INTEGRABLE SYSTEMS IN THE INFINITE GENUS LIMIT

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Dedicated with great pleasure to Sergio Albeverio on the occasion of his 60th birthday

ABSTRACT. We provide an elementary approach to integrable systems associated with hyperelliptic curves of infinite genus. In particular, we explore the extent to which the classical Burchnall-Chaundy theory generalizes in the infinite genus limit, and systematically study the effect of Darboux transformations for the KdV hierarchy on such infinite genus curves. Our approach applies to complex-valued periodic solutions of the KdV hierarchy and naturally identifies the Riemann surface familiar from standard Floquet theoretic considerations with a limit of Burchnall-Chaundy curves.

1. INTRODUCTION

Ever since Marchenko's 1974 treatment of the periodic KdV problem (cf. [55] and the references therein) and especially after the work by McKean and Trubowitz in 1976 and 1978 [59], [60], it seemed natural to consider infinite genus limits of completely integrable systems. In fact, in a series of papers from 1982–1985, Levitan [50], [51], [52], [53] addressed the Jacobi inversion problem on infinite genus hyperelliptic curves and constructed classes of quasi periodic and almost periodic potentials. The corresponding constructions of KdV flows can be found in Ch. 12 of his monograph [54]. These investigations of almost periodic Schrödinger operators were continued by Kotani and Krishna [49], Craig [12], and many others (see, e.g., the recent work by Sodin and Yuditskii [74] and the references therein). In the past five years a resurgence of interest in these problems appears to have taken place, as demonstrated by papers of Egorova [24], Feldman, Knörrer, and Trubowitz [28], [29], Müller, Schmidt, and Schrader [63], [64], Schmidt [71], Zakharevich [77], and Merkl [61], [62]. The various approaches in these papers, however, are quite different. Some focus on the construction of theta functions associated with infinite period matrices for the underlying infinite genus curve (cf. [28], [29], [63], [64]) and then derive KdV, KP, etc., solutions in analogy to the Its-Matveev formula [47] in the finite genus case, whereas other authors focus on divisors corresponding to potentials (cf. [61], [62], [71], [77]) and hence devote their attention to extensions of the Riemann-Roch theorem to infinite genus curves. In either approach, the traditional use of Lax pairs to define the underlying infinite genus Riemann surface is avoided as it formally enforces the use of differential expressions of infinite order. Indeed, stationary algebro-geometric KdV solutions V by definition correspond to a commuting Lax pair (P_{2n+1}, L), $[P_{2n+1}, L] = 0$, where $L = -\frac{d^2}{dx^2} + V(x)$ is a second-order Schrödinger differential expression with potential $V(x)$ and P_{2n+1}

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is a differential expression of order $2n+1$, $n \in \mathbb{N} \cup \{0\}$ (cf. Section 2 for details). The Burchnall-Chaundy polynomial associated with (P_{2n+1}, L) then is of the form $P_{2n+1}^2 + R_{2n+1}(L) = 0$ for some polynomial $R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m)$, $\{E_m\}_{m=0,\dots,2n} \subset \mathbb{C}$, of degree $2n+1$ in z, naturally defining the hyperelliptic curve $\mathcal{K}_n$ of (arithmetic) genus n,

$$\mathcal{K}_n \colon y^2 = \prod_{m=0}^{2n}(z - E_m). \tag{1.1}$$

Thus, infinite genus KdV potentials V formally correspond to infinite-order differential expressions P_∞, not too popular a subject! However, a moment of reflection reveals that the situation at hand can be controlled as follows. In the finite genus case, P_{2n+1} restricted to the two-dimensional nullspace $\ker(L-z)$, $z \in \mathbb{C}$ of L, just becomes a first-order differential expression of the type,

$$P_{2n+1}\big|_{\ker(L-z)} = \big(F_n(z,x)\frac{d}{dx} - (1/2)F_{n,x}(z,x)\big)\big|_{\ker(L-z)}, \tag{1.2}$$

where $F_n(z,x)$ is a recursively defined polynomial of degree n with respect to $z \in \mathbb{C}$ (cf. Section 2 for details). The corresponding stationary nth KdV equation satisfied by V is then of the type

$$\text{s-KdV}_n(V) = (1/2)F_{n,xxx}(0,x) - 2V(x)F_{n,x}(0,x) - V_x(x)F_n(0,x) = 0. \tag{1.3}$$

Hence to describe infinite genus situations, where the KdV potential $V(x)$ corresponds to an infinite genus curve $\mathcal{K}_\infty$ of the type

$$\mathcal{K}_\infty : y^2 = (E_0 - z)\prod_{m\in\mathbb{N}}(1 - (z/E_m)), \quad \{E_m\}_{m\in\mathbb{N}} \subset \mathbb{C}, \ \sum_{m\in\mathbb{N}}|E_m|^{-1} < \infty, \tag{1.4}$$

one simply needs to replace the polynomial $F_n(z,x)$ by an appropriate entire function $F_\infty(z,x)$ and this simple recipe is a guiding principle for this paper.

As a result we obtain a canonical stationary KdV_∞ equation satisfied by any KdV potential V associated with an infinite genus curve $\mathcal{K}_\infty$ in (1.4). In complete analogy to the finite genus case (1.3), this equation is of the type

$$\text{s-KdV}_\infty(V) = (1/2)F_{\infty,xxx}(0,x) - 2V(x)F_{\infty,x}(0,x) - V_x(x)F_\infty(0,x) = 0. \tag{1.5}$$

As a concrete application of our formalism we show that any complex-valued periodic $C^1(\mathbb{R})$ potential V satisfies an equation of the type (1.5). Moreover, these considerations naturally identify the underlying Riemann surface (generically of infinite genus), obtained from Floquet theoretic considerations, with the infinite genus limit of Burchnall-Chaundy curves.

Next, a quick outline of the content of each section. Section 2 reviews a construction of the KdV hierarchy; the infinite genus limit is discussed in detail in our principal Section 3; the effect of Darboux transformations on infinite genus curves of the type (1.4) is studied in our final Section 4.

Although this paper focuses only on the case of the KdV hierarchy, combining the methods of this paper with the polynomial recursion approach developed in [7], [35], [38], [40] immediately extends all results to other completely integrable hierarchies associated with hyperelliptic curves such as the Toda lattice, sine-Gordon, classical Boussinesq, and AKNS (nonlinear Schrödinger) hierarchies.

2. The KdV hierarchy

In this section we briefly recall an explicit construction of the KdV hierarchy.

We start by introducing a polynomial recursion formalism following Al'ber [1], [2] (see also [18, Ch. 12], [32], [33]) presented in detail in [41], [45] (see also [36], [39], [46]).

Suppose $V\colon \mathbb{C} \to \mathbb{C}_\infty$ (with $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$) is meromorphic and consider the Schrödinger operator

$$L = -\frac{d^2}{dx^2} + V(x), \quad x \in \mathbb{C}. \tag{2.1}$$

Introducing $\{f_j(x)\}_{j\in\mathbb{N}_0}$, with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, recursively by

$$f_0 = 1, \quad f_{j,x} = -(1/4)f_{j-1,xxx} + Vf_{j-1,x} + (1/2)V_x f_{j-1}, \quad j \in \mathbb{N}, \tag{2.2}$$

one finds explicitly,

$$\begin{aligned}
f_0 &= 1,\\
f_1 &= (1/2)V + c_1,\\
f_2 &= -(1/8)V_{xx} + (3/8)V^2 + c_1(1/2)V + c_2,\\
f_3 &= (1/32)V_{xxxx} - (5/16)VV_{xx} - (5/32)V_x^2 + (5/16)V^3\\
&\quad + c_2(1/2)V + c_1(-(1/8)V_{xx} + (3/8)V^2) + c_3, \text{ etc.},
\end{aligned} \tag{2.3}$$

where $\{c_\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{C}$ denote integration constants. The f_j are well-known to be differential polynomials in V (see, e.g., [26]). Given V and f_j one defines differential expressions P_{2n+1} of order $2n+1$,

$$P_{2n+1} = \sum_{j=0}^{n} \big(f_{n-j}(x)\frac{d}{dx} - (1/2)f_{n-j,x}(x)\big)L^j, \quad n \in \mathbb{N}_0, \tag{2.4}$$

and one verifies

$$[P_{2n+1}, L] = 2f_{n+1,x}, \quad n \in \mathbb{N}_0, \tag{2.5}$$

with $[\,\cdot\,,\,\cdot\,]$ the commutator. The stationary KdV hierarchy, denoted by s-KdV$_n(\cdot)$, $n \in \mathbb{N}_0$, is then defined in terms of the stationary Lax relations

$$-[P_{2n+1}, L] = -2f_{n+1,x} = \text{s-KdV}_n(V) = 0, \quad n \in \mathbb{N}_0. \tag{2.6}$$

Explicitly, one finds for the first few s-KdV equations

$$\begin{aligned}
n = 0:&\quad \text{s-KdV}_0(V) = -V_x = 0,\\
n = 1:&\quad \text{s-KdV}_1(V) = (1/4)V_{xxx} - (3/2)VV_x - c_1V_x = 0,\\
n = 2:&\quad \text{s-KdV}_2(V) = -(1/16)V_{xxxxx} + (5/8)VV_{xxx} + (5/4)V_xV_{xx} - (15/8)V^2V_x\\
&\quad - c_2V_x + c_1((1/4)V_{xxx} - (3/2)VV_x) = 0, \text{ etc.}
\end{aligned} \tag{2.7}$$

$V(x)$ is called an *algebro-geometric* KdV *potential* if it satisfies one (and hence infinitely many) of the equations in the stationary KdV hierarchy (2.6).

Introducing the fundamental polynomial $F_n(z,x)$ of degree n in $z \in \mathbb{C}$,

$$F_n(z,x) = \sum_{j=0}^{n} f_{n-j}(x)z^j = \prod_{j=1}^{n}(z - \mu_j(x)), \quad \{\mu_j(x)\}_{j=1,\dots,n} \subset \mathbb{C},\ x \in \mathbb{C}, \tag{2.8}$$

equations (2.2) and (2.6), that is, $f_{n+1,x}(x) = 0$, $x \in \mathbb{C}$, imply

$$F_{n,xxx}(z,x) - 4(V(x) - z)F_{n,x}(z,x) - 2V_x(x)F_n(z,x) = 0. \tag{2.9}$$

Multiplying (2.9) by $F_n(z,x)$ and integrating yields

$$(1/2)F_{n,xx}(z,x)F_n(z,x) - (1/4)F_{n,x}(z,x)^2 - (V(x)-z)F_n(z,x)^2 = R_{2n+1}(z), \tag{2.10}$$

where the integration constant $R_{2n+1}(z)$ is a monic polynomial in z of degree $2n+1$, and hence of the form

$$R_{2n+1}(z) = \prod_{m=0}^{2n}(z-E_m), \quad \{E_m\}_{m=0,\dots,2n} \subset \mathbb{C}. \tag{2.11}$$

Introducing the algebraic eigenspace,

$$\ker(L-z) = \{\psi\colon \mathbb{C} \to \mathbb{C}_\infty \text{ meromorphic} \mid (L-z)\psi = 0\}, \quad z \in \mathbb{C} \tag{2.12}$$

one verifies

$$P_{2n+1}\big|_{\ker(L-z)} = \big(F_n(z,x)\frac{d}{dx} - (1/2)F_{n,x}(z,x)\big)\big|_{\ker(L-z)} \tag{2.13}$$

and

$$\begin{aligned}\big(P_{2n+1}\big|_{\ker(L-z)}\big)^2 &= \big(-(1/2)F_{n,xx}(z,x)F_n(z,x) + (1/4)F_{n,x}(z,x)^2 \\ &\quad + (V(x)-z)F_n(z,x)^2\big)\big|_{\ker(L-z)} = -R_{2n+1}(z)\big|_{\ker(L-z)}.\end{aligned} \tag{2.14}$$

In agreement with a celebrated result by Burchnall–Chaundy [8], [9], [10] (see also [31], [68], [69], [73], [76]), this yields

$$P_{2n+1}^2 + R_{2n+1}(L) = 0. \tag{2.15}$$

Equation (2.15) naturally leads to the hyperelliptic curve $\mathcal{K}_n$ defined by

$$\mathcal{K}_n\colon y^2 = R_{2n+1}(z), \quad R_{2n+1}(z) = \prod_{m=0}^{2n}(z-E_m). \tag{2.16}$$

The one-point compactification of $\mathcal{K}_n$ by joining P_∞, the point at infinity, is then denoted by $\overline{\mathcal{K}_n}$. A general point $P \in \overline{\mathcal{K}_n}\backslash\{P_\infty\}$ will be denoted by $P=(z,y)$, where $y^2 = R_{2n+1}(z)$.

For future purposes we define the involution $*$ on $\overline{\mathcal{K}_n}$ (i.e., the sheet exchange map) by

$$*\colon \overline{\mathcal{K}_n} \to \overline{\mathcal{K}_n}, \quad P=(z,y) \mapsto P^* = (z,-y), \quad P_\infty^* = P_\infty. \tag{2.17}$$

Returning to (2.8)–(2.11) and (2.16), we introduce the polynomial $H_{n+1}(z,x)$ of degree $n+1$ in $z \in \mathbb{C}$,

$$H_{n+1}(z,x) = (1/2)F_{n,xx}(z,x) - (V(x)-z)F_n(z,x) \tag{2.18}$$

and note that (2.10) then can be rewritten as

$$R_{2n+1}(z) + (1/4)F_{n,x}(z,x)^2 = F_n(z,x)H_{n+1}(z,x), \quad z,x \in \mathbb{C}. \tag{2.19}$$

Next we define the following meromorphic function $\phi(P,x)$ on $\overline{\mathcal{K}_n}$ by

$$\phi(P,x) = \frac{iy(P) + (1/2)F_{n,x}(z,x)}{F_n(z,x)} \tag{2.20}$$

$$= \frac{H_{n+1}(z,x)}{-iy(P) + (1/2)F_{n,x}(z,x)}, \quad P=(z,y) \in \overline{\mathcal{K}_n},\ x \in \mathbb{C}, \tag{2.21}$$

where $y(P)$ denotes the meromorphic function on $\overline{\mathcal{K}_n}$ obtained by solving $y^2 = R_{2n+1}(z)$ with $P = (z, y)$. By (2.8), (2.19), and (2.21), the poles of $\phi(P,x)$ on $\overline{\mathcal{K}_n}$ are given by

$$\hat{\mu}_j(x) = (\mu_j(x), -(i/2)F_{n,x}(\mu_j(x),x)),\ 1 \le j \le n, \text{ and by } P_\infty. \tag{2.22}$$

Given $\phi(P,x)$, we introduce the meromorphic Baker-Akhiezer function $\psi(P,x,x_0)$ on $\overline{\mathcal{K}_n}\backslash\{P_\infty\}$ by

$$\psi(P,x,x_0) = \exp\left(\int_{x_0}^{x} dx'\, \phi(P,x')\right), \quad P = (z,y) \in \overline{\mathcal{K}_n}\backslash\{P_\infty\},\ x, x_0 \in \mathbb{C}, \tag{2.23}$$

choosing a smooth non-selfintersecting path from x_0 to x which avoids singularities of $\phi(P,x)$. Straightforward computations then yield the Riccati-type equation

$$\phi_x(P,x) + \phi(P,x)^2 = V(x) - z, \tag{2.24}$$

for $\phi(P,x)$ and show that $\psi(P,x,x_0)$ satisfies the Schrödinger equation,

$$(L - z(P))\psi(P,\,\cdot\,,x_0) = 0. \tag{2.25}$$

Moreover, one computes

$$\psi(P,x,x_0)\psi(P^*,x,x_0) = F_n(z,x)/F_n(z,x_0), \tag{2.26}$$

$$W(\psi(P,\,\cdot\,,x_0),\psi(P^*,\,\cdot\,,x_0)) = -i2y(P)/F_n(z,x_0), \tag{2.27}$$

where $W(f,g)(x) = f(x)g'(x) - f'(x)g(x)$ denotes the Wronskian of f and g.

Given $\psi(P,x,x_0)$ we can define the formal diagonal Green's function $g(P,x)$, $P \in \overline{\mathcal{K}_n}\backslash\{P_\infty\}$, of L by

$$g(P,x) = \frac{\psi(P,x,x_0)\psi(P^*,x,x_0)}{W(\psi(P,\,\cdot\,,x_0),\psi(P^*,\,\cdot\,,x_0))} = \frac{iF_n(z,x)}{2y(P)} \tag{2.28}$$

$$= \frac{i}{2}\frac{\prod_{j=1}^{n}(z-\mu_j(x))}{\big(\prod_{m=0}^{2n}(z-E_m)\big)^{1/2}}, \quad P = (z,y) \in \overline{\mathcal{K}_n}\backslash\{P_\infty\},\ x \in \mathbb{C}, \tag{2.29}$$

and observe that $g(P,x)$ satisfies the universal differential equations

$$-g_{xxx}(P,x) + 4(V(x)-z)g_x(P,x) + 2V'(x)g(P,x) = 0 \tag{2.30}$$

and

$$-2g_{xx}(P,x)g(P,x) + g_x(P,x)^2 + 4(V(x)-z)g(P,x)^2 = 1 \tag{2.31}$$

after inserting (2.28) into (2.9) and (2.10). The universal equation (2.30) was known to Appell in 1880 [3].

Moreover, comparing terms of order z^{2n} in (2.10) yields the trace formula for $V(x)$,

$$V(x) = E_0 + \sum_{j=1}^{n}(E_{2j-1} + E_{2j} - 2\mu_j(x)). \tag{2.32}$$

Next we turn to the time-dependent case. In order to define the time-dependent KdV hierarchy one introduces a deformation parameter $t_r \in \mathbb{C}$ into $V(x)$, that is, one systematically makes the replacements $V(x) \to V(x,t_r)$, $L \to L(t_r)$, $P_{2r+1} \to P_{2r+1}(t_r)$, $f_j(x) \to f_j(x,t_r)$, $F_n(z,x) \to F_n(z,x,t_r)$, $\mu_j(x) \to \mu_j(x,t_r)$, $\phi(P,x) \to \phi(Px,t_r)$, etc., and then postulates the Lax commutator representation,

$$0 = \frac{d}{dt_r}L(t_r) - [P_{2r+1}(t_r), L(t_r)] = V_{t_r} - 2f_{r+1,x} = \mathrm{KdV}_r(V), \quad r \in \mathbb{N}_0. \tag{2.33}$$

Varying $r \in \mathbb{N}_0$ defines the time-dependent KdV hierarchy, denoted by $\mathrm{KdV}_r(\cdot)$. Explicitly, one obtains for the first few KdV equations,

$$\begin{aligned}
&\mathrm{KdV}_0(V) = V_{t_0} - V_x = 0,\\
&\mathrm{KdV}_1(V) = V_{t_1} + (1/4)V_{xxx} - (3/2)VV_x - c_1 V_x = 0, \qquad (2.34)\\
&\mathrm{KdV}_2(V) = V_{t_2} - (1/16)V_{xxxxx} + (5/8)VV_{xxx} + (5/4)V_x V_{xx} - (15/8)V^2 V_x\\
&\qquad\qquad - c_2 V_x + c_1((1/4)V_{xxx} - (3/2)VV_x) = 0, \text{ etc.,}
\end{aligned}$$

where, of course, $\mathrm{KdV}_1(V)$ is *the* KdV equation.

In terms of $F_n(z,x,t_r)$, (2.33) can be rewritten as

$$V_{t_r}(x,t_r) = -(1/2)F_{n,xxx}(z,x,t_r) + 2(V(x)-z)F_{n,x}(z,x,t_r) + V_x(x)F_n(z,x,t_r). \tag{2.35}$$

The time-dependent Baker-Akhiezer function (i.e., the analog of (2.23)) is defined by

$$\begin{aligned}
\psi(P,x,x_0,t_r,t_{0,r}) = \exp\bigg(\int_{x_0}^{x} dx'\,\phi(P,x',t_r) + \int_{t_{0,r}}^{t_r} ds\,(\widetilde{F}_r(z,x_0,s)\phi(P,x_0,s)\\
- (1/2)\widetilde{F}_{r,x}(z,x_0,s))\bigg), \quad P=(z,y)\in\overline{\mathcal{K}_n}\backslash\{P_\infty\},\ x,x_0,t_r,t_{0,r}\in\mathbb{C}
\end{aligned} \tag{2.36}$$

with $\phi(P,x,t_r)$ given as in (2.20), (2.21) (replacing $F_n(z,x)$ by $F_n(z,x,t_r)$ and $H_{n+1}(z,x)$ by $H_{n+1}(z,x,t_r)$, respectively) and one verifies that $\psi(\cdot,x,x_0,t_r,t_{0,r})$ is meromorphic on $\mathcal{K}_n$, $(L(t_r)-z(P))\psi(P,\cdot,x_0,t_r,t_{0,r})=0$, and (cf. [39, Ch. 1], [41]),

$$\phi_{t_r}(P,x,t_r) = \partial_x(\widetilde{F}_r(z,x,t_r)\phi(P,x,t_r) - (1/2)\widetilde{F}_{r,x}(z,x,t_r)), \tag{2.37}$$

$$\begin{aligned}
\psi_{t_r}(P,x,x_0,t_r,t_{0,r}) &= (\widetilde{P}_{2r+1}(t_r)\psi)(P,x,x_0,t_r,t_{0,r}) \qquad (2.38)\\
&= \widetilde{F}_r(z,x,t_r)\psi_x(P,x,x_0,t_r,t_{0,r}) - (1/2)\widetilde{F}_{r,x}(z,x,t_r)\psi(P,x,x_0,t_r,t_{0,r}),
\end{aligned}$$

$$F_{n,t_r}(z,x,t_r) = \widetilde{F}_r(z,x,t_r)F_{n,x}(z,x,t_r) - \widetilde{F}_{r,x}(z,x,t_r)F_n(z,x,t_r). \tag{2.39}$$

Finally, we briefly consider the algebro-geometric initial value problem. Fix $\Omega\subset\mathbb{C}^2$, Ω open, $t_{0,r}\in\mathbb{C}$, then on the level of Lax differential expressions this initial value problem reads

$$[P_{2n+1}(t_{0,r}), L(t_{0,r})] = 0, \quad (x,t_{0,r})\in\Omega, \tag{2.40}$$

$$\frac{d}{dt_r}L(t_r) - [\widetilde{P}_{2r+1}(t_r), L(t_r)] = 0, \quad (x,t_r)\in\Omega, \tag{2.41}$$

where the integration constants $(c_1,\dots,c_n)$ in P_{2n+1} and those in $\widetilde{P}_{2r+1}$, denoted by $(\tilde{c}_1,\dots,\tilde{c}_r)$, are independent of each other. Similarly, we denote by $\tilde{f}_j$, $1\le j\le r$, $\tilde{f}_{r+1,x}$ $\widetilde{F}_r$, $\widetilde{\mathrm{KdV}}_r(\cdot)$, the corresponding quantities obtained from f_j, $1\le j\le r$, $f_{r+1,x}$ F_r, $\mathrm{KdV}_r(\cdot)$, replacing the set of integration constants $(c_1,\dots,c_n)$ by $(\tilde{c}_1,\dots,\tilde{c}_r)$. Since KdV flows are expected to be isospectral we next replace (2.40), (2.41) by

$$[P_{2n+1}(t_r), L(t_r)] = 0, \quad (x,t_r)\in\Omega, \tag{2.42}$$

$$\frac{d}{dt_r}L(t_r) - [\widetilde{P}_{2r+1}(t_r), L(t_r)] = 0, \quad (x,t_r)\in\Omega. \tag{2.43}$$

In terms of the polynomial formalism, (2.42) and (2.43) become $((x,t_r)\in\Omega)$

$$(1/2)F_{n,xxx}(0,x,t_r) - 2V(x)F_{n,x}(0,x,t_r) - V_x(x,t_r)F_n(0,x,t_r) = 0, \tag{2.44}$$

$$V_{t_r}(x,t_r) = -\tfrac{1}{2}\widetilde{F}_{r,xxx}(0,x,t_r) + 2V(x)\widetilde{F}_{r,x}(0,x,t_r) + V_x(x,t_r)\widetilde{F}_r(0,x,t_r). \quad (2.45)$$

(2.44) and (2.45) are of course equivalent to

$$\text{s-KdV}_n(V) = -2f_{n+1,x} = 0, \quad (x,t_r) \in \Omega, \quad (2.46)$$

$$\widetilde{\text{KdV}}_r(V) = V_{t_r} - 2\tilde{f}_{r+1,x} = 0, \quad (x,t_r) \in \Omega. \quad (2.47)$$

Hence $V(x,t_r)$, for each fixed t_r (with $(x,t_r) \in \Omega$) satisfies the stationary nth KdV equation, while $V(x,t_r)$, $(x,t_r) \in \Omega$, satisfies the rth time-dependent KdV equation.

Remark 2.1. We emphasize that in the general case where $V(x,t_r)$ is complex-valued, (2.46), (2.47) does not necessarily have solutions for any distribution of $\{E_m\}_{m=0,\dots,2n} \subset \mathbb{C}$ as demonstrated by Birnir [6]. Unique solvability of the isospectral deformation problem (2.46), (2.47) (or equivalently, (2.44), (2.45)) can be reduced to unique solvability of the standard first-order system of Dubrovin equations (see, e.g., [15], [18, Sect. 12.3], [22], [33], [36], [41], [54, Chs. 10, 12], [55, Ch. 4]). As long as the curve $\mathcal{K}_n$ is nonsingular (i.e., $E_m \neq E_{m'}$ for $m \neq m'$) and none of the $\mu_j(x,t_r)$ collide (i.e., $\mu_j(x,t_r) \neq \mu_{j'}(x,t_r)$ for $(x,t_r) \in \Omega$, $\Omega \in \mathbb{C}^2$ open), the Dubrovin system is uniquely solvable on $\mathcal{K}_n$ and then $V(x,t_r)$, $(x,t_r) \in \Omega$, is recovered from the time-dependent trace formula

$$V(x,t_r) = E_0 + \sum_{j=1}^{n}(E_{2j-1} + E_{2j} - 2\mu_j(x,t_r)). \quad (2.48)$$

The corresponding $F_n(z,x,t_r)$, $\widetilde{F}_r(z,x,t_r)$, $(x,t_r) \in \Omega$, then satisfy the isospectral deformation equations (2.44), (2.45). The case of collisions between the $\mu_j(x,t_r)$ is more subtle but can also be treated as shown by Birnir [5], [6]. In particular, in the case of nonsingular curves $\mathcal{K}_n$ with real branch points $\{E_m\}_{m=0,\dots,2n} \subset \mathbb{R}$, $E_m \neq E_{m'}$ for $m \neq m'$ no collisions between any of the $\mu_j(x,t_r)$ for $(x,t_r) \in \mathbb{R}^2$ can ever occur due to the strict separation of all spectral gaps by spectral bands. In this case, (2.46), (2.47) yield a smooth solution $V(x,t_r)$, $(x,t_r) \in \mathbb{R}^2$, expressable in terms of the Riemann theta function $\theta(\underline{z})$ associated with $\overline{\mathcal{K}_n}$ by the Its-Matveev formula [47]

$$V(x,t_r) = \Lambda_0 - 2\partial_x^2 \ln(\theta(\underline{A} + \underline{B}x + \underline{C}_r t_r)) \quad (2.49)$$

for appropriate $\Lambda_0 \in \mathbb{R}$, $\underline{A}, \underline{B}, \underline{C}_r \in \mathbb{R}^n$, depending on the homology basis chosen on $\overline{\mathcal{K}_n}$.

3. The infinite genus limit

In this section we will extend the approach of Section 2 to a special class of infinite genus curves whose branch points accumulate at $+\infty$. In particular, we will explore the extent to which the Burchnall-Chaundy results generalize to the transcendental hyperelliptic curves at hand.

We start with the stationary (i.e., time-independent) case. Our point of departure is a rewriting of the diagonal Green's function $g(P,x)$ in the finite genus case (2.29) in the form

$$g(P,x) = \frac{1}{2}\frac{\prod_{j=1}^{n}((\mu_j(x)-z)/(E_{2j-1}E_{2j})^{1/2})}{\big((E_0 - z)\prod_{m=1}^{2n}(1-(z/E_m))\big)^{1/2}} = \frac{\check{F}_n(z,x)}{2y(P)}, \quad (3.1)$$

(for $P=(z,y)\in\breve{\mathcal{K}}_n$) where

$$\breve{F}_n(z,x)=\prod_{j=1}^{n}((\mu_j(x)-z)/(E_{2j-1}E_{2j})^{1/2}) \tag{3.2}$$

and $\breve{\mathcal{K}}_n$ denotes the hyperelliptic curve

$$\breve{\mathcal{K}}_n : y^2=\breve{R}_{2n+1}(z),\quad \breve{R}_{2n+1}(z)=(E_0-z)\prod_{m=1}^{2n}(1-(z/E_m)). \tag{3.3}$$

For simplicity we assume that $E_m\neq 0$, $m=1,\dots,2n$ and that $\breve{R}_{2n+1}(z)$ has a simple zero at $z=E_0$. According to our convention in Section 2, the products in (3.1) repeat each factor associated with $\mu_j(x)$ and E_m according to their multiplicities. Introducing the following Hypothesis 3.1 will then enable us to treat the case $n=\infty$ in (3.1)–(3.3).

Hypothesis 3.1. *Let $\mathcal{I}\subseteq\mathbb{R}$ be open and $x\in\mathcal{I}$.*
(i) *Assume $\{E_m\}_{m\in\mathbb{N}_0}\subset\mathbb{C}$ such that $\sum_{m\in\mathbb{N}}|E_m|^{-1}<\infty$, $\mathrm{Re}(E_m)\underset{m\to\infty}{\longrightarrow}+\infty$, $\mathrm{Im}(E_m)\underset{m\to\infty}{\longrightarrow}0$, ordering the E_m according to their absolute values counting multiplicity.*
(ii) *Suppose $\{\mu_j(x)\}_{j\in\mathbb{N}}\subset\mathbb{C}$ such that $\{1-\mu_j(x)(E_{2j-1}E_{2j})^{-1/2}\}_{j\in\mathbb{N}}\in\ell^1(\mathbb{N})$ for all $x\in\mathcal{I}$, where we agree to take the principal branch of $(E_{2j-1}E_{2j})^{1/2}$ (such that $\arg((E_{2j-1}E_{2j})^{1/2})\underset{j\to\infty}{\longrightarrow}0$).*
(iii) *For all $x\in\mathcal{I}$, assume $\{E_{2j-1}+E_{2j}-2\mu_j(x)\}_{j\in\mathbb{N}}\in\ell^1(\mathbb{N})$ and $\{(E_{2j-1}E_{2j}-\mu_j(x)^2)/E_{2j}\}_{j\in\mathbb{N}}\in\ell^1(\mathbb{N})$.*

Hypothesis 3.1 (i) and the Cauchy-Schwarz inequality then yield $\sum_{j\in\mathbb{N}}|E_{2j-1}E_{2j}|^{-1/2}\leq(\sum_{j\in\mathbb{N}}|E_{2j-1}|^{-1})^{1/2}(\sum_{j\in\mathbb{N}}|E_{2j}|^{-1})^{1/2}<\infty$. Thus, assuming Hypotheses 3.1 (i), (ii) for the remainder of this paper, one can define the transcendental (infinite genus) hyperelliptic curve

$$\mathcal{K}_\infty : y^2=R_\infty(z),\quad R_\infty(z)=(E_0-z)\prod_{m\in\mathbb{N}}(1-(z/E_m)). \tag{3.4}$$

In analogy to Section 2, points $P\in\mathcal{K}_\infty$ are denoted by $P=(z,y)$, where $y^2=R_\infty(z)$ and the involution $*$ on $\mathcal{K}_\infty$ is defined as in (2.17) by

$$*\colon \mathcal{K}_\infty\to\mathcal{K}_\infty,\quad P=(z,y)\mapsto P^*=(z,-y). \tag{3.5}$$

In addition we define the diagonal Green's function,

$$g(P,x)=\frac{1}{2}\frac{\prod_{j\in\mathbb{N}}((\mu_j(x)-z)/(E_{2j-1}E_{2j})^{1/2})}{\big((E_0-z)\prod_{m\in\mathbb{N}}(1-(z/E_m))\big)^{1/2}}=\frac{F_\infty(z,x)}{2y(P)} \tag{3.6}$$

(for $P=(z,y)\in\mathcal{K}_\infty$), with absolutely convergent infinite products in (3.4) and (3.6). Here

$$F_\infty(z,x)=\prod_{j\in\mathbb{N}}((\mu_j(x)-z)/(E_{2j-1}E_{2j})^{1/2}), \tag{3.7}$$

and as in (3.2) and (3.3) we asume that in the infinite products (3.4) and (3.7) each factor associated with $\mu_j(x)$ and E_m is repeated according to its multiplicity. For simplicity we suppose again that $R_\infty(z)$ has a simple zero at E_0. This can always be achieved by a simple translation $V(x)\to V(x)+z_0$ for some $z_0\in\mathbb{C}$.

Taking (3.6)–(3.4) as the model for diagonal Green's functions of differential expressions $L = -\frac{d^2}{dx^2} + V(x)$ associated with infinite genus curves of the type (3.4), we now postulate the following hypothesis.

Hypothesis 3.2. *Assume Hypothesis* 3.1 *(i), (ii) and suppose $x \in \mathcal{I}$. Define $\mathcal{K}_\infty$, $g(P,x)$, $P \in \mathcal{K}_\infty$, $F_\infty(z,x)$, and $R_\infty(z)$ as in* (3.6)–(3.4). *Denote by $C_\varepsilon = \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > 0, \, |\arg(z)| < \varepsilon\}$ the cone along the positive real axis with apex at the origin and opening angle ε, $0 < \varepsilon < \pi/2$.*
(i) *Suppose that $V \in C^1(\mathcal{I})$ and $g(P,\cdot) \in C^3(\mathcal{I})$ for all $P \in \mathcal{K}_\infty$.*
(ii) *Assume that $g(P,x)$ satisfies*

$$-2g_{xx}(P,x)g(P,x) + g_x(P,x)^2 + 4(V(x)-z)g(P,x)^2 = 1. \tag{3.8}$$

(iii) *Suppose that for each $x \in \mathcal{I}$,*

$$g(P,x) = (i/2)z(P)^{-1/2} + (i/4)z(P)^{-3/2}V(x) + o(z(P)^{-3/2}) \tag{3.9}$$

as $z \to \infty$, $z \in \mathbb{C}\backslash C_\varepsilon$.
(iv) *Assume that as $z \to \infty$, $z \in \mathbb{C}\backslash C_\varepsilon$,*

$$g(P,x) = (i/2)z(P)^{-1/2} + o(z(P)^{-1/2}), \tag{3.10}$$

$$g_x(P,x) = o(z(P)^{-1/2}), \tag{3.11}$$

uniformly with respect to x as long as x varies in compact subintervals of $\mathcal{I}$.

For the time being we prefer to develop the theory for $x \in \mathcal{I}$ rather than $x \in \mathbb{R}$ (or $x \in \mathbb{C}$) in order to include a possible meromorphic behavior of $V(x)$. Moreover, we are not necessarily restricting ourselves to smooth functions V (such as $V \in C^\infty(\mathcal{I})$, etc.).

Differentiating (3.8) with respect to x yields the linear third-order differential equation

$$-g_{xxx}(P,x) + 4(V(x)-z)g_x(P,x) + 2V'(x)g(P,x) = 0, \quad P \in \mathcal{K}_\infty, \, x \in \mathcal{I}. \tag{3.12}$$

Given Hypothesis 3.2, we introduce the entire function $H_\infty(z,x)$ with respect to z, by

$$H_\infty(z,x) = (1/2)F_{\infty,xx}(z,x) + (z - V(x))F_\infty(z,x), \quad z \in \mathbb{C}, \, x \in \mathcal{I}. \tag{3.13}$$

Inserting $g(P,x) = F_\infty(z,x)/(2y(P))$ (cf. (3.6) and (3.4)) into (3.8) and (3.12) then yields ($z \in \mathbb{C}$, $x \in \mathcal{I}$)

$$-\tfrac{1}{2}F_{\infty,xx}(z,x)F_\infty(z,x) + (V(x)-z)F_\infty(z,x)^2 + \tfrac{1}{4}F_{\infty,x}(z,x)^2 = R_\infty(z), \tag{3.14}$$

and

$$-F_{\infty,xxx}(z,x) + 4(V(x)-z)F_{\infty,x}(z,x) + 2V'(x)F_\infty(z,x) = 0. \tag{3.15}$$

Combining (3.13) and (3.14) results in

$$R_\infty(z) + (1/4)F_{\infty,x}(z,x)^2 = F_\infty(z,x)H_\infty(z,x), \quad z \in \mathbb{C}, \, x \in \mathcal{I}. \tag{3.16}$$

Thus, we may define the following fundamental meromorphic function $\phi(P,x)$ on $\mathcal{K}_\infty$,

$$\phi(P,x) = \frac{-y(P) + (1/2)F_{\infty,x}(z,x)}{F_\infty(z,x)} \tag{3.17}$$

$$= \frac{H_\infty(z,x)}{y(P) + (1/2)F_{\infty,x}(z,x)}, \quad P = (z,y) \in \mathcal{K}_\infty, \, x \in \mathcal{I}, \tag{3.18}$$

where $y(P)$ denotes the analytic function on $\mathcal{K}_\infty$ obtained by solving $y^2 = R_\infty(z)$ with $P = (z, y)$ (cf. (3.4)). By (3.7), (3.16), and (3.18) we can identify the pole positions $\{\hat{\mu}_j(x)\}_{j\in\mathbb{N}}$ of $\phi(P, x)$ on $\mathcal{K}_\infty$ as

$$\hat{\mu}_j(x) = (\mu_j(x), -(1/2)F_{\infty,x}(\mu_j(x), x)), \quad j \in \mathbb{N}. \tag{3.19}$$

In addition, we introduce the analog of the stationary Baker-Akhiezer function $\psi(P, x, x_0)$ on $\mathcal{K}_\infty$ by

$$\psi(P, x, x_0) = \exp\left(\int_{x_0}^{x} dx'\, \phi(P, x')\right), \quad P = (z, y) \in \mathcal{K}_\infty,\ x, x_0 \in \mathcal{I}, \tag{3.20}$$

choosing a smooth non-selfintersecting path from x_0 to x which avoids singularities of $\phi(P, x)$.

Lemma 3.3. *Assume Hypothesis* 3.2 *(i),(ii) and let* $P = (z, y) \in \mathcal{K}_\infty$, $x, x_0 \in \mathcal{I}$. *Then* $\phi(P, x)$ *satisfies*

$$\phi_x(P, x) + \phi(P, x)^2 = V(x) - z, \tag{3.21}$$

$$\phi(P, x)\phi(P^*, x) = H_\infty(z, x)/F_\infty(z, x), \tag{3.22}$$

$$\phi(P, x) + \phi(P^*, x) = F_{\infty,x}(z, x)/F_\infty(z, x), \tag{3.23}$$

$$\phi(P, x) - \phi(P^*, x) = -2y(P)/F_\infty(z, x). \tag{3.24}$$

$\psi(\cdot, x, x_0)$ *is meromorphic on* $\mathcal{K}_\infty$ *and satisfies*

$$(L - z(P))\psi(P, \cdot\,, x_0) = 0, \quad L = -\frac{d^2}{dx^2} + V(x), \tag{3.25}$$

$$\psi(P, x, x_0)\psi(P^*, x, x_0) = F_\infty(z, x)/F_\infty(z, x_0), \tag{3.26}$$

$$\psi_x(P, x, x_0)\psi_x(P^*, x, x_0) = H_\infty(z, x)/F_\infty(z, x_0), \tag{3.27}$$

$$\psi(P, x, x_0)\psi_x(P^*, x, x_0) + \psi(P^*, x, x_0)\psi_x(P, x, x_0) = F_{\infty,x}(z, x)/F_\infty(z, x_0), \tag{3.28}$$

$$W(\psi(P, \cdot\,, x_0), \psi(P^*, \cdot\,, x_0)) = 2y(P)/F_\infty(z, x_0). \tag{3.29}$$

Moreover, one obtains

$$g(P, x) = \frac{F_\infty(z, x)}{2y(P)} = \frac{\psi(P, x, x_0)\psi(P^*, x, x_0)}{W(\psi(P, \cdot\,, x_0), \psi(P^*, \cdot\,, x_0))}. \tag{3.30}$$

Proof. (3.14) and (3.17) immediately yield (3.21) and (3.22). (3.23) and (3.24) are clear from (3.17). (3.20) and (3.21) yield (3.25), and (3.20) and (3.23) prove (3.26). $\phi = \psi_x/\psi$ and (3.22), (3.26) then yield (3.27)–(3.29). (3.30) follows from (3.26) and (3.29). Finally, by (3.20), $\psi(\cdot, x, x_0)$ is clearly meromorphic on $\mathcal{K}_\infty$ away from the poles $\hat{\mu}_j(x')$ of $\phi(\cdot, x')$. By (3.17) and (3.19) one concludes

$$\phi(P, x') \underset{P\to\hat{\mu}_j(x')}{=} \frac{d}{dx'}\ln(F_\infty(z, x')) + O(1) \text{ as } z \to \mu_j(x'), \quad j \in \mathbb{N}, \tag{3.31}$$

and hence $\psi(\cdot, x, x_0)$ is meromorphic on $\mathcal{K}_\infty$. $\square$

Next, we derive a trace formula representation of $V(x)$ in terms of $\{\mu_j(x)\}_{j\in\mathbb{N}}$.

Lemma 3.4. *Assume Hypotheses* 3.1 *and* 3.2 *(i)–(iii). Then*

$$V(x) = E_0 + \sum_{j\in\mathbb{N}}(E_{2j-1} + E_{2j} - 2\mu_j(x)), \quad x \in \mathcal{I}. \tag{3.32}$$

Proof. By Hypothesis 3.1 (i)–(iii) one infers for $x \in \mathcal{I}$,

$$\begin{aligned}
&\ln\Big(\Big(\prod_{j\in\mathbb{N}}(\mu_j(x)-z)^2(E_{2j-1}E_{2j})^{-1}\Big/\prod_{m\in\mathbb{N}}(1-(z/E_m))\Big)^{1/2}\Big)\\
&=(1/2)\sum_{j\in\mathbb{N}}\ln\bigg(1+\frac{E_{2j-1}+E_{2j}-2\mu_j(x)}{z(1-(E_{2j-1}/z))(1-(E_{2j}/z))}\\
&\qquad\qquad+\frac{\mu_j(x)^2-E_{2j-1}E_{2j}}{z^2(1-(E_{2j-1}/z))(1-(E_{2j}/z))}\bigg)\\
&=(1/2)z^{-1}\sum_{j\in\mathbb{N}}(E_{2j-1}+E_{2j}-2\mu_j(x))+o(z^{-1}) \text{ as } z\to\infty,\ z\in\mathbb{C}\backslash C_\varepsilon. \qquad (3.33)
\end{aligned}$$

In connection with the $o(z^{-1})$ term in (3.33) we used $[\mu_j(x)^2 - E_{2j-1}E_{2j}]/[z^2(1-(E_{2j-1}/z))(1-(E_{2j}/z))] = z^{-1}[\mu_j(x)^2 - E_{2j-1}E_{2j}]/[(1-(E_{2j-1}/z))(z-E_{2j})]$, Hypothesis 3.2 (iii), and the dominated convergence theorem (for discrete measures). Together with

$$g(P,x)=\Big(\frac{1}{4(E_0-z)}\prod_{j\in\mathbb{N}}(\mu_j(x)-z)^2(E_{2j-1}E_{2j})^{-1}\Big/\prod_{m\in\mathbb{N}}(1-(z/E_m))\Big)^{1/2},\quad P=(z,y), \qquad (3.34)$$

(3.33) yields

$$\begin{aligned}
g(P,x)&=(i/2)z(P)^{-1/2}+(i/4)z(P)^{-3/2}\Big(E_0+\sum_{j\in\mathbb{N}}(E_{2j-1}+E_{2j}-2\mu_j(x)\Big)\\
&\quad+o(z(P)^{-3/2}) \text{ as } z\to\infty,\ z\in\mathbb{C}\backslash C_\varepsilon \qquad (3.35)
\end{aligned}$$

and hence (3.32) comparing (3.35) and (3.9). ☐

The trace formula (3.32) for $V(x)$, although well-known in the real-valued (infinite genus) periodic case and in the algebro-geometric complex-valued case, appears to be new in the present context of general complex-valued potentials associated with infinite genus curves.

Lemma 3.5. *Assume Hypotheses* 3.1 *and* 3.2*. In addition, let* $x, x_0 \in \mathcal{I}$, $P = (z,y) \in \mathcal{K}_\infty$, $z \in \mathbb{C}\backslash C_\varepsilon$*. Then, as* $z \to \infty$, $z \in \mathbb{C}\backslash C_\varepsilon$,

$$\phi(P,x) = iz(P)^{1/2} + o(z(P)^{1/2}), \qquad (3.36)$$

uniformly with respect to x *as long as* x *varies in compact subintervals of* $\mathcal{I}$.

Proof. It suffices to combine (3.6), (3.10), (3.11), and (3.17). ☐

Next we state the following result.

Lemma 3.6. *Assume Hypothesis* 3.2 *(i), (ii) and let* $P = (z,y) \in \mathcal{K}_\infty$*. Then*

$$\big[g(P,x)\frac{d}{dx}-(1/2)g_x(P,x),L\big]\big|_{\ker(L-z)}=0, \qquad (3.37)$$

$$\big((g(P,x)\frac{d}{dx}-(1/2)g_x(P,x))\big|_{\ker(L-z)}\big)^2=(1/4)\big|_{\ker(L-z)}. \qquad (3.38)$$

Proof. (3.37) is a direct consequence of (3.12), and (3.8) immediately implies (3.38). ☐

Equations (3.37) and (3.38) are of course the analogs of

$$\big[\breve F_n(z,x)\frac{d}{dx}-(1/2)\breve F_{n,x}(z,x),L\big]\big|_{\ker(L-z)}=0, \tag{3.39}$$

$$\big((\breve F_n(z,x)\frac{d}{dx}-(1/2)\breve F_{n,x}(z,x))\big|_{\ker(L-z)}\big)^2=\breve R_{2n+1}(z)\big|_{\ker(L-z)} \tag{3.40}$$

(cf. (2.6), (2.13), (2.14)), which result from restriction of

$$[\breve P_{2n+1},L]=0, \tag{3.41}$$

$$\breve P_{2n+1}^2=-\breve R_{2n+1}(L) \tag{3.42}$$

to $\ker(L-z)$, with

$$\breve P_{2n+1}=\breve F_n(L,x)\frac{d}{dx}-(1/2)\breve F_{n,x}(L,x), \tag{3.43}$$

$$\breve P_{2n+1}\big|_{\ker(L-z)}=\big(\breve F_n(z,x)\frac{d}{dx}-(1/2)\breve F_{n,x}(z,x)\big)\big|_{\ker(L-z)} \tag{3.44}$$

(cf. (2.6), (2.15)). Thus, denoting $\mathcal{J}=\{1,\dots,n\}$, in the case of finite genus n and $\mathcal{J}=\mathbb{N}$ in the infinite genus case, and by $\#(J)$ the cardinality of $\mathcal{J}$, the universal differential equation (3.8) for $g(P,x)$ together with the (reflectionless) ansatz

$$g(P,x)=\frac{1}{2}\frac{\prod_{j=1}^{\#(\mathcal{J})}((\mu_j(x)-z)/(E_{2j-1}E_{2j})^{1/2})}{\big((E_0-z)\prod_{m=1}^{2\#(\mathcal{J})}(1-(z/E_m))\big)^{1/2}},\quad P=(z,y)\in\mathcal{K}_{\#(\mathcal{J})}, \tag{3.45}$$

then leads to algebraically integrable systems associated with hyperelliptic curves $\mathcal{K}_{\#(J)}: y^2=(E_0-z)\prod_{m=1}^{2\#(\mathcal{J})}(1-(z/E_m))$ of (arithmetic) genus $\#(J)$ and thus unifies the finite and infinite genus cases at hand. In particular, introducing the infinite-order differential expression

$$P_\infty=F_\infty(L,x)(d/dx)-(1/2)F_{\infty,x}(L,x), \tag{3.46}$$

$$P_\infty\big|_{\ker(L-z)}=\big(F_\infty(z,x)(d/dx)-(1/2)F_{\infty,x}(z,x)\big)\big|_{\ker(L-z)}, \tag{3.47}$$

Lemma 3.6 implies

$$\begin{aligned}&\big[P_\infty,L\big]\big|_{\ker(L-z)} &(3.48)\\&=\big(-(1/2)F_{\infty,xxx}(z,x)+2(V(x)-z)F_{\infty,x}(z,x)+V'(x)F_\infty(z,x)\big)\big|_{\ker(L-z)}=0,\end{aligned}$$

$$\big(P_\infty\big|_{\ker(L-z)}\big)^2=R_\infty(L)\big|_{\ker(L-z)}, \tag{3.49}$$

with $F_\infty(z,x)$ and $R_\infty(z)$ defined in (3.7) and (3.4). In analogy to (2.6), (2.8), and (2.9) we thus define the stationary s-KdV$_\infty$ equation by

$$\text{s-KdV}_\infty(V)=(1/2)F_{\infty,xxx}(0,x)-2V(x)F_{\infty,x}(0,x)-V'(x)F_\infty(0,x)=0. \tag{3.50}$$

Remark 3.7. We emphasize that the formalism in this section applies to general periodic (complex-valued) potentials $V(x)$ satisfying $V\in C^1(\mathbb{R})$. Let $\omega>0$ denote the period of V. By standard Floquet theory one introduces a fundamental system of solutions $s(z,x,x_0)$, $c(z,x,x_0)$ of $-\psi''(z,x)+(V(x)-z)\psi(z,x)=0$ defined by

$$s(z,x_0,x_0)=0,\quad s_x(z,x_0,x_0)=1, \tag{3.51}$$

$$c(z,x_0,x_0)=1,\quad c_x(z,x_0,x_0)=0,\quad z\in\mathbb{C},\ x,x_0\in\mathbb{R}. \tag{3.52}$$

The Floquet discriminant $\Delta(z)$ is then given by

$$\Delta(z)=(c(z,x_0+\omega,x_0)+s_x(z,x_0+\omega,x_0))/2,\quad z\in\mathbb{C} \tag{3.53}$$

and one verifies (cf. [55, Sect. 3.4], [70])

$$\Delta(z)^2 - 1 = \omega^2(E_0 - z)\prod_{m\in\mathbb{N}}\big((E_{2m-1}-z)(E_{2m}-z)(\omega/\pi)^4 m^{-4}\big), \tag{3.54}$$

$$E_{\substack{2m-1\\2m}} = (m\pi\omega^{-1} + c_1 m^{-1} + c_2 m^{-2} \mp \delta_m m^{-2} + \varepsilon_m^{\mp} m^{-3})^2, \tag{3.55}$$

$$s(z, x+\omega, x) = \omega\prod_{j\in\mathbb{N}}\big((\mu_j(x) - z)(\omega/\pi)^2 j^{-2}\big), \tag{3.56}$$

$$\mu_j(x) = (m\pi\omega^{-1} + c_1 j^{-1} + c_2 j^{-2} + s_j(x) j^{-2})^2, \tag{3.57}$$

where $\{\delta_m\}_{m\in\mathbb{N}}$, $\{\varepsilon_m^{\mp}\}_{m\in\mathbb{N}}$, $\{s_m(x)\}_{m\in\mathbb{N}} \in \ell^2(\mathbb{N})$. This then shows that Hypothesis 3.1 is satisfied. Moreover, one obtains

$$g(z,x) = -\frac{s(z,x+\omega,x)}{2(\Delta(z)^2-1)^{1/2}}, \tag{3.58}$$

$$F_\infty(z,x) = -Cs(z,x+\omega,x), \quad R_\infty(z) = C^2(\Delta(z)^2 - 1) \tag{3.59}$$

for some constant $C \in \mathbb{R}$. Hypothesis 3.2 (i), (ii) then follows from standard Floquet theory and Hypothesis 3.2 (iii), (iv) from iterating the Volterra integral equations for $s(z,x,x_0)$ and $c(z,x,x_0)$, integrating by parts, and applying the Riemann-Lebesgue lemma. Hence the Burchnall-Chaundy theory for the algebro-geometric case naturally extends to the class of infinite genus curves $\mathcal{K}_\infty$ treated in this section. Moreover, every periodic $V \in C^1(\mathbb{R})$ satisfies a stationary equation of the type (3.50).

For the time-dependent case one can closely follow the last part of Section 2 and hence we merely sketch the corresponding steps. First we note that (2.33)–(2.35) of course still apply and hence it remains to consider the initial value problem with stationary s-KdV$_\infty$ initial data. In complete analogy to (2.44)–(2.47) one then introduces ($(x,t_r)\in\Omega$)

$$(1/2)F_{\infty,xxx}(0,x,t_r) - 2V(x,t_r)F_{\infty,x}(0,x,t_r) - V_x(x,t_r)F_\infty(0,x,t_r) = 0, \tag{3.60}$$

$$V_{t_r}(x,t_r) = -(1/2)\widetilde{F}_{r,xxx}(0,x,t_r) + 2V(x,t_r)\widetilde{F}_{r,x}(0,x,t_r) + V_x(x,t_r)\widetilde{F}_r(0,x,t_r). \tag{3.61}$$

In analogy to Section 2, (3.60) and (3.61) are equivalent to

$$\text{s-KdV}_\infty(V) = 0, \quad (x,t_r)\in\Omega, \tag{3.62}$$

$$\widetilde{\text{KdV}}_r(V) = V_{t_r} - 2\tilde{f}_{r+1,x} = 0, \quad (x,t_r)\in\Omega. \tag{3.63}$$

Hence $V(x,t_r)$, for each fixed t_r (with $(x,t_r)\in\Omega$) satisfies the stationary s-KdV$_\infty$ equation, while $V(x,t_r)$, $(x,t_r)\in\Omega$, satisfies the time-dependent KdV$_r$ equation.

The analog of the time-dependent Baker-Akhiezer function (2.36) then is still given by

$$\psi(P,x,x_0,t_r,t_{0,r}) = \exp\bigg(\int_{x_0}^{x} dx'\,\phi(P,x',t_r) + \int_{t_{0,r}}^{t_r} ds\,(\widetilde{F}_r(z,x_0,s)\phi(P,x_0,s) \\ - (1/2)\widetilde{F}_{r,x}(z,x_0,s))\bigg), \quad P=(z,y)\in\mathcal{K}_\infty,\ (x,t_r),(x_0,t_{0,r})\in\Omega, \tag{3.64}$$

with $\phi(P,x,t_r)$ given as in (3.17), (3.18) (replacing $F_\infty(z,x)$ by $F_\infty(z,x,t_r)$ and $H_\infty(z,x)$ by $H_\infty(z,x,t_r)$, respectively). As in Section 2, $\psi(\cdot,x,x_0,t_r,t_{0,r})$ is meromorphic on $\mathcal{K}_\infty$ and $(L(t_r)-z(P))\psi(P,\cdot,x_0,t_r,t_{0,r})=0$. Next we formulate the analogs of (2.37)–(2.39) in the following Lemma.

Lemma 3.8. *Assume Hypothesis* 3.1 *and* 3.2 *(replacing $\mu_j(x)$, $g(P,x)$, and the open interval $\mathcal{I}\subseteq\mathbb{R}$ by $\mu_j(x,t_r)$, $g(P,x,t_r)$, and the open subset $\Omega\subseteq\mathbb{R}^2$, respectively) supposing $\partial_x^k V\in C(\Omega)$, $k=0,\dots,2r+1$, $V_{t_r}\in C(\Omega)$. In addition, let $P=(z,y)\in\mathcal{K}_\infty$, $(x,t_r),(x_0,t_{0,r})\in\Omega$. Then Lemmas* 3.3 *and* 3.4 *apply to $\phi(P,x,t_r)$, $\psi(P,x,x_0,t_r,t_{0,r})$, $g(P,x,t_r)$, and $V(x,t_r)$. Moreover, one has*

$$\phi_{t_r}(P,x,t_r)=\partial_x(\widetilde F_r(z,x,t_r)\phi(P,x,t_r)-(1/2)\widetilde F_{r,x}(z,x,t_r)), \tag{3.65}$$

$$\begin{aligned}\psi_{t_r}(P,x,x_0,t_r,t_{0,r})&=(\widetilde P_{2r+1}(t_r)\psi)(P,x,x_0,t_r,t_{0,r}) \\ &=\underline{\widetilde F}_r(z,x,t_r)\psi_x(P,x,x_0,t_r,t_{0,r})-(1/2)\widetilde F_{r,x}(z,x,t_r)\psi(P,x,x_0,t_r,t_{0,r}),\end{aligned} \tag{3.66}$$

$$F_{\infty,t_r}(z,x,t_r)=\widetilde F_r(z,x,t_r)F_{\infty,x}(z,x,t_r)-\widetilde F_{r,x}(z,x,t_r)F_\infty(z,x,t_r). \tag{3.67}$$

Proof. We only need to prove (3.65)–(3.67). Starting from

$$\begin{aligned}V_{t_r}(x,t_r)=&-(1/2)\widetilde F_{r,xxx}(z,x,t_r)+2(V(x,t_r)-z)\widetilde F_{r,x}(z,x,t_r)\\ &+V_x(x,t_r)\widetilde F_r(z,x,t_r),\end{aligned} \tag{3.68}$$

which is equivalent to (3.60) due to the recursion (2.2), one computes from (3.21),

$$\begin{aligned}\partial_{t_r}(\phi_x+\phi^2)&=\phi_{t_r,x}+2\phi\phi_{t_r}=V_{t_r}\\ &=(\widetilde F_r\phi-(1/2)\widetilde F_r)_{xx}+2\phi(\widetilde F_r\phi-(1/2)\widetilde F_r)_x.\end{aligned} \tag{3.69}$$

Thus,

$$(\partial_x+2\phi)(\phi_{t_r}-(\widetilde F_r\phi-(1/2)\widetilde F_r)_x)=0 \tag{3.70}$$

and hence

$$\phi_{t_r}=(\widetilde F_r\phi-(1/2)\widetilde F_r)_x+C\exp\Big(-2\int^x dx'\,\phi\Big), \tag{3.71}$$

where C is independent of x (but may depend on P and t_r). Next, combining (3.36) and the Riccati equation (3.21), one observes that ϕ_x has an asymptotic expansion of the type

$$\phi_x(P,x,t_r)=o(z(P)) \tag{3.72}$$

uniformly with respect to (x,t_r) as long as (x,t_r) varies in compact subsets of Ω. Inserting (3.72) into (3.71) and integrating with respect to t_r then contradicts the asymptotic expansion (3.36) for ϕ unless $C=0$. This proves (3.65). (3.66) is then obvious from (3.64) and (3.65). Using (3.24) and (3.65), one obtains

$$\begin{aligned}\phi_{t_r}(P)-\phi_{t_r}(P^*)&=2y(P)F_\infty^{-2}F_{\infty,t_r}\\ &=\partial_x(\widetilde F_r(\phi(P)-\phi(P^*)))=2y(P)F_\infty^{-2}(\widetilde F_rF_{\infty,x}-\widetilde F_{r,x}F_\infty)\end{aligned} \tag{3.73}$$

and hence (3.67). ∎

Remark 3.9. As in Remark 2.1 we emphasize that the isospectral deformation problem (3.62), (3.63) (or equivalently, (3.60), (3.61)) is not necessarily well-posed. Unique solvability of (3.62), (3.63) can again be reduced to unique solvability of the corresponding first-order system of Dubrovin equations. In the present infinite genus context this system is of the form

$$\mu_{j,x}(x,t_r) = \frac{-2y(\hat{\mu}_j(x,t_r))}{\prod_{\ell\in\mathbb{N}\setminus\{j\}}(\mu_\ell(x,t_r)-\mu_j(x,t_r))/(E_{2\ell-1}E_{2\ell})^{1/2}}, \tag{3.74}$$

$$\begin{aligned}\mu_{j,t_r}(x,t_r) &= \widetilde{F}_r(\mu_j(x,t_r),x,t_r)\mu_{j,x}(x,t_r) \\ &= \frac{-2y(\hat{\mu}_j(x,t_r))}{\prod_{\ell\in\mathbb{N}\setminus\{j\}}(\mu_\ell(x,t_r)-\mu_j(x,t_r))/(E_{2\ell-1}E_{2\ell})^{1/2}}\widetilde{F}_r(\mu_j(x,t_r),x,t_r)\end{aligned} \tag{3.75}$$

for $j\in\mathbb{N}$, with initial data for (3.74), (3.75) on $\mathcal{K}_\infty$ given by

$$\{\hat{\mu}_1(x_0,t_{0,r})\}_{j\in\mathbb{N}}\subset\mathcal{K}_\infty. \tag{3.76}$$

Here $\hat{\mu}_j(x,t_r))\in\mathcal{K}_\infty$ is defined as in (3.19) (replacing x by x,t_r)

$$\hat{\mu}_j(x,t_r) = (\mu_j(x,t_r), -(1/2)F_{\infty,x}(\mu_j(x,t_r),x,t_r)), \quad j\in\mathbb{N}, \tag{3.77}$$

and (3.74) is then seen to be an immediate consequence of the analog of (3.7), that is,

$$F_\infty(z,x,t_r) = \prod_{j\in\mathbb{N}}((\mu_j(x,t_r)-z)/(E_{2j-1}E_{2j})^{1/2}), \tag{3.78}$$

and of (3.77), while (3.75) follows from (3.67), taking $z=\mu_j(x,t_r)$ in either case. As long as the curve $\mathcal{K}_\infty$ is nonsingular (i.e., $E_m\neq E_{m'}$ for $m\neq m'$) and none of the $\mu_j(x,t_r)$ collide (i.e., for $j\neq j'$, $\mu_j(x,t_r)\neq\mu_{j'}(x,t_r)$ for $(x,t_r)\in\Omega$, $\Omega\in\mathbb{C}^2$ open), the Dubrovin system is uniquely solvable on $\mathcal{K}_\infty$ and $V(x,t_r)$, $(x,t_r)\in\Omega$, is then recovered from the time-dependent trace formula

$$V(x,t_r) = E_0 + \sum_{j\in\mathbb{N}}(E_{2j-1}+E_{2j}-2\mu_j(x,t_r)). \tag{3.79}$$

The corresponding $F_\infty(z,x,t_r)$, $\widetilde{F}_r(z,x,t_r)$, $(x,t_r)\in\Omega$, then satisfy the isospectral deformation equations (3.62), (3.63). Such ideas, in the case of nonsingular curves $\mathcal{K}_\infty$ with real branch points $\{E_m\}_{m\in\mathbb{N}}\subset\mathbb{R}$, $E_m\neq E_{m'}$ for $m\neq m'$ have been applied by Trubowitz [75] in the stationary periodic case, by Kotani and Krishna [49] in some almost periodic stationary cases, and in great generality by Sodin and Yuditskii [74] (avoiding Dubrovin equations and directly working with the corresponding infinite dimensional Jacobi inversion problem) in the case $r=0$. The general case of the KdV hierarchy is discussed by Egorova [24], Levitan [50], [51], [52], [53], [54, Ch. 11, 12], and in the periodic case by Marchenko [55, Sect. 4.3].

The isospectral deformation problem (3.62), (3.63) clearly deserves further study. For instance, suppose $V(x,0)$ is real-analytic (or elliptic) and of infinite genus type satisfying (3.62) (resp. (3.60)). A natural question to ask is whether the solution $V(x,t_r)$ of (3.62), (3.63) is real-analytic (or elliptic) with respect to x for each $t_r\in\mathbb{R}$. Similarly, on the stationary level, one might ask whether the isospectral torus of a real-analytic (or elliptic) and infinite genus type potential $V(x)$ satisfying (3.50) consists of only real-analytic (or elliptic) potentials. (In the real-valued finite genus case and in the case of real-valued (infinite genus) periodic potentials the answer is well-known to be affirmative, cf. [11], [75].)

We did not consider the infinite genus limit $n\to\infty$ of the theta function representation (i.e., the Its-Matveev formula) (2.49) of $V(x,t_r)$ in this paper but plan to return to this topic elsewhere.

4. Darboux-type transformations

In this section we study the effects of Darboux transformations of the KdV hierarchy on the transcendental hyperelliptic curves introduced in Section 3.

Since the the algebro-geometric (finite genus) case has recently been treated in detail in [37], we exclusively focus on the infinite genus case in this section. Moreover, since time plays no active role in the following arguments, we shall suppress t_r in what follows and without loss of generality work within the stationary formalism.

Assuming Hypothesis 3.2, we start by introducing Darboux transformations in connection with the differential expression $L = -d^2/dx^2 + V$ and the curve $\mathcal{K}_\infty$ defined in (3.4). Define (for $P \in \mathcal{K}_\infty$)

$$\psi(P,x,x_0,\sigma) = \begin{cases} \frac{1}{2}(1+\sigma)\psi(P,x,x_0) + \frac{1}{2}(1-\sigma)\psi(P^*,x,x_0), & \sigma \in \mathbb{C}, \\ \psi(P,x,x_0) - \psi(P^*,x,x_0), & \sigma = \infty, \end{cases} \tag{4.1}$$

pick $Q_0 = (z_0, y_0) \in \mathcal{K}_\infty$, and introduce the differential expressions

$$A_\sigma(Q_0) = \frac{d}{dx} + \phi(Q_0,x,\sigma), \quad A_\sigma^+(Q_0) = -\frac{d}{dx} + \phi(Q_0,x,\sigma), \quad \sigma \in \mathbb{C}_\infty, \tag{4.2}$$

where

$$\phi(P,x,\sigma) = \psi_x(P,x,x_0,\sigma)/\psi(P,x,x_0,\sigma), \quad P \in \mathcal{K}_\infty, \, \sigma \in \mathbb{C}_\infty. \tag{4.3}$$

One verifies (cf. (3.21))

$$L = A_\sigma(Q_0)A_\sigma^+(Q_0) + z_0 = -\frac{d^2}{dx^2} + V, \tag{4.4}$$

with

$$V(x) = \phi(Q_0,x,\sigma)^2 + \phi_x(Q_0,x,\sigma) + z_0, \tag{4.5}$$

independent of the choice of $\sigma \in \mathbb{C}_\infty$. Interchanging the order of the differential expressions $A_\sigma(Q_0)$ and $A_\sigma^+(Q_0)$ in (4.4) then yields

$$\widehat{L}_\sigma(Q_0) = A_\sigma^+(Q_0)A_\sigma(Q_0) + z_0 = -\frac{d^2}{dx^2} + \widehat{V}_\sigma(x,Q_0), \tag{4.6}$$

with

$$\begin{aligned} \widehat{V}_\sigma(x,Q_0) &= \phi(Q_0,x,\sigma)^2 - \phi_x(Q_0,x,\sigma) + z_0 \\ &= V(x) - 2(\ln(\psi(Q_0,x,x_0,\sigma)))_{xx}, \quad \sigma \in \mathbb{C}_\infty. \end{aligned} \tag{4.7}$$

The transformation

$$V(x) \mapsto \widehat{V}_\sigma(x,Q_0), \quad Q_0 \in \mathcal{K}_\infty, \, \sigma \in \mathbb{C}_\infty \tag{4.8}$$

is usually called the Darboux transformation (also Crum-Darboux transformation or single commutation method) and goes back to at least Jacobi [48] and Darboux [14]. While we only aim at its properties from an algebraic point of view, its analytic properties in connection with spectral deformations (isospectral and non-isospectral ones) have received enormous attention in the context of spectral theory (especially, regarding the insertion of eigenvalues into spectral gaps), inverse spectral theory, and Bäcklund tranformations for the (time-dependent) KdV hierarchy. A complete bibliography in this context being impossible, we just refer to [13], [16], [17], [23, Ch. 4], [25], [27], [30], [34], [42], [43], [44], [45], [56], [57], [58], [72], and the extensive literature therein. From a historical point of view it is very interesting to note that Drach [19], [20], [21] in his 1919 studies of Darboux transformations (being a student of Darboux) not only introduced a set of nonlinear differential equations for V, which today can be identified with the stationary KdV hierarchy, but also studied the effect of Darboux transformations on the underlying hyperelliptic curve. As a consequence, he seems to have been the first to explicitly establish the connection between integrable systems and spectral theory. For recent treatments of this connection see, for instance, [4, Ch. 3] and [39].

Assuming that $\psi \in \ker(L-z)$,

$$L\psi(z) = z\psi(z), \tag{4.9}$$

one infers $A_\sigma^+(Q_0)\psi(z) \in \ker(\widehat{L}_\sigma(Q_0)-z)$,

$$\widehat{L}_\sigma(Q_0)(A_\sigma^+(Q_0)\psi(z)) = zA_\sigma^+(Q_0)\psi(z), \tag{4.10}$$

and

$$\begin{aligned} W(A_\sigma^+(Q_0)\psi_1(z), A_\sigma^+(Q_0)\psi_2(z)) = (z-z_0)W(\psi_1(z),\psi_2(z)),& \\ \psi_1(z),\psi_2(z) \in \ker(L-z).& \end{aligned} \tag{4.11}$$

Since

$$(A_\sigma^+(Q_0)\psi(P,\,\cdot\,,x_0))(x) = \big(\phi(Q_0,x,\sigma)-\phi(P,x)\big)\psi(P,x,x_0) \tag{4.12}$$

for $P \in \mathcal{K}_\infty\backslash\{Q_0\}$, we define

$$\begin{aligned} \hat\psi_\sigma(P,x,x_0,Q_0) &= (A_\sigma^+(Q_0)\psi(P,\,\cdot\,,x_0))(x) \\ &= \big(\phi(Q_0,x,\sigma)-\phi(P,x)\big)\psi(P,x,x_0), \quad P \in \mathcal{K}_\infty\backslash\{Q_0\},\, \sigma \in \mathbb{C}_\infty. \end{aligned} \tag{4.13}$$

Then

$$(\widehat{L}_\sigma(Q_0)\hat\psi_\sigma(P,\,\cdot\,,x_0,Q_0))(x) = z\hat\psi_\sigma(P,x,x_0,Q_0) \tag{4.14}$$

for $P=(z,y) \in \mathcal{K}_\infty\backslash\{Q_0\}$, and we define in analogy to (3.6) the diagonal Green's function $\hat g_\sigma(P,x,Q_0)$ of $\widehat{L}_\sigma(Q_0)$ by

$$\hat g_\sigma(P,x,Q_0) = \frac{\hat\psi_\sigma(P,x,x_0,Q_0)\hat\psi_\sigma(P^*,x,x_0,Q_0)}{W(\hat\psi_\sigma(P,\,\cdot\,,x_0,Q_0),\hat\psi_\sigma(P^*,\,\cdot\,,x_0,Q_0))}, \quad P=(z,y) \in \mathcal{K}_\infty\backslash\{Q_0\}. \tag{4.15}$$

Lemma 4.1. *Assume* s-KdV$_\infty(V)=0$ *and let* $Q_0=(z_0,y_0) \in \mathcal{K}_\infty$, $P=(z,y) \in \mathcal{K}_\infty\backslash\{Q_0\}$, $\sigma \in \mathbb{C}_\infty$. *Then the diagonal Green's function* $\hat g_\sigma(P,x,Q_0)$ *in* (4.15) *explicitly reads*

$$\begin{aligned} \hat g_\sigma(P,x,Q_0) &= \frac{(\phi(P,x)-\phi(Q_0,x,\sigma))(\phi(P^*,x)-\phi(Q_0,x,\sigma))F_\infty(z,x)}{2(z-z_0)y(P)} && (4.16)\\ &= \frac{H_\infty(z,x)+\phi(Q_0,x,\sigma)^2F_\infty(z,x)-\phi(Q_0,x,\sigma)F_{\infty,x}(z,x)}{2(z-z_0)y(P)} && (4.17)\\ &= \frac{\widehat{F}_{\sigma,\infty}(z,x,Q_0)}{2\hat y(P)}, && (4.18) \end{aligned}$$

where $\hat y(P)$ *denotes the meromorphic solution on* $\widehat{\mathcal{K}}_{\sigma,\infty}(Q_0)$ *obtained upon solving* $y^2=\widehat{R}_{\sigma,\infty}(z,Q_0)$, $P=(z,y)$ *for some entire function* $\widehat{R}_{\sigma,\infty}(z,Q_0)$ *of the type* (3.4) *and* $\widehat{F}_{\sigma,\infty}(z,x,Q_0)$ *denotes an entire function with respect to* z *of the type* (3.7). *In particular, the Darboux transformation* (4.8), $V(x)\mapsto \widehat{V}_\sigma(x,Q_0)$ *maps the class of* KdV$_\infty$ *potentials into itself.*

Proof. (4.16) and (4.17) follow upon use of $\phi(P,x) = \psi_x(P,x,x_0)/\psi(P,x,x_0)$, (3.22), (3.23), (3.26)–(3.29), and (4.11). Since the numerator in (4.17) is entire in z, and

$$\hat{g}_\sigma(P,x,Q_0) = (i/2)\zeta(P) + O(\zeta(P)^2) \text{ as } P \to P_\infty \tag{4.19}$$

again by (4.17), one concludes (4.18). By inspection, $\widehat{F}_{\sigma,\infty}(z,x,Q_0)$ satisfies equation (3.14) (and hence (3.50)) with $V(x)$ replaced by $\widehat{V}_\sigma(x,Q_0)$ and $R_\infty(z)$ by $\widehat{R}_{\sigma,\infty}(z,Q_0)$. Consequently, $V(x)$ being a KdV_∞ potential implies that $\widehat{V}_\sigma(x,Q_0)$ is one as well. $\square$

The following theorem, the principal result of this section, will clarify the relation between $\mathcal{K}_\infty$ and $\widehat{\mathcal{K}}_{\sigma,\infty}(Q_0)$ depending on (Q_0,σ).

Theorem 4.2. *Suppose* s-$\mathrm{KdV}_\infty(V) = 0$ *and let* $Q_0 = (z_0,y_0) \in \mathcal{K}_\infty$ *and* $\sigma \in \mathbb{C}_\infty$. *Then the transcendental hyperelliptic curve* $\widehat{\mathcal{K}}_{\sigma,\infty}(Q_0)$ *associated with* $\widehat{V}_\sigma(x,Q_0)$ *is of the type*

$$\widehat{\mathcal{K}}_{\sigma,\infty}(Q_0)\colon \widehat{\mathcal{F}}_{\sigma,\infty}(z,y,Q_0) = y^2 - \widehat{R}_{\sigma,\infty}(z,Q_0) = 0, \tag{4.20}$$

with

$$\widehat{R}_{\sigma,\infty}(z,Q_0) = \begin{cases} (z-z_0)^2 R_\infty(z), & \sigma \in \mathbb{C}_\infty \backslash \{-1,1\} \text{ and } y_0 \neq 0, \\ (z-z_0)^2 R_\infty(z), & \sigma = \infty \text{ and } y_0 = 0, \\ R_\infty(z), & \sigma \in \{-1,1\} \text{ and } y_0 \neq 0, \\ R_\infty(z), & \sigma \in \mathbb{C},\ y_0 = 0, \text{ and } R_{\infty,z}(z_0) \neq 0, \\ (z-z_0)^{-2} R_\infty(z), & \sigma \in \mathbb{C},\ y_0 = 0, \text{ and } R_{\infty,z}(z_0) = 0, \end{cases} \tag{4.21}$$

and $R_\infty(z) = \prod_{m=0}^{2n}(1-(z/E_m))$.

Proof. Our starting point will be (4.16) and a careful case distinction taking into account whether or not Q_0 is a branch point, and distinguishing the cases $\sigma \in \mathbb{C}\backslash\{-1,1\}$, $\sigma \in \{-1,1\}$, and $\sigma = \infty$.

Case (i). $\sigma \in \mathbb{C}_\infty \backslash \{-1,1\}$ and $y_0 \neq 0$: One computes from (4.1) and (4.3),

$$\phi(Q_0,x,\sigma) = \begin{cases} \frac{(1+\sigma)\psi_x(Q_0,x,x_0)+(1-\sigma)\psi_x(Q_0^*,x,x_0)}{(1+\sigma)\psi(Q_0,x,x_0)+(1-\sigma)\psi(Q_0^*,x,x_0)}, & \sigma \in \mathbb{C}\backslash\{-1,1\}, \\ \frac{\psi_x(Q_0,x,x_0)-\psi_x(Q_0^*,x,x_0)}{\psi(Q_0,x,x_0)-\psi(Q_0^*,x,x_0)}, & \sigma = \infty, \end{cases} \tag{4.22}$$

and upon comparison with $\phi(Q_0,x) \neq \phi(Q_0^*,x)$,

$$\phi(Q_0,x) = \frac{\psi_x(Q_0,x,x_0)}{\psi(Q_0,x,x_0)}, \quad \phi(Q_0^*,x) = \frac{\psi_x(Q_0^*,x,x_0)}{\psi(Q_0^*,x,x_0)}, \tag{4.23}$$

one concludes that no cancellations can occur in (4.16), proving the first statement in (4.21).

Case (ii). $\sigma = \infty$ and $y_0 = 0$: Combining (3.17), (3.20), (3.24), and (4.1) one computes

$$\begin{aligned} &\phi(Q_0,x,\infty) = \lim_{P\to Q_0} \phi(P,x,\infty) \\ &= \lim_{P\to Q_0} \left(\frac{\phi(P,x)\exp\big(\int_{x_0}^x dx'\,\phi(P,x')\big) - \phi(P^*,x)\exp\big(\int_{x_0}^x dx'\,\phi(P^*,x')\big)}{\exp\big(\int_{x_0}^x dx'\,\phi(P,x')\big) - \exp\big(\int_{x_0}^x dx'\,\phi(P^*,x')\big)} \right) \\ &= \phi(Q_0,x) \end{aligned}$$

$$+\lim_{P\to Q_0}\left(\frac{\phi(P,x)-\phi(P^*,x)}{\exp\big(\int_{x_0}^x dx'\,\phi(P,x')\big)-\exp\big(\int_{x_0}^x dx'\,\phi(P^*,x')\big)}\times\right.$$
$$\left.\times\exp\bigg(\int_{x_0}^x dx'\,\phi(P^*,x')\bigg)\right)$$
$$=\phi(Q_0,x)+\exp\bigg(\int_{x_0}^x dx'\,\phi(Q_0,x')\bigg)\times$$
$$\times\lim_{P\to Q_0}\left(\frac{-2y(P)/F_\infty(z,x)}{\exp\big(-y(P)\int_{x_0}^x\frac{dx'}{F_\infty(z,x')}\big)-\exp\big(y(P)\int_{x_0}^x\frac{dx'}{F_\infty(z,x')}\big)}\times\right.$$
$$\left.\times\exp\bigg(-\frac{1}{2}\int_{x_0}^x dx'\,\frac{F_{\infty,x}(z,x')}{F_\infty(z,x')}\bigg)\right)$$
$$=\phi(Q_0,x)$$
$$+\psi(Q_0,x,x_0)\frac{1}{F_\infty(z_0,x)\psi(Q_0,x,x_0)}\lim_{P\to Q_0}\left(\frac{2y(P)}{2y(P)\int_{x_0}^x\frac{dx'}{F_\infty(z,x')}+O(y(P)^2)}\right)$$
$$=\phi(Q_0,x)+\bigg(F_\infty(z_0,x)\int_{x_0}^x\frac{dx'}{F_\infty(z,x')}\bigg)^{-1},\quad x\in\mathbb{C}\backslash\{x_0\},\tag{4.24}$$

using $\lim_{P\to Q_0}y(P)=y(Q_0)=y_0=0$. From

$$\phi(Q_0,x)=\frac{1}{2}\frac{F_{\infty,x}(z_0,x)}{F_\infty(z_0,x)}\tag{4.25}$$

one concludes again that no cancellations can occur in (4.16). Thus, the second statement in (4.21) holds.

The remainder of the proof requires a more refined argument, the basis of which will be derived next. Writing

$$y(P)^2=R_\infty(z)\underset{z\to z_0}{=}y_0^2+\hat{y}_1(z-z_0)+\hat{y}_2(z-z_0)^2+O((z-z_0)^3),\tag{4.26}$$

a comparison of the powers $(z-z_0)^0$ and $(z-z_0)^1$ in (3.7) yields

$$2F_{\infty,xx}(z_0,x)F_\infty(z_0,x)-F_{\infty,x}(z_0,x)^2-4VF_\infty(z_0,x)^2+4y_0^2=0\tag{4.27}$$

and

$$\begin{aligned}&\dot{F}_{\infty,xx}(z_0,x)F_\infty(z_0,x)+F_{\infty,xx}(z_0,x)\dot{F}_\infty(z_0,x)-F_{\infty,x}(z_0,x)\dot{F}_{\infty,x}(z_0,x)\\&-4VF_\infty(z_0,x)\dot{F}_\infty(z_0,x)+2F_\infty(z_0,x)^2+2\hat{y}_1=0.\end{aligned}\tag{4.28}$$

Inserting (4.27) into (4.28), a little algebra proves the basic identity

$$\begin{aligned}&F_\infty(z_0,x)^2(\dot{F}_\infty(z_0,x)/F_\infty(z_0,x))_{xx}+F_{\infty,x}(z_0,x)F_\infty(z_0,x)(\dot{F}_\infty(z_0,x)/F_\infty(z_0,x))_x\\&+2F_\infty(z_0,x)^2-4y_0^2(\dot{F}_\infty(z_0,x)/F_\infty(z_0,x))+2\hat{y}_1=0.\end{aligned}\tag{4.29}$$

Case (iii). $\sigma\in\{-1,1\}$ and $y_0\neq 0$: Then (4.1) yields

$$\phi(Q_0,x,1)=\phi(Q_0,x),\quad \phi(Q_0,x,-1)=\phi(Q_0^*,x),\tag{4.30}$$

with $\phi(Q_0,x)\neq\phi(Q_0^*,x)$ since $y_0\neq 0$. In this case there is a cancellation in (4.16). For instance, choosing $\sigma=1$ one computes from (3.7) and (3.17),

$$\phi(P,x)-\phi(Q_0,x,1)=\phi(P,x)-\phi(Q_0,x)$$

$$\underset{P\to Q_0}{=} \frac{-(y(P)-y_0)}{F_\infty(z_0,x)} + y_0 \frac{\dot F_\infty(z_0,x)}{F_\infty(z_0,x)^2}(z-z_0) + \frac{1}{2}\bigg(\frac{\dot F_\infty(z_0,x)}{F_\infty(z_0,x)}\bigg)_x (z-z_0) + O((z-z_0)^2)$$

$$\underset{P\to Q_0}{=} c_1(x)(z-z_0) + O((z-z_0)^2) \tag{4.31}$$

since

$$y(P)-y_0 \underset{P\to Q_0}{=} y_1(z-z_0) + O((z-z_0)^2), \quad \hat y_1 = 2y_0y_1. \tag{4.32}$$

It remains to show that $c_1(x)$ does not vanish identically in $x\in\mathbb{C}$. Arguing by contradiction we assume

$$0 = c_1(x) \tag{4.33}$$
$$= \frac{-y_1}{F_\infty(z_0,x)} + \frac{y_0}{F_\infty(z_0,x)}\frac{\dot F_\infty(z_0,x)}{F_\infty(z_0,x)} + \frac{1}{2}\bigg(\frac{\dot F_\infty(z_0,x)}{F_\infty(z_0,x)}\bigg)_x, \quad x\in\mathbb{C}.$$

Differentiating (4.33) with respect to x and inserting the ensuing expression for $(\dot F_\infty(z_0,x)/F_\infty(z_0,x))_{xx}$ and the one for $(\dot F_\infty(z_0,x)/F_\infty(z_0,x))_x$ from (4.33) into (4.29) then results in the contradiction

$$0 = 2F_\infty(z_0,x)^2, \quad x\in\mathbb{C}. \tag{4.34}$$

Moreover, since

$$\phi(P^*,x)-\phi(Q_0,x,x,1) = \phi(P^*,x)-\phi(Q_0,x) \underset{P\to Q_0}{=} \frac{2y_0}{F_\infty(z_0,x)} + O(z-z_0), \tag{4.35}$$

one concludes that precisely one factor of $z-z_0$ cancels in (4.16). Hence, the third relation in (4.21) holds. The case $\sigma=-1$ in treated analogously.

Case (iv). $\sigma\in\mathbb{C}$, $y_0=0$, and $R_{\infty,z}(z_0)\neq 0$: Taking into account that $\phi(Q_0,x,\sigma)=\phi(Q_0,x)$ (using (4.3) and $Q_0=Q_0^*$) is independent of $\sigma\in\mathbb{C}$, (3.7) and (3.17) yield

$$\big(\phi(P,x)-\phi(Q_0,x)\big)\big(\phi(P^*,x)-\phi(Q_0,x)\big) \underset{P\to Q_0}{=} \frac{-y_1^2(z-z_0)}{F_\infty(z_0,x)^2} + O((z-z_0)^2) \tag{4.36}$$

since

$$y(P) \underset{P\to Q_0}{=} y_1(z-z_0)^{1/2} + O((z-z_0)^{3/2}). \tag{4.37}$$

Thus we infer again that precisely one factor of $z-z_0$ cancels in (4.16). Hence, the fourth relation in (4.21) is proved.

Case (v). $\sigma\in\mathbb{C}$, $y_0=\hat y_1=0$, and $\hat y_2\neq 0$ (cf. (4.26)): One calculates as in (4.36),

$$\big(\phi(P,x)-\phi(Q_0,x)\big)\big(\phi(P^*,x)-\phi(Q_0,x)\big)$$
$$\underset{P\to Q_0}{=} \bigg(\frac{-y_1^2}{F_\infty(z_0,x)^2} + \frac{1}{4}\bigg(\bigg(\frac{\dot F_\infty(z_0,x)}{F_\infty(z_0,x)}\bigg)_x\bigg)^2\bigg)(z-z_0)^2 + O((z-z_0)^3)$$
$$\underset{P\to Q_0}{=} c_2(x)(z-z_0)^2 + O((z-z_0)^3) \tag{4.38}$$

since

$$y(P) \underset{P \to Q_0}{=} y_1(z - z_0) + O((z - z_0)^2). \tag{4.39}$$

Next we show that $c_2(x)$ does not vanish identically in $x \in \mathbb{C}$. Arguing again by contradiction we suppose that

$$0 = c_2(x) = \frac{-y_1^2}{F_\infty(z_0, x)^2} + \frac{1}{4}\bigg(\bigg(\frac{\dot{F}_\infty(z_0, x)}{F_\infty(z_0, x)}\bigg)_x\bigg)^2, \quad x \in \mathbb{C}. \tag{4.40}$$

Thus

$$\bigg(\frac{\dot{F}_\infty(z_0, x)}{F_\infty(z_0, x)}\bigg)_x = \frac{C}{F_\infty(z_0, x)} \tag{4.41}$$

for some constant $C \in \mathbb{C}$. Insertion of (4.41) and its x-derivative into (4.29) then again yields the contradiction

$$0 = 2F_\infty(z_0, x)^2, \quad x \in \mathbb{C}. \tag{4.42}$$

Hence the last relation in (4.21) holds in this case.

Case (vi). $\sigma \in \mathbb{C}$, $y_0 = \hat{y}_1 = \hat{y}_2 = 0$ (cf. (4.26)): As in (4.38) one obtains

$$\begin{aligned}\big(\phi(P, x) - \phi(Q_0, x)\big)\big(\phi(P^*, x) - \phi(Q_0, x)\big) \underset{P \to Q_0}{=} \frac{1}{4}\bigg(\bigg(\frac{\dot{F}_\infty(z_0, x)}{F_\infty(z_0, x)}\bigg)_x\bigg)^2 (z - z_0)^2 \\ + O((z - z_0)^3)\end{aligned} \tag{4.43}$$

since

$$y(P) \underset{P \to Q_0}{=} O((z - z_0)^{3/2}). \tag{4.44}$$

The remainder of the proof of case (vi) is now just a special case of case (v) (with $y_1 = C = 0$). ☐

These results show, in particular, that Darboux transformations do not change the local structure of the original curve $y^2 = R_\infty(z)$, except, possibly near the point Q_0.

Remark 4.3. In the finite genus case, the analog of Theorem 4.2 was first derived by purely algebro-geometric means by Ehlers and Knörrer [25] in 1982. An elementary but rather lengthy derivation of the finite genus case was provided by Ohmiya [66] in 1997 (based on two other papers [65], [67]). The present proof of Theorem 4.2 is patterned after the finite genus treatment in [37], which seems to be the only elementary and short one available at this point.

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