Twisted duality of the CAR–Algebra

Hellmut Baumgärtel, Matthias Jurke and Fernando Lledó

Mathematical Institute, University of Potsdam,
Am Neuen Palais 10, Postfach 601 553,
D–14415 Potsdam, Germany.

baumg@rz.uni-potsdam.de
lledo@rz.uni-potsdam.de

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Abstract

We give a complete proof of the twisted duality property $\mathcal{M}(\mathfrak{q})' = \widetilde{Z}\,\mathcal{M}(\mathfrak{q}^\perp)\,\widetilde{Z}^*$ of the (self–dual) CAR–Algebra in any Fock representation. The proof is based on the natural Halmos decomposition of the (reference) Hilbert space when two suitable closed subspaces have been distinguished. We use Modular Theory and techniques developed by Kato concerning pairs of projections in some essential steps of the proof.

1 Introduction

Twisted duality is a structural property of the von Neumann algebra obtained from the CAR–Algebra in any Fock representation. The (self–dual) CAR–Algebra is uniquely given once a separable Hilbert space $\mathfrak{h}$ and an anti–unitary involution Γ are specified [3, 4]. Now for any Γ–invariant subspace $\mathfrak{q}$ of $\mathfrak{h}$ and any Fock state characterized by a so–called basis projection P we can canonically construct a von Neumann algebra $\mathcal{M}(\mathfrak{q})$. Twisted duality means that the equation

$$\mathcal{M}(\mathfrak{q})' = \widetilde{Z}\,\mathcal{M}(\mathfrak{q}^\perp)\,\widetilde{Z}^* \tag{1}$$

holds, where $\widetilde{Z}$ is a certain unitary twist operator to be introduced in the following section and $\mathfrak{q} \oplus \mathfrak{q}^\perp = \mathfrak{h}$. Thus in order to formulate duality in our context one needs to distinguish two closed subspaces $\mathfrak{q}$ and $\mathfrak{p} := P\mathfrak{h}$ in the reference Hilbert space $\mathfrak{h}$. Now the study of two closed subspaces of a Hilbert space has a long and interesting history in functional analysis (e.g. [15, 13, 22, 30, 6, 11, 26]) as well as applications in mathematical physics [5]. We will see that to prove (1) the analysis of the relative position of these subspaces will be essential. In particular, given $\mathfrak{p}$ and $\mathfrak{q}$ as before we can canonically consider the Halmos decomposition [22] $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$, where $\mathfrak{h}_0$ is the maximal subspace where the orthoprojections corresponding to the closed subspaces commute [11, Section III]. Our proof of (1) in the general case is based on the corresponding property of the extremal situations when $\mathfrak{h}_0 = \{0\}$ (generic position case) resp. $\mathfrak{h}_1 = \{0\}$ (abelian case). The case where $\mathfrak{p}$ and $\mathfrak{q}$ are in generic position is the mathematically richest one, since it allows to use Modular Theory as well as techniques of Kato [26] for pairs of projections. In this context we will characterize the bicontinuity of different mappings that naturally appear here, e.g. the Tomita operator restricted to the one–particle Hilbert space. Further, we discuss systematically the relation of a mapping φ (and the components of its polar decomposition) used by Araki and Dell'Antonio [2, 14], to the modular objects given in our case. The mapping φ is introduced by these authors to study the type of certain local von Neumann algebras.

The equation (1) appears naturally in the context of algebraic quantum field theory, in particular in relation with Haag duality, which is one of its central concepts (see e.g. [16, 17, 31, 21, 7]). To prove this strengthening of Einstein causality for a net of Neumann algebras indexed by suitable space–time regions in the context of the CCR–Algebra (bosonic systems), one usually proceeds in two steps [2, 18, 29, 27, 23]: first one shows the so–called *abstract* duality

$$\mathcal{M}(\mathfrak{m}^{\perp_\sigma}) = \mathcal{M}(\mathfrak{m})' ,$$

where $\mathcal{M}(\mathfrak{m})'$ is the commutant of the von Neumann algebra $\mathcal{M}(\mathfrak{m})$ which is generated by the Weyl operators associated to the closed real subspace $\mathfrak{m}$ of the one–particle Hilbert space and $\mathfrak{m}^{\perp_\sigma}$ denotes the symplectic complement of $\mathfrak{m}$. This result is then essentially used in a second step in order to reduce the proof of Haag duality to the discussion of certain real subspaces $\mathfrak{m}(\mathcal{O})$ associated to sufficiently regular space–time regions $\mathcal{O}$. In the context of the CAR–Algebra (fermionic systems), and taking into account that now the generators of the algebra will anti–commute if the corresponding elements of the reference space are mutually orthogonal, one can adapt the notions of duality (cf. (1)) and Haag duality. (In the following we will avoid the use of the adjective 'abstract'). By our knowledge there is only one brief proof of twisted duality (1) in the fermionic case [20], which seems to be not so well–known (actually we found this reference after completing the present analysis) and that uses similar arguments as in the CCR–case. Although we will consider in Section 5 similar techniques as in this reference (mainly Modular Theory) there are good reasons for giving this new proof. The main difference w.r.t. [20] is that we will use the self–dual approach to the CAR–Algebra [3, 4] and will consistently work with *complex* Hilbert subspaces. This is not only a matter of elegance, but the *explicit* use of a basis projection in order to specify the Fock states of the CAR–Algebra will allow to consider the natural Halmos decomposition of the reference Hilbert space. From this we will be able to apply the strategy and techniques mentioned in the previous paragraph based on the distinguished pair of subspaces. Finally, the twisted duality property (1) of the CAR–Algebra can be applied to the fermionic free nets defined in [8, 24, 28] in order to prove Haag duality for these models. In the mentioned references the authors present a direct way to construct nets of local C*– or von Neumann–algebras associated to massive (resp. massless) models for any half–integer spin (resp. helicity) value. The nets for these models are naturally characterized by a net of local Γ–invariant linear subspaces $\mathcal{O} \mapsto \mathfrak{q}(\mathcal{O})$ of the corresponding reference Hilbert space.

The paper is structured in 7 sections: In Section 2 we will state basic results concerning the CAR–Algebra that will be used later on. In the next section we will consider the Halmos decomposition of a Hilbert space and state necessary and sufficient conditions on the subspaces $\mathfrak{p}$ and $\mathfrak{q}$ in order that the Modular Theory is well defined for $\mathcal{M}(\mathfrak{q})$ and its Fock vacuum vector Ω. In Section 4 we will systematically analyze the context defined by two projections: the first one being the orthoprojection Q onto $\mathfrak{q}$ and the other one being the basis projection P. The main goal here is to relate the objects that appear in the polar decomposition of the mapping φ and of the Tomita operator S corresponding to $(\mathcal{M}(\mathfrak{q}), \Omega)$ and restricted to the one–particle space $\mathfrak{p}$. These results will be applied in the next section where (1) is proved in the case where $\mathfrak{p}$ and $\mathfrak{q}$ are in generic position. Section 6 contains the proof of twisted duality in the other extremal case where P and Q commute. Finally, in the last section we give a prove of (1) in the most general situation, i.e. for any Γ–invariant closed subspace $\mathfrak{q}$ and any Fock state. The proof is based on the results of the previous two sections.

2 Basic structure of the CAR–Algebra

In order to establish our notation we will begin this section collecting some standard results concerning the CAR–Algebra that will be needed later on. For proofs and further results we refer to [3, 4]. In the following subsections we will also consider additional structure of this algebra necessary for our proof of twisted duality.

2.1 Theorem *Let $\mathfrak{h}$ be a complex Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and let Γ be an anti–unitary involution on it, i.e. $\langle\Gamma f, \Gamma h\rangle = \langle h, f\rangle$, for all $f, h \in \mathfrak{h}$. Then $\mathrm{CAR}(\mathfrak{h}, \Gamma)$ denotes the algebraically unique C*–algebra generated by $\mathbb{1}$ and $a(f)$, $f \in \mathfrak{h}$, such that the following relations hold:*

(i) The mapping $\mathfrak{h} \ni f \mapsto a(f)$ is antilinear.

(ii) For any $f \in \mathfrak{h}$ one has $a(f)^* = a(\Gamma f)$.

(iii) For any $f, h \in \mathfrak{h}$ the equation $a(f)a(h)^* + a(h)^*a(f) = \langle f, h\rangle\,\mathbb{1}$ holds.

Next we define a pure state of the preceding C*–algebra. An orthoprojection P of $\mathfrak{h}$ is called a *basis projection* if it satisfies the relation $P + \Gamma P \Gamma = \mathbb{1}$. These projections characterize uniquely so–called *Fock states* ω_P by means of the equation

$$\omega_P\Big(a(f)^* a(f)\Big) := 0 \qquad \text{for any} \quad f \in P\mathfrak{h}\,. \tag{2}$$

The antisymmetric Fock space is given by

$$\mathfrak{F} := \mathop{\oplus}_{n=0}^{\infty} \Big(\mathop{\wedge}^{n} P\mathfrak{h}\Big)\,. \tag{3}$$

In order to specify the Fock representation $\pi(a(f))$ of the generators $a(f)$ we need to introduce the usual annihilation and creation operators on $\mathfrak{F}$.

$$\begin{aligned}
c(p)\,\Omega &:= 0\,,\\
c(p)\,(p_1 \wedge \ldots \wedge p_n) &:= \sum_{r=1}^{n} (-1)^{r-1}\, \langle p, p_r\rangle_{\mathfrak{h}}\; p_1 \wedge \ldots \wedge \widehat{p}_r \wedge \ldots \wedge p_n\,,\\
c(p)^*\,\Omega &:= p\,,\\
c(p)^*\,(p_1 \wedge \ldots \wedge p_n) &:= p \wedge p_1 \wedge \ldots \wedge p_n\,,
\end{aligned}$$

where Ω is the Fock vacuum in the subspace corresponding to $n = 0$ in the definition (3) and $p, p_1, \ldots, p_n \in P\mathfrak{h}$. $\widehat{p}_r$ means that the vector p_r is omitted in the wedge product. Finally, the Fock representation π is defined by

$$\pi(a(f)) := c(P\Gamma f)^* + c(Pf)\,, \quad f \in \mathfrak{h}\,.$$

In the rest of the paper we assume that a basis projection P is given and when no confusion arises we will also write simply $a(f)$ instead of $\pi(a(f))$. To prove twisted duality in Section 5 we will need an explicit formula for the vector $a(f_n) \cdot \ldots \cdot a(f_1)\,\Omega$. Let n, k, p be natural numbers with $2p + k = n$ and define the following subset of the symmetric group $\mathfrak{S}_n$:

$$\begin{aligned}
\mathfrak{S}_{n,p} := \Bigg\{ &\begin{pmatrix} n & n-1 & \cdots & n-2p+2 & n-2p+1 & k & \cdots & 1 \\ \alpha_1 & \beta_1 & \cdots & \alpha_p & \beta_p & j_1 & \ldots & j_k \end{pmatrix} \in \mathfrak{S}_n \\
&\Big|\; \alpha_1 > \ldots > \alpha_p\,,\ \alpha_l > \beta_l\,,\ l = 1, \ldots, p \quad \text{and} \quad n \geq j_1 > j_2 > \ldots j_k \geq 1 \Big\}\,.
\end{aligned}$$

Note that $\mathfrak{S}_{n,p}$ contains $\begin{pmatrix} n \\ n-2p \end{pmatrix} \dfrac{(2p)!}{p!\,2^p}$ elements.

2.2 Proposition *For $f_1, \ldots, f_n \in \mathfrak{h}$ the equation*

$$\Big(a(f_n) \cdot \ldots \cdot a(f_1)\Big)\, \Omega = \sum_{\substack{\pi \,\in\, \mathfrak{S}_{n,\,p} \\ 0 \,\leq\, 2p \,\leq\, n}} (\operatorname{sgn} \pi) \prod_{l=1}^{p} \langle P f_{\alpha_l} \,,\, P\Gamma f_{\beta_l} \rangle \, P\Gamma f_{j_1} \wedge \ldots \wedge P\Gamma f_{j_k}$$

holds, where the indices $\alpha_l, \beta_l, j_1, \ldots, j_k$ are given in the defintion of $\mathfrak{S}_{n,p}$ and where for $n = 2p$ in the preceding sum one replaces the wedge product by the vacuum Ω.

Proof: See appendix. ∎

Let Z be the implementation on $\mathfrak{F}$ of the even–oddness automorphism associated to the Bogoljubov unitarity $-\mathbb{1}$ [4, p. 76]. It satisfies $Z = Z^* = Z^{-1}$ and therefore its spectral decomposition is simply given by

$$Z = E^+ - E^- . \tag{4}$$

Let further $X = X_{\text{even}} + X_{\text{odd}}$ be the unique decomposition of any $X \in \mathrm{CAR}(\mathfrak{h}, \Gamma)$ into its even and odd parts. The following result will be used in Section 7.

2.3 Lemma *Let π be a Fock representation of $\mathrm{CAR}(\mathfrak{h}, \Gamma)$ and $Z = E^+ - E^-$ as before. Then for any $X \in \mathrm{CAR}(\mathfrak{h}, \Gamma)$ we have*

$$\begin{aligned} E^+ \pi(X_{\text{even}}) E^- &= E^- \pi(X_{\text{even}}) E^+ = 0 \\ E^+ \pi(X_{\text{odd}}) E^+ &= E^- \pi(X_{\text{odd}}) E^- = 0 \,. \end{aligned}$$

Proof: Recall that $Z_0 \pi(X_{\text{even}}) = \pi(X_{\text{even}}) Z_0$. Multiplying from the left by E^+ and from the right by E^- we get

$$E^+ \pi(X_{\text{even}}) E^- = -E^+ \pi(X_{\text{even}}) E^- ,$$

which implies the first two eqs. Similarly we obtain the eqs. corresponding to the odd part. ∎

2.1 Direct sums and tensor products

Let $(\mathfrak{h}_k, \Gamma_k)$, $k = 0, 1$, be two Hilbert spaces with anti–unitary involutions and let P_k be two basis projections for the corresponding spaces. We denote by π_k the Fock representations of $\mathrm{CAR}(\mathfrak{h}_k, \Gamma_k)$ on the Fock spaces $\mathfrak{F}_k$. Z_k are the implementers of the respective even–oddness automorphisms, $k = 0, 1$. Motivated by [14, p. 82] (cf. also [19, p. 219]) we consider:

2.4 Proposition *With the preceding notation put $\mathfrak{h} := \mathfrak{h}_0 \oplus \mathfrak{h}_1$ and $\Gamma := \Gamma_0 \oplus \Gamma_1$ and consider $\mathrm{CAR}(\mathfrak{h}, \Gamma)$. The representation defined on $\mathfrak{F}_0 \otimes \mathfrak{F}_1$ by*

$$\pi\Big(a(f_0 \oplus f_1)\Big) := \pi_0\Big(a(f_0)\Big) \otimes \mathbb{1} + Z_0 \otimes \pi_1\Big(a(f_1)\Big) , \quad f_k \in \mathfrak{h}_k \,, k = 0, 1 \,, \tag{5}$$

is the Fock representation of $\mathrm{CAR}(\mathfrak{h}, \Gamma)$ w.r.t. the basis projection $P := P_0 \oplus P_1$ on $(\mathfrak{h}, \Gamma)$. Further, the even–oddness automorphism of $\mathrm{CAR}(\mathfrak{h}, \Gamma)$ is implemented on $\mathfrak{F}_0 \otimes \mathfrak{F}_1$ by $Z := Z_0 \otimes Z_1$.

Proof: We check first that the definition (5) is compatible with the algebraic structure of $\mathrm{CAR}(\mathfrak{h}, \Gamma)$. Indeed, for any $f_k, h_k \in \mathfrak{h}_k$, $k = 0, 1$,

$$\pi\Big(a(f_0 \oplus f_1)\Big)^* = \pi_0\Big(a(\Gamma_0 f_0)\Big) \otimes \mathbb{1} + Z_0 \otimes \pi_1\Big(a(\Gamma_1 f_1)\Big) = \pi\Big(a(\Gamma(f_0 \oplus f_1))\Big)$$

and

$$
\begin{aligned}
\pi\Big(a(f_0\oplus f_1)\Big)\pi\Big(a(h_0\oplus h_1)\Big)^* + \pi\Big(a(h_0\oplus h_1)\Big)^*\pi\Big(a(f_0\oplus f_1)\Big) \\
&= \pi_0\Big(a(f_0)a(h_0)^*\Big)\otimes \mathbb{1} + \mathbb{1}\otimes\pi_1\Big(a(f_1)a(h_1)^*\Big) \\
&\quad +\pi_0\Big(a(h_0)^*a(f_0)\Big)\otimes \mathbb{1} + \mathbb{1}\otimes\pi_1\Big(a(h_1)^*a(f_1)\Big) \\
&\quad +\underbrace{Z_0\pi_0\Big(a(h_0)^*\Big)\otimes\pi_1\Big(a(f_1)\Big) + \pi_0\Big(a(h_0)^*\Big)Z_0\otimes\pi_1\Big(a(f_1)\Big)}_{=0} \\
&\quad +\underbrace{Z_0\pi_0\Big(a(f_0)\Big)\otimes\pi_1\Big(a(h_1)^*\Big) + \pi_0\Big(a(f_0)\Big)Z_0\otimes\pi_1\Big(a(h_1)^*\Big)}_{=0} \\
&= \langle f_0\oplus f_1, h_0\oplus h_1\rangle\, \mathbb{1}\otimes\mathbb{1}\,.
\end{aligned}
$$

Consider next the vector state associated to $\Omega_0\otimes\Omega_1$, where Ω_k are the Fock vacua in $\mathfrak{F}_k$, $k=0,1$. In this case we have

$$
\begin{aligned}
\Big\langle\Omega_0\otimes\Omega_1, \pi\Big(a(f_0\oplus f_1)\Big)^*\pi\Big(a(f_0\oplus f_1)\Big)\Omega_0\otimes\Omega_1\Big\rangle \\
&= \Big\|\pi_0\Big(a(f_0)\Big)\Omega_0\otimes\Omega_1 + \Omega_0\otimes\pi_1\Big(a(f_1)\Big)\Omega_1\Big\|^2 \\
&= \|P_0\Gamma_0 f_0\|^2 + \|P_1\Gamma_1 f_1\|^2 = \|(\mathbb{1}-P)(f_0\oplus f_1)\|^2\,.
\end{aligned}
$$

Thus by the uniqueness of the characterizing condition (2) of a Fock state, together with the fact that the vectors $\pi\Big(a(f_0\oplus f_1)\Big)\Omega_0\otimes\Omega_1$, $f_k\in\mathfrak{h}_k$, generate the whole space $\mathfrak{F}_0\otimes\mathfrak{F}_1$, we have that π is actually the Fock representation of $\mathrm{CAR}(\mathfrak{h},\Gamma)$ w.r.t. the basis projection P with Fock vacuum $\Omega_0\otimes\Omega_1$.

We still need to show that $Z_0\otimes Z_1$ implements the even–oddness automorphism of $\mathrm{CAR}(\mathfrak{h},\Gamma)$ on $\mathfrak{F}_0\otimes\mathfrak{F}_1$. Note that $\alpha := \mathrm{ad}(Z_0\otimes Z_1)$ satisfies $\alpha^2=\mathrm{id}$ and it is enough to consider the action of α on the generators. Therefore the equations

$$
(Z_0\otimes Z_1)\pi\Big(a(f_0\oplus f_1)\Big)(Z_0\otimes Z_1) = Z_0\pi_0\Big(a(f_0)\Big)Z_0\otimes Z_1^2 + Z_0^3\otimes Z_1\pi_1\Big(a(f_1)\Big)Z_1 = \pi\Big(-a(f_0\oplus f_1)\Big)\,,
$$

conclude the proof. ■

2.5 Corollary *For $X_0\in\mathrm{CAR}(\mathfrak{h}_0,\Gamma_0)$ and $X_1 = X_{1,\mathrm{even}} + X_{1,\mathrm{odd}}\in\mathrm{CAR}(\mathfrak{h}_1,\Gamma_1)$ we have*

$$
\begin{aligned}
\pi(X_0) &= \pi_0(X_0)\otimes\mathbb{1} \quad \text{and} \\
\pi(X_1) &= \mathbb{1}\otimes\pi_1(X_{1,\mathrm{even}}) + Z_0\otimes\pi_1(X_{1,\mathrm{odd}}) \\
&= E_0^+\otimes\pi_1(X_1) + E_0^-\otimes Z_1\pi_1(X_1)Z_1\,,
\end{aligned}
$$

where $Z_0 = E_0^+ - E_0^-$.

2.2 The subspace $\mathfrak{q}$ and twisted causality

Let $(\mathfrak{h},\Gamma)$ be as in Theorem 2.1 and denote by $\mathfrak{q}$ a closed Γ–invariant subspace of $\mathfrak{h}$. We can naturally associate with the subspace $\mathfrak{q}$ a von Neumann algebra that acts on the Fock space characterized by the basis projection P:

$$
\mathcal{M}(\mathfrak{q}) := \Big(\{a(q)\mid q\in\mathfrak{q}\}\Big)'' \subset \mathcal{L}(\mathfrak{F})\,. \tag{6}
$$

In order to be able to formulate causality or duality in the context of the CAR–Algebra it is necessary to introduce a so–called *twist operator* $\widetilde{Z}$ on $\mathfrak{F}$. Indeed, this operator allows to express orthogonality relations of subspaces of $\mathfrak{h}$ in terms of the commutant of a suitable von Neumann algebra. Recalling the definition of Z before Eq. (4) we define the twist operator as usual [10, Eq. (26)]

$$\widetilde{Z} := \frac{1}{1+i}(\mathbb{1} + iZ) \quad \text{and} \quad \widetilde{Z} a(f) \widetilde{Z}^* = iZa(f)\,,\ f \in \mathfrak{h}\,. \tag{7}$$

Putting

$$\eta(n) := \begin{cases} 1, & \text{if} \quad n \quad \text{even}\,, \\ -i, & \text{if} \quad n \quad \text{odd}\,, \end{cases}$$

we may describe the action of $\widetilde{Z}$ and $\widetilde{Z}^*$ on $\mathfrak{F}$ by $\widetilde{Z}(p_1 \wedge \ldots \wedge p_n) = \eta(n)\, p_1 \wedge \ldots \wedge p_n$ and $\widetilde{Z}^*(p_1 \wedge \ldots \wedge p_n) = \overline{\eta(n)}\, p_1 \wedge \ldots \wedge p_n$. Further, considering the spectral projections $E^\pm$ of Z we also have the relation

$$\widetilde{Z} = E^+ - iE^-\,. \tag{8}$$

The algebra $\widetilde{Z}\,\mathcal{M}(\mathfrak{q})\,\widetilde{Z}^*$ is usually called 'twisted' algebra. It is now immediate to check the following inclusion, which expresses the twisted causality property in the present context.

2.6 Proposition *Let* $\mathfrak{q}$ *be a closed* Γ*–invariant subspace of* $\mathfrak{h}$ *as before. Then the inclusion* $\widetilde{Z}\,\mathcal{M}(\mathfrak{q}^\perp)\,\widetilde{Z}^* \subseteq \mathcal{M}(\mathfrak{q})'$ *holds.*

2.7 Remark Twisted duality strengthens this relation by turning the preceding inclusion to an equality for *any* closed Γ–invariant subspace $\mathfrak{q}$ of $\mathfrak{h}$.

2.8 Remark Writing now explicitly the Fock representation and considering Eq. (7) as well as the even–odd grading of the CAR–Algebra we have the following ways of generating the twisted von Neumann algebra:

$$\begin{aligned} \widetilde{Z}\,\mathcal{M}(\mathfrak{q}^\perp)\,\widetilde{Z}^* &= \Big\{ Z\pi(a(q^\perp)) \mid q^\perp \in \mathfrak{q}^\perp \Big\}'' \\ &= \Big\{ \pi(Y_{\text{even}}) + iZ\pi(Y_{\text{odd}})) \mid Y = Y_{\text{even}} + Y_{\text{odd}} \in \mathrm{CAR}(\mathfrak{q}^\perp, \Gamma\lceil\mathfrak{q}^\perp) \Big\}''\,. \end{aligned}$$

The second 'finer' generation of the twisted von Neumann algebra using explicitly the graded elements of the C*–algebra will be important for the proof of the main theorem in Section 7.

3 The Halmos decomposition of a Hilbert space

From the preceding subsection we see that to formulate the duality property in the present context one needs to distinguish two closed subspaces of the reference space $\mathfrak{h}$: the subspace $\mathfrak{p}$ (one–particle Hilbert space) which is given by the basis projection P and the Γ–invariant subspace $\mathfrak{q}$ to which we associate the orthoprojection Q. Therefore it is natural to consider the Halmos decomposition [22] of $\mathfrak{h}$ w.r.t. $\mathfrak{p}$ and $\mathfrak{q}$, which is given by

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1\,, \tag{9}$$

where $\mathfrak{h}_0 = (\mathfrak{p} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{q}^\perp) \oplus (\mathfrak{p}^\perp \cap \mathfrak{q}) \oplus (\mathfrak{p}^\perp \cap \mathfrak{q}^\perp)$ and $\mathfrak{h}_1 = \mathfrak{h} \ominus \mathfrak{h}_0$. $\mathfrak{h}_0$ is actually the maximal subspace where P and Q commute [11, Section III]. In $\mathfrak{h}_1$ the subspaces $\mathfrak{p}$ and $\mathfrak{q}$ are said to be in *generic position* [22], in the sense that the preceding four mutual intersections of the subspaces $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{p}^\perp$ and $\mathfrak{q}^\perp$ are equal to $\{0\}$.

In the present context the decomposition (9) is also natural because it allows to separate the general situation into simpler 'pieces' on $\mathfrak{h}_0$ and $\mathfrak{h}_1$:

3.1 Lemma *Consider* $(\mathfrak{h}, \Gamma)$ *and* P,Q *as before. Let further* R_0 *be the orthoprojection onto the subspace* $\mathfrak{h}_0$ *in (9). Then we have*

$$R_0 P = P R_0\,, \quad R_0 Q = Q R_0 \quad \text{and} \quad R_0 \Gamma = \Gamma R_0\,.$$

Proof: The first two equations are clear from the form of $\mathfrak{h}_0$ given after (9) and recall that R_0 is the maximal orthoprojection with the property $R_0\, PQ = R_0\, QP$. Now from

$$\Gamma R_0 \Gamma\, PQ = \Gamma R_0\, (\mathbb{1} - P) Q \Gamma = \Gamma R_0 \Gamma\, QP$$

and the maximality of R_0 we must have $\Gamma R_0 \Gamma \leq R_0$. Finally, from the Γ–invariance of $\mathfrak{h}$ and since $\Gamma^2 = \mathbb{1}$ we get $R_0 \Gamma = \Gamma R_0$. ■

3.2 Remark The preceding result allows to consider the following restrictions:

$$P_k := P \lceil \mathfrak{h}_k\,, \quad Q_k := Q \lceil \mathfrak{h}_k\,, \quad \text{and} \quad \Gamma_k := \Gamma \lceil \mathfrak{h}_k\,, \;\; k = 0, 1\,.$$

Further, P_k is a basis projection on $(\mathfrak{h}_k, \Gamma_k)$ characterizing the Fock representation π_k of $\mathrm{CAR}(\mathfrak{h}_k, \Gamma_k)$, $k = 0, 1$. This possibility to split off the general situation in an abelian piece and a generic position piece will be essential for the proof of our main theorem in Section 7.

3.1 Modular Theory

Let $(\mathfrak{h}, \Gamma)$, P and $\mathfrak{q}$ be given as in Subsection 2.2 and denote by $\mathfrak{p} := P\mathfrak{h}$ the corresponding one–particle Hilbert space. We will give in this subsection necessary and sufficient conditions on the subspaces $\mathfrak{p}$ and $\mathfrak{q}$ in order that the Fock vacuum Ω becomes a cyclic and separating vector for the von Neumann algebra $\mathcal{M}(\mathfrak{q})$ (cf. [19, p. 234]). We will see that in the case where the subspaces $\mathfrak{p}$ and $\mathfrak{q}$ are in generic position we can use the results in Modular Theory [25, 12] for the pair $(\mathcal{M}(\mathfrak{q}), \Omega)$. These techniques will be essentially used in the next two sections for the proof of the twisted duality property in the generic position context.

3.3 Lemma *Let* $\mathfrak{q} \subset \mathfrak{h}$ *be a closed* Γ*–invariant subspace. Then the following conditions are equivalent:*

(i) $q \in \mathfrak{q}$ *and* $Pq = 0$ *implies* $q = 0$.

(ii) $P(\mathfrak{q}^\perp)$ *is a dense submanifold of* $\mathfrak{p}$.

(iii) $\mathfrak{q} \cap \mathfrak{p} = \{0\}$.

Proof: To show (iii)⇒(i) take $q \in \mathfrak{q}$ with $Pq = 0$. From $\Gamma P \Gamma = \mathbb{1} - P$ and the Γ–invariance of $\mathfrak{q}$ we have that $\Gamma q \in \mathfrak{q} \cap \mathfrak{p} = \{0\}$, thus $q = 0$. The other implications (i)⇒(ii)⇒(iii) are checked similarly. ■

3.4 Proposition *Let* $\mathfrak{q}$ *be a closed* Γ*–invariant subspace of* $\mathfrak{h}$ *as before. Then we have:*

(i) The vacuum vector Ω *is cyclic for* $\mathcal{M}(\mathfrak{q})$ *iff* $P\mathfrak{q}$ *is a dense submanifold of* $\mathfrak{p}$.

(ii) The vacuum vector Ω *is separating for* $\mathcal{M}(\mathfrak{q})$ *iff* $P(\mathfrak{q}^\perp)$ *is a dense submanifold of* $\mathfrak{p}$.

Proof: (i) We assume that Ω is cyclic for $\mathcal{M}(\mathfrak{q})$ and let $p \in \mathfrak{p}$ be a vector satisfying $p \perp P\mathfrak{q}$. From this, from Proposition 2.2 and from the structure of the Fock space $\mathfrak{F}$ (recall Eq. (3)) we have

$$\begin{aligned} p &\perp \operatorname{span}\{a(q_1)\cdot \ldots \cdot a(q_n)\Omega \mid q_1,\ldots q_n \in \mathfrak{q}\,,\ n \in \mathbb{N}\}\,, \quad \text{thus} \\ p &\perp \{A\Omega \mid A \in \mathcal{M}(\mathfrak{q})\}\,. \end{aligned}$$

Now since Ω is cyclic for $\mathcal{M}(\mathfrak{q})$ we must have $p = 0$.

Assume now that $P\mathfrak{q} \subset \mathfrak{p}$ is a dense submanifold, so that $\overset{\infty}{\underset{n=0}{\oplus}} \left(\overset{n}{\wedge} P\mathfrak{q}\right) \subset \mathfrak{F}$ is also dense (here $\overset{\infty}{\underset{n=0}{\oplus}}$ denotes the algebraic direct sum). From Proposition 2.2 we obtain the inclusions

$$\overset{\infty}{\underset{n=0}{\oplus}} \left(\overset{n}{\wedge} P\mathfrak{q}\right) \subset \mathcal{M}(\mathfrak{q})\,\Omega \subset \mathfrak{F}\,,$$

which imply that Ω is cyclic for $\mathcal{M}(\mathfrak{q})$.

(ii) Suppose now that Ω is a separating vector for $\mathcal{M}(\mathfrak{q})$. We show that this implies part (i) of Lemma 3.3. So, let $q \in \mathfrak{q}$ satisfy $Pq = 0$. Since $0 = Pq = a(\Gamma q)\Omega$ and since Ω is separating for $\mathcal{M}(\mathfrak{q})$ we must have $a(\Gamma q) = 0$, which implies $q = 0$.

Finally, assume that $P(\mathfrak{q}^\perp)$ is a dense submanifold of $\mathfrak{p}$. By part (i) of the present theorem applied to $\mathfrak{q}^\perp$, which is also Γ–invariant, we know that Ω is cyclic for $\mathcal{M}(\mathfrak{q}^\perp)$. Further, since $\widetilde{Z}\Omega = \Omega$ we have that Ω is also cyclic for $\widetilde{Z}\,\mathcal{M}(\mathfrak{q}^\perp)\,\widetilde{Z}^*$ and consequently by Proposition 2.6 also for $\mathcal{M}(\mathfrak{q})' \supseteq \widetilde{Z}\,\mathcal{M}(\mathfrak{q}^\perp)\,\widetilde{Z}^*$. This shows that Ω is separating for $\mathcal{M}(\mathfrak{q})$. ■

By the preceding result we know that if $\mathfrak{q}$ is a closed Γ–invariant subspace of $\mathfrak{h}$ where $P\mathfrak{q}$ as well as $P(\mathfrak{q}^\perp)$ are dense submanifolds of $\mathfrak{p}$, then the Modular Theory is well–defined for the pairs $(\mathcal{M}(\mathfrak{q}), \Omega)$ and $(\mathcal{M}(\mathfrak{q}^\perp), \Omega)$. Denote by S and T the Tomita operators corresponding to these pairs respectively. We will next study their action on the submanifolds $P\mathfrak{q}$ and $P(\mathfrak{q}^\perp)$.

3.5 Lemma *With the preceding notation we have for $q \in \mathfrak{q}$ and $q^\perp \in \mathfrak{q}^\perp$:*

(i) $S(Pq) = P\Gamma q$.

(ii) $T(Pq^\perp) = P\Gamma q^\perp = -S^*(Pq^\perp)$.

Proof: The first two equations follow by direct computation:

$$S(Pq) = S\Big(a(\Gamma q)\,\Omega\Big) = a(\Gamma q)^*\,\Omega = a(q)\,\Omega = P\Gamma q\,,$$

and similarly for T. To prove the last equation recall that S^* is actually the Tomita operator of $\mathcal{M}(\mathfrak{q})' \supseteq \widetilde{Z}\,\mathcal{M}(\mathfrak{q}^\perp)\,\widetilde{Z}^*$. Note further that

$$i\left(\widetilde{Z}a(\Gamma q^\perp)\widetilde{Z}^*\right)\Omega = i\,\widetilde{Z}a(\Gamma q^\perp)\,\Omega = i\,\left(\frac{1-i}{1+i}\right) Pq^\perp = Pq^\perp\,.$$

From this we can finally check,

$$S^*(Pq^\perp) = -i\,S^*\Big(\widetilde{Z}a(\Gamma q^\perp)\widetilde{Z}^*\,\Omega\Big) = -i\,\widetilde{Z}a(q^\perp)\widetilde{Z}^*\,\Omega = -P\Gamma q^\perp = -T(Pq^\perp)\,,$$

and the proof is concluded. ■

4 Pairs of projections in generic position and the mapping φ

In this section we will consider the mathematically richest situation which appears when the closed subspaces $\mathfrak{p}$ and $\mathfrak{q}$ are in generic position. Let P and Q be the corresponding orthoprojections satisfying as usual the relations $\Gamma P\Gamma = \mathbb{1} - P = P^\perp$ and $Q\Gamma = \Gamma Q$. Motivated by Proposition 3.4 we will also assume here that

$$\mathfrak{p} \cap \mathfrak{q} = \{0\} = \mathfrak{p} \cap \mathfrak{q}^\perp \,, \quad \text{where } \mathfrak{p} := P\mathfrak{h} \,, \tag{10}$$

which directly implies using the basis projection property:

$$\mathfrak{p}^\perp \cap \mathfrak{q} = \{0\} = \mathfrak{p}^\perp \cap \mathfrak{q}^\perp \,, \quad \text{where } \mathfrak{p}^\perp = P^\perp\mathfrak{h} \,. \tag{11}$$

Thus in the notation of Section 3 we have the extremal case where $\mathfrak{p}$ and $\mathfrak{q}$ are already in generic position and the Halmos decomposition (9) is trivial in the sense that $\mathfrak{h}_0 = \{0\}$.

Note that the properties like $\mathfrak{p} \cap \mathfrak{q} = \{0\}$ can be also expressed by the corresponding projections P and Q, because the orthoprojection onto the intersection $\mathfrak{p} \cap \mathfrak{q}$ is given by

$$\text{s-}\lim_{n\to\infty} (PQ)^n = \text{s-}\lim_{n\to\infty} (QP)^n \,.$$

Note further that by Lemma 3.3 the intersection assumptions in (10) are equivalent to the density conditions in Proposition 3.4.

4.1 Remark The following useful density statements are immediate consequences of the assumption that $\mathfrak{p}$ and $\mathfrak{q}$ are in generic position. If $\mathfrak{r} \subseteq \mathfrak{q}$ (or $\mathfrak{r} \subseteq \mathfrak{q}^\perp$) is a dense linear submanifold in $\mathfrak{q}$ (resp. in $\mathfrak{q}^\perp$), then $P\mathfrak{r}$ is dense in $\mathfrak{p}$ and $P^\perp\mathfrak{r}$ is dense in $\mathfrak{p}^\perp$. The same holds if Q and P are interchanged. Thus we have for example that $Q\mathfrak{p}^\perp$ is dense in $\mathfrak{q}$, $PQ\mathfrak{p}$ is dense in $\mathfrak{p}$ etc.

We will begin next a systematic analysis of a mapping φ that can be naturally defined in the present context. Put

$$\begin{aligned} \mathfrak{H}_\varphi &:= \Big\{(q, q^\perp) \in \mathfrak{q} \times \mathfrak{q}^\perp \mid P^\perp(q + q^\perp) = 0\Big\} \\ \mathfrak{H}_\rho &:= \Big\{(q, q^\perp) \in \mathfrak{q} \times \mathfrak{q}^\perp \mid Pq = Pq^\perp\Big\} \,. \end{aligned}$$

4.2 Lemma *The sets $\mathfrak{H}_\varphi$ and $\mathfrak{H}_\rho$ are graphs of linear, injective and closed mappings $\varphi, \rho \colon \mathfrak{q} \to \mathfrak{q}^\perp$ with dense domains and dense images. The graphs $\operatorname{gra}\varphi = \mathfrak{H}_\varphi$ and $\operatorname{gra}\rho = \mathfrak{H}_\rho$ can be parametrized by $\mathfrak{p}$ resp. $\mathfrak{p}^\perp$ as*

$$\begin{aligned} \operatorname{gra}\varphi &:= \Big\{(Qp, Q^\perp p) \mid p \in \mathfrak{p}\Big\} \\ \operatorname{gra}\rho &:= \Big\{(Qp^\perp, -Q^\perp p^\perp) \mid p^\perp \in \mathfrak{p}^\perp\Big\} \,, \end{aligned} \tag{12}$$

where the domains and images are given explicitly. Moreover, the equation $\rho^{-1} = \varphi^*$ holds.

Proof: We consider first the mapping φ and one can similarly argue for ρ. First note that the assignment $\varphi(q) := q^\perp$ if $P^\perp(q + q^\perp) = 0$ is a well–defined linear map. Indeed, if $(q, q_1^\perp) \in \mathfrak{H}_\varphi \ni (q, q_2^\perp)$, then $q_1^\perp = q_2^\perp$, because in this case $q_1^\perp - q_2^\perp \in \mathfrak{q}^\perp \cap \mathfrak{p} = \{0\}$. Injectivity is proved analogously.

Next we show Eq. (12). Let $\mathfrak{p} \ni p = Qp + Q^\perp p$, so that $P^\perp(Qp + Q^\perp p) = 0$ and we have $Q\mathfrak{p} \subseteq \operatorname{dom}\varphi$ as well as $Q^\perp\mathfrak{p} \subseteq \operatorname{ima}\varphi$. To show the reverse inclusions take $q_0 \in \operatorname{dom}\varphi$, i.e. $P^\perp(q_0 + q_0^\perp) = 0$ for some $q_0^\perp \in \mathfrak{q}^\perp$. But this implies that $q_0 = Q(q_0 + q_0^\perp) \in Q\mathfrak{p}$ and $q_0^\perp = Q^\perp(q_0 + q_0^\perp) \in Q^\perp\mathfrak{p}$ and Eq. (12) is proved. Note that by the preceding remark the domain and image of φ are dense in $\mathfrak{q}$ resp. $\mathfrak{q}^\perp$ and it is easy to see that $\operatorname{gra}\varphi$ is closed.

Finally, it remains to show that $\rho^{-1} = \varphi^*$. Recall first that

$$\operatorname{gra}\rho^{-1} = \Big\{(Q^\perp p^\perp, -Qp^\perp) \mid p^\perp \in \mathfrak{p}^\perp\Big\}.$$

Using the isometric mapping $U\colon\ \mathfrak{q}\times\mathfrak{q}^\perp \to \mathfrak{q}^\perp\times\mathfrak{q}$ given by $U(q,q^\perp) := (q^\perp,-q)$ we may use the well known formula (see [1, p. 124])

$$\operatorname{gra}\varphi^* = (\mathfrak{q}^\perp\times\mathfrak{q}) \ominus U(\operatorname{gra}\varphi).$$

Therefore $(q^\perp, q) \in \operatorname{gra}\varphi^*$ iff $\Big\langle (q^\perp,q),(Q^\perp p,-Qp)\Big\rangle = 0$ for all $p\in\mathfrak{p}$ iff $\langle q^\perp, p\rangle = \langle q,p\rangle$ for all $p\in\mathfrak{p}$ iff $p^\perp := q^\perp - q \in \mathfrak{p}^\perp$ iff $(q^\perp,q)\in\operatorname{gra}\rho^{-1}$. ∎

4.3 Remark Note that the preceding lemma depends only on the assumption that P and Q are in generic position (the different role of P and Q w.r.t. Γ, i.e. $Q\Gamma = \Gamma Q$ and $\Gamma P\Gamma = P^\perp$, is not used). This means that the preceding lemma remains true if we do the following replacements

$$Q\to P \qquad\text{and}\qquad P\to Q^\perp.$$

Then we obtain:

4.4 Corollary *Put*

$$\mathfrak{H}_\lambda := \Big\{(p,p^\perp)\in\mathfrak{p}\times\mathfrak{p}^\perp \mid Q^\perp p = Q^\perp p^\perp\Big\}.$$

Then $\mathfrak{H}_\lambda$ is the graph of a linear injective closed mapping $\lambda\colon\ \mathfrak{p}\to\mathfrak{p}^\perp$ with dense domain and image. $\operatorname{gra}\lambda$ *can be parametrized by* $\mathfrak{q}$*:*

$$\operatorname{gra}\lambda = \Big\{(Pq, -P^\perp q) \mid q\in\mathfrak{q}\Big\}.$$

The parametrization of φ and ρ in terms of $\mathfrak{p}$ resp. $\mathfrak{p}^\perp$ suggests to consider the following mappings:

$$Q\ :\ \mathfrak{p}^\perp \longrightarrow \mathfrak{q} \tag{13}$$
$$Q^\perp\colon\ \mathfrak{p}^\perp \longrightarrow \mathfrak{q}^\perp \tag{14}$$
$$Q\ :\ \mathfrak{p} \longrightarrow \mathfrak{q} \tag{15}$$
$$Q^\perp\colon\ \mathfrak{p} \longrightarrow \mathfrak{q}^\perp, \tag{16}$$

where the first two are related to ρ and the last two are related to φ. For example, the parametrization of ρ in terms of $\mathfrak{p}^\perp$ means that ρ can be seen as the composition of the following mappings:

$$QP^\perp\mathfrak{h} \longrightarrow P^\perp\mathfrak{h} \longrightarrow Q^\perp P^\perp\mathfrak{h}.$$

Later we will also need to consider the mappings

$$P\colon\ \mathfrak{q}^\perp \longrightarrow \mathfrak{p} \tag{17}$$
$$P\colon\ \mathfrak{q} \longrightarrow \mathfrak{p}. \tag{18}$$

Due to the fact that $\mathfrak{q}$ and $\mathfrak{p}$ are in generic position the mappings (13)–(18) are bounded, injective and their images are dense in the corresponding final subspaces.

We will next give a criterion for the bicontinuity of the mappings (13)–(18). First note that because P is a basis projection

$$\|PQ\| = \|QP\| = \|(\mathbb{1}-P)Q\| = \|Q(\mathbb{1}-P)\| =: \delta$$

and $0<\delta\le 1$. So we can distinguish between the two cases: $\delta<1$ and $\delta=1$.

4.5 Proposition *Let P, Q and δ given as before. If $\delta < 1$, then the mappings (13)–(18) are bicontinuous, in particular their images coincide with the corresponding final spaces. Moreover, the relations*

$$\|P - Q\| = \|(\mathbb{1} - Q)P\| = \|(\mathbb{1} - Q)(\mathbb{1} - P)\| = \delta$$

hold.

Proof: This result is a special case of Theorem 6.34 in [26, p. 56]. Note that the second alternative stated in Kato's result cannot appear in the present situation, as a consequence of the fact that $\mathfrak{p}$ and $\mathfrak{q}$ are in generic position. ■

4.6 Remark This situation corresponds to the case where the index of P and Q is 0 (cf. [6, Theorem 3.3]).

4.7 Proposition *Let P, Q and δ given as before. If $\delta = 1$, then the inverse mappings of (13)–(18) are unbounded and densely defined, i.e. the images of (13)–(18) are nontrivial proper dense sets in the corresponding final subspaces.*

Proof: We will only show the assertion for the mapping (18), since one can easily adapt the following arguments to the other cases. Put $A := QP^\perp Q\restriction\mathfrak{q} \in \mathcal{L}(\mathfrak{q})$, so that $A = A^*$ and $A \geq 0$. From

$$\operatorname{spr} A = \|A\| = \|QP^\perp P^\perp Q\| = \|P^\perp Q\|^2 = \delta^2 = 1$$

we obtain $1 \in \operatorname{spec} A$. However, 1 is not an eigenvalue of A, because $Aq = q$, $q \in \mathfrak{q}$, implies $\operatorname{s-}\lim\limits_{n\to\infty} (QP)^n q = q$ and this means $q \in \mathfrak{q} \cap \mathfrak{p}^\perp = \{0\}$. Thus $\ker(\mathbb{1}_\mathfrak{q} - A) = \{0\}$ or $(\mathbb{1}_\mathfrak{q} - A)^{-1}$ exists and is unbounded since $1 \notin \operatorname{res} A$. Therefore $\vartheta := \operatorname{dom}(\mathbb{1}_\mathfrak{q} - A)^{-1}$ is a proper dense subset in $\mathfrak{q}$ and this means $\operatorname{ima}(\mathbb{1}_\mathfrak{q} - A) = \vartheta = \operatorname{ima}(Q - QP^\perp Q) = \operatorname{ima}(QPQ)$. Finally, from the polar decomposition of PQ,

$$PQ = \operatorname{sgn}(PQ) \cdot (QPQ)^{\frac{1}{2}},$$

we have that $\operatorname{sgn}(PQ)$ maps $\operatorname{ima}(QPQ)^{\frac{1}{2}}$ isometrically onto $\operatorname{ima}(PQ) = P\mathfrak{q}$. Thus $P\mathfrak{q}$ is a proper dense set in $\mathfrak{p}$, i.e. $P\colon \mathfrak{q} \to \mathfrak{p}$ is unbounded invertible. ■

4.8 Remark (i) Note that if $\dim \mathfrak{h} < \infty$, then the case $\|PQ\| = 1$ is not possible, since the corresponding operators can not have continuous spectrum. It is easy to show that in this case $\mathfrak{p} \cap \mathfrak{q} = \{0\}$ iff $\|PQ\| < 1$.

(ii) Note also that $\|(\mathbb{1} - P)Q\| = 1$ implies $\|(\mathbb{1} - Q)P\| = 1$, because otherwise by Proposition 4.5 $\|(\mathbb{1} - Q)P\| < 1$ implies $\|(\mathbb{1} - P)Q\| < 1$.

4.9 Proposition *Let the projections P, Q and the mappings φ, ρ be given as before. Then $\varphi, \rho\colon\ \mathfrak{q} \to \mathfrak{q}^\perp$ are bicontinuous iff $\|PQ\| < 1$.*

Proof: Suppose that $\|PQ\| < 1$, so that by Proposition 4.5 we have that the mappings (13)–(18) are bicontinuous. But as mentioned before we know that φ as well as ρ can be seen as composition of the mappings

$$\begin{aligned} \varphi&\colon\ QP\mathfrak{h} \longrightarrow P\mathfrak{h} \longrightarrow Q^\perp P\mathfrak{h} \\ \rho&\colon\ QP^\perp\mathfrak{h} \longrightarrow P^\perp\mathfrak{h} \longrightarrow Q^\perp P^\perp\mathfrak{h}, \end{aligned}$$

hence they must be bicontinuous.

In the case that ρ and φ are bicontinuous, then $\operatorname{dom}\rho = \operatorname{dom}\varphi = \mathfrak{q}$ and $\operatorname{ima}\rho = \operatorname{ima}\varphi = \mathfrak{q}^\perp$. Finally, Proposition 4.7 implies $\|PQ\| < 1$. ■

Motivated by Lemma 3.5 we will analyze next the antilinear mappings defined by the following graphs:

$$\begin{aligned}\mathrm{gra}\,\beta &:= \Big\{(Pq, P\Gamma q) \in \mathfrak{p}\times\mathfrak{p} \mid q \in \mathfrak{q}\Big\}\\ \mathrm{gra}\,\alpha &:= \Big\{(Pq^\perp, -P\Gamma q^\perp) \in \mathfrak{p}\times\mathfrak{p} \mid q^\perp \in \mathfrak{q}^\perp\Big\}.\end{aligned}$$

(Note that the r.h.s. of the preceding equations define indeed graphs of antilinear mappings, because the assignments $q \to Pq$ and $q^\perp \to Pq^\perp$ are injective.)

4.10 Lemma *The mappings α, β defined by the preceding graphs are anti–linear, injective and closed with dense domains and images* $\mathrm{dom}\,\alpha = \mathrm{ima}\,\alpha = P(\mathfrak{q}^\perp)$, $\mathrm{dom}\,\beta = \mathrm{ima}\,\beta = P\mathfrak{q}$. *Further, we have* $\alpha^2 = \mathrm{id}$, $\beta^2 = \mathrm{id}$ *on* $P(\mathfrak{q}^\perp)$ *resp.* $P\mathfrak{q}$ *and* $\alpha = \beta^*$.

Proof: We will only prove the last equation, because the other statements follow immediately from the definition. Now by definition we have $(p_0, p_1) \in \mathrm{gra}\,\beta^*$ iff $\langle p_0, P\Gamma q\rangle = \langle Pq, p_1\rangle$ for all $q \in \mathfrak{q}$ iff $q^\perp := p_0 - \Gamma p_1 \in \mathfrak{q}^\perp$ iff $(p_0, p_1) \in \mathrm{gra}\,\alpha$. ■

4.11 Remark (i) Recall Subsection 3.1 and denote by S the Tomita operator associated to $(\mathcal{M}(\mathfrak{q}), \Omega)$. Then from the preceding result we have $S\lceil\mathfrak{p} \supseteq \beta$ and $S^*\lceil\mathfrak{p} \supseteq \alpha$.

(ii) Using the mappings (17) and (18) we can now state similarly as in Proposition 4.9 a criterion for the bicontinuity of α, β: the mappings α, β are bicontinuous iff $\|PQ\| < 1$.

4.12 Proposition *The mapping $\beta^*\beta$: $\mathfrak{p} \to \mathfrak{p}$ is a densely defined linear positive self–adjoint operator on $\mathfrak{p}$ with graph*

$$\mathrm{gra}\,\beta^*\beta = \Big\{(PQp, PQ^\perp p) \mid p \in \mathfrak{p}\Big\}.$$

Moreover, $(\beta^*\beta)^{-1} = \beta\beta^* = \alpha^*\alpha$.

Proof: We will compute first the domain of $\beta^*\beta$. Recalling that $\beta^* = \alpha$ we have

$$\begin{aligned}\mathrm{dom}\,(\beta^*\beta) &= \Big\{Pq \mid q \in \mathfrak{q} \text{ and } P\Gamma q \in \mathrm{dom}\,\alpha = P(\mathfrak{q}^\perp)\Big\}\\ &= \Big\{Pq \mid q \in \mathfrak{q} \text{ and } P\Gamma q = Pq^\perp \text{ for some } q^\perp \in \mathfrak{q}^\perp\Big\}\\ &= \Big\{Pq \mid q \in \mathfrak{q} \text{ and } \Gamma q \in \mathrm{dom}\,\rho = Q(\mathfrak{p}^\perp)\Big\}\\ &= \Big\{Pq \mid q \in \mathfrak{q} \text{ and } q \in \Gamma Q(\mathfrak{p}^\perp) = Q(\Gamma\mathfrak{p}^\perp) = Q\mathfrak{p}\Big\}\\ &= PQ\mathfrak{p} \quad = \quad PQP\mathfrak{h},\end{aligned}$$

which is dense in $\mathfrak{p}$. Furthermore, since $P\Gamma Qp = -PQ^\perp\Gamma p$, $p \in \mathfrak{p}$ (recall $P\Gamma p = 0$, $p \in \mathfrak{p}$), we have

$$\Big(\beta^*\beta\Big)(PQp) = \alpha\Big(P\Gamma Qp\Big) = -\alpha\Big(PQ^\perp\Gamma p\Big) = P\Gamma Q^\perp\Gamma p = PQ^\perp p\,, \quad p \in \mathfrak{p}\,.$$

The last equations concerning the inverse of $\beta^*\beta$ follow from the preceding computation and from the fact that $\alpha^2 = \mathrm{id}$ and $\beta^2 = \mathrm{id}$ on the corresponding domains (recall Lemma 4.10). ■

In the following we will use the notation

$$\Delta_{\mathfrak{p}} := \beta^*\beta\,,$$

where $\mathrm{dom}\,\Delta_{\mathfrak{p}} = PQ\mathfrak{p}$ is dense in $\mathfrak{p}$. Note further that $\Delta_{\mathfrak{p}}^{-\frac{1}{2}} = |\alpha|$ and thus $\mathrm{dom}\,\Delta_{\mathfrak{p}}^{-\frac{1}{2}} = P(\mathfrak{q}^\perp)$.

Next we will calculate the graph of the positive self–adjoint operator $\varphi^*\varphi$: $\mathfrak{q} \to \mathfrak{q}$.

4.13 Proposition *The graph of $\varphi^*\varphi$ is given by*

$$\operatorname{gra}\varphi^*\varphi = \Big\{(QPq, QP^\perp q) \mid q \in \mathfrak{q}\Big\}.$$

Proof: We begin computing $\operatorname{dom}(\varphi^*\varphi)$. Since by Lemma 4.2 $\varphi^* = \rho^{-1}$ we have

$$\begin{aligned}
\operatorname{dom}(\varphi^*\varphi) &= \Big\{Qp \mid p \in \mathfrak{p} \text{ and } Q^\perp p \in \operatorname{dom}\rho^{-1} = Q^\perp \mathfrak{p}^\perp\Big\} \\
&= \Big\{Qp \mid p \in \mathfrak{p} \text{ and } Q^\perp p = Q^\perp p^\perp \text{ for some } p^\perp \in \mathfrak{p}^\perp\Big\} \\
&= \Big\{Qp \mid p = Pq \text{ for some } q \in \mathfrak{q}\Big\} \\
&= QP\mathfrak{q} \quad = \quad QPQ\mathfrak{h}\,,
\end{aligned}$$

where for the third equation we have used Corollary 4.4. Using again this corollary we can calculate

$$\rho^{-1}\Big(\varphi(QPq)\Big) = \rho^{-1}(Q^\perp Pq) = \rho^{-1}\Big(Q^\perp(-P^\perp q)\Big) = QP^\perp q\,, \quad q \in \mathfrak{q}\,,$$

and the proof is concluded. ■

Now we can relate φ and $\varphi^*\varphi$ with $\Delta_\mathfrak{p}$ just computing the orthogonal decomposition of $\varphi(q)$, $q \in \operatorname{dom}\varphi$, resp. $\varphi^*\varphi(q)$, $q \in \operatorname{dom}\varphi^*\varphi$, w.r.t. $\mathfrak{h} = \mathfrak{p} \oplus \mathfrak{p}^\perp$.

4.14 Corollary *Using the notation before we have the following formulas:*

$$\begin{aligned}
\varphi(Qp) &= \Delta_\mathfrak{p}(PQp) - P^\perp Qp\,, \quad p \in \mathfrak{p}\,, \qquad (19)\\
\Big(\varphi^*\varphi\Big)(QPq) &= \Delta_\mathfrak{p}(PQPq) + \Gamma\,\Delta_\mathfrak{p}^{-1}(P\Gamma QPq)\,, \quad q \in \mathfrak{q}\,.
\end{aligned}$$

Proof: From Lemma 4.2 as well as Proposition 4.12 we have

$$\varphi(Qp) = Q^\perp p = PQ^\perp p + P^\perp Q^\perp p = \Delta_\mathfrak{p}(PQp) - P^\perp Qp\,, \quad p \in \mathfrak{p}\,.$$

Further, from the preceding proposition we also have for any $q \in \mathfrak{q}$

$$\begin{aligned}
\Big(\varphi^*\varphi\Big)(QPq) &= QP^\perp q \quad = \quad PQP^\perp q + P^\perp QP^\perp q \\
&= PQ^\perp Pq + \Gamma PQP\Gamma q \\
&= \Delta_\mathfrak{p}(PQPq) + \Gamma\,\Delta_\mathfrak{p}^{-1}(PQ^\perp P\Gamma q) \\
&= \Delta_\mathfrak{p}(PQPq) + \Gamma\,\Delta_\mathfrak{p}^{-1}(PQP^\perp\Gamma q) \\
&= \Delta_\mathfrak{p}(PQPq) + \Gamma\,\Delta_\mathfrak{p}^{-1}(P\Gamma QPq)\,,
\end{aligned}$$

which proves the second formula. ■

We will give next two formulas in terms of $\Delta_\mathfrak{p}^{\frac{1}{2}}$ for the components of the polar decomposition of φ. Denote $\varphi = \operatorname{sgn}\varphi\cdot|\varphi|$, where as usual $|\varphi| := (\varphi^*\varphi)^{\frac{1}{2}}$. Recall from the results in this section that

$$\begin{aligned}
\operatorname{dom}\varphi &= \operatorname{dom}|\varphi| = Q\mathfrak{p} \supseteq QP\mathfrak{q} = \operatorname{dom}(\varphi^*\varphi) \\
\operatorname{dom}\beta &= \operatorname{dom}|\beta| = P\mathfrak{q} \supseteq PQ\mathfrak{p} = \operatorname{dom}\Delta_\mathfrak{p}\,.
\end{aligned}$$

4.15 Theorem *With the notation above we have:*

$$|\varphi|(q) \;=\; \Delta_{\mathfrak{p}}^{\frac{1}{2}}(Pq) + \Gamma\,\Delta_{\mathfrak{p}}^{-\frac{1}{2}}(P\Gamma q)\,, \quad q \in \mathrm{dom}\,\varphi^*\varphi = QP\mathfrak{q}\,, \tag{20}$$

$$\mathrm{sgn}\,\varphi(q) \;=\; \Delta_{\mathfrak{p}}^{\frac{1}{2}}(Pq) - \Gamma\,\Delta_{\mathfrak{p}}^{\frac{1}{2}}(P\Gamma q)\,, \quad q \in \mathfrak{q}\,. \tag{21}$$

Moreover $\mathrm{sgn}\,\varphi$ *is an isometry of* $\mathfrak{q}$ *onto* $\mathfrak{q}^\perp$, *i.e.* $(\mathrm{sgn}\,\varphi)^*\mathrm{sgn}\,\varphi = Q$ *and* $\mathrm{sgn}\,\varphi(\mathrm{sgn}\,\varphi)^* = Q^\perp$.

Proof: From the explicit knowledge of all the domains of the mappings used before it is easily seen that the formulas are well–defined. Further, recall Proposition 4.13 and Corollary 4.14 and the fact that the two terms of the r.h.s. of the above formulas correspond to the decomposition of $\mathfrak{h}$ in terms of $P\mathfrak{h}$ and $P^\perp\mathfrak{h}$. Appling now for $q \in \mathrm{dom}\,\varphi^*\varphi$ twice the r.h.s. of (20) we get

$$\begin{aligned}\Delta_{\mathfrak{p}}^{\frac{1}{2}} P\Big(\Delta_{\mathfrak{p}}^{\frac{1}{2}}(Pq) + \Gamma\,\Delta_{\mathfrak{p}}^{-\frac{1}{2}}(P\Gamma q)\Big) &+ \Gamma\,\Delta_{\mathfrak{p}}^{-\frac{1}{2}} P\Gamma\Big(\Delta_{\mathfrak{p}}^{\frac{1}{2}}(Pq) + \Gamma\,\Delta_{\mathfrak{p}}^{-\frac{1}{2}}(P\Gamma q)\Big)\\ &= \;\Delta_{\mathfrak{p}}(Pq) + \Gamma\Delta_{\mathfrak{p}}^{-1}(P\Gamma q) \;=\; \Big(\varphi^*\varphi\Big)(q)\,,\end{aligned}$$

which shows the first formula. To prove the second one note first that for $q' = |\varphi|(q)$, $q \in \mathrm{dom}\,\varphi^*\varphi$, and using Eq. (20) as well as Corollary 4.14 we obtain from a similar calculation as before that

$$\Big(\Delta_{\mathfrak{p}}^{\frac{1}{2}} P - \Gamma\,\Delta_{\mathfrak{p}}^{\frac{1}{2}} P\Gamma\Big)|\varphi|(q') = \Big(\Delta_{\mathfrak{p}} P - P^\perp\Big)(q') = \varphi(q')\,.$$

Thus the r.h.s. and the l.h.s. of (21) coincide on the dense subspace $\mathrm{ima}\,\varphi^*\varphi$. Finally, the fact that the r.h.s. is also well defined for all $q \in \mathfrak{q}$ (recall that $\mathrm{dom}\,\Delta_{\mathfrak{p}}^{\frac{1}{2}} = \mathrm{dom}\,\beta = P\mathfrak{q}$) and that $\mathrm{sgn}\,\varphi$ maps isometrically the dense subspace $\mathrm{ima}\,|\varphi| \subseteq \mathfrak{q}$ onto the dense subspace $\mathrm{ima}\,\varphi = Q^\perp\mathfrak{p} \subseteq \mathfrak{q}^\perp$ proves formula (21). Therefore Q is the initial projection of $\mathrm{sgn}\,\varphi$ and $Q^\perp$ is the corresponding final projection. ■

4.16 Remark Recall that $\mathrm{dom}\,\varphi^*\varphi$ is a core for $|\varphi|$ and note that the r.h.s. of formula (20) *can not* be extended to the whole $\mathrm{dom}\,|\varphi|$.

Finally, we consider the mapping

$$\mathfrak{q} \ni q \mapsto Wq := (\mathbb{1} + \Delta_{\mathfrak{p}})^{\frac{1}{2}}\, Pq \in \mathfrak{p}\,. \tag{22}$$

4.17 Lemma W *is an isometry from* $\mathfrak{q}$ *onto* $\mathfrak{p}$.

Proof: First choose $q_1, q_2 \in Q\mathfrak{p}$ which is dense in $\mathfrak{q}$. Now using

$$\langle Pq_1, \Delta_{\mathfrak{p}}(Pq_2)\rangle = \langle Pq_1, \beta^*\beta(Pq_2)\rangle = \langle \beta(Pq_2), \beta(Pq_1)\rangle = \langle P^\perp q_1, P^\perp q_2\rangle$$

we obtain

$$\Big\langle(\mathbb{1} + \Delta_{\mathfrak{p}})^{\frac{1}{2}} Pq_1, (\mathbb{1} + \Delta_{\mathfrak{p}})^{\frac{1}{2}} Pq_2\Big\rangle = \langle Pq_1, (\mathbb{1} + \Delta_{\mathfrak{p}})Pq_2\rangle = \langle q_1, q_2\rangle\,, q_1, q_2 \in Q\mathfrak{p}\,.$$

Further, for any $p \in \mathfrak{p}$ we have from Proposition 4.12 that $(\mathbb{1} + \Delta_{\mathfrak{p}})(PQp) = PQp + PQ^\perp p = p$. This implies that $\mathrm{ima}\,(\mathbb{1} + \Delta_{\mathfrak{p}})^{\frac{1}{2}} = \mathfrak{p}$, since

$$\mathfrak{p} = \mathrm{ima}\,(\mathbb{1} + \Delta_{\mathfrak{p}}) \subseteq \mathrm{ima}\,(\mathbb{1} + \Delta_{\mathfrak{p}})^{\frac{1}{2}} \subseteq \mathfrak{p}\,.$$

Therefore (22) is the isometric extension of $W\restriction Q\mathfrak{p}$. ■

Using now the isometry W we conclude this section showing the unitary equivalence of $|\varphi|$ and $\Delta_{\mathfrak{p}}^{\frac{1}{2}}$.

4.18 Theorem *With the preceding notation we have*

$$W\,|\varphi|(q) = \Delta_{\mathfrak{p}}^{\frac{1}{2}}\,W(q)\,,\quad q\in \mathrm{dom}\,|\varphi|\,.$$

Proof: For any $q\in \mathrm{dom}\,\varphi^*\varphi$, which is a core of $|\varphi|$, we may use (20) and in this case

$$W\,|\varphi|(q) = (\mathbb{1}+\Delta_{\mathfrak{p}})^{\frac{1}{2}}P\Big(\Delta_{\mathfrak{p}}^{\frac{1}{2}}(Pq)+\Gamma\,\Delta_{\mathfrak{p}}^{-\frac{1}{2}}(P\Gamma q)\Big) = (\mathbb{1}+\Delta_{\mathfrak{p}})^{\frac{1}{2}}\,\Delta_{\mathfrak{p}}^{\frac{1}{2}}(Pq) = \Delta_{\mathfrak{p}}^{\frac{1}{2}}W(q)\,.$$

But this implies that $W|\varphi|W^*\subseteq \Delta_{\mathfrak{p}}^{\frac{1}{2}}$ and since the l.h.s. as well as the r.h.s. of the preceding inclusion are self–adjoint operators, we must actually have the equality $W|\varphi|W^*=\Delta_{\mathfrak{p}}^{\frac{1}{2}}$. ∎

5 Twisted duality. The generic position case

We begin the proof of twisted duality considering first one of the extremal cases that may appear in the Halmos decomposition (9). For this assume that $(\mathfrak{h},\Gamma)$, P and Q are given as in the preceding section. In particular $\mathfrak{p}$ and $\mathfrak{q}$ are in generic position, so that by Proposition 3.4 the Modular Theory is well defined for the pair $(\mathcal{M}(\mathfrak{q}),\Omega)$. Denote as usual by $S=J\Delta^{\frac{1}{2}}$ the polar decomposition of the Tomita operator.

We prove first that the different modular objects leave the n–particle submanifolds $\overset{n}{\wedge}(P\mathfrak{q})$ invariant. This fact is well known in the context of CCR algebras [27], where one can use the so–called exponential vectors which are specially well–adapted to the Weyl operators.

5.1 Proposition *Let $q_1,\ldots,q_n\in\mathfrak{q}$ and $q_1^\perp,\ldots,q_n^\perp\in\mathfrak{q}^\perp$. Then the following equations hold*

$$\begin{aligned} S(Pq_1\wedge\ldots\wedge Pq_n) &= P\Gamma q_n\wedge\ldots\wedge P\Gamma q_1 = S(Pq_n)\wedge\ldots\wedge S(Pq_1)\\ S^*(Pq_1^\perp\wedge\ldots\wedge Pq_n^\perp) &= S^*(Pq_n^\perp)\wedge\ldots\wedge S^*(Pq_1^\perp)\end{aligned}$$

Proof: The proof is done by induction on the number of vectors in the wedge product. For $n=1$ the above equations are trivially satisfied (cf. Lemma 3.5). We will now concentrate on the first formula since one can argue similarly for S^*. Suppose that the first expression holds for a number of vectors $\leq n-1$. Then applying this induction hypothesis as well as Proposition 2.2 we get

$$\begin{aligned} & S(P\Gamma q_n\wedge\ldots\wedge P\Gamma q_1)\\ &= S\Bigg(a(q_n)\cdot\ldots\cdot a(q_1)\,\Omega-\sum_{\substack{\pi\,\in\,\mathfrak{S}_{n,p}\\ p\,\geq\,1}}(\mathrm{sgn}\,\pi)\prod_{l=1}^{p}\langle Pq_{\alpha_l}\,,\,P\Gamma q_{\beta_l}\rangle\,P\Gamma q_{j_1}\wedge\ldots\wedge P\Gamma q_{j_k}\Bigg)\\ &= a(\Gamma q_1)\cdot\ldots\cdot a(\Gamma q_n)\,\Omega-\sum_{\substack{\pi\,\in\,\mathfrak{S}_{n,p}\\ p\,\geq\,1}}(\mathrm{sgn}\,\pi)\prod_{l=1}^{p}\langle P\Gamma q_{\beta_l}\,,\,Pq_{\alpha_l}\rangle\,Pq_{j_k}\wedge\ldots\wedge Pq_{j_1} \end{aligned}\tag{23}$$

and recall that the indices sepcified by $\pi\in\mathfrak{S}_{n,p}$ satisfy $\alpha_1>\ldots>\alpha_p$, $\alpha_l>\beta_l$, $l=1,\ldots,p$ and $n\geq j_1>j_2>\ldots>j_k\geq 1$. But we can now apply again Proposition 2.2 to the first term of the preceding sum and we get

$$a(\Gamma q_1)\cdot\ldots\cdot a(\Gamma q_n)\,\Omega = Pq_1\wedge\ldots\wedge Pq_n+\sum_{\substack{\pi'\\ p\,\geq\,1}}(\mathrm{sgn}\,\pi')\prod_{l=1}^{p}\langle P\Gamma q_{\alpha_l'}\,,\,Pq_{\beta_l'}\rangle\,Pq_{j_1'}\wedge\ldots\wedge Pq_{j_k'}\,,\tag{24}$$

where now $\pi' = \begin{pmatrix} 1 & 2 & \cdots & 2p-1 & 2p & n-k+1 & \cdots & n \\ \alpha_1' & \beta_1' & \cdots & \alpha_p' & \beta_p' & j_1' & \cdots & j_k' \end{pmatrix}$ with $\alpha_l' < \beta_l'$, $l = 1, \ldots, p$, $1 \leq j_1' < j_2' < \ldots < j_k' \leq n$ and we may reorganize the scalar products such that $\beta_1' < \ldots < \beta_p'$. We can next associate bijectively these permutations with elements in $\mathfrak{S}_{n,p}$ by means of

$$\pi' \mapsto \pi_0 = \begin{pmatrix} n & n-1 & \cdots & n-2p+2 & n-2p+1 & k & \cdots & 1 \\ \beta_p' & \alpha_p' & \cdots & \beta_1' & \alpha_1' & j_k' & \cdots & j_1' \end{pmatrix} \in \mathfrak{S}_{n,p} \,.$$

Since $\operatorname{sgn} \pi' = \operatorname{sgn} \pi_0'$ we have inserting (24) in (23) that all vectors with particle number less than n cancel so that

$$S(P\Gamma q_n \wedge \ldots \wedge P\Gamma q_1) = Pq_1 \wedge \ldots \wedge Pq_n$$

and the proof is concluded. ■

Recall from Remark 4.11 (i) $S\lceil\mathfrak{p} \supseteq \beta$ and $S^*\lceil\mathfrak{p} \supseteq \alpha$. We will show next that actually the equality holds. Let P_n denote the projection of $\mathfrak{F}$ onto the n–particle subspace $\overset{n}{\wedge}\mathfrak{p}$. Then the family of orthoprojections $\{P_n\}_{n\in\mathbb{N}}$ is mutually orthogonal and $\sum_{n=0}^{\infty} P_n = \mathbb{1}_{\mathfrak{F}}$. Further we define the operator S_{fin} by

$$\begin{aligned} \operatorname{dom} S_{\mathrm{fin}} &:= \operatorname{span}\Big\{a(q_1)\cdot\ldots\cdot a(q_n)\Omega \mid q_1,\ldots,q_n \in \mathfrak{q}\,,\ n \in \mathbb{N}\cup\{0\}\Big\} \\ S_{\mathrm{fin}} x &:= Sx\,, \quad x \in \operatorname{dom} S_{\mathrm{fin}}\,. \end{aligned}$$

5.2 Lemma $\operatorname{dom} S_{\mathit{fin}}$ *is a core for the Tomita operator* S.

Proof: Put $\mathcal{C} := \operatorname{dom} S_{\mathrm{fin}}$ and denote by $\mathcal{A}(\mathfrak{q})$ the $*$–algebra generated by $\{a(q) \mid q \in \mathfrak{q}\}$, so that $\mathcal{M}(\mathfrak{q}) = \mathcal{A}(\mathfrak{q})''$. Further recall that if $S_0(M\Omega) := M^*\Omega$, $M \in \mathcal{M}(\mathfrak{q})$, then the graph of the Tomita operator $S = \operatorname{clo} S_0$ can be written as

$$\operatorname{gra} S = \operatorname{clo}_w(\operatorname{gra} S_0) = \operatorname{clo}_w\Big\{(M\Omega, M^*\Omega) \mid M \in \mathcal{M}(\mathfrak{q})\Big\}\,,$$

where clo_w denotes the closure in the weak operator topology. Thus to prove the core property of $\mathcal{C}$, i.e. $\operatorname{clo}(S\lceil\mathcal{C}) = S$, we need to show that

$$\operatorname{clo}_w\Big\{(M\Omega, M^*\Omega) \mid M \in \mathcal{M}(\mathfrak{q})\Big\} = \operatorname{clo}_w\Big\{(A\Omega, A^*\Omega) \mid A \in \mathcal{A}(\mathfrak{q})\Big\}\,.$$

Now for each $M \in \mathcal{M}(\mathfrak{q})$ there exists a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}(\mathfrak{q})$ with $|\langle x, (M - A_n)y\rangle| < \epsilon$ for all $x, y \in \mathfrak{F}$ if n is sufficiently large. This implies that for all $(x,y) \in \mathfrak{F}\times\mathfrak{F}$, we have

$$\Big|\Big\langle (M\Omega, M^*\Omega), (x,y)\Big\rangle - \Big\langle (A_n\Omega, A_n^*\Omega), (x,y)\Big\rangle\Big| = \Big|\Big\langle (M - A_n)\Omega, x\Big\rangle - \Big\langle (M^* - A_n^*)\Omega, y\Big\rangle\Big| \leq 2\epsilon\,,$$

where we have used that in the weak operator topology if $A_n \to M$, then $A_n^* \to M^*$. Therefore if $(M_l\Omega, M_l^*\Omega) \xrightarrow{l} (x_0, y_0)$ weakly, then for each l there exists a sequence $\{A_{(n,l)}\}_{n\in\mathbb{N}} \subset \mathcal{A}(\mathfrak{q})$ such that $(A_{(n,l)}\Omega, A_{(n,l)}^*\Omega) \xrightarrow{n} (M_l\Omega, M_l^*\Omega)$ weakly. Finally, the estimate

$$\begin{aligned} &\Big|\Big\langle (A_{(n,l)}\Omega, A_{(n,l)}^*\Omega), (x,y)\Big\rangle - \Big\langle (x_0,y_0), (x,y)\Big\rangle\Big| \\ &\qquad\leq \Big|\Big\langle (A_{(n,l)}\Omega, A_{(n,l)}^*\Omega), (x,y)\Big\rangle - \Big\langle (M_l\Omega, M_l^*\Omega), (x,y)\Big\rangle\Big| \\ &\qquad\quad + \Big|\Big\langle (M_l\Omega, M_l^*\Omega), (x,y)\Big\rangle - \Big\langle (x_0,y_0), (x,y)\Big\rangle\Big| \end{aligned}$$

completes the proof. ■

From Propositions 5.1 and 2.2 we obtain further

$$\begin{aligned} P_n \operatorname{dom} S_{\mathrm{fin}} &\subseteq \operatorname{dom} S_{\mathrm{fin}}\,, \quad n = 0,1,\ldots\,, \\ P_n \operatorname{dom} S_{\mathrm{fin}} &\subset P_n\mathfrak{F} \text{ is dense}\,, \quad n = 0,1,\ldots\,, \\ P_n S_{\mathrm{fin}} P_n &= S_{\mathrm{fin}} P_n\,, \quad n = 0,1,\ldots\,. \end{aligned}$$

Note that for $x \in \operatorname{dom} S_{\rm fin}$ the series $x = \sum_{n=0}^{\infty} P_n x$ is a finite sum. Further we consider the operators $S_{\rm fin}(n) : P_n\mathfrak{F} \to P_n\mathfrak{F}$ by $\operatorname{dom} S_{\rm fin}(n) := P_n \operatorname{dom} S_{\rm fin}$ and $S_{\rm fin}(n)x := S_{\rm fin}x \,,\, x \in \operatorname{dom} S_{\rm fin}(n)$. Recall from Lemma 4.10 that $S_{\rm fin}(1) = \beta$ is a closed densely defined operator from $\mathfrak{p} = P_1\mathfrak{F}$ into $\mathfrak{p}$ and by the preceding arguments we have

$$S_{\rm fin} = \mathop{\oplus}_{n=0}^{\infty} S_{\rm fin}(n) \qquad \text{(algebraic direct sum)}. \tag{25}$$

We can now state:

5.3 Proposition *The Tomita operator S and its adjoint S^* can be restricted to the n–particles subspaces $P_n\mathfrak{F}$. We have*

$$S\lceil P_n\mathfrak{F} = \operatorname{clo} S_{\mathit{fin}}(n) \quad \text{and} \quad S^*\lceil P_n\mathfrak{F} = \operatorname{clo} S^*_{\mathit{fin}}(n)\,, \quad n = 0, 1, \ldots .$$

In particular $S\lceil\mathfrak{p} = \beta$ and $S^*\lceil\mathfrak{p} = \alpha$.

Proof: From (25) we obtain immediately that

$$\operatorname{gra} S = \mathop{\oplus}_{n=0}^{\infty} \operatorname{clo} \operatorname{gra} S_{\rm fin}(n) = \mathbb{C}\Omega \oplus \operatorname{gra} S_{\rm fin}(1) \oplus \sum_{n=2}^{\infty} \operatorname{clo} \operatorname{gra} S_{\rm fin}(n) \qquad \text{(Hilbert sum)}. \tag{26}$$

Note that $\operatorname{clo} \operatorname{gra} S_{\rm fin}(n) = \operatorname{gra} \operatorname{clo} S_{\rm fin}(n)$, $n \geq 2$ and that now the direct sum (26), in contrast to (25), is the Hilbert sum of these subspaces. Eq. (26) implies immediately the assertion that S can be restricted to $P_n\mathfrak{F}$ and that the restriction coincides with $\operatorname{clo} S_{\rm fin}(n)$, in particular $S\lceil\mathfrak{p} = \beta$. The statements concerning S^* are shown analogously. ■

5.4 Corollary *Let $S = J\Delta^{\frac{1}{2}}$ be the polar decomposition of the Tomita operator. The modular operator $\Delta = S^*S$ and the modular conjugation J can be restricted to the respective n–particle subspaces. In particular we have:*

(i) Modular operator: $\Delta\lceil\mathfrak{p} = \Delta_{\mathfrak{p}}$ (recall from the preceding section that $\Delta_{\mathfrak{p}} = \beta^*\beta$), $\operatorname{dom} \Delta\lceil P_n\mathfrak{F} = \overset{n}{\wedge} \operatorname{dom} \Delta_{\mathfrak{p}}$ and

$$\Delta(p_1 \wedge \ldots \wedge p_n) = (\Delta_{\mathfrak{p}}\, p_1) \wedge \ldots \wedge (\Delta_{\mathfrak{p}}\, p_n)\,, \quad p_1, \ldots, p_n \in \operatorname{dom} \Delta_{\mathfrak{p}} = PQ\mathfrak{p}\,. \tag{27}$$

(ii) Modular conjugation:

$$J(p_1 \wedge \ldots \wedge p_n) = (Jp_n) \wedge \ldots \wedge (Jp_1)\,, \quad p_1, \ldots, p_n \in \mathfrak{p}\,. \tag{28}$$

Proof: Note first that from Proposition 5.3 and Lemma 4.10 we have $S^*S\lceil\mathfrak{p} = \alpha\beta = \beta^*\beta$. Further Eq. (27) follows from Propositions 5.1 and 5.3. Next, applying formula (28) to the n–particle vectors $\Delta^{\frac{1}{2}}(p_1\wedge\ldots\wedge p_n)$, where $p_k := \Delta_{\mathfrak{p}}^{\frac{1}{2}}\, p'_k$, $p'_k \in \operatorname{dom} \Delta_{\mathfrak{p}}$, $k = 1, \ldots, n$, we obtain $S(p_1\wedge\ldots\wedge p_n)$. Thus Eq. (28) coincides with $J\lceil P_n\mathfrak{F}$ on the dense set $(\operatorname{ima} \Delta_{\mathfrak{p}}) \wedge \ldots \wedge (\operatorname{ima} \Delta_{\mathfrak{p}})$, which concludes the proof. ■

For the next definition recall the formulas concerning $\operatorname{sgn} \varphi$ in Theorem 4.15.

5.5 Definition *Define the following anti–linear isometry from $\mathfrak{q}$ onto $\mathfrak{q}^\perp$:*

$$Vq := -i(\Gamma \operatorname{sgn} \varphi)(q) = i\Big(\Delta_{\mathfrak{p}}^{\frac{1}{2}} P\Gamma q - \Gamma\, \Delta_{\mathfrak{p}}^{\frac{1}{2}} Pq\Big)\,, \quad q \in \mathfrak{q}\,.$$

5.6 Theorem *With the preceding definition we have for any $q \in \mathfrak{q}$*

$$J\, a(q)\, J = \widetilde{Z}\, a(Vq)\, \widetilde{Z}^* . \tag{29}$$

Proof: Since $\overset{\infty}{\underset{n=0}{\oplus}} \overset{n}{\wedge} P\mathfrak{q}$ (algebraic direct sum) is dense in $\mathfrak{F}$ and the operators on both sides of Eq. (29) are bounded it is enough to show the preceding relation for the n–particle vectors in $\overset{n}{\wedge} P\mathfrak{q}$. For $q_1, \ldots, q_n \in \mathfrak{q}$ and using Corollary 5.4 (ii) the l.h.s. of the equation reads

$$\begin{aligned}
&\Big(J\, a(q)\, J\Big)(Pq_1 \wedge \ldots \wedge Pq_n) \\
&\quad = \quad Pq_1 \wedge \ldots \wedge Pq_n \wedge JP\Gamma q + \sum_{r=n}^{1} (-1)^{n-r} \langle JPq, Pq_r \rangle \; Pq_1 \wedge \ldots \wedge \widehat{Pq_r} \wedge \ldots \wedge Pq_n .
\end{aligned}$$

To compute the r.h.s. recall the definition of the function η in Subsection 2.2.

$$\begin{aligned}
&\Big(\widetilde{Z}\, a(Vq)\, \widetilde{Z}^*\Big)(Pq_1 \wedge \ldots \wedge Pq_n) \\
&\quad = \quad -i\overline{\eta(n)}\, \widetilde{Z} \cdot \Big(- c(\Delta_{\mathfrak{p}}^{\frac{1}{2}} Pq)^* Pq_1 \wedge \ldots \wedge Pq_n + c(\Delta_{\mathfrak{p}}^{\frac{1}{2}} P\Gamma q) Pq_1 \wedge \ldots \wedge Pq_n \Big) \\
&\quad = \quad Pq_1 \wedge \ldots \wedge Pq_n \wedge \Delta_{\mathfrak{p}}^{\frac{1}{2}} Pq + \sum_{r=1}^{n} (-1)^{n-r} \langle \Delta_{\mathfrak{p}}^{\frac{1}{2}} P\Gamma q, Pq_r \rangle \; Pq_1 \wedge \ldots \wedge \widehat{Pq_r} \wedge \ldots \wedge Pq_n ,
\end{aligned}$$

where for the last equation we have used that $-i\overline{\eta(n)}\eta(n+1) = -i\overline{\eta(n)}\eta(n-1) = (-1)^{n+1}$. Finally, the equality of both sides follows from the fact that $\Delta_{\mathfrak{p}}^{\frac{1}{2}} Pq = JSPq = JP\Gamma q$, $q \in \mathfrak{q}$. ■

5.7 Remark As expected we can relate the mapping V defined before with the mapping j (the modular conjugation restricted to the one–particle Hilbert space) used in [20, p. 738]. Indeed, considering the projection onto the one–particle Hilbert space we have the relation

$$J(Pq) = \Delta_{\mathfrak{p}}^{\frac{1}{2}} S(Pq) = \Delta_{\mathfrak{p}}^{\frac{1}{2}} (P\Gamma q) = -iPV(q) , \quad q \in \mathfrak{q} .$$

5.8 Theorem *(Twisted Duality) Let $\mathcal{M}(\mathfrak{q})$ be the von Neumann algebra given at the beginning of this section. Then*

$$\mathcal{M}(\mathfrak{q})' = \widetilde{Z}\, \mathcal{M}(\mathfrak{q}^\perp)\, \widetilde{Z}^* .$$

Proof: From the last theorem and using standard results in Modular Theory we have

$$\begin{aligned}
\mathcal{M}(\mathfrak{q})' &= J\, \mathcal{M}(\mathfrak{q})\, J \;=\; J\, \{a(q) \mid q \in \mathfrak{q}\}''\, J \\
&= \{J\, a(q)\, J \mid q \in \mathfrak{q}\}'' \\
&= \{\widetilde{Z}\, a(Vq)\, \widetilde{Z}^* \mid q \in \mathfrak{q}\}'' \;=\; \widetilde{Z}\, \{a(Vq) \mid q \in \mathfrak{q}\}''\, \widetilde{Z}^* \\
&= \widetilde{Z}\, \mathcal{M}(\mathfrak{q}^\perp)\, \widetilde{Z}^* ,
\end{aligned}$$

where we have used that V is an anti–linear isometry from $\mathfrak{q}$ onto $\mathfrak{q}^\perp$. ■

5.9 Remark If one does not want to bother about domain questions, there is an alternative way to show the preceding result. Indeed, one can first prove the statements in this section for finite dimensional subspaces $\mathfrak{q}_n$ of $\mathfrak{q}$ and then apply the AFD–property of $\mathcal{M}(\mathfrak{q})$ as in the proof of [9, Theorem 15.1.3].

6 Twisted duality. The abelian case

We will prove in the present section the twisted duality condition in the other extremal case that may appear in the Halmos decomposition (9). Let $(\mathfrak{h},\Gamma)$ and P,Q with $\Gamma P\Gamma = P^\perp$ and $\Gamma Q = Q\Gamma$ be given as in Subsection 2.2 and consider the complementary situation to the generic position case, namely

$$PQ = QP$$

(i.e. in the notation of Section 3 we have now that $\mathfrak{h}_1 = \{0\}$). In this situation we may now consider the following restrictions to $\mathfrak{q} = Q\mathfrak{h}$ and $\mathfrak{q}^\perp = Q^\perp\mathfrak{h}$:

$$P_{\mathfrak{q}} := P\lceil\mathfrak{q}\,,\ \Gamma_{\mathfrak{q}} := \Gamma\lceil\mathfrak{q}\,,\ P_{\mathfrak{q}^\perp} := P\lceil\mathfrak{q}^\perp\,,\ \Gamma_{\mathfrak{q}^\perp} := \Gamma\lceil\mathfrak{q}^\perp\,.$$

Note that $P_{\mathfrak{q}}$ and $P_{\mathfrak{q}^\perp}$ are basis projections for $(\mathfrak{q},\Gamma_{\mathfrak{q}})$ and $(\mathfrak{q}^\perp,\Gamma_{\mathfrak{q}^\perp})$, respectively. We denote the corresponding Fock representations by $\pi_{\mathfrak{q}}$ and $\pi_{\mathfrak{q}^\perp}$ which act on the anti-symmetric Fock spaces $\mathfrak{F}_{\mathfrak{q}}$ resp. $\mathfrak{F}_{\mathfrak{q}^\perp}$ over the one–particle spaces $\mathfrak{p}\cap\mathfrak{q}$ resp. $\mathfrak{p}\cap\mathfrak{q}^\perp$. Therefore we can now directly apply Proposition 2.4 to the following decomposition of the reference space $\mathfrak{h} = \mathfrak{q}\oplus\mathfrak{q}^\perp$.

6.1 Theorem *(Twisted Duality) Let $(\mathfrak{h},\Gamma)$ and P,Q with $QP = PQ$ be given as above. Further let $\mathcal{M}(\mathfrak{q})$ and $\widetilde{Z}$ be the von Neumann algebra and the twist operator specified in Subsection 2.2. Then*

$$\mathcal{M}(\mathfrak{q})' = \widetilde{Z}\,\mathcal{M}(\mathfrak{q}^\perp)\,\widetilde{Z}^*\,. \tag{30}$$

Proof: Note first that from Proposition 2.4 and the irreducibility of the Fock representation we have

$$\mathcal{M}(\mathfrak{q}) = \{\pi(a(q))\mid q\in\mathfrak{q}\}'' = \{\pi_{\mathfrak{q}}(a(q))\otimes\mathbb{1}\mid q\in\mathfrak{q}\}'' = \mathcal{L}(\mathfrak{F}_{\mathfrak{q}})\otimes\mathbb{C}\mathbb{1}\,.$$

Therefore using again standard results from [25, Section 11.2] we obtain for the l.h.s. of (30)

$$\mathcal{M}(\mathfrak{q})' = \mathbb{C}\mathbb{1}\otimes\mathcal{L}(\mathfrak{F}_{\mathfrak{q}^\perp})\,.$$

We show now that the same result is true for the r.h.s. of (30). Recall from Proposition 2.4 that the implementer Z of the even–oddness automorphism of CAR$(\mathfrak{h},\Gamma)$ on $\mathfrak{F}_{\mathfrak{q}}\otimes\mathfrak{F}_{\mathfrak{q}^\perp}$ is just given by $Z = Z_{\mathfrak{q}}\otimes Z_{\mathfrak{q}^\perp}$, where $Z_{\mathfrak{q}}$ and $Z_{\mathfrak{q}^\perp}$ implement this automorphism of the smaller algebras. Therefore we get

$$\begin{aligned}\widetilde{Z}\,\mathcal{M}(\mathfrak{q}^\perp)\,\widetilde{Z}^* &= \left\{Z\pi(a(q^\perp))\mid q^\perp\in\mathfrak{q}^\perp\right\}'' = \left\{\left(Z_{\mathfrak{q}}\otimes Z_{\mathfrak{q}^\perp}\right)\left(Z_{\mathfrak{q}}\otimes\pi_{\mathfrak{q}^\perp}(a(q^\perp))\right)\mid q^\perp\in\mathfrak{q}^\perp\right\}''\\ &= \mathbb{C}\mathbb{1}\otimes\left(\widetilde{Z_{\mathfrak{q}^\perp}}\,\{\pi_{\mathfrak{q}^\perp}(a(q^\perp))\mid q^\perp\in\mathfrak{q}^\perp\}''\,\widetilde{Z_{\mathfrak{q}^\perp}}^*\right) = \mathbb{C}\mathbb{1}\otimes\mathcal{L}(\mathfrak{F}_{\mathfrak{q}^\perp})\,,\end{aligned}$$

where $\widetilde{Z_{\mathfrak{q}^\perp}}$ is the twist operator on $\mathfrak{F}_{\mathfrak{q}^\perp}$ and we have used again the irreducibility of the Fock representation. ∎

7 Twisted duality. The general case

We are now in a position to give the proof of twisted duality in the most general situation. Let $(\mathfrak{h},\Gamma)$ be a Hilbert space with anti–unitary involution Γ, P *any* basis projection and $\mathfrak{q}$ *any* closed Γ–invariant subspace in $\mathfrak{h}$, to which we associate the orthoprojection Q. From Lemma 3.1 and Remark 3.2 we can consider the following restrictions w.r.t. the Halmos decompostion $\mathfrak{h} = \mathfrak{h}_0\oplus\mathfrak{h}_1$ given in Eq. (9):

$$\mathfrak{q}_k := Q\lceil\mathfrak{h}_k \quad\text{and}\quad \mathfrak{q}_k^\perp := Q^\perp\lceil\mathfrak{h}_k\,,$$

with $\mathfrak{h}_k = \mathfrak{q}_k \oplus \mathfrak{q}_k^\perp$, $k = 0, 1$. Now using the notation of Remark 3.2 and the twisted duality results in the generic position and the abelian situations (Sections 5 and 6, respectively) we have already shown the following equalities:

$$\begin{aligned} \mathcal{M}(\mathfrak{q}_0)' &= \widetilde{Z_0}\,\mathcal{M}(\mathfrak{q}_0^\perp)\,\widetilde{Z_0}^* \subset \mathcal{L}(\mathfrak{F}_0) && (31)\\ \mathcal{M}(\mathfrak{q}_1)' &= \widetilde{Z_1}\,\mathcal{M}(\mathfrak{q}_1^\perp)\,\widetilde{Z_1}^* \subset \mathcal{L}(\mathfrak{F}_1)\,, && (32)\end{aligned}$$

where $\mathfrak{F}_k$ are the Fock spaces given by $P_k := P\lceil\mathfrak{h}_k$ on which the Fock representations π_k of $\mathrm{CAR}(\mathfrak{h}_k, \Gamma_k)$, $k = 0, 1$, act. $\widetilde{Z_0}$ and $\widetilde{Z_1}$ are the twist operators defined in (7) from the corresponding implementers Z_0 and Z_1 of the even–oddness automorphisms in the respective Fock spaces. To be able to apply the preceding equations to the present most general case we will write the Fock representation π of $\mathrm{CAR}(\mathfrak{h}, \Gamma)$ associated to the basis projection P over the space $\mathfrak{F}_0 \otimes \mathfrak{F}_1$ as in Proposition 2.4.

The proof of our main theorem later on will be based on the computation of the commutants given in the following two lemmas.

7.1 Lemma *With the preceding notation we have*

$$\begin{aligned} \Big\{\pi(a(q_0)) \mid q_0 \in \mathfrak{q}_0\Big\}' &= \widetilde{Z_0}\,\mathcal{M}(\mathfrak{q}_0^\perp)\,\widetilde{Z_0}^*\ \overline{\otimes}\ \mathcal{L}(\mathfrak{F}_1)\\ &= \left\{\begin{pmatrix} E_0^+ M E_0^+ & iE_0^+ M E_0^- \\ -iE_0^- M E_0^+ & E_0^- M E_0^- \end{pmatrix} \mid M \in \mathcal{M}(\mathfrak{q}_0^\perp)\right\}\ \overline{\otimes}\ \mathcal{L}(\mathfrak{F}_1)\,. \end{aligned}$$

Proof: Using Proposition 2.4 and standard relations for the tensor product of von Neumann algebras [25, Section 11.2] we get

$$\begin{aligned} \Big\{\pi(a(q_0)) \mid q_0 \in \mathfrak{q}_0\Big\}' &= \Big\{\pi_0(a(q_0)) \otimes \mathbb{1} \mid q_0 \in \mathfrak{q}_0\Big\}' = \Big\{\pi_0(a(q_0)) \mid q_0 \in \mathfrak{q}_0\Big\}'\ \overline{\otimes}\ \mathcal{L}(\mathfrak{F}_1)\\ &= \mathcal{M}(\mathfrak{q}_0)'\ \overline{\otimes}\ \mathcal{L}(\mathfrak{F}_1) = \widetilde{Z_0}\,\mathcal{M}(\mathfrak{q}_0^\perp)\,\widetilde{Z_0}^*\ \overline{\otimes}\ \mathcal{L}(\mathfrak{F}_1)\,, \end{aligned}$$

where for the last equation we have used Eq. (31).

Finally, the last equation is obtained writing $\widetilde{Z_0} = E_0^+ - iE_0^-$ (cf. Eq. (8)) in the first term of the preceding tensor product. ■

For the calculation of the other commutant needed, take into account the decomposition $\mathfrak{F}_0 = \mathfrak{F}_0^+ \oplus \mathfrak{F}_0^-$, where $\mathfrak{F}_0^\pm := E_0^\pm \mathfrak{F}_0$ and $Z_0 = E_0^+ - E_0^-$ (cf. Eq. (4)). In particular any $C \in \mathcal{L}(\mathfrak{F}_0 \otimes \mathfrak{F}_1) \cong \mathcal{L}(\mathfrak{F}_0^+ \otimes \mathfrak{F}_1 \oplus \mathfrak{F}_0^- \otimes \mathfrak{F}_1)$ can be written as

$$\begin{pmatrix} C_{++} & C_{+-} \\ C_{-+} & C_{--} \end{pmatrix}\,, \quad \text{where } C_{jk} \in \mathcal{L}(\mathfrak{F}_0^k \otimes \mathfrak{F}_1 \to \mathfrak{F}_0^j \otimes \mathfrak{F}_1)\,,\ j,k = +,-\,.$$

7.2 Lemma *With the preceding conventions we have*

$$\{\pi(a(q_1)) \mid q_1 \in \mathfrak{q}_1\}' = \begin{pmatrix} \mathcal{L}(\mathfrak{F}_0^+)\ \overline{\otimes}\ \widetilde{Z_1}\mathcal{M}(\mathfrak{q}_1^\perp)\widetilde{Z_1}^* & \mathcal{L}(\mathfrak{F}_0^- \to \mathfrak{F}_0^+)\ \overline{\otimes}\ (-i\widetilde{Z_1}\mathcal{M}(\mathfrak{q}_1^\perp)\widetilde{Z_1}) \\ \mathcal{L}(\mathfrak{F}_0^+ \to \mathfrak{F}_0^-)\ \overline{\otimes}\ (i\widetilde{Z_1}^*\mathcal{M}(\mathfrak{q}_1^\perp)\widetilde{Z_1}^*) & \mathcal{L}(\mathfrak{F}_0^-)\ \overline{\otimes}\ \widetilde{Z_1}\mathcal{M}(\mathfrak{q}_1^\perp)\widetilde{Z_1}^* \end{pmatrix}\,.$$

Proof: First note that from Corollary 2.5, $\pi(a(q_1))$ can be written as a diagonal operator

$$\pi(a(q_1)) = \begin{pmatrix} \mathbb{1}_+ \otimes \pi_1(a(q_1)) & 0 \\ 0 & \mathbb{1}_- \otimes Z_1\pi_1(a(q_1))Z_1 \end{pmatrix}\,, q_1 \in \mathfrak{q}_1\,,$$

where $\mathbb{1}_\pm$ are the unit operators on $\mathfrak{F}_0^\pm$. Therefore, with the notation above we have that $C \in \{\pi(a(q_1)) \mid q_1 \in \mathfrak{q}_1\}'$ iff

$$\begin{aligned}
&\begin{pmatrix} C_{++} & C_{+-} \\ C_{-+} & C_{--} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1}_+ \otimes \pi_1(a(q_1)) & 0 \\ 0 & \mathbb{1}_- \otimes Z_1\pi_1(a(q_1))Z_1 \end{pmatrix} \\
&\quad = \begin{pmatrix} \mathbb{1}_+ \otimes \pi_1(a(q_1)) & 0 \\ 0 & \mathbb{1}_- \otimes Z_1\pi_1(a(q_1))Z_1 \end{pmatrix} \cdot \begin{pmatrix} C_{++} & C_{+-} \\ C_{-+} & C_{--} \end{pmatrix}, \quad q_1 \in \mathfrak{q}_1 .
\end{aligned}$$

We can now rewrite the preceding equation as

$$C_{++}(\mathbb{1}_+ \otimes \pi_1(a(q_1))) = (\mathbb{1}_+ \otimes \pi_1(a(q_1)))C_{++}, \quad q_1 \in \mathfrak{q}_1, \tag{33}$$

$$C_{--}(\mathbb{1}_- \otimes \pi_1(a(q_1))) = (\mathbb{1}_- \otimes \pi_1(a(q_1)))C_{--}, \quad q_1 \in \mathfrak{q}_1, \tag{34}$$

$$C_{+-}(\mathbb{1}_- \otimes Z_1\pi_1(a(q_1))Z_1) = (\mathbb{1}_+ \otimes \pi_1(a(q_1)))C_{+-}, \quad q_1 \in \mathfrak{q}_1, \tag{35}$$

$$C_{-+}(\mathbb{1}_+ \otimes \pi_1(a(q_1))) = (\mathbb{1}_- \otimes Z_1\pi_1(a(q_1))Z_1)C_{-+}, \quad q_1 \in \mathfrak{q}_1 . \tag{36}$$

Eqs. (33) and (34) imply simply that

$$\begin{aligned}
C_{++} &\in \Big(\{\mathbb{1}_+\} \otimes \{\pi_1(a(q_1))\}\Big)' = \{\mathbb{1}_+\}' \,\overline{\otimes}\, \{\pi_1(a(q_1))\}' = \mathcal{L}(\mathfrak{F}_0^+) \,\overline{\otimes}\, \widetilde{Z_1}\,\mathcal{M}(\mathfrak{q}_1^\perp)\,\widetilde{Z_1}^* \\
C_{--} &\in \Big(\{\mathbb{1}_-\} \otimes \{\pi_1(a(q_1))\}\Big)' = \{\mathbb{1}_-\}' \,\overline{\otimes}\, \{\pi_1(a(q_1))\}' = \mathcal{L}(\mathfrak{F}_0^-) \,\overline{\otimes}\, \widetilde{Z_1}\,\mathcal{M}(\mathfrak{q}_1^\perp)\,\widetilde{Z_1}^*,
\end{aligned}$$

where we have used Eq. (32). To obtain C_{+-} we need an isometry $K\colon \mathfrak{F}_0^+ \to \mathfrak{F}_0^-$. Multiplying Eq. (35) from the right by $(K \otimes Z_1)$ and using $\mathbb{1}_- K = K\mathbb{1}_+$ we obtain

$$D_{+-}(\mathbb{1}_+ \otimes \pi_1(a(q_1))) = (\mathbb{1}_+ \otimes \pi_1(a(q_1)))D_{+-}, \quad q_1 \in \mathfrak{q}_1,$$

where $D_{+-} := C_{+-}(K \otimes Z_1) \in \mathcal{L}(\mathfrak{F}_0^+) \,\overline{\otimes}\, \mathcal{L}(\mathfrak{F}_1)$. This implies as before that $D_{+-} \in \mathcal{L}(\mathfrak{F}_0^+) \,\overline{\otimes}\, \widetilde{Z_1}\,\mathcal{M}(\mathfrak{q}_1^\perp)\,\widetilde{Z_1}^*$. Now from the definition of D_{+-} and from the relation $\widetilde{Z_1}^* Z_1 = -i\widetilde{Z_1}$ we get finally

$$C_{+-} \in \mathcal{L}(\mathfrak{F}_0^- \to \mathfrak{F}_0^+) \,\overline{\otimes}\, (-i\widetilde{Z_1}\mathcal{M}(\mathfrak{q}_1^\perp)\widetilde{Z_1}).$$

In a similar way we obtain the corresponding relation for C_{-+}. ■

7.3 Theorem *(Twisted Duality) Let* $(\mathfrak{h}, \Gamma)$*,* P *and* $\mathfrak{q}$ *be given as in the beginning of this section. Then*

$$\mathcal{M}(\mathfrak{q})' = \widetilde{Z}\,\mathcal{M}(\mathfrak{q}^\perp)\,\widetilde{Z}^* .$$

Proof: From Proposition 2.6 it is enough to show the inclusion

$$\mathcal{M}(\mathfrak{q})' \subseteq \widetilde{Z}\,\mathcal{M}(\mathfrak{q}^\perp)\,\widetilde{Z}^* . \tag{37}$$

Now, for the l.h.s. of the preceding inclusion we have

$$\begin{aligned}
\mathcal{M}(\mathfrak{q})' &= \{\pi(a(q_0 + q_1)) \mid q_0 + q_1 \in \mathfrak{q} = \mathfrak{q}_0 \oplus \mathfrak{q}_1\}' \\
&= \Big(\{\pi(a(q_0)) \mid q_0 \in \mathfrak{q}_0\} \cup \{\pi(a(q_1)) \mid q_1 \in \mathfrak{q}_1\}\Big)' \\
&= \{\pi(a(q_0)) \mid q_0 \in \mathfrak{q}_0\}' \cap \{\pi(a(q_1)) \mid q_1 \in \mathfrak{q}_1\}' \\
&= \left(\left\{\begin{pmatrix} E_0^+ M E_0^+ & iE_0^+ M E_0^- \\ -iE_0^- M E_0^+ & E_0^- M E_0^- \end{pmatrix} \,\Big|\, M \in \mathcal{M}(\mathfrak{q}_0^\perp)\right\} \,\overline{\otimes}\, \mathcal{L}(\mathfrak{F}_1)\right) \\
&\quad \cap \begin{pmatrix} \mathcal{L}(\mathfrak{F}_0^+) \,\overline{\otimes}\, \widetilde{Z_1}\mathcal{M}(\mathfrak{q}_1^\perp)\widetilde{Z_1}^* & \mathcal{L}(\mathfrak{F}_0^- \to \mathfrak{F}_0^+) \,\overline{\otimes}\, (-i\widetilde{Z_1}\mathcal{M}(\mathfrak{q}_1^\perp)\widetilde{Z_1}) \\ \mathcal{L}(\mathfrak{F}_0^+ \to \mathfrak{F}_0^-) \,\overline{\otimes}\, (i\widetilde{Z_1}^*\mathcal{M}(\mathfrak{q}_1^\perp)\widetilde{Z_1}^*) & \mathcal{L}(\mathfrak{F}_0^-) \,\overline{\otimes}\, \widetilde{Z_1}\mathcal{M}(\mathfrak{q}_1^\perp)\widetilde{Z_1}^* \end{pmatrix},
\end{aligned}$$

where for the last equation we have used Lemmas 7.1 and 7.2. Now applying [32, Corollary 5.10] to the preceding intersection of tensor products we obtain finally

$$\begin{aligned}
\mathcal{M}(\mathfrak{q})' &= \left(\left\{\begin{pmatrix} E_0^+ M E_0^+ & iE_0^+ M E_0^- \\ -iE_0^- M E_0^+ & E_0^- M E_0^- \end{pmatrix} \mid M \in \mathcal{M}(\mathfrak{q}_0^\perp)\right\} \cap \mathcal{L}(\mathfrak{F}_0)\right) \\
&\quad \overline{\otimes} \left(\mathcal{L}(\mathfrak{F}_1) \cap \begin{pmatrix} \widetilde{Z}_1 \mathcal{M}(\mathfrak{q}_1^\perp) \widetilde{Z}_1^{\,*} & (-i\widetilde{Z}_1 \mathcal{M}(\mathfrak{q}_1^\perp)\widetilde{Z}_1) \\ (i\widetilde{Z}_1^{\,*} \mathcal{M}(\mathfrak{q}_1^\perp)\widetilde{Z}_1^{\,*}) & \widetilde{Z}_1 \mathcal{M}(\mathfrak{q}_1^\perp) \widetilde{Z}_1^{\,*} \end{pmatrix}\right) \\
&= \mathrm{clo}_w \left\{\begin{pmatrix} E_0^+ M E_0^+ \otimes N_1 & iE_0^+ M E_0^- \otimes N_2 \\ -iE_0^- M E_0^+ \otimes N_3 & E_0^- M E_0^- \otimes N_4 \end{pmatrix} \mid M \in \mathcal{M}(\mathfrak{q}_0^\perp)\,, \right. \\
&\qquad\qquad \left. N_1, N_4 \in \widetilde{Z}_1 \mathcal{M}(\mathfrak{q}_1^\perp) \widetilde{Z}_1^{\,*}\,,\ N_2, N_3^* \in -i\widetilde{Z}_1 \mathcal{M}(\mathfrak{q}_1^\perp)\widetilde{Z}_1 \right\}\,.
\end{aligned} \tag{38}$$

To analyize the first factors of the previous tensor products recall Lemma 7.1 and that from Remark 2.8 we have

$$\widetilde{Z_0}\, \mathcal{M}(\mathfrak{q}_0^\perp)\, \widetilde{Z_0}^{\,*} = \left\{\pi(Y_{0,\mathrm{even}}) + iZ_0\pi(Y_{0,\mathrm{odd}})) \mid Y_0 = Y_{0,\mathrm{even}} + Y_{0,\mathrm{odd}} \in \mathrm{CAR}(\mathfrak{q}_0^\perp, \Gamma_0)\right\}''$$

as well as that Lemma 2.3 implies

$$\begin{aligned}
E_0^+ \pi_0(Y_{0,\mathrm{even}}) E_0^- &= E_0^- \pi_0(Y_{0,\mathrm{even}}) E_0^+ = 0 \\
Z_0 \pi_0(Y_{0,\mathrm{odd}}) &= E_0^+ \pi_0(Y_{0,\mathrm{odd}}) E_0^- - E_0^- \pi_0(Y_{0,\mathrm{odd}}) E_0^+\,.
\end{aligned}$$

Therefore, applying again Remark 2.8 to the terms $\widetilde{Z_1} \mathcal{M}(\mathfrak{q}_1^\perp) \widetilde{Z_1}^{\,*}$ (respectively $-i\widetilde{Z_1} \mathcal{M}(\mathfrak{q}_1^\perp)\widetilde{Z_1}$) we obtain that the tensor product (38) is generated by the elements

$$\pi_0(Y_{0,\mathrm{even}}) \otimes \Big(\pi_1(Y_{1,\mathrm{even}}) + iZ_1\pi_1(Y_{1,\mathrm{odd}})\Big) \text{ and } Z_0\pi_0(Y_{0,\mathrm{odd}}) \otimes \Big(iZ_1\pi_1(Y_{1,\mathrm{even}}) + \pi_1(Y_{1,\mathrm{odd}})\Big)\,, \tag{39}$$

where $Y_k = Y_{k,\mathrm{even}} + Y_{k,\mathrm{odd}} \in \mathrm{CAR}(\mathfrak{q}_k^\perp, \Gamma_k)$, $k = 0, 1$. Now the r.h.s. of the inclusion (37) can also be written using Proposition 2.4 and Corollary 2.5 as

$$\begin{aligned}
&\widetilde{Z}\, \mathcal{M}(\mathfrak{q}^\perp)\, \widetilde{Z}^* = \{\pi(Y_{\mathrm{even}}) + iZ\pi(Y_{\mathrm{odd}})) \mid Y_{\mathrm{even}} + Y_{\mathrm{odd}} \in \mathrm{CAR}(\mathfrak{q}^\perp, \Gamma)\}'' \\
&= \Big\{\pi_0(Y_{0,\mathrm{even}}) \otimes \mathbb{1} + \mathbb{1} \otimes \pi_1(Y_{1,\mathrm{even}}) + i(Z_0 \otimes Z_1)\Big(\pi_0(Y_{0,\mathrm{odd}}) \otimes \mathbb{1} + Z_0 \otimes \pi_1(Y_{1,\mathrm{odd}})\Big) \\
&\qquad \mid Y_{k,\mathrm{even}} + Y_{k,\mathrm{odd}} \in \mathrm{CAR}(\mathfrak{q}_k^\perp, \Gamma_k), k = 0, 1\Big\}'' \\
&= \Big\{\pi_0(Y_{0,\mathrm{even}}) \otimes \mathbb{1} + \mathbb{1} \otimes \pi_1(Y_{1,\mathrm{even}}) + iZ_0\pi_0(Y_{0,\mathrm{odd}}) \otimes Z_1 + i\mathbb{1} \otimes Z_1\pi_1(Y_{1,\mathrm{odd}}) \\
&\qquad \mid Y_{k,\mathrm{even}} + Y_{k,\mathrm{odd}} \in \mathrm{CAR}(\mathfrak{q}_k^\perp, \Gamma_k), k = 0, 1\Big\}''\,.
\end{aligned}$$

Thus to show the inclusion (37) we still need to check that the generators (39) are contained in $\widetilde{Z}\, \mathcal{M}(\mathfrak{q}^\perp)\, \widetilde{Z}^*$. Therefore the relations

$$\begin{aligned}
\pi_0(Y_{0,\mathrm{even}}) \otimes \pi_1(Y_{1,\mathrm{even}}) &= (\pi_0(Y_{0,\mathrm{even}}) \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes \pi_1(Y_{1,\mathrm{even}})) \in \widetilde{Z}\, \mathcal{M}(\mathfrak{q}^\perp)\, \widetilde{Z}^* \\
\pi_0(Y_{0,\mathrm{even}}) \otimes Z_1\pi_1(Y_{1,\mathrm{odd}}) &= (\pi_0(Y_{0,\mathrm{even}}) \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes Z_1\pi_1(Y_{1,\mathrm{odd}})) \in \widetilde{Z}\, \mathcal{M}(\mathfrak{q}^\perp)\, \widetilde{Z}^* \\
Z_0\pi_0(Y_{0,\mathrm{odd}}) \otimes Z_1\pi_1(Y_{1,\mathrm{even}}) &= (Z_0\pi_0(Y_{0,\mathrm{odd}}) \otimes Z_1) \cdot (\mathbb{1} \otimes \pi_1(Y_{1,\mathrm{even}})) \in \widetilde{Z}\, \mathcal{M}(\mathfrak{q}^\perp)\, \widetilde{Z}^* \\
Z_0\pi_0(Y_{0,\mathrm{odd}}) \otimes \pi_1(Y_{1,\mathrm{odd}}) &= (Z_0\pi_0(Y_{0,\mathrm{odd}}) \otimes Z_1) \cdot (\mathbb{1} \otimes Z_1\pi_1(Y_{1,\mathrm{odd}})) \in \widetilde{Z}\, \mathcal{M}(\mathfrak{q}^\perp)\, \widetilde{Z}^*
\end{aligned}$$

conclude the proof. ∎

Appendix

We will give in this appendix the proof of Proposition 2.2. Recall the notation and results of Section 2.

Proposition *For $f_1, \ldots, f_n \in \mathfrak{h}$ the equation*

$$\Big(a(f_n) \cdot \ldots \cdot a(f_1)\Big)\, \Omega = \sum_{\substack{\pi \,\in\, \mathfrak{S}_{n,\,p} \\ 0 \,\leq\, 2p \,\leq\, n}} (\operatorname{sgn} \pi) \prod_{l=1}^{p} \langle P f_{\alpha_l} \,,\, P\Gamma f_{\beta_l} \rangle \, P\Gamma f_{j_1} \wedge \ldots \wedge P\Gamma f_{j_k}$$

holds, where the indices $\alpha_l, \beta_l, j_1, \ldots, j_k$ are given in the defintion of $\mathfrak{S}_{n,p}$ and where for $n = 2p$ in the preceding sum one replaces the wedge product by the vacuum Ω.

Proof: The proof is done by induction on the number of generators of the CAR–Algebra. For $n = 1$ the above formula is immediately veryfied using the definition of creation and annihilation operators. Assume that it holds for n generators and we proof that it is also true for $n + 1$ generators. Take $f_{n+1}, f_n, \ldots, f_1 \in \mathfrak{h}$ and from the preceding assumption as well as the results stated in Section 2 we have

$$\begin{aligned}
& a(f_{n+1})\, \Big(a(f_n) \cdot \ldots \cdot a(f_1)\, \Omega\Big) \\
&\quad = \Big(c(P\Gamma f_{n+1})^* + c(P f_{n+1})\Big)\, \Big((a(f_n) \cdot \ldots \cdot a(f_1)\, \Omega\Big) \\
&\quad = \sum_{\substack{\pi \,\in\, \mathfrak{S}_{n,\,p} \\ 0 \,\leq\, 2p \,\leq\, n}} (\operatorname{sgn} \pi) \prod_{l=1}^{p} \langle P f_{\alpha_l} \,,\, P\Gamma f_{\beta_l} \rangle \, P\Gamma f_{n+1} \wedge P\Gamma f_{j_1} \wedge \ldots \wedge P\Gamma f_{j_k} \qquad (40) \\
&\qquad + \sum_{\substack{\pi \,\in\, \mathfrak{S}_{n,\,p} \\ 0 \,\leq\, 2p \,\leq\, n}} \sum_{r=1}^{k} (\operatorname{sgn} \pi)\, (-1)^{r-1} \, \langle P f_{n+1} \,,\, P\Gamma f_{j_r} \rangle \,\cdot \\
&\qquad \cdot \prod_{l=1}^{p} \langle P f_{\alpha_l} \,,\, P\Gamma f_{\beta_l} \rangle \, P\Gamma f_{j_1} \wedge \ldots \wedge \widehat{P\Gamma f_{j_r}} \wedge \ldots \wedge P\Gamma f_{j_k} \,.
\end{aligned}$$

We will determine how many terms with particle number k' appear in the preceding sum. For this let $p' \in \mathbb{N}$ be such that $2p' + k' = n + 1$. Now the first term in the above formula contributes by means of expressions where $k = k' - 1$ (hence $p = p'$) and there are

$$\binom{n}{n - 2p} \frac{(2p)!}{p!\, 2^p} = \binom{n}{n - 2p'} \frac{(2p')!}{p'!\, 2^{p'}}$$

such summands. Further, the second term contributes by means of expressions where $k = k' + 1$ (hence $p = p' - 1$) and there are now

$$k \binom{n}{n - 2p} \frac{(2p)!}{p!\, 2^p} = \binom{n}{n - 2p' + 1} \frac{(2p')!}{p'!\, 2^{p'}}$$

such summands. Altogether we obtain

$$\binom{n}{n - 2p'} \frac{(2p')!}{p'!\, 2^{p'}} + \binom{n}{n - 2p' + 1} \frac{(2p')!}{p'!\, 2^{p'}} = \binom{n+1}{n + 1 - 2p'} \frac{(2p')!}{p'!\, 2^{p'}}$$

terms with particle number k' and this coincides with the number of elements in $\mathfrak{S}_{n+1,\,p'}$. To conclude the proof we still need to show that each term in the sum carries the correct sign. For the summands in (40) this follows from

$$\begin{aligned}
&\operatorname{sgn}\begin{pmatrix} n & n-1 & \cdots & n-2p+2 & n-2p+1 & k & \cdots & 1 \\ \alpha_1 & \beta_1 & \cdots & \alpha_p & \beta_p & j_1 & \cdots & j_k \end{pmatrix} \\
&\qquad = \operatorname{sgn}\begin{pmatrix} n+1 & n & \cdots & 1 \\ n+1 & \alpha_1 & \cdots & j_k \end{pmatrix} \\
&\qquad = \operatorname{sgn}\begin{pmatrix} n+1 & n & \cdots & n-2p+3 & n-2p+2 & k+1 & k & \cdots & 1 \\ \alpha_1 & \beta_1 & \cdots & \alpha_p & \beta_p & n+1 & j_1 & \cdots & j_k \end{pmatrix}.
\end{aligned}$$

For the remaining terms we have

$$\begin{aligned}
&(-1)^{r-1}\operatorname{sgn}\begin{pmatrix} n & n-1 & \cdots & k & \dots & k+2-r & k+1-r & k-r & \cdots & 1 \\ \alpha_1 & \beta_1 & \cdots & j_1 & \cdots & j_{r-1} & j_r & j_{r+1} & \cdots & j_k \end{pmatrix} \\
&\qquad = (-1)^{r-1}\operatorname{sgn}\begin{pmatrix} n+1 & n & \cdots & 1 \\ n+1 & \alpha_1 & \cdots & j_k \end{pmatrix} \\
&\qquad = \operatorname{sgn}\begin{pmatrix} n+1 & n & n-1 & n-2 & \cdots & k & \cdots & k+1-r & k-r & \cdots & 1 \\ n+1 & j_r & \alpha_1 & \beta_1 & \cdots & j_1 & \cdots & j_{r-1} & j_{r+1} & \cdots & j_k \end{pmatrix}.
\end{aligned}$$

Therefore we have shown that

$$\Big(a(f_{n+1})\cdot\ldots\cdot a(f_1)\Big)\,\Omega = \sum_{\substack{\pi\,\in\,\mathfrak{S}_{n+1,\,p'} \\ 0\,\le\,2p'\,\le\,n+1}} (\operatorname{sgn}\pi)\prod_{l=1}^{p'}\langle Pf_{\alpha_l}\,,\,P\Gamma f_{\beta_l}\rangle\, P\Gamma f_{j_1}\wedge\ldots\wedge P\Gamma f_{j_k}$$

which concludes the proof. ■

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