Spectral decomposition for the Dirac system associated to the DSII equation

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Abstract

A new (scalar) spectral decomposition is found for the Dirac system in two dimensions associated to the focusing Davey–Stewartson II (DSII) equation. Discrete spectrum in the spectral problem corresponds to eigenvalues embedded into a two-dimensional essential spectrum. We show that these embedded eigenvalues are structurally unstable under small variations of the initial data. This instability leads to the decay of localized initial data into continuous wave packets prescribed by the nonlinear dynamics of the DSII equation.

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1 Introduction

Gravity-capillary surface wave packets are described by the Davey–Stewartson (DS) system [1] which is integrable by inverse scattering tranform in the limit of shallow water [2]. In this paper, we study the focusing DSII equation which can be written in a complex form,

$$\begin{aligned} iu_t + u_{zz} + u_{\bar{z}\bar{z}} + 4(g+\bar{g})u &= 0, \\ 2g_{\bar{z}} - \left(|u|^2\right)_z &= 0, \end{aligned} \tag{1.1}$$

where $z = x+iy$, $\bar{z} = x - iy$, $u(z,\bar{z},t)$ and $g(z,\bar{z},t)$ are complex functions. This equation appears as the compatibility condition for the two-dimensional Dirac system,

$$\varphi_{1\bar{z}} = -u\varphi_2, \quad \varphi_{2z} = \bar{u}\varphi_1, \tag{1.2}$$

coupled to the equations for the time evolution of the eigenfunctions,

$$\begin{aligned} i\varphi_{1t} + \varphi_{1zz} + u\varphi_{2\bar{z}} - u_{\bar{z}}\varphi_2 + 4g\varphi_1 &= 0, \\ -i\varphi_{2t} + \varphi_{2\bar{z}\bar{z}} + \bar{u}_z\varphi_1 - \bar{u}\varphi_{1z} + 4\bar{g}\varphi_2 &= 0. \end{aligned} \tag{1.3}$$

The DSII equation was solved formally through the $\bar{\partial}$ problem of complex analysis by Fokas and Ablowitz [3] and Beals and Coifman [4]. Rigorous results on existence and uniqueness of solutions of the initial-value problem were established under a small-norm assumption [5]. The small-norm assumption was used to eliminate homogeneous solutions of equations of the inverse scattering which correspond to bound states and radially symmetric localized waves (lumps) of the DSII equation. When the potential in the linear system becomes weakly localized (in L^2 but not in L^1), homogeneous solutions may exist and the analysis developed in Ref. [5] is not applicable.

The lump solutions were included formally in [3], where their weak decay rate was found, $u \sim \mathrm{O}(R^{-1})$ as $R = \sqrt{x^2+y^2} \to \infty$. This result is only valid for complexified solutions of the DSII equation (when $|u|^2$ is considered to be complex). The reality conditions were incorporated in the work of Arkadiev et al. [6] where lumps were shown to decay like $u \sim \mathrm{O}(R^{-2})$. Multi-lump solutions were expressed as a ratio of two determinants [6], or, in a special case, as a ratio of two polynomials [7] but their dynamical role was left out of consideration.

Recently, structural instability of a single lump of the DSII equation was reported by Gadyl'shin and Kiselev [8, 9]. The authors used methods of perturbation theory based on completeness of *squared* eigenfunctions of the Dirac system [10, 11]. A similar conclusion was announced by Yurov who studied Darboux transformation of the Dirac system [12, 13].

In this paper, we present an alternative solution of the problem of stability of multi-lump solutions of the DSII equation. The approach generalizes our recent work on spectral decomposition of a linear time-dependent Schrödinger equation with weakly localized (not in L^1) potentials [14]. We find a new spectral decomposition in terms of *single* eigenfunctions of the Dirac system. Surprisely enough, the two-component Dirac system in two dimensions has a scalar spectral decomposition.

In contrast, we recall that the Dirac system in one dimension (the so-called AKNS system) has a well-known 2 x 2 matrix spectral decomposition [15].

Using the scalar spectral decomposition, we associate the multi-lump potentials with eigenvalues embedded into a two-dimensional essential spectrum of the Dirac system. Eigenvalues embedded into a one-dimensional essential spectrum occur for instance for the time-dependent Schrödinger problem [14, 16]. They were found to be structurally unstable under a small variation of the potential. Depending on the sign of the variation, they either disappear or become resonant poles in the complex spectral plane which correspond to lump solutions of the KPI equation [14].

For the Dirac system in two dimensions, the multi-lump potentials and embedded eigenvalues are more exotic. The discrete spectrum of the Dirac system is separated from the continuous spectrum contribution in the sense that the spectral data satisfy certain constraints near the embedded eigenvalues. These constraints are met for special solutions of the DSII equation such as lumps, but may not be satisfied for a generic combination of lumps and radiative waves. As a result, embedded eigenvalues of the Dirac system generally disappear under a local disturbance of the initial data. Physically, this implies that a localized initial data of the DSII equation decays into radiation except for the cases where the data reduce to special solutions such as lumps.

The paper is organized as follows. Elements of inverse scattering for the Dirac system are reviewed in Section 2, where we find that the discrete spectrum of the Dirac system is prescribed by certain constraints on the spectral data. Spectral decomposition is described in Section 3 with the proof of orthogonality and completeness relations through a proper adjoint problem. The perturbation theory for lumps is developed in Section 4 where some of previous results [8, 9] are recovered. Section 5 contains concluding remarks. Appendix A provides a summary of formulas of the complex $\bar{\partial}$-analysis used in proofs of Section 3.

2 Spectral Data and Inverse Scattering

Here we review some results on the Dirac system (1.2) and discard henceforth the time dependence of u, g and $\boldsymbol{\varphi}$. The potential $u(z,\bar{z})$ is assumed to be non-integrable ($u \notin L^1$) with the boundary conditions, $u \sim \mathrm{O}(|z|^{-2})$ as $|z| \to \infty$.

2.1 Essential Spectrum of the Dirac System

We define the fundamental matrix solution of Eq. (1.2) in the form [6],

$$\boldsymbol{\varphi} = \left[\boldsymbol{\mu}(z,\bar{z},k,\bar{k}) e^{ikz}, \quad \boldsymbol{\chi}(z,\bar{z},k,\bar{k}) e^{-i\bar{k}\bar{z}} \right], \tag{2.1}$$

where k is a spectral parameter, $\boldsymbol{\mu}(z,\bar{z},k,\bar{k})$ and $\boldsymbol{\chi}(z,\bar{z},k,\bar{k})$ satisfy the system,

$$\mu_{1\bar{z}} = -u\mu_2, \qquad \mu_{2z} = -ik\mu_2 + \bar{u}\mu_1, \tag{2.2}$$

$$\chi_{1\bar{z}} = i\bar{k}\chi_1 - u\chi_2, \qquad \chi_{2z} = \bar{u}\chi_1. \tag{2.3}$$

It follows from Eqs. (2.2) and (2.3) that $\boldsymbol{\mu}$ and $\boldsymbol{\chi}$ are related by the symmetry constraint,

$$\boldsymbol{\chi}(z, \bar{z}, k, \bar{k}) = \boldsymbol{\sigma}\bar{\boldsymbol{\mu}}(z, \bar{z}, k, \bar{k}), \qquad \boldsymbol{\sigma} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

We impose the boundary conditions for $\boldsymbol{\mu}(z, \bar{z}, k, \bar{k})$,

$$\lim_{|k|\to\infty} \boldsymbol{\mu}(z, \bar{z}, k, \bar{k}) = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.5}$$

Solutions of Eq. (2.2) with boundary conditions (2.5) can be expressed through the Green's functions as Fredholm's inhomogeneous integral equations [3, 4],

$$\mu_1(z, \bar{z}, k, \bar{k}) = 1 - \frac{1}{2\pi i}\iint \frac{dz' \wedge d\bar{z}'}{z' - z}(u\mu_2)(z', \bar{z}'), \tag{2.6}$$

$$\mu_2(z, \bar{z}, k, \bar{k}) = \frac{1}{2\pi i}\iint \frac{dz' \wedge d\bar{z}'}{\bar{z}' - \bar{z}}(\bar{u}\mu_1)(z', \bar{z}')e^{-ik(z-z')-i\bar{k}(\bar{z}-\bar{z}')}. \tag{2.7}$$

Values of k for which the homogeneous system associated to Eqs. (2.6) and (2.7) has bounded solutions are called *eigenvalues* of the discrete spectrum of the Dirac system. Let us suppose that the homogeneous solutions (eigenvalues) are not supported by the potential $u(z, \bar{z})$. We evaluate the departure from analyticity of $\boldsymbol{\mu}$ in the k plane by calculating the derivative $\partial\boldsymbol{\mu}/\partial\bar{k}$ directly from the system (2.6)-(2.7) [3] as

$$\frac{\partial\boldsymbol{\mu}}{\partial\bar{k}} = b(k, \bar{k})\mathbf{N}_\mu(z, \bar{z}, k, \bar{k}). \tag{2.8}$$

Here $b(k, \bar{k})$ is the spectral data,

$$b(k, \bar{k}) = \frac{1}{2\pi}\iint dz \wedge d\bar{z}\,(\bar{u}\mu_1)(z, \bar{z})e^{i(kz+\bar{k}\bar{z})}, \tag{2.9}$$

and $\mathbf{N}_\mu(z, \bar{z}, k, \bar{k})$ is a solution of Eq. (2.2) which is linearly independent of $\boldsymbol{\mu}(z, \bar{z}, k, \bar{k})$ and satisfies the boundary condition,

$$\lim_{|k|\to\infty} \mathbf{N}_\mu(z, \bar{z}, k, \bar{k})e^{i(kz+\bar{k}\bar{z})} = \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.10}$$

This solution can be expressed through the Fredholm's inhomogeneous equations,

$$N_{1\mu}(z, \bar{z}, k, \bar{k}) = \frac{1}{2\pi i}\iint \frac{dz' \wedge d\bar{z}'}{z' - z}(uN_{2\mu})(z', \bar{z}'), \tag{2.11}$$

$$N_{2\mu}(z, \bar{z}, k, \bar{k}) = e^{-i(kz+\bar{k}\bar{z})} + \frac{1}{2\pi i}\iint \frac{dz' \wedge d\bar{z}'}{\bar{z}' - \bar{z}}(\bar{u}N_{1\mu})(z', \bar{z}')e^{-ik(z-z')-i\bar{k}(\bar{z}-\bar{z}')}. \tag{2.12}$$

The following reduction formula [3] connects $\mathbf{N}_\mu(z,\bar z,k,\bar k)$ and $\boldsymbol{\mu}(z,\bar z,k,\bar k)$,

$$\mathbf{N}_\mu(z,\bar z,k,\bar k) = \boldsymbol{\chi}(z,\bar z,k,\bar k)e^{-i(kz+\bar k\bar z)} = \boldsymbol{\sigma}\bar{\boldsymbol{\mu}}(z,\bar z,k,\bar k)e^{-i(kz+\bar k\bar z)}. \tag{2.13}$$

If the potential $u(z,\bar z)$ has the boundary values $u \sim \mathrm{O}(|z|^{-2})$ as $|z| \to \infty$, ($u \notin L^1$), then, the integral kernel in Eq. (2.9) is not absolutely integrable, while Eqs. (2.6) and (2.7) are still well-defined. We specify complex integration in the z-plane of a non-absolutely integrable function $f(z,\bar z)$ according to the formula,

$$\iint dz \wedge d\bar z f(z,\bar z) = \lim_{R\to\infty} \iint_{|z|\le R} dz \wedge d\bar z f(z,\bar z). \tag{2.14}$$

The same formula is valid for integrating eigenfunctions of the Dirac system in the k plane as well. In Section 3, we use (2.14) when computing the inner products and completeness relations for the Dirac system and its adjoint.

2.2 The Discrete Spectrum

Suppose here that integral equations (2.6) and (2.7) have homogeneous solutions at an eigenvalue $k = k_j$. The discrete spectrum associated to multi-lump potentials was introduced in Refs. [3, 6]. Here we review their approach and give a new result (Proposition 2.1) which clarifies the role of discrete spectrum in the spectral problem (2.2).

For the discrete spectrum associated to the multi-lump potentials, an isolated eigenvalue $k = k_j$ has double multiplicity with the corresponding two bound states $\mathbf{\Phi}_j(z,\bar z)$ and $\mathbf{\Phi}'_j(z,\bar z)$ [6]. The bound state $\mathbf{\Phi}_j(z,\bar z)$ is a solution of the homogeneous equations,

$$\begin{aligned}\Phi_{1j}(z,\bar z) &= -\frac{1}{2\pi i}\int\int \frac{dz'\wedge d\bar z'}{z'-z}(u\Phi_{2j})(z',\bar z'), && (2.15)\\ \Phi_{2j}(z,\bar z) &= \frac{1}{2\pi i}\int\int \frac{dz'\wedge d\bar z'}{\bar z'-\bar z}(\bar u\Phi_{1j})(z',\bar z')e^{-ik_j(z-z')-i\bar k_j(\bar z-\bar z')}, && (2.16)\end{aligned}$$

with the boundary conditions as $|z| \to \infty$,

$$\mathbf{\Phi}_j(z,\bar z) \to \frac{\mathbf{e}_1}{z}. \tag{2.17}$$

Equivalently, this boundary condition can be written as renormalization conditions for Eqs. (2.15) and (2.16),

$$\begin{aligned}\frac{1}{2\pi i}\iint dz\wedge d\bar z(u\Phi_{2j})(z,\bar z) &= 1, && (2.18)\\ \frac{1}{2\pi i}\iint dz\wedge d\bar z(\bar u\Phi_{1j})(z,\bar z)e^{i(k_jz+\bar k_j\bar z)} &= 0. && (2.19)\end{aligned}$$

The other (degenerate) bound state $\mathbf{\Phi}'_j(z,\bar{z})$ can be expressed in terms of $\mathbf{\Phi}_j(z,\bar{z})$ using Eq. (2.13),

$$\mathbf{\Phi}'_j(z,\bar{z}) = \boldsymbol{\sigma}\bar{\mathbf{\Phi}}_j(z,\bar{z})e^{-i(k_j z+\bar{k}_j\bar{z})}. \tag{2.20}$$

The behaviour of the eigenfunction $\boldsymbol{\mu}(z,\bar{z},k,\bar{k})$ near the eigenvalue $k=k_j$ becomes complicated due to the fact that the double eigenvalue is embedded into the two-dimensional essential spectrum of the Dirac system (2.2). We prove the following result.

Proposition 2.1. For smooth data $b(k,\bar{k})\in C^1$ at $k\neq k_j$, the eigenfunction $\boldsymbol{\mu}(z,\bar{z},k,\bar{k})$ has a pole singularity at $k\to k_j$ only if

$$b_0 = \frac{1}{2\pi}\iint dz\wedge d\bar{z}\ \left(\bar{z}\bar{u}\Phi_{1j}\right)(z,\bar{z})\ e^{i(k_j z+\bar{k}_j\bar{z})} = 0. \tag{2.21}$$

Proof. Suppose $\boldsymbol{\mu}(z,\bar{z},k,\bar{k})$ has a pole singularity at $k=k_j$. Then, it can be shown from Eq. (2.2) that the meromorphic continuation of $\boldsymbol{\mu}(z,\bar{z},k,\bar{k})$ is given by the limiting relation,

$$\lim_{k\to k_j}\left[\boldsymbol{\mu}(z,\bar{z},k,\bar{k}) - \frac{i\mathbf{\Phi}_j(z,\bar{z})}{k-k_j}\right] = (z+z_j)\mathbf{\Phi}_j(z,\bar{z}) + c_j\mathbf{\Phi}'_j(z,\bar{z}), \tag{2.22}$$

where z_j, c_j are some constants. Using Eqs. (2.8), (2.9), and (2.13), we find the differential relation for $b(k,\bar{k})$,

$$\frac{\partial b}{\partial\bar{k}} = \frac{b(k,\bar{k})}{2\pi}\iint dz\wedge d\bar{z}\left(\bar{u}\bar{\mu}_2\right)(z,\bar{z}) - \frac{1}{2\pi i}\iint dz\wedge d\bar{z}\left(\bar{z}\bar{u}\mu_1\right)(z,\bar{z})e^{i(kz+\bar{k}\bar{z})}.$$

In the limit $k\to k_j$, this equation reduces with the help of Eqs. (2.18) and (2.22) to the form,

$$\frac{\partial b}{\partial\bar{k}} = \frac{b(k,\bar{k})}{\bar{k}-\bar{k}_j} - \frac{b_0}{k-k_j},$$

where b_0 is given in Eq. (2.21). The reduced equation exhibits the limiting behavior of $b(k,\bar{k})$ as $k\to k_j$,

$$b(k,\bar{k}) \to -b_0\frac{\bar{k}-\bar{k}_j}{k-k_j}\ln|\bar{k}-\bar{k}_j|. \tag{2.23}$$

On the other hand, it follows from Eqs. (2.13), (2.20), and (2.22) that $\mathbf{N}_\mu(z,\bar{z},k,\bar{k})$ has the limiting behavior,

$$\mathbf{N}_\mu(z,\bar{z},k,\bar{k}) \to \frac{-i\mathbf{\Phi}'_j(z,\bar{z})}{\bar{k}-\bar{k}_j}. \tag{2.24}$$

According to Eqs. (2.23) and (2.24), the right-hand-side of Eq. (2.8) is of order $\mathrm{O}(b_0|k-k_j|^{-1}\ln|k-k_j|)$ as $k\to k_j$. On the other hand, the left-hand-side of Eq. (2.8) must be of order $O(1)$ in the limit $k \to k_j$ according to Eq. (2.22). Therefore, the eigenfunction $\boldsymbol{\mu}(z,\bar{z},k,\bar{k})$ has a pole at $k = k_j$ only if the constraint $b_0 = 0$ holds. $\square$

The limiting relation (2.22) was introduced by Arkadiev *et al* [6]. However, the authors did not notice that the discrete spectrum is supported only by potentials which satisfy the additional constraint (2.21). In particular, such potentials include the multi-lump solutions for which $b(k,\bar{k}) = 0$ everywhere in the k-plane.

2.3 Expansion Formulas for Inverse Scattering

Combining Eq. (2.8) for the essential spectrum and Eq. (2.22) for the discrete spectrum, we reconstruct the eigenfunction $\boldsymbol{\mu}(z,\bar{z},k,\bar{k})$ [3, 6],

$$\boldsymbol{\mu}(z,\bar{z},k,\bar{k}) = \mathbf{e}_1 + \sum_{j=1}^{n} \frac{i\boldsymbol{\Phi}_j(z,\bar{z})}{k-k_j} + \frac{1}{2\pi i}\iint \frac{dk' \wedge d\bar{k}'}{k'-k} b(k',\bar{k}')\mathbf{N}_\mu(z,\bar{z},k',\bar{k}'), \tag{2.25}$$

where n is number of distinct eigenvalues k_j of double multiplicity. At $k \to k_j$, this system is coupled with the algebraic system for the bound states,

$$(z+z_j)\boldsymbol{\Phi}_j(z,\bar{z}) + c_j\boldsymbol{\Phi}'_j(z,\bar{z}) = \mathbf{e}_1 + \sum_{l\neq j} \frac{i\boldsymbol{\Phi}_l(z,\bar{z})}{k_j-k_l} + \frac{1}{2\pi i}\iint \frac{dk \wedge d\bar{k}}{k-k_j} b(k,\bar{k})\mathbf{N}_\mu(z,\bar{z},k,\bar{k}). \tag{2.26}$$

Expansion (2.25) can be related to the inverse scattering transform for the potential $u(z,\bar{z})$ [3, 6]. It follows from Eq. (2.2) that the eigenfunction $\boldsymbol{\mu}(z,\bar{z},k,\bar{k})$ has the asymptotic expansion as $|k| \to \infty$,

$$\boldsymbol{\mu}(z,\bar{z},k,\bar{k}) = \mathbf{e}_1 + \frac{1}{ik}\boldsymbol{\mu}_\infty(z,\bar{z}) + \mathrm{O}(|k|^{-2}), \tag{2.27}$$

where $\mu_{2\infty}(z,\bar{z}) = \bar{u}(z,\bar{z})$ and

$$\mu_{1\infty}(z,\bar{z}) = -\frac{1}{2\pi i}\int\int \frac{dz' \wedge d\bar{z}'}{z'-z}(|u|^2)(z',\bar{z}').$$

We deduce from Eqs. (2.25) and (2.27) that the potential $\bar{u}(z,\bar{z})$ is expressed through the eigenfunctions of the Dirac system in the form [3],

$$\bar{u}(z,\bar{z}) = -\sum_{j=1}^{n} \Phi_{2j}(z,\bar{z}) - \frac{1}{2\pi}\iint dk \wedge d\bar{k}\; b(k,\bar{k})\; N_{2\mu}(z,\bar{z},k,\bar{k}). \tag{2.28}$$

Formulas (2.6) to (2.28) constitute a standard framework for the inverse scattering transform of the DSII equation with a new relation (2.21). The existence and uniqueness of solutions of the

Fredholm integral equations (2.6) and (2.7) and the $\bar\partial$ problem (2.8) and (2.25) were proved in [4] and [5] under the small-norm assumption for the potential $u(x,y)$,

$$\left(\sup_{(x,y)\in\mathbf{R}^2} |u|(x,y) \right) \left(\iint |u(x,y)| dx dy \right) < \frac{\pi}{8}.$$

In this case $n = 0$ and $b(k,\bar k) \neq 0$. The nonlinear two-dimensional Fourier transform associated to this scheme was discussed in Examples 8-10 of Chapter 7.7 of Ref. [17]. Indeed, the connection formula (2.28) implies that there is a scalar spectral decomposition of $\bar u(z,\bar z)$ through $N_{2\mu}(z,\bar z,k,\bar k)$ for $n = 0$. In order to close the decomposition, one could use Eqs. (2.9) and (2.13) to construct a "completeness relation" for the expansion of $\delta(z'-z)$ in the form,

$$\delta(z'-z) = -\frac{1}{2\pi^2 i} \iint dk \wedge d\bar k \bar N_{2\mu}(z',\bar z',k,\bar k) N_{2\mu}(z,\bar z,k,\bar k).$$

However, we show in Proposition 3.4 below that a completeness theorem for Eq. (2.2) is different and is based on the set of eigenfunctions of the adjoint Dirac system.

3 Basis for a Scalar Spectral Decomposition

In this section, we specify the adjoint problem for the Dirac system (2.2) and establish orthogonality and completeness relations.

3.1 The Adjoint System

The adjoint system for Eq. (2.2) is

$$\mu^a_{1z} = ik\mu^a_1 - u\mu^a_2, \quad \mu^a_{2\bar z} = \bar u \mu^a_1, \tag{3.1}$$

which provides the balance equation,

$$i(k'-k)\mu^a_1(k')\mu_2(k) = \frac{\partial}{\partial z}\left[\mu^a_1(k')\mu_2(k)\right] - \frac{\partial}{\partial \bar z}\left[\mu^a_2(k')\mu_1(k)\right]. \tag{3.2}$$

The system (3.1) admits plane solutions $\boldsymbol{\mu}^a(z,\bar z,k,\bar k)$ and oscillatory-type solutions $\mathbf{N}^a_\mu(z,\bar z,k,\bar k)$ with the boundary conditions,

$$\lim_{|k|\to\infty} \boldsymbol{\mu}^a(z,\bar z,k,\bar k) \;=\; \mathbf{e}_2, \tag{3.3}$$

$$\lim_{|k|\to\infty} \mathbf{N}^a_\mu(z,\bar z,k,\bar k) e^{-i(kz+\bar k\bar z)} \;=\; \mathbf{e}_1. \tag{3.4}$$

The adjoint eigenfunctions $\mathbf{N}^a_\mu(z,\bar z,k,\bar k)$ can be expressed through the Green functions,

$$N^a_{1\mu}(z,\bar z) = e^{i(kz+\bar k\bar z)} - \frac{1}{2\pi i}\iint \frac{dz'\wedge d\bar z'}{\bar z'-\bar z}(uN^a_{2\mu})(z',\bar z')e^{ik(z-z')+i\bar k(\bar z-\bar z')}, \tag{3.5}$$

$$N^a_{2\mu}(z,\bar z) = \frac{1}{2\pi i}\iint \frac{dz'\wedge d\bar z'}{z'-z}(\bar u N^a_{1\mu})(z',\bar z'). \tag{3.6}$$

They are related to the adjoint eigenfunctions $\boldsymbol{\mu}^a(z,\bar z,k,\bar k)$ by the formula,

$$\mathbf{N}^a_\mu(z,\bar z,k,\bar k) = -\boldsymbol{\sigma}\bar{\boldsymbol{\mu}}^a(z,\bar z,k,\bar k)e^{i(kz+\bar k\bar z)}. \tag{3.7}$$

Using this representation, we prove the following result.

Lemma 3.1. The spectral data $b(k,\bar k)$ is expressed in terms of the adjoint eigenfunctions as

$$b(k,\bar k) = \frac{1}{2\pi}\iint dz\wedge d\bar z(\bar u N^a_{1\mu})(z,\bar z). \tag{3.8}$$

Proof. Multiplying Eq. (3.5) by $\bar u\mu_1(k)$, integrating over $dz\wedge d\bar z$ and using Eq. (2.7), we express $b(k,\bar k)$ defined in Eq. (2.9) in the form,

$$b(k,\bar k) = \frac{1}{2\pi}\iint dz\wedge d\bar z\left[\bar u\mu_1(k)N^a_{1\mu}(k) - u\mu_2(k)N^a_{2\mu}(k)\right]. \tag{3.9}$$

On the other hand, multiplying Eq. (2.6) by $uN^a_{1\mu}(k)$, integrating over $dz\wedge d\bar z$, and using Eqs. (3.6) and (3.9), we get Eq. (3.8). $\square$

Suppose now that $k=k_j$ is an isolated double eigenvalue of Eq. (2.2) with the bound states $\boldsymbol{\Phi}_j(z,\bar z)$ and $\boldsymbol{\Phi}'_j(z,\bar z)$ given by Eqs. (2.15) – (2.20). Suppose also that $k=k^a_j$ is an eigenvalue of the adjoint system (3.1) with the adjoint bound states $\boldsymbol{\Phi}^a_j(z,\bar z)$ and $\boldsymbol{\Phi}^{a\prime}_j(z,\bar z)$.

Lemma 3.2. If k_j is a double eigenvalue of the Dirac system (2.2), then k_j is also a double eigenvalue of the adjoint system (3.1).

Proof. We use Eq. (3.2) with $\boldsymbol{\mu}=\boldsymbol{\Phi}_j(z,\bar z)$ and $\boldsymbol{\mu}^a=\boldsymbol{\Phi}^a_j(z,\bar z)$ at $k=k_j$ and $k'=k^a_j$ and integrate over $dz\wedge d\bar z$ with the help of Eq. (A.3) of Appendix A. The contour contribution of the integral vanishes due to the boundary conditions (2.17) and (3.12) and the resulting expression is

$$(k^a_j-k_j)\iint dz\wedge d\bar z\;(\Phi^a_{1j}\Phi_{2j})(z,\bar z) = 0.$$

The relation $k^a_j=k_j$ follows from this formula if the integral is non-zero at $k^a_j=k_j$ (which is proved below in Eq. (3.22)). The other possibility is when $k^a_j\neq k_j$ but Φ^a_{1j} is orthogonal to Φ_{2j}. We do not consider such a non-generic situation. The other bound state $\boldsymbol{\Phi}^{a\prime}_j$ at $k^a_j=k_j$ can be defined using the symmetry relation (see Eq. (3.15) below). □

The adjoint bound state $\mathbf{\Phi}^a_j(z,\bar{z})$ solves the homogeneous equations,

$$\Phi^a_{1j}(z,\bar{z}) = -\frac{1}{2\pi i}\iint \frac{dz'\wedge d\bar{z}'}{\bar{z}'-\bar{z}}(u\Phi^a_{2j})(z',\bar{z}')e^{ik_j(z-z')+i\bar{k}_j(\bar{z}-\bar{z}')}, \tag{3.10}$$

$$\Phi^a_{2j}(z,\bar{z}) = \frac{1}{2\pi i}\iint \frac{dz'\wedge d\bar{z}'}{z'-z}(\bar{u}\Phi^a_{1j})(z',\bar{z}') \tag{3.11}$$

with the boundary condition as $|z|\to\infty$,

$$\mathbf{\Phi}^a_j(z,\bar{z}) \to \frac{\mathbf{e}_2}{z}, \tag{3.12}$$

and the normalization conditions,

$$-\frac{1}{2\pi i}\int\int dz\wedge d\bar{z}(\bar{u}\Phi^a_{1j})(z,\bar{z}) = 1, \tag{3.13}$$

$$\frac{1}{2\pi i}\int\int dz\wedge d\bar{z}(u\Phi^a_{2j})(z,\bar{z})e^{-i(k_jz+\bar{k}_j\bar{z})} = 0. \tag{3.14}$$

In addition, the bound state $\mathbf{\Phi}^{a\prime}_j(z,\bar{z})$ is related to $\mathbf{\Phi}(z,\bar{z})$ according to the symmetry formula,

$$\mathbf{\Phi}^{a\prime}_j(z,\bar{z}) = -\boldsymbol{\sigma}\bar{\mathbf{\Phi}}^a_j(z,\bar{z})e^{i(k_jz+\bar{k}_j\bar{z})}. \tag{3.15}$$

Using Eqs. (3.1)–(3.15), we see that the adjoint eigenfunction $\boldsymbol{\mu}^a(z,\bar{z},k,\bar{k})$ satisfies relations similar to those for $\boldsymbol{\mu}(z,\bar{z},k,\bar{k})$,

$$\frac{\partial\boldsymbol{\mu}^a}{\partial\bar{k}} = -\bar{b}(k,\bar{k})\mathbf{N}^a_\mu(z,\bar{z},k,\bar{k}) \tag{3.16}$$

and

$$\lim_{k\to k_j}\left[\boldsymbol{\mu}^a(z,\bar{z},k,\bar{k}) + \frac{i\mathbf{\Phi}^a_j(z,\bar{z})}{k-k_j}\right] = (z+z_j)\mathbf{\Phi}^a_j(z,\bar{z}) - \bar{c}_j\mathbf{\Phi}^{a\prime}_j(z,\bar{z}). \tag{3.17}$$

The expansions for inverse scattering transform of the adjoint eigenfunctions can be found in the form,

$$\boldsymbol{\mu}^a(z,\bar{z},k,\bar{k}) = \mathbf{e}_2 - \sum_{j=1}^{n}\frac{i\mathbf{\Phi}^a_j(z,\bar{z})}{k-k_j} - \frac{1}{2\pi i}\iint\frac{dk'\wedge d\bar{k}'}{k'-k}\bar{b}(k',\bar{k}')\mathbf{N}^a_\mu(z,\bar{z},k',\bar{k}') \tag{3.18}$$

and

$$(z'+z_j)\mathbf{\Phi}^a_j(z',\bar{z}')-\bar{c}_j\mathbf{\Phi}^{a\prime}_j(z',\bar{z}') = \mathbf{e}_2-\sum_{l\neq j}\frac{i\mathbf{\Phi}^a_l(z',\bar{z}')}{k_j-k_l}-\frac{1}{2\pi i}\iint\frac{dk\wedge d\bar{k}}{k-k_j}\bar{b}(k,\bar{k})\mathbf{N}^a_\mu(z,\bar{z},k,\bar{k}). \tag{3.19}$$

3.2 Orthogonality and Completeness Relations

Using the Dirac system (2.2) and its adjoint system (3.1), we prove the orthogonality and completeness relations for the set of eigenfunctions $S = [N_{2\mu}(k,\bar{k}), \{\Phi_{2j}\}_{j=1}^n]$ and its adjoint set $S^a = [N_{1\mu}^a(k,\bar{k}), \{\Phi_{1j}^a\}_{j=1}^n]$.

Proposition 3.3. The eigenfunctions $N_{2\mu}(z,\bar{z},k,\bar{k})$ and $\Phi_{2j}(z,\bar{z})$ are orthogonal to the eigenfunctions $N_{1\mu}^a(z,\bar{z},k,\bar{k})$ and $\Phi_{1j}^a(z,\bar{z})$ as follows

$$\langle N_{1\mu}^a(k')|N_{2\mu}(k)\rangle_z = -2\pi^2 i\delta(k'-k), \tag{3.20}$$

$$\langle N_{1\mu}^a(k)|\Phi_{2j}\rangle_z = \langle \Phi_{1j}^a|N_{2\mu}(k)\rangle_z = 0, \tag{3.21}$$

$$\langle \Phi_{1l}^a|\Phi_{2j}\rangle_z = 2\pi i\delta_{jl}, \tag{3.22}$$

where the inner product is defined as

$$\langle g(k')|f(k)\rangle_z = \iint dz\wedge d\bar{z}\; g(z,\bar{z},k',\bar{k}')\; f(z,\bar{z},k,\bar{k}).$$

Proof. Using Eqs. (2.12) and (3.5), we expand the inner product in Eq. (3.20) as

$$\langle N_{1\mu}^a(k')|N_{2\mu}(k)\rangle_z = I_0$$
$$+\iint dz\wedge d\bar{z}\left(uN_{2\mu}^a\right)(z,k')e^{-i(k'z+\bar{k}'\bar{z})}I_1(z) - \iint dz\wedge d\bar{z}\left(\bar{u}N_{1\mu}\right)(z,k)e^{i(kz+\bar{k}\bar{z})}I_1(z)$$
$$+\frac{1}{2\pi i}\iint dz\wedge d\bar{z}\left(uN_{2\mu}^a\right)(z,k')e^{-i(k'z+\bar{k}'\bar{z})}\iint \frac{dz'\wedge d\bar{z}'}{\bar{z}'-\bar{z}}\left(\bar{u}N_{1\mu}\right)(z',k)e^{i(kz'+\bar{k}\bar{z}')}\left[I_1(z)-I_1(z')\right],$$

where

$$I_0 = \iint dz\wedge d\bar{z}\; e^{i(k'-k)z+i(\bar{k}'-\bar{k})\bar{z}} = -2\pi^2 i\delta(k'-k) \tag{3.23}$$

and

$$I_1(z) = \frac{1}{2\pi i}\iint \frac{dz'\wedge d\bar{z}'}{\bar{z}'-\bar{z}}e^{i(k'-k)z'+i(\bar{k}'-\bar{k})\bar{z}'} = \frac{1}{i(k'-k)}e^{i(k'-k)z+i(\bar{k}'-\bar{k})\bar{z}}. \tag{3.24}$$

The integrals I_0 and $I_1(z)$ are computed in Appendix A. Using these formulas, we find the inner product in Eq. (3.20) in the form,

$$\langle N_{1\mu}^a(k')|N_{2\mu}(k)\rangle_z = -2\pi^2 i\delta(k'-k) + \frac{1}{i(k'-k)}\mathcal{R}(k,k'),$$

where the residual term $\mathcal{R}(k,k')$ is expressed in the form,

$$\mathcal{R}(k,k') = \iint dz\wedge d\bar{z}\left[uN_{2\mu}^a(k')N_{2\mu}(k) - \bar{u}N_{1\mu}^a(k')N_{1\mu}(k)\right],$$

with the help of Eqs. (2.12) and (3.5). We show that $\mathcal{R}(k,k')=0$ by multiplying Eq. (3.6) by $\bar{u}N_{1\mu}(k)$, integrating over $dz\wedge d\bar{z}$ and using Eq. (2.11).

The zero inner products in Eqs. (3.21) and (3.22) for $j\neq l$ are obtained in a similar way with the help of the Fredholm's equations for eigenfunctions $\mathbf{\Phi}_j$, $\mathbf{\Phi}_j^a$, $\mathbf{N}_\mu$, and $\mathbf{N}_\mu^a$. In order to find the non-zero inner product (3.22) for $j=l$ we evaluate the following integral by using the same integral equations,

$$\begin{aligned}\iint dz\wedge d\bar{z}\ \Phi_{1j}^a\mu_2(k) &= \frac{1}{i(k_j-k)}\iint dz\wedge d\bar{z}\left[u\Phi_{2j}^a\mu_2(k)-\bar{u}\Phi_{1j}^a\mu_1(k)\right]\\ &= \frac{1}{i(k-k_j)}\iint dz\wedge d\bar{z}\ \bar{u}\Phi_{1j}^a.\end{aligned}$$

Using Eq. (3.13), the right-hand-side identifies to $\frac{1}{i(k-k_j)}$. Substituting Eq. (2.25) in the left-hand-side and using the zero inner products (3.21) and (3.22), we find Eq. (3.22) for $j=l$. □

Proposition 3.4. The eigenfunctions $N_{2\mu}(z,\bar{z},k,\bar{k})$ and $\Phi_{2j}(z,\bar{z})$ are complete with respect to the adjoint eigenfunctions $N_{1\mu}^a(z,\bar{z},k,\bar{k})$ and $\Phi_{1j}^a(z,\bar{z})$ according to the identity,

$$\delta(z'-z)=-\frac{1}{2\pi^2 i}\iint dk\wedge d\bar{k}N_{1\mu}^a(z',\bar{z}',k,\bar{k})N_{2\mu}(z,\bar{z},k,\bar{k})-\frac{1}{\pi}\sum_{j=1}^{n}\Phi_{1j}^a(z',\bar{z}')\Phi_{2j}(z,\bar{z}). \quad (3.25)$$

Proof. Using the symmetry relations (2.13) and (3.7), we express the integral in Eq. (3.25) as

$$\langle N_{1\mu}^a(z')|N_{2\mu}(z)\rangle_k=\iint dk\wedge d\bar{k}\ \bar{\mu}_2^a(z',\bar{z}',k,\bar{k})\bar{\mu}_1(z,\bar{z},k,\bar{k})e^{ik(z'-z)+i\bar{k}(\bar{z}'-\bar{z})}. \quad (3.26)$$

We use Eqs. (2.25), (2.26), (3.18), and (3.19) and find the pole decomposition for the integrand in Eq. (3.26),

$$\begin{aligned}\bar{\mu}_2^a(z')\bar{\mu}_1(z) = 1 &+\sum_{j=1}^{n}\frac{i}{\bar{k}-\bar{k}_j}\left[\bar{c}_j\bar{\Phi}_{2j}^a(z',\bar{z}')\bar{\Phi}_{1j}'(z,\bar{z})+c_j\bar{\Phi}_{2j}^{a\prime}(z',\bar{z}')\bar{\Phi}_{1j}(z,\bar{z})\right]\\ &+\sum_{j=1}^{n}\frac{i}{\bar{k}-\bar{k}_j}(\bar{z}-\bar{z}')\bar{\Phi}_{2j}^a(z',\bar{z}')\bar{\Phi}_{1j}(z,\bar{z})+\sum_{j=1}^{n}\frac{\bar{\Phi}_{2j}^a(z',\bar{z}')\bar{\Phi}_{1j}(z,\bar{z})}{(\bar{k}-\bar{k}_j)^2}\\ &+\frac{1}{2\pi i}\iint\frac{dk'\wedge d\bar{k}'}{\bar{k}'-\bar{k}}\left[\bar{\mu}_2^a(z')\bar{b}\bar{N}_{1\mu}(z)-\bar{\mu}_1(z)b\bar{N}_{2\mu}^a(z')\right](k',\bar{k}'). \end{aligned}\quad (3.27)$$

We substitute (3.27) into Eq. (3.26) and reduce the integral to the form,

$$\langle N_{1\mu}^a(z')|N_{2\mu}(z)\rangle_k=I_0-2\pi\sum_{j=1}^{n}\left[\bar{c}_j\bar{\Phi}_{2j}^a(z',\bar{z}')\bar{\Phi}_{1j}'(z,\bar{z})+c_j\bar{\Phi}_{2j}^{a\prime}(z',\bar{z}')\bar{\Phi}_{1j}(z,\bar{z})\right]\ I_1(k_j)$$

$$+\sum_{j=1}^{n}\bar{\Phi}^a_{2j}(z',\bar{z}')\bar{\Phi}_{1j}(z,\bar{z})\left[2\pi(\bar{z}'-\bar{z})I_1(k_j)+I_2(k_j)\right]$$

$$-\iint dk\wedge d\bar{k}I_1(k)\left[\bar{\mu}^a_2(z')\bar{b}\bar{N}_{1\mu}(z)-\bar{\mu}_1(z)b\bar{N}^a_{2\mu}(z')\right](k,\bar{k})e^{-ik(z'-z)-i\bar{k}(\bar{z}'-\bar{z})}, \tag{3.28}$$

where the integrals I_0 and $I_1(k)$ are given in Eqs. (3.23) and (3.24) respectively, with z and k interchanged, while the integral $I_2(k_j)$ is defined by

$$I_2(k_j)=\lim_{\epsilon\to 0}\iint_{|k-k_j|\geq\epsilon}\frac{dk\wedge d\bar{k}}{(\bar{k}-\bar{k}_j)^2}e^{ik(z'-z)+i\bar{k}(\bar{z}'-\bar{z})}. \tag{3.29}$$

The integral $I_2(k)$ is found in Appendix A in the form, $I_2(k_j)=-2\pi(\bar{z}'-\bar{z})I_1(k_j)$, such that the third term in Eq. (3.28) vanishes. In order to express the second term in Eq. (3.28) we use Eqs. (2.26), (3.19), (2.20), and (3.15) and derive the relation,

$$-\sum_{j=1}^{n}\left[\bar{c}_j\bar{\Phi}^a_{2j}(z',\bar{z}')\bar{\Phi}'_{1j}(z,\bar{z})+c_j\bar{\Phi}^{a\prime}_{2j}(z'\bar{z}')\bar{\Phi}_{1j}(z,\bar{z})\right]e^{ik_j(z'-z)+i\bar{k}_j(\bar{z}'-\bar{z})}=$$

$$\sum_{j=1}^{n}\left[\bar{c}_j\Phi^{a\prime}_{1j}(z',\bar{z}')\Phi_{2j}(z,\bar{z})+c_j\Phi^a_{1j}(z',\bar{z}')\Phi'_{2j}(z,\bar{z})\right]=$$

$$(z'-z)\sum_{j=1}^{n}\Phi^a_{1j}(z',\bar{z}')\Phi_{2j}(z,\bar{z})+\frac{1}{2\pi i}\sum_{j=1}^{n}\iint\frac{dk\wedge d\bar{k}}{k-k_j}\left[\Phi_{2j}(z)\bar{b}N^a_{1\mu}(z')+\Phi^a_{1j}(z')bN_{2\mu}(z)\right](k,\bar{k}). \tag{3.30}$$

Using this expression and Eqs. (3.23), (3.24), and (3.29), we rewrite Eq. (3.28) in the form,

$$\langle N^a_{1\mu}(z')|N_{2\mu}(z)\rangle_k=-2\pi^2 i\delta(z'-z)-2\pi i\sum_{j=1}^{n}\Phi^a_{1j}(z',\bar{z}')\Phi_{2j}(z,\bar{z})$$

$$+\frac{i}{z'-z}\iint dk\wedge d\bar{k}\frac{\partial}{\partial\bar{k}}\left[\left(\mu^a_1(z')+\sum_{j=1}^{n}\frac{i\Phi^a_{1j}(z')}{k-k_j}\right)\left(\mu_2(z)-\sum_{j=1}^{n}\frac{i\Phi_{2j}(z)}{k-k_j}\right)\right].$$

The last integral vanishes according to Eq. (A.3) of Appendix A and the boundary conditions (2.5) and (3.3). □

Our main result for the spectral decomposition associated to the Dirac system (2.2) follows from the above orthogonality and completeness relations.

Proposition 3.5. An arbitrary scalar function $f(z,\bar{z})$ satisfying the condition $f(z,\bar{z})\sim \mathrm{O}(|z|^{-2})$ as $|z|\to\infty$ can be decomposed through the set $S=[N_{2\mu}(z,\bar{z},k,\bar{k}),\{\Phi_{2j}(z,\bar{z})\}_{j=1}^{n}]$.

Proof. The spectral decomposition is defined through the orthogonality relations (3.20)–(3.22) as

$$f(z,\bar{z}) = \iint dk \wedge d\bar{k} \alpha(k,\bar{k}) N_{2\mu}(z,\bar{z},k,\bar{k}) + \sum_{j=1}^{n} \alpha_j \Phi_{2j}(z,\bar{z}), \tag{3.31}$$

where

$$\alpha(k,\bar{k}) = -\frac{1}{4\pi^2}\langle N^a_{1\mu}(k)|f\rangle_z, \quad \alpha_j = \frac{1}{2\pi i}\langle \Phi^a_{1j}|f\rangle_z. \tag{3.32}$$

Provided the condition on $f(z,\bar{z})$ is satisfied, we interchange integration with respect to $dz \wedge d\bar{z}$ and $dk \wedge d\bar{k}$ and use the completeness formula (3.25). □

The spectral decomposition presented here is different from that of Kiselev [10, 11]. In the latter approach, the function $f(z,\bar{z})$ is spanned by *squared* eigenfunctions of the original problem (1.2) defined according to oscillatory-type behaviour at infinity. In our approach, we transformed the system (1.2) to the form (2.2) and defined the oscillatory-type eigenfunctions according to the *single* eigenfunctions $\mathbf{N}_\mu(z,\bar{z},k,\bar{k})$. We also notice that the (degenerate) bound states $\mathbf{\Phi}'_j(z,\bar{z})$ are not relevant for the spectral decomposition, although they appear implicitly through the meromorphic contributions of the eigenfunctions $\mathbf{N}_\mu(z,\bar{z},k,\bar{k})$ at $k = k_j$ (see Section 4).

4 Perturbation Theory for a Single Lump

We use the scalar spectral decomposition based on Eq. (3.31) and develop a perturbation theory for multi-lump solutions of the DSII equation. We present formulas in the case of a single lump ($n = 1$), the case of multi-lump potentials can be obtained by summing along the indices j, l occuring in the expressions below.

The single-lump potential $u(z,\bar{z})$ has the form [6],

$$u(z,\bar{z}) = \frac{c_j}{|z+z_j|^2 + |c_j|^2} e^{i(k_j z + \bar{k}_j \bar{z})}, \tag{4.1}$$

where c_j, z_j are complex parameters. The associated bound states follow from Eqs. (2.26) and (3.19) as

$$\mathbf{\Phi}_j(z,\bar{z}) = \frac{1}{|z+z_j|^2 + |c_j|^2} \begin{bmatrix} \bar{z} + \bar{z}_j \\ -\bar{c}_j e^{-i(k_j z + \bar{k}_j \bar{z})} \end{bmatrix}, \tag{4.2}$$

$$\mathbf{\Phi}^a_j(z,\bar{z}) = \frac{1}{|z+z_j|^2 + |c_j|^2} \begin{bmatrix} c_j e^{i(k_j z + \bar{k}_j \bar{z})} \\ \bar{z} + \bar{z}_j \end{bmatrix}. \tag{4.3}$$

We first consider a general perturbation to the single lump subject to the localization condition, $\Delta u \sim \mathrm{O}(|z|^{-2})$ as $|z| \to \infty$. We then derive explicit formulas for a special form of the perturbation term $\Delta u(z,\bar{z})$.

4.1 General Perturbation of a Single Lump

Suppose the potential is specified as $u^\epsilon = u(z,\bar z) + \epsilon \Delta u(z,\bar z)$, where $u(z,\bar z)$ is given by Eq. (4.1) and $\Delta u(z,\bar z)$ is a perturbation term. Two bound states $\mathbf{\Phi}_j(z,\bar z)$ and $\mathbf{\Phi}'_j(z,\bar z)$ are supported by a single-lump potential $u(z,\bar z)$ at a single point $k = k_j$. The spectral decomposition given by Eq. (3.31) provides a basis for expansion of $\mu_2^\epsilon(z,\bar z,\kappa,\bar\kappa)$ at $k=\kappa$,

$$\mu_2^\epsilon(z,\bar z,\kappa,\bar\kappa) = \iint dk\wedge d\bar k\; \alpha(k,\bar k) N_{2\mu}(z,\bar z,k,\bar k) + \alpha_j \Phi_{2j}(z,\bar z) \tag{4.4}$$

where $\alpha(k,\bar k)$ and α_j are defined by Eq. (3.32) and depend on the parameter κ. The other component $\mu_1^\epsilon(z,\bar z,\kappa,\bar\kappa)$ can be expressed from Eq. (2.2) as

$$\mu_1^\epsilon(z,\bar z,\kappa,\bar\kappa) = \iint dk\wedge d\bar k\; \alpha(k,\bar k) N_{1\mu}(z,\bar z,k,\bar k) + \alpha_j \Phi_{1j}(z,\bar z) + \epsilon \Delta\mu_1(z,\bar z), \tag{4.5}$$

where the remainder term $\Delta\mu_1(z,\bar z)$ solves the equation,

$$(\Delta\mu_1)_{\bar z} = -\Delta u \mu_2^\epsilon.$$

We write the solution of this equation in the form,

$$\Delta\mu_1(z,\bar z) = A - \frac{1}{2\pi i}\iint \frac{dz'\wedge d\bar z'}{z'-z}\left(\Delta u \mu_2^\epsilon\right)(z',\bar z'), \tag{4.6}$$

subject to the boundary condition as $|z|\to\infty$,

$$\Delta\mu_1(z,\bar z) \to A + \mathrm{O}(z^{-1}),$$

where A is an arbitrary constant. Using the explicit representation (4.6), we transform Eq. (2.2) into the system of integral equations for $\alpha(k,\bar k)$ and α_j,

$$\alpha(k,\bar k) = \frac{\epsilon}{4\pi^2 i(k-\kappa)}\left[\iint dk'\wedge d\bar k'\; K(k,\bar k,k',\bar k')\alpha(k',\bar k') + K_j(k,\bar k)\alpha_j + R(k,\bar k)A\right] + \mathrm{O}(\epsilon^2), \tag{4.7}$$

$$\alpha_j = \frac{\epsilon}{2\pi(k_j-\kappa)}\left[\iint dk\wedge d\bar k\; P_j(k,\bar k)\alpha(k,\bar k) + K_{jl}\alpha_l + R_j A\right] + \mathrm{O}(\epsilon^2), \tag{4.8}$$

where

$$K(k,\bar k,k',\bar k') = \langle \mathbf{N}^a_\mu(k)|\mathbf{N}_\mu(k')\rangle_{\Delta u}, \quad K_j(k,\bar k) = \langle \mathbf{N}^a_\mu(k)|\mathbf{\Phi}_j\rangle_{\Delta u},$$
$$P_j(k,\bar k) = \langle \mathbf{\Phi}^a_j|\mathbf{N}_\mu(k)\rangle_{\Delta u}, \quad K_{jl} = \langle \mathbf{\Phi}^a_j|\mathbf{\Phi}_l\rangle_{\Delta u},$$

and the scalar product for squared eigenfunction is defined as [10, 11]

$$\langle \mathbf{f}(k)|\mathbf{g}(k')\rangle_h = \iint dz\wedge d\bar z\left[\bar h(z,\bar z) f_1(z,\bar z,k,\bar k) g_1(z,\bar z,k',\bar k') + h(z,\bar z) f_2(z,\bar z,k,\bar k) g_2(z,\bar z,k',\bar k')\right].$$

The non-homogeneous terms $R(k,\bar{k})$ and R_j can be computed exactly as

$$
\begin{aligned}
R(k,\bar{k}) &= \iint dz\wedge d\bar{z}\,\left(\bar{u}N^a_{1\mu}\right)(z,\bar{z}) = 2\pi b(k,\bar{k}),\\
R_j &= \iint dz\wedge d\bar{z}\left(\bar{u}\Phi^a_{1j}\right)(z,\bar{z}) = -2\pi i,
\end{aligned}
$$

where $b(k,\bar{k})=0$ if $n\neq 0$. We solve the system of equations (4.7) and (4.8) asymptotically for $\kappa = k_j + \epsilon\Delta\kappa$ and $\Delta\kappa \sim \mathrm{O}(1)$. The leading order behaviour of the integral kernels follows from the asymptotic representation (2.24) as $k\to k_j$,

$$
K(k,\bar{k},k',\bar{k}')\to\frac{\bar{K}_{jj}}{(\bar{k}-\bar{k}_j)(\bar{k}'-\bar{k}_j)},\quad K_j(k,\bar{k})\to\frac{iP_{jj}}{\bar{k}-\bar{k}_j},\quad P_j(k,\bar{k})\to\frac{i\bar{P}_{jj}}{\bar{k}-\bar{k}_j},\tag{4.9}
$$

where

$$
\bar{K}_{jj} = \langle\mathbf{\Phi}^{a\prime}_j|\mathbf{\Phi}'_j\rangle_{\Delta u},\quad P_{jj} = \langle\mathbf{\Phi}^{a\prime}_j|\mathbf{\Phi}_j\rangle_{\Delta u},\quad \bar{P}_{jj} = -\langle\mathbf{\Phi}^{a}_j|\mathbf{\Phi}'_j\rangle_{\Delta u}.\tag{4.10}
$$

Here we have used the symmetry constraints (2.20) and (3.15). The leading order of $\alpha(k,\bar{k})$ as $k\to k_j$ follows from Eq. (4.7) as

$$
\alpha(k,\bar{k})\to -\frac{\epsilon\Delta\bar{\kappa}\beta_j}{2\pi(k-\kappa)(\bar{k}-\bar{k}_j)},
$$

where β_j is not yet defined. We use Eq. (A.4) of Appendix A to compute the integral term,

$$
\iint\frac{dk\wedge d\bar{k}}{(k-\kappa)(\bar{k}-\bar{k}_j)^2} = \frac{2\pi i}{\bar{k}_j-\bar{\kappa}},
$$

and reduce the system of integral equations (4.7) and (4.8) to an algebraic system as $k\to k_j$,

$$
\begin{aligned}
-2\pi\Delta\kappa\alpha_j &= K_{jj}\alpha_j - \bar{P}_{jj}\beta_j - 2\pi i A, &(4.11)\\
2\pi\Delta\bar{\kappa}\beta_j &= -P_{jj}\alpha_j - \bar{K}_{jj}\beta_j. &(4.12)
\end{aligned}
$$

If $P_{jj}\neq 0$, the determinant of the above system is strictly positive. Therefore, homogeneous solutions at $A=0$ (bound states) are absent for $\epsilon\neq 0$. This result indicates that the double eigenvalue at $k=k_j$ disappears under a generic perturbation of the potential $u(z,\bar{z})$ with $P_{jj}\neq 0$ (see also Ref. [8]).

For $A\neq 0$, we find inhomogeneous solutions of Eqs. (4.11) and (4.12),

$$
\alpha_j = \frac{2\pi i A\left(\bar{K}_{jj}+2\pi\Delta\bar{\kappa}\right)}{|K_{jj}+2\pi\Delta\kappa|^2+|P_{jj}|^2},\quad \beta_j = \frac{-2\pi i A P_{jj}}{|K_{jj}+2\pi\Delta\kappa|^2+|P_{jj}|^2}.\tag{4.13}
$$

The eigenfunction $\boldsymbol{\mu}^\epsilon(z,\bar{z},\kappa,\bar{\kappa})$ given by Eqs. (4.4) and (4.5) satisfies the boundary condition (2.5) if $A=\epsilon^{-1}$ and has the following asymptotic representation,

$$\boldsymbol{\mu}^\epsilon(z,\bar{z},\kappa,\bar{\kappa}) = \mathbf{e}_1 + \frac{2\pi i\left[2\pi(\bar{\kappa}-\bar{k}_j)+\epsilon\bar{K}_{jj}\right]\boldsymbol{\Phi}_j(z,\bar{z})}{|2\pi(\kappa-k_j)+\epsilon K_{jj}|^2+|\epsilon P_{jj}|^2} - \frac{2\pi i\epsilon P_{jj}\boldsymbol{\Phi}'_j(z,\bar{z})}{|2\pi(\kappa-k_j)+\epsilon K_{jj}|^2+|\epsilon P_{jj}|^2} + \Delta\boldsymbol{\mu}^\epsilon(z,\bar{z}), \tag{4.14}$$

where the term $\Delta\boldsymbol{\mu}^\epsilon(z,\bar{z})$ is not singular in the limit $\epsilon\to 0$ and $\kappa\to k_j$.

In the limit $\epsilon\to 0$, $\kappa\neq k_j$, we find a meromorphic expansion for $\mu^\epsilon(z,\bar{z},\kappa,\bar{\kappa})$ as

$$\boldsymbol{\mu}^\epsilon(z,\bar{z},\kappa,\bar{\kappa}) = \mathbf{e}_1 + \frac{i\boldsymbol{\Phi}_j(z,\bar{z})}{\kappa-k_j} + \epsilon\left[\frac{K_{jj}\boldsymbol{\Phi}_j(z,\bar{z})}{2\pi i(\kappa-k_j)^2} + \frac{P_{jj}\boldsymbol{\Phi}'_j(z,\bar{z})}{2\pi i|\kappa-k_j|^2}\right] + \mathrm{O}(\epsilon^2). \tag{4.15}$$

It is clear that the double pole can be incorporated by shifting the eigenvalue k_j to

$$k_j^\epsilon = k_j - \frac{\epsilon K_{jj}}{2\pi}.$$

The other double-pole term in the expansion (4.15) has a non-analytic behaviour in the k-plane and leads to the appearance of the spectral data $b^\epsilon(\kappa,\bar{\kappa})=\epsilon\Delta b(\kappa,\bar{\kappa})$ which measures the departure of $\boldsymbol{\mu}^\epsilon(z,\bar{z},\kappa,\bar{\kappa})$ from analyticity according to Eq. (2.8). We find from Eqs. (2.9) and (4.15) that the spectral data $\Delta b(\kappa,\bar{\kappa})$ has the following singular behaviour as $\kappa\to k_j$

$$\Delta b(\kappa,\bar{\kappa}) \to \frac{-P_{jj}}{2\pi|\kappa-k_j|^2}. \tag{4.16}$$

Thus, if $P_{jj}\neq 0$ the analyticity of $\boldsymbol{\mu}^\epsilon(z,\bar{z},\kappa,\bar{\kappa})$ is destroyed and the lump disappears. This conclusion as well as the analytical solution (4.13) agree with the results of Gadyl'shin and Kiselev [8, 9] where the transformation of a single lump into decaying wave packet was also studied.

In the other limit $\epsilon\neq 0$ and $\kappa\to k_j^\epsilon$ we find another expansion from Eq. (4.14),

$$\boldsymbol{\mu}^\epsilon(z,\bar{z},\kappa,\bar{\kappa}) = \mathbf{e}_1 - \frac{2\pi i}{\epsilon\bar{P}_{jj}}\boldsymbol{\Phi}'_j(z,\bar{z}) + \mathrm{O}(\kappa-k_j^\epsilon). \tag{4.17}$$

We conclude that the eigenfunction $\boldsymbol{\mu}^\epsilon(z,\bar{z},\kappa,\bar{\kappa})$ is now free of pole singularities [8, 9]. We summarize the main result in the form of a proposition.

Proposition 4.1. Suppose $u(z,\bar{z})$ is given by Eq. (4.1) and $\Delta u(z,\bar{z})$ satisfies the constraint,

$$P_{jj} = \langle\boldsymbol{\Phi}_j^{a\prime}|\boldsymbol{\Phi}_j\rangle_{\Delta u} \neq 0.$$

Then, the potential $u^\epsilon = u(z,\bar{z})+\epsilon\Delta u(z,\bar{z})$ does not support embedded eigenvalues of the Dirac system (2.2) for $\epsilon\neq 0$.

4.2 Explicit Solution for a Particular Perturbation

Here we specify $c_j = ce^{i\theta}$, where c and θ are real, and consider a particular perturbation $\Delta u(z,\bar{z})$ to the lump $u(z,\bar{z})$ (4.1) in the form,

$$\Delta u(z,\bar{z}) = Q(z,\bar{z}) e^{i(k_j z + \bar{k}_j \bar{z} + \theta)},$$

where $Q(z,\bar{z})$ is a real function. Using Eqs. (4.10), (4.2), and (4.3), we find explicitly the matrix elements K_{jj} and P_{jj},

$$K_{jj} = \iint dz \wedge d\bar{z} \frac{c(\bar{z}+\bar{z}_j)[Q(z,\bar{z}) - \bar{Q}(z,\bar{z})]}{[|z+z_j|^2 + c^2]^2} = 0,$$

$$P_{jj} = \iint dz \wedge d\bar{z} \frac{|z+z_j|^2 \bar{Q}(z,\bar{z}) + c^2 Q(z,\bar{z})}{[|z+z_j|^2 + c^2]^2} = \frac{1}{2c} \iint dz \wedge d\bar{z} \left(u \Delta \bar{u} + \bar{u} \Delta u \right).$$

The element P_{jj} can be seen as a correction to the field energy,

$$N = \frac{i}{2} \iint dz \wedge d\bar{z} |u^{\epsilon}|^2(z,\bar{z}) = N_0 + i\epsilon c P_{jj} + \mathrm{O}(\epsilon^2),$$

where $N_0 = \pi$ is the energy of the single lump solution (independent of the lump parameters k_j and c_j). Thus, a perturbation which leads to the destruction of a single lump, that is with $P_{jj} \neq 0$, changes necessarily the value for the lump energy N_0.

5 Concluding Remarks

The main result of our paper is the prediction of structural instability of multi-lump potentials in the Dirac system associated to the DSII equation. The multi-lump potentials correspond to eigenvalues embedded into a two-dimensional continuous spectrum with the spectral data $b(k,\bar{k})$ satisfying the additional constraint (2.21). In this case, there is no interaction between lumps and continuous radiation. However, a generic initial perturbation induces coupling between the lumps and radiation and, as a result of their interaction, the embedded eigenvalues disappear. This result indicates that the localized multi-lump solutions decay into continuous wave packets in the nonlinear dynamics of the DSII equation (see also Refs. [8, 9]).

This scenario is different from the two types of bifurcations of embedded eigenvalues discussed in our previous paper [14]. The type I bifurcation arises from the edge of the essential spectrum when the limiting bounded (non-localized) eigenfunction is transformed into a localized bound state. The type II bifurcation occurs when an embedded eigenvalue splits off the essential spectrum. Both situations persist in the spectral plane when the essential spectrum is one-dimensional and covers either a half-axis or the whole axis. However, in the case of the DSII equation, the essential spectrum is the whole spectral plane and embedded eigenvalue can not split off the essential spectrum. As a result, they disappear due to their structural instability.

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Appendix A. Formulas of the ∂-analysis

Here we reproduce some formulas of the complex $\bar{\partial}$-analysis [17] to compute the integrals I_0, $I_1(z)$ and $I_2(z)$ defined in Eqs. (3.23), (3.24), and (3.29). We define the complex integration in the z-plane by

$$\iint dz \wedge d\bar{z} f(z,\bar{z}) = -\iint d\bar{z} \wedge dz f(z,\bar{z}),$$

where $dz \wedge d\bar{z} = -2i dx dy$. The complex $\delta(z)$ distribution is defined by

$$\iint dz \wedge d\bar{z} f(z)\delta(z-z_0) = -2if(z_0), \tag{A.1}$$

where $\delta(z) = \delta(x)\delta(y)$. In particular, the δ-distribution appears in the $\bar{\partial}$-analysis according to the relation [17],

$$\frac{\partial}{\partial \bar{z}}\left[\frac{1}{z-z_0}\right] = \pi\delta(z-z_0). \tag{A.2}$$

Computing the integral I_0, we get the formula,

$$I_0 = \iint dz \wedge d\bar{z} e^{ikz+i\bar{k}\bar{z}} = -2i\iint dx dy e^{2i\mathrm{Re}(k)x-2i\mathrm{Im}(k)y} = -2\pi^2 i\delta(k),$$

which proves the identity (3.23).

Using the Green's theorem [17], one has the integration identity,

$$\iint_D dz \wedge d\bar{z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_2}{\partial \bar{z}}\right) = \int_C \left(f_1 d\bar{z} + f_2 dz\right), \tag{A.3}$$

where D is a domain of the complex plane and C its boundary. The generalized Cauchy's formula has the form [17],

$$f(z,\bar{z}) = \frac{1}{2\pi i}\int_C \frac{f(z',\bar{z}')dz'}{z'-z} + \frac{1}{2\pi i}\iint_D \frac{dz' \wedge d\bar{z}'}{z'-z}\frac{\partial f}{\partial \bar{z}'}, \tag{A.4}$$

or, equivalently,

$$f(z,\bar{z}) = -\frac{1}{2\pi i}\int_C \frac{f(z',\bar{z}')d\bar{z}'}{\bar{z}'-\bar{z}} + \frac{1}{2\pi i}\iint_D \frac{dz' \wedge d\bar{z}'}{\bar{z}'-\bar{z}}\frac{\partial f}{\partial z'}. \tag{A.5}$$

In order to find the integral $I_1(z)$ we use Eq. (A.5) with

$$f(z,\bar{z}) = \frac{1}{ik}e^{i(kz+\bar{k}\bar{z})}, \quad k \neq 0$$

and choose the domain D to be a large ball of radius R (see Eq. (2.14)). The boundary value integral vanishes since

$$\lim_{R\to\infty}\int_{|z|=R}\frac{d\bar{z}}{\bar{z}}e^{i(kz+\bar{k}\bar{z})}=-2\pi i\lim_{R\to\infty}J_0(2|k|R)=0, \qquad (A.6)$$

where $J_0(z)$ is the Bessel function. Equation (A.5) for the function $f(z,\bar{z})$ then reduces to Eq. (3.24).

In order to compute the integral $I_2(z_0)$, we apply Eq. (A.3) with $f_1=0$ and

$$f_2(z,\bar{z})=\frac{1}{\bar{z}-\bar{z}_0}e^{i(kz+\bar{k}\bar{z})}.$$

The domain D is chosen as above. The boundary value integral vanishes again,

$$\lim_{R\to\infty}\int_{|z|=R}\frac{dz}{\bar{z}}e^{i(kz+\bar{k}\bar{z})}=-2\pi i\lim_{R\to\infty}J_{-2}(2|k|R)=0, \qquad (A.7)$$

where $J_{-2}(z)$ is the Bessel function. Equation (A.3) for the function $f_2(z,\bar{z})$ reduces to Eq. (3.29).

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