any discrete or continuous chiral invariance of V is preserved under the RG transformation. In other words, if V obeys (4), then the same holds for $F_L(V)$ and $R_L(V)$.

2 Estimate on the renormalization flow

We now give a norm estimate on the renormalization group transformation $R_L = S_L \circ F_L$ built from two separate norm estimates, an estimate on the scale transformation S_L and an estimate on the fluctuation integral F_L. It will serve as a template for the refined estimates presented thereafter.

2.1 Banach space $\mathbb{V}_{h,\kappa}$

We consider potentials of the following general (power series) type. Let $V(\Psi)$ be given by an infinite sum $V(\Psi) = \sum_{f=1}^{\infty} V_f(\Psi)$ of f–point vertices

$$V_f(\Psi) = \int \mathrm{d}X_1 \Psi(X_1) \cdots \int \mathrm{d}X_f \Psi(X_f)\, V_f(X_1, \ldots, X_f), \tag{21}$$

where the vertices are distributional kernels. We will restrict our attention to vertex functions of the general form

$$V_f(X_1, \ldots, X_f) = \sum_{l=0}^{k-1} \overline{V}_{f,l}(X_1, \ldots, X_f) \sum_{\pi \in \mathfrak{S}_f} \prod_{i=1}^{l} \delta(x_{\pi(1)} - x_{\pi(1+i)}), \tag{22}$$

[3] The scaled renormalization group is best thought of as a block spin transformation on lattice theories, which live on an infinite unit lattice, but encode exact continuum information.

where $\overline{V}_{f,l} \in L^1_{loc}(\mathbb{X} \times \cdots \times \mathbb{X}, \mathbb{C})$, and has the usual properties of a fermionic theory (anti-symmetry, Euclidean covariance). We also assume that V is even, that is, $V_f(\Psi) = 0$ for $f \in 2\mathbb{N}+1$. Then $R_L(V)$ is also even. Temporarily, Ψ denotes the fermionic field without derivatives. Later, we will encorporate derivative fields by enlarging Λ to an appropriate multiplet. Let $\|V\|_{h,\kappa} = \sum_{f=1}^{\infty} h^f \, \|V_f\|_\kappa$, where

$$\|V_f\|_\kappa = \sup_{x_0 \in \mathbb{R}^2} \int \mathrm{d}X_1 \cdots \mathrm{d}X_f \, \delta(x_0 - x_1) \, |V_f(X_1, \ldots, X_f)| \, \exp\Big(\kappa \, \mathcal{L}(x_1, \ldots, x_f)\Big). \tag{23}$$

Here $\mathcal{L}(x_1, \ldots, x_f)$ denotes the tree distance of $(x_1, \ldots, x_f)$. The tree distance on an f-tuple is defined as

$$\mathcal{L}(x_1, \ldots, x_n) = \inf_{\tau \in \mathcal{T}_n} \sum_{b \in \tau} \|x_{b_1} - x_{b_2}\|, \tag{24}$$

where $\mathcal{T}_n$ is the set of trees on $\{1, 2, \ldots, n\}$, and where $b = (b_1, b_2) \in \tau$ are the bonds of τ. The potentials with $\|V\|_{h,\kappa} < \infty$ form a Banach space $\mathbb{V}_{h,\kappa}$. It depends on two parameters h and κ, where h can be thought of as an inverse radius of convergence in field space and κ as an inverse exponential rate of decay. We will show that there exists a choice such that the action of R_L is well–defined on a suitable ball around zero in $\mathbb{V}_{h,\kappa}$.

2.2 Estimate on S_L

Let $S_L(V_f)(\Psi) = V_f\Big(S_L(\Psi)\Big)$. Then [4] $\|S_L(V_f)\|_\kappa = L^{2-f\sigma} \, \|V_f\|_{L^{-1}\kappa}$, so S_L performs the following simple scale transformation on our norm

$$\|S_L(V)\|_{h,\kappa} = L^2 \, \|V\|_{L^{-\sigma}h, L^{-1}\kappa}. \tag{25}$$

Derivatives produce additional inverse powers of L. Because $V_f(\Psi) = 0$ for $f \in 2\mathbb{N}+1$,

$$\|S_L(V)\|_{h,\kappa} \leq L^{1+3\epsilon} \, \|V\|_{L^{-\frac{\epsilon}{4}}h, L^{-1}\kappa}, \tag{26}$$

at least under the wasteful condition that $0 < \epsilon < \frac{2}{3}$. Here we saved a small amount of the scale factor to control an anticipated shift of the field, which will come about in the integral over fluctuations.

2.3 Estimate on F_L

As shown in the Appendix, the one–scale propagator C_L satisfies

$$|C_L(X,Y)| \leq O(1) \, L^{-2\sigma} \, \exp\Big(- L^{-1} \, \|x - y\|\Big) \tag{27}$$

[4] The exponent $2 - f\sigma$ is an old friend from perturbative renormalization theory, namely the scaling dimension of V_f.

(here and in the following, $O(1)$ denotes constants which are independent of L), and it has a Gram representation

$$\mathbf{C}_{1,L}(x,\sigma,a;x',\sigma',a') = \langle \varphi_L(x,\sigma,a) | \tilde{\varphi}_L(x',\sigma',a') \rangle \tag{28}$$

where $\|\varphi_L(x,\sigma,a)\|$ and $\|\tilde{\varphi}_L(x',\sigma',a')\|$ are $O(1)$ uniformly in X. The fluctuation transformation is defined as follows. In an expansion in the fields, $F_L(V)(\Psi) = \sum_{f=1}^{\infty} F_L(V)_f(\Psi)$ with

$$\begin{aligned} F_L(V)_f(\Psi) &= \int \mathrm{d}Z_1 \Psi(Z_1) \cdots \int \mathrm{d}Z_f \Psi(Z_f) \\ \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{f_1=1}^{\infty} \sum_{e_1=0}^{f_1-1} \binom{f_1}{e_1} \cdots &\sum_{f_n=1}^{\infty} \sum_{e_n=0}^{f_n-1} \binom{f_n}{e_n} \delta_{f,e_1+\cdots e_n} \, \Theta_{f,1} \, (-1)^{\alpha_n(f_1,e_1,\ldots,f_n,e_n)} \\ &\int \mathrm{d}Y_{1,1} \cdots \mathrm{d}Y_{1,i_1} \, V_{f_1}(X_{1,1},\ldots,X_{1,e_1},Y_{1,1},\ldots,Y_{1,i_1}) \\ &\cdots \int \mathrm{d}Y_{n,1} \cdots \mathrm{d}Y_{n,i_n} \, V_{f_n}(X_{n,1},\ldots,X_{n,e_n},Y_{n,1},\ldots,Y_{n,i_n}) \\ &\Big\langle \Phi(Y_{1,1}) \cdots \Phi(Y_{1,i_1}); \cdots ; \Phi(Y_{n,1}) \cdots \Phi(Y_{n,i_n}) \Big\rangle_{C_L}^{T}, \end{aligned} \tag{29}$$

where $(Z_1,\ldots,Z_f) = (X_{1,1},\ldots,X_{1,e_1},\ldots,X_{n,1},\ldots,X_{n,e_n})$, $(-1)^{\alpha_n}$ is a sign factor, and where we use the notation $f_l = e_l + i_l$. (f_l is the power of fields of the l'th vertex, e_l is the number of external fields chosen therefrom, and i_l is the number of the remaining internal fields.) The fluctuation integral produces effective vertices $F_L(V)_f(Z_1,\ldots,Z_f)$ as the anti-symmetrized kernels given by the integrand of the expression (29). They are infinite sums of convolutions of the original kernels with propagators. The norm estimates in the remainder of this section imply that these infinite sums converge if $\|V\|_{h,\kappa}$ is small enough.

2.3.1 Estimate on partially truncated correlators

The cumulant expansion (29) involves partially truncated correlators. They obey the following beautiful bound due to Gawedzki and Kupianen [3] and Lesniewski [4]. There are positive constants κ_1, C_1 and C_2, all independent of L, such that

$$\begin{aligned} &\left| \Big\langle \Phi(Y_{1,1}) \cdots \Phi(Y_{1,i_1}); \cdots ; \Phi(Y_{n,1}) \cdots \Phi(Y_{n,i_n}) \Big\rangle_{C_L}^{T} \right| \\ \leq n! \, C_1^{i_1+\cdots+i_n} &\Big(L^{-2\sigma} C_2 \Big)^{n-1} \exp\Big(-L^{-1} \kappa_1 \mathcal{L}(y_{1,1},\ldots,y_{1,i_i} \mid \cdots \mid y_{n,1},\ldots,y_{n,i_n}) \Big) \end{aligned} \tag{30}$$

In an accompanying paper, we derive these bounds, and the norm bounds that follow from them, in a simplified way [5]. Here $\mathcal{L}(\underline{y_1} \mid \ldots \mid \underline{y_n})$ denotes the inter-tuple tree distance of the tuples $\underline{y_l} = (y_{l,1},\ldots,y_{l,i_l})$ defined as

$$\mathcal{L}(\underline{y_1} \mid \ldots \mid \underline{y_n}) = \inf_{j_l \in \{1,\ldots,i_l\}} \mathcal{L}(y_{1,j_1},\ldots,y_{n,j_n}). \tag{31}$$

One selects a point in each tuple and computes the ordinary tree distance for this selection. The selection with a minimal tree distance defines the inter-tuple tree distance.

It will be important that κ_1, C_1, and C_2 do not depend on L because we shall use a large L argument later on. The constant C_1 is proportional to the Gram constant (see the Appendix).

2.3.2 Estimates on tree distances

Since the inter-tuple distance is a tree distance with respect to one particular tree, which may or may not be the minimal one, we have that

$$\sum_{l=1}^{n} \mathcal{L}(\underline{x_l}, \underline{y_l}) + \mathcal{L}(\underline{y_1} \mid \dots \mid \underline{y_n}) \geq \mathcal{L}(\underline{x_1}, \underline{y_1}, \dots, \underline{x_n}, \underline{y_n}). \tag{32}$$

In addition to (32), we need a bound which tells how tree distances behave under the removal of points. [5] There exists a constant α, with $1 < \alpha \leq 2$, such that

$$\alpha \, \mathcal{L}(\underline{x_1}, \underline{y_1}, \dots, \underline{x_n}, \underline{y_n}) \geq \mathcal{L}(\underline{x_1}, \dots, \underline{x_n}). \tag{33}$$

Simple examples show that (33) cannot hold with $\alpha = 1$. For $\alpha = 2$, (33) is readily proved by grouping the removed points into trees and reconnecting the connected components in a way that can be estimated by twice the length of the removed trees.

Let

$$\kappa = L^{-1} \kappa_1 \text{ and } L \geq 4 \geq 2\alpha. \tag{34}$$

Then

$$\frac{\kappa}{L} \mathcal{L}(\underline{x_1}, \dots, \underline{x_n}) - \frac{\kappa_1}{L} \mathcal{L}(\underline{y_1} \mid \dots \mid \underline{y_n}) \leq \kappa \sum_{l=1}^{n} \mathcal{L}(\underline{x_l}, \underline{y_l}) - \frac{\kappa_1}{2\,L} \mathcal{L}(\underline{x_1}, \underline{y_1}, \dots, \underline{x_n}, \underline{y_n}) \tag{35}$$

The estimate (35) is the only property of tree distances which we need in our bound for the fluctuation integral.

2.3.3 Estimate on the f-vertex

We proceed under the assumption (34). Return to (29). From the estimates (30) and (35), it follows that

$$\begin{aligned} \|F_L(V)_f\|_{L^{-1}\kappa} \leq & \sum_{n=1}^{\infty} \left(L^{2-2\sigma} \, C_2 \, C_3 \right)^{n-1} \\ & \sum_{f_1=1}^{\infty} \sum_{e_1=0}^{f_1-1} \binom{f_1}{e_1} C_1^{i_1} \, \|V_{f_1}\|_\kappa \cdots \sum_{f_n=1}^{\infty} \sum_{e_n=0}^{f_n-1} \binom{f_n}{e_n} C_1^{i_n} \, \|V_{f_n}\|_\kappa \, \delta_{f, e_1 + \dots e_n}. \end{aligned} \tag{36}$$

For each vertex, one chooses a point to anchor its tree. One then pulls out the vertex norms. The remaining integral over the anchors is estimated using the spared exponential decay,

$$\sup_{x_0} \int \mathrm{d}^2 x_1 \cdots \int \mathrm{d}^2 x_n \, \delta(x_0 - x_1) \, \exp\left(-\frac{\kappa_1}{2\,\alpha\,L} \mathcal{L}(x_1, \dots, x_n) \right) \leq \left(L^2 \, C_3 \right)^{n-1}. \tag{37}$$

[5] For this reason, Gawedzki and Kupiainen use a different tree distance in [3].

To obtain a bound on the (h,κ)-norm from this, we have to sum over the powers of fields. This yields the geometric series

$$\|F_L(V)\|_{L^{-\sigma}h,L^{-1}\kappa} \le \sum_{n=1}^{\infty}\left(L^{2-2\sigma}C_2C_3\right)^{n-1}\left(\|V\|_{L^{-\sigma}h+C_1,\kappa}\right)^{n} \tag{38}$$

which converges if $q\|V\|_{L^{-\sigma}h+C_1,\kappa} < 1$, where $q = L^{2-2\sigma}C_2C_3$; then

$$\|F_L(V)\|_{L^{-\sigma}h,L^{-1}\kappa} = \frac{\|V\|_{L^{-\sigma}h+C_1,\kappa}}{1-q\|V\|_{L^{-\sigma}h+C_1,\kappa}}. \tag{39}$$

This shows that the RG transform is well–defined on a ball of potentials that are analytic in the fields.

2.4 Estimate on R_L

Let h satisfy

$$h = L^{-\frac{\epsilon}{4}}h + C_1. \tag{40}$$

Both h and κ now depend on L. Let $V(\Psi)$ then be an element of the ball $B_r = \left\{V(\Psi)\in\mathbb{V}_{h,\kappa}\,\middle|\,\|V\|_{h,\kappa}\le r\right\}$ with sufficiently small radius r. Then (26) and (39) imply together that $R_L : B_r \to B_{f_L(r)}$ with a flow of radii given by

$$f_L(r) = \frac{L^{1+3\epsilon}r}{1-qr} \tag{41}$$

Unfortunately, this bound is not sufficient for an iteration of R_L because small potentials tend to grow. [6] This behavior indicates the necessity of renormalization. The factor L in (41) will be removed by restricting to a subspace of potentials with vanishing mass vertex.

3 Two point vertex

The scaling dimension of an $2f$–vertex is $2 - f\sigma$. Because all vertices with an odd number of fields f vanish, the lowest non-vanishing vertex is a two point vertex. Its scaling dimension is $1+\epsilon$, which is also the scaling dimension of a local mass vertex. The two point vertex is the most relevant vertex of our flow. In this section, we will split it into a local and a non-local part. The non-local part will have an improved scaling dimension. The local part is zero for chirally invariant interactions.

[6] The largest eigenvalue of the linearized renormalization group is here $L^{1+\epsilon}$. The extra factor of $L^{2\epsilon}$ is due to the non-linear corrections.

3.1 Localization operator L

The localization operator amounts to a Taylor expansion with remainder in momentum space. For the purposes of this paper, a lowest order expansion of the two point vertex suffices. [7] In real space, we define $\mathbf{L}$ by the decomposition

$$V_2(\Psi) = \int \mathrm{d}X_1 \int \mathrm{d}X_2\, \Psi(X_1) \Big(\mathbf{L}(V_2)(X_1, X_2) + (\mathbf{1} - \mathbf{L})(V_2)(X_1, X_2)\Big)\, \Psi(X_2) \tag{42}$$

into a local part

$$\mathbf{L}(V_2)\Big((x_1, \lambda_1), (x_2, \lambda_2)\Big) = \delta(x_1 - x_2) \int \mathrm{d}^2 y_2\, V_2\Big((x_1, \lambda_1), (y_2, \lambda_2)\Big) \tag{43}$$

and a non-local part

$$\begin{aligned}&(\mathbf{1} - \mathbf{L})(V_2)\Big((x_1, \lambda_1), (x_2, \lambda_2)\Big)\\ &= \sum_{\mu=1}^{2} \int_0^1 \mathrm{d}t\, t^{-2}\, V_2\left((x_1, \lambda_1), \left(x_1 + \frac{x_2 - x_1}{t}\right), \lambda_2\right) \frac{x_2^\mu - x_1^\mu}{t} \frac{\partial}{\partial x_2^\mu}.\end{aligned} \tag{44}$$

The t-integral converges at $t = 0$ because the vertex decays exponentially fast at infinity. This splitting follows from a Taylor expansion with remainder term of the second field

$$\Psi(x_2, \lambda_2) = \Psi(x_1, \lambda_2) + \int_0^1 \mathrm{d}t\, (x_2 - x_1) \cdot (\partial\Psi)(x_1 + t(x_2 - x_1), \lambda_2) \tag{45}$$

around the position of the first one. After a change of integration variables, one obtains an expression of the form

$$\begin{aligned} V_2(\Psi) &= \int \mathrm{d}^2 x \sum_{\lambda_1, \lambda_2} \Psi(x, \lambda_1)\, m_{\lambda_1, \lambda_2}\, \Psi(x, \lambda_2) \\ &+ \sum_{\mu} \int \mathrm{d}X_1 \int \mathrm{d}X_2 \Psi(X_1)\, V_{2,\mu}(X_1, X_2)\, (\partial_\mu \Psi)(X_2) \end{aligned} \tag{46}$$

3.2 Redefinition of the norm $\|V_2\|_\kappa$

Let us represent the two point vertex as in (46). Then we may redefine its norm into $|||V_2|||_\kappa = \|\mathbf{L}V_2\| + \|(\mathbf{1} - \mathbf{L})V_2\|_\kappa$ with $\|\mathbf{L}V_2\| = \sum_{\lambda_1, \lambda_2} |m_{\lambda_1, \lambda_2}|$ and

$$\|(\mathbf{1} - \mathbf{L})V_2\|_\kappa = \kappa \sup_{x_0} \int \mathrm{d}X_1 \int \mathrm{d}X_2 \sum_{\mu} |V_{2,\mu}(X_1, X_2)| \exp\Big(\kappa\, \|x_1 - x_2\|\Big), \tag{47}$$

which is the old norm of the non-local part times κ. For the higher vertices, we use the old norm. We can now redo the above estimates with this redefined norm.

[7] In the case when $\epsilon = 0$, one has to expand the two point vertex to third order and the four point vertex to first order, as is done in [3] and [8]. The formulas are immediate generalizations of those presented here.

3.2.1 Estimate on S_L

The remainder term has an improved scaling dimension. S_L and $\mathbf{L}$ commute. Therefore, the local term scales according to $\|\mathbf{L}S_L V_2\| = L^{1+\epsilon}\,\|\mathbf{L}V_2\|$, while the non-local remainder scales as

$$\|(\mathbf{1}-\mathbf{L})S_L V_2\|_\kappa = L^\epsilon\,\|(\mathbf{1}-\mathbf{L})V_2\|_{L^{-1}\kappa} \tag{48}$$

because of its derivative field. In our model, the local mass term is zero because of the chiral symmetry (4). The net gain of the localization procedure is a factor of L^{-1} for the redefined norm, since

$$|||S_L V|||_{h,\kappa} = L^\epsilon\, h^2\, |||V_2|||_{L^{-1}\kappa} + \sum_{n=2}^{\infty} L^{2-n(1-\epsilon)}\, h^{2n}\, |||V_{2n}|||_{L^{-1}\kappa}. \tag{49}$$

This gives the following refinement of (26). If $0 < \epsilon < \frac{2}{3}$, then

$$|||S_L V|||_{h,\kappa} \leq L^{3\epsilon}\, |||V|||_{L^{-\frac{\epsilon}{4}}h,L^{-1}\kappa} \tag{50}$$

3.2.2 Estimate on F_L

The norm of the non-local term can be bounded by the norm of the non-differentiated vertex. By definition

$$\|(\mathbf{1}-\mathbf{L})V_2\|_\kappa = \kappa\,\sup_{x_1}\int \mathrm{d}^2x_2 \sum_{\lambda_1,\lambda_2,\mu} \mathrm{e}^{\kappa\|x_1-x_2\|}$$
$$\left|\int_0^1 \mathrm{d}t\, t^{-2}\, V_2\left((x_1,\lambda_1),\left(x_1+\frac{x_2-x_1}{t},\lambda_2\right)\right)\,\frac{x_2^\mu-x_1^\mu}{t}\right| \tag{51}$$

it follows that

$$\|(\mathbf{1}-\mathbf{L})V_2\|_\kappa \leq \sup_{x_1}\int \mathrm{d}^2x_2 \sum_{\lambda_1,\lambda_2} \mathrm{e}^{\kappa\|x_1-x_2\|}\,\Big|V_2\Big((x_1,\lambda_1),(x_2,\lambda_2)\Big)\Big|$$
$$\kappa\sum_\mu |x_1^\mu - x_2^\mu|\int_0^1 \mathrm{d}t\,\mathrm{e}^{t\,\kappa\,\|x_1-x_2\|}. \tag{52}$$

For our convenience, we define $\|x\| = \sum_\mu |x^\mu|$. Then we have the promised estimate

$$\|(\mathbf{1}-\mathbf{L})V_2\|_\kappa \leq \|V_2\|_\kappa. \tag{53}$$

Consequently, we find the following estimate for the effective non-local two point vertex

$$\|(\mathbf{1}-\mathbf{L})S_L F_L(V)_2\|_\kappa = L^\epsilon\,\|(\mathbf{1}-\mathbf{L})F_L(V)_2\|_{L^{-1}\kappa} \leq L^\epsilon\,\|F_L(V)_2\|_{L^{-1}\kappa} \tag{54}$$

computed as the image of one renormalization group transformation. The local mass term is zero by the chiral invariance.

We now do the estimate of the fluctuation step exactly as in Section 2. This is possible because (47) is of the same form as the old norm up to a factor κ. For each factor V_2 we pick up a factor κ^{-1}. But we also get one derivative field for each factor V_2. Fortunately,

$$\left|\frac{\partial}{\partial x^\mu} C_L(X,Y)\right| \le O(1)\, L^{-2\sigma-1}\, \exp\Big(-L^{-1}\,\|x-y\|\Big) \tag{55}$$

comes with an additional factor L^{-1}, which compensates the L-factor in κ^{-1}. The remaining constant is easily accommodated since it is of the order $O(1)$. We shift it into a modified cumulant bound. Thus C_1 and C_2 are now understood to be redefined such that the cumulant bound holds for the enlarged multiplet Ψ, which includes derivative fields.

3.3 Estimate on the massless renormalization group

Summing over powers of the field, we get

$$|||R_L(V)|||_{h,\kappa} \le L^\epsilon\, h^2\, \|F_L(V)_2\|_{L^{-1}\kappa} + \sum_{n=2}^{\infty} L^{2-n(1-\epsilon)}\, h^{2n}\, \|F_L(V)_{2n}\|_{L^{-1}\kappa}. \tag{56}$$

The largest scale factor is now $L^{2\epsilon}$. For $0<\epsilon<\frac{2}{3}$, it follows that

$$|||R_L(V)|||_{h,\kappa} \le L^{3\epsilon} \sum_{n=1}^{\infty} \Big(L^{-\frac{\epsilon}{4}}\, h\Big)^{2n}\, \|F_L(V)_{2n}\|_{L^{-1}\kappa}. \tag{57}$$

As before, $h = L^{-\frac{\epsilon}{4}} h + C_1$. Then we can sum the series as above. The result is the estimate

$$|||R_L(V)|||_{h,\kappa} \le L^{3\epsilon} \frac{|||V|||_{h,\kappa}}{1-q|||V|||_{h,\kappa}} \tag{58}$$

with $q = L^{2-2\sigma}\, O(1)$. Thus, also in the massless renormalization group, small potentials tend to grow. But the pace is reduced.

In the following, we shall only work with $|||\cdot|||$; for simplicity, we denote it by the usual norm symbol $\|\cdot\|$.

4 Invariant ball

We turn our attention from points in the space of chirally invariant even potentials to parametrized continuous curves $V(\Psi|g)$, $g \in [0,\gamma]$, which are of the form

$$V(\Psi|g) = g\, \mathcal{O}_{GN}(\Psi) + g^{\frac{7}{4}}\, \mathcal{V}(\Psi|g). \tag{59}$$

Here $\mathcal{O}_{GN}(\Psi)$ denotes the normal ordered Gross-Neveu vertex and $\mathcal{V}(\Psi|g) = O(g^{\frac{1}{4}})$ denotes a second order correction to it. For any fixed g, the potentials of the type (59) form a linear space. We shall estimate the remainder in

$$\|\mathcal{V}\|_{\gamma,h,\kappa} = \sup_{g\in[0,\gamma]} \|\mathcal{V}(\cdot|g)\|_{h,\kappa}. \tag{60}$$

The additional parameter γ denotes the maximal admissible value of the coupling constant g in our estimates.

4.1 Step β-function

The linearization of R_L at the free field fixed point $V^\star(\Psi) = 0$ is the first term of the cumulant expansion

$$DR_L(V)(\Psi) = \int \mathrm{d}\mu_{C_L}(\Phi)\, V(S_L\Psi + \Phi) - \text{const.} \tag{61}$$

The normal ordered Gross-Neveu vertex is an eigenvector of the linearized renormalization group

$$DR_L(\mathcal{O}_{GN})(\Psi) = L^{2\epsilon}\, \mathcal{O}_{GN}(\Psi) \tag{62}$$

with eigenvalue $L^{2\epsilon}$. For $\epsilon > 0$, it is a relevant perturbation. We use the inverse of the eigenvalue in (62) to define our step β-function as the linear function

$$\beta_L(g) = L^{-2\epsilon}\, g. \tag{63}$$

4.2 Extended renormalization group

We then define an extended renormalization group transformation as the composition $T_L = \beta_L \circ R_L$ of a linear coupling transformation $\beta_L(V)(\Psi|g) = V(\Psi|\beta_L(g))$ and the renormalization group R_L. The additional step β-function turns the Gross-Neveu vertex into a fixed point

$$DT_L(g\mathcal{O}_{GN})(\Psi) = g\mathcal{O}_{GN}(\Psi) \tag{64}$$

of the linearized extended renormalization group. Our desire is a non-linear extension thereof. For this purpose, we consider the transformation of the second order correction

$$\mathcal{T}_L(\mathcal{V})(\Psi|g) = g^{-\frac{7}{4}}\, \beta_L S_L \mathcal{F}_L(\mathcal{V})(\Psi|g) = g^{-\frac{7}{4}} \Big(T_L(V)(\Psi|g) - g\, \mathcal{O}_{GN}(\Psi) \Big) \tag{65}$$

4.2.1 Estimate on β_L

The flow of the coupling constant yields an extra small factor. It will turn out to be sufficient to renormalize the theory. We have that

$$\|\mathcal{T}_L(\mathcal{V})\|_{\gamma,h,\kappa} = L^{-\frac{7\epsilon}{2}} \sup_{g\in[0,L^{-2\epsilon}\gamma]} g^{-\frac{7}{4}}\, \|S_L\mathcal{F}_L(\mathcal{V})(\cdot|g)\|_{h,\kappa} \tag{66}$$

Since $L > 1$ and $\epsilon > 0$, we have that $[0, L^{-2\epsilon}\gamma] \subset [0,\gamma]$ and therefore

$$\|\mathcal{T}_L(\mathcal{V})\|_{\gamma,h,\kappa} = L^{-\frac{7\epsilon}{2}} \sup_{g\in[0,\gamma]} g^{-\frac{7}{4}}\, \|S_L\mathcal{F}_L(\mathcal{V})(\cdot|g)\|_{h,\kappa}. \tag{67}$$

In the following, we do not need the small scale factors coming with terms of higher order than g^2.

4.2.2 Estimate on S_L

As a payoff of our general massless estimate (50) it follows that the right hand side of (67) itself can be further estimated by

$$\|\mathcal{T}_L(\mathcal{V})\|_{\gamma,h,\kappa} \leq L^{-\frac{\epsilon}{2}} \sup_{g\in[0,\gamma]} g^{-\frac{7}{4}} \|\mathcal{F}_L(\mathcal{V})(\cdot|g)\|_{L^{-\frac{\epsilon}{4}}h,L^{-1}\kappa} \tag{68}$$

The prefactor $L^{-\frac{\epsilon}{2}} < 1$ will become responsible for the contraction property.

4.2.3 Estimate on $\mathcal{F}_L$

In this renormalization group, we track the transformation of the non-linear corrections to a pure Gross–Neveu vertex. By its definition (65), the subtracted fluctuation step reads, in a selfexplanatory notation,

$$\begin{aligned}\mathcal{F}_L(\mathcal{V})(\Psi|g) &= g^{\frac{7}{4}} \Big\langle \mathcal{V}(S_L\Psi + \Phi\cdot|g)\Big\rangle_{C_L} \\ &+\sum_{n=2}^{\infty}\frac{1}{n!}\Big\langle\prod_{i=1}^{n}\Big[g\,\mathcal{O}_{GN}(S_L\Psi+\Phi)+g^{\frac{7}{4}}\,\mathcal{V}(S_L\Psi+\Phi);\Big]\Big\rangle^T_{C_L}.\end{aligned} \tag{69}$$

The estimate of (69) goes exactly as in Section 2. With the choice (40), the result is

$$\begin{aligned}\|\mathcal{F}_L(\mathcal{V})(\cdot|g)\|_{L^{-\frac{\epsilon}{4}}h,L^{-1}\kappa} &\leq g^{\frac{7}{4}}\|\mathcal{V}(\cdot|g)\|_{h,\kappa} \\ &+\sum_{n=2}^{\infty}\Big(L^{2-2\sigma}\,C_2\,C_3\Big)^{n-1}\Big(g\,\|\mathcal{O}_{GN}\|_{h,\kappa}+g^{\frac{7}{4}}\,\|\mathcal{V}(\cdot|g)\|_{h,\kappa}\Big)^n\end{aligned} \tag{70}$$

When this estimate is plugged into (68), the factor $g^{-\frac{7}{4}}$ cancels, and

$$\begin{aligned}\|\mathcal{T}_L(\mathcal{V})\|_{\gamma,h,\kappa} &\leq L^{-\frac{\epsilon}{2}} \sup_{g\in[0,\gamma]}\Big\{\|\mathcal{V}(\cdot|g)\|_{h,\kappa} \\ &+\sum_{n=2}^{\infty}\Big(L^{2-2\sigma}\,C_2\,C_3\,g^{\frac{1}{4}}\Big)^{n-1}\,g^{\frac{3}{4}(n-2)}\,\Big(\|\mathcal{O}_{GN}\|_{h,\kappa}+g^{\frac{3}{4}}\,\|\mathcal{V}(\cdot|g)\|_{h,\kappa}\Big)^n\Big\}\end{aligned} \tag{71}$$

We have chosen h and κ to depend on L. We now also choose γ to depend on L. We demand that $\gamma \leq 1$ be so small that

$$L^{2-2\sigma}\,C_2\,C_3\,\gamma^{\frac{1}{4}} \leq \frac{1}{2\,\|\mathcal{O}_{GN}\|_{h,\kappa}}. \tag{72}$$

With this, we have the following estimate on the extended renormalization group

$$\|\mathcal{T}_L(\mathcal{V})\|_{\gamma,h,\kappa} \leq L^{-\frac{\epsilon}{2}}\left\{\|\mathcal{V}\|_{\gamma,h,\kappa}+2\,\|\mathcal{O}_{GN}\|_{h,\kappa}\,\sum_{n=2}^{\infty}\left(\frac{1}{2}+\frac{\|\mathcal{V}\|_{\gamma,h,\kappa}}{2\,\|\mathcal{O}_{GN}\|_{h,\kappa}}\right)^n\right\}. \tag{73}$$

4.3 Invariant ball

The only parameter which has not been fixed yet is L. This last parameter in our map can be used to find an invariant ball of second order perturbations. Let

$$B = \left\{ \mathcal{V} \in \mathbb{V}_{\gamma,h,\kappa} \middle| \|\mathcal{V}\|_{\gamma,h,\kappa} \le \frac{\|\mathcal{O}_{GN}\|_{h,\kappa}}{2} \right\} \tag{74}$$

Let L be so large such that

$$L^{-\frac{\epsilon}{2}} \le \frac{1}{10}. \tag{75}$$

Then (73) implies that the ball of second order perturbations (74) is invariant under the extended renormalization group transformation $\mathcal{T}_L$.

5 Contraction property

The last property to be shown is that any pair of points in the invariant ball move closer under an extended renormalization group transformation.

5.1 Estimates on β_L and S_L

The treatment of β_L and S_L remains the same as in the previous section. The result is

$$\begin{aligned} &\|\mathcal{T}_L(\mathcal{V}_1) - \mathcal{T}_L(\mathcal{V}_2)\|_{\gamma,h,\kappa} \\ &\le L^{-\frac{\epsilon}{2}} \sup_{g\in[0,\gamma]} g^{-\frac{7}{4}} \|\mathcal{F}_L(\mathcal{V}_1)(\cdot|g) - \mathcal{F}_L(\mathcal{V}_2)(\cdot|g)\|_{L^{-\frac{\epsilon}{4}}h, L^{-1}\kappa} \end{aligned} \tag{76}$$

5.2 Estimate on $\mathcal{F}_L$

The difference on the right hand side of (76) leads to a cancellation of the $\mathcal{V}$-independent term, as is best seen from the formula

$$\begin{aligned} &\mathcal{F}_L(\mathcal{V}_1)(\Psi|g) - \mathcal{F}_L(\mathcal{V}_2)(\Psi|g) = g^{\frac{7}{4}} \Big\langle \mathcal{V}_1(\cdot|g) - \mathcal{V}_2(\cdot|g) \Big\rangle_{C_L}(\Psi) \\ &+ \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \int_0^1 \mathrm{d}s \, \Big\langle \Big[g\,\mathcal{O}_{GN} + g^{\frac{7}{4}}\,\mathcal{V}_2 + s\,g^{\frac{7}{4}} \Big(\mathcal{V}_1(\cdot|g) - \mathcal{V}_2(\cdot|g)\Big); \Big]^{n-1} \\ &\qquad ; g^{\frac{7}{4}} \Big(\mathcal{V}_1(\cdot|g) - \mathcal{V}_2(\cdot|g)\Big) \Big\rangle^T_{C_L}(\Psi) \end{aligned} \tag{77}$$

In complete analogy to (70), we conclude that the following estimate holds

$$\begin{aligned} &\|\mathcal{F}_L(\mathcal{V}_1)(\Psi|g) - \mathcal{F}_L(\mathcal{V}_2)(\Psi|g)\|_{L^{-\frac{\epsilon}{4}}h, L^{-1}\kappa} \le g^{\frac{7}{4}} \, \|\mathcal{V}_1(\cdot|g) - \mathcal{V}_2(\cdot|g)\|_{h,\kappa} \\ &+ g^{\frac{7}{4}} \, \|\mathcal{V}_1(\cdot|g) - \mathcal{V}_2(\cdot|g)\|_{h,\kappa} \sum_{n=2}^{\infty} \Big(L^{2-2\sigma} C_2 C_3 g^{\frac{1}{4}} \Big)^{n-1} n \, g^{\frac{3}{4}(n-1)} \\ &\int_0^1 \mathrm{d}s \, \Big(\|\mathcal{O}_{GN}\|_{h,\kappa} + g^{\frac{3}{4}} \, \|(1-s)\mathcal{V}_1(\cdot|g) + s\mathcal{V}_2(\cdot|g)\|_{h,\kappa} \Big)^{n-1} \end{aligned} \tag{78}$$

On top of (and consistent with) the above choices of h, κ, γ, and L, we demand that γ be so small that $n\gamma^{\frac{3}{4}(n-1)} \le \left(\frac{4}{3}\right)^2$ for all $n \in \{2,3,4,\dots\}$. Then we have that

$$\|\mathcal{T}_L(\mathcal{V}_1) - \mathcal{T}_L(\mathcal{V}_2)\|_{\gamma,h,\kappa} \le L^{-\frac{\epsilon}{2}} \left\{ 1 + \left(\frac{4}{3}\right)^2 \sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n \right\} \|\mathcal{V}_1 - \mathcal{V}_2\|_{\gamma,h,\kappa} \tag{79}$$

But $L^{-\frac{\epsilon}{2}} \le \frac{1}{10}$, so (79) implies that

$$\|\mathcal{T}_L(\mathcal{V}_1) - \mathcal{T}_L(\mathcal{V}_2)\|_{\gamma,h,\kappa} \le \frac{1}{2} \|\mathcal{V}_1 - \mathcal{V}_2\|_{\gamma,h,\kappa}. \tag{80}$$

Thus our extended RG transformation $\mathcal{T}_L$ is indeed a contraction mapping on the ball B.

6 Conclusions

In this paper, we have constructed the renormalized Gross-Neveu trajectory as an invariant curve in the unstable manifold of the free field fixed point. We have chosen a parametrization, in which the renormalization group acts on the curve in a normal form. The normal form of super-renormalizable models is a linear step β-function. It can be used in models whose differential β-function

$$\dot{\beta}(g) = \partial_L \beta_L(g)|_{L=1} = \beta_1 g + \beta_2 g^2 + \beta_3 g^3 \cdots \tag{81}$$

has a non-vanishing coefficient β_1 (and is regular enough for the first coefficent to be leading). In our model $\beta_1 = -2\epsilon$.

In the non-deformation limit $\epsilon = 0$, the model stays renormalizable due to the sign of the second order correction. Its construction is slightly different from the super-renormalizable case. The normal form of the differential β-function is now cubic, that is, $\dot{\beta}(g) = \beta_2 g^2 + \beta_3 g^3$ with the well known universal constants β_2 and β_3. As L is increased, the coupling flows logarithmically rather than powerlike. Therefore, we cannot extract inverse powers of L from the flowing coupling. One deals with this situation in the usual way by imposing renormalization conditions the non-irrelevant vertices, namely the Gross-Neveu vertex and the wave function vertex. (The mass vertex is still forbidden.) But the infinite series of higher monomials in the fields ψ can be treated exactly as in this paper.

An interesting extension of the present work would be to gain complete control of the renormalized trajectory all the way from the ultraviolet to the expected infrared fixed point. In our model, this would require control of the large coupling limit.

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A Propagator properties

The decay properties follow immediately from the integral representation

$$\mathbf{C}_{1,L}(x,a,\sigma;x',a',\sigma') = \delta_{a,a'}\frac{\mathrm{i}}{4\pi\,\Gamma(1+\frac{\epsilon}{2})}\int_1^{L^2}\mathrm{d}\alpha\;\alpha^{\frac{\epsilon}{2}-1}\not{\partial}_{\sigma,\sigma'}\mathrm{e}^{-\frac{(x-x')^2}{4\alpha}}. \tag{82}$$

The Gram representation holds by the Fourier representation (12) of $\mathbf{C}_{1,L}$: with the spectral decomposition

$$\gamma_\mu = \sum_\rho \lambda_\rho|\mu,\rho\rangle\langle\mu,\rho|, \tag{83}$$

we have

$$(\gamma_\mu)_{\sigma,\sigma'} = \sum_\rho \lambda_\rho\langle\sigma|\mu,\rho\rangle\langle\mu,\rho|,\sigma'\rangle. \tag{84}$$

Thus

$$\varphi_L(x,a,\sigma)(\rho,p,\mu,c) = \delta_{a,c}\,\mathrm{e}^{-\mathrm{i}px}\,|\lambda_\rho p_\mu|^{\frac{1}{2}}\,\langle\sigma|\mu\rho\rangle f_L(p) \tag{85}$$

$$\tilde{\varphi}_L(x',a',\sigma')(\rho,p,\mu,c) = \delta_{c,a'}\,\mathrm{e}^{-\mathrm{i}px'}\,\frac{\lambda_\rho p_\mu}{|\lambda_\rho p_\mu|^{\frac{1}{2}}}\,\langle\sigma'|\mu\rho\rangle f_L(p) \tag{86}$$

with

$$f_L(p) = \left(\frac{\hat{\chi}(p)-\hat{\chi}(Lp)}{|p|^{1+\frac{\epsilon}{2}}}\right)^{1/2}. \tag{87}$$

The norms of φ and $\tilde{\varphi}$ are bounded by

$$(|\lambda_1|+|\lambda_2|)\int_{\mathbb{R}^2}\frac{\mathrm{d}^2p}{|p|^{1+\epsilon}}(\hat{\chi}(p)-\hat{\chi}(L\,p)) \le 2\int_{\mathbb{R}^2}\frac{\mathrm{d}^2p}{|p|^{1+\epsilon}}\hat{\chi}(p) \tag{88}$$

Because $\hat{\chi}(p)\le O(1)|p|^\epsilon\mathrm{e}^{-p^2}$ for $|p|\ge 1$, the integral converges at infinity. The integral over the unit disk is finite for $\epsilon<1$.

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