# DARBOUX-TYPE TRANSFORMATIONS AND HYPERELLIPTIC CURVES

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ABSTRACT. We systematically study Darboux-type transformations for the KdV and AKNS hierarchies and provide a complete account of their effects on hyperelliptic curves associated with algebro-geometric solutions of these hierarchies.

## 1. INTRODUCTION

Since the early days of completely integrable evolution equations in the late sixties, Darboux-type transformations (cf., e.g., [42] and the references therein) played an important role and turned out to be an integral part of (auto-)Bäcklund transformations between soliton equations. The canonical example in this context is Miura's transformation [49] between the Korteweg-de Vries (KdV) hierarchy and modified Korteweg-de Vries (mKdV) hierarchy. As described in (2.35)–(2.42), Miura's transformation, disregarding the time variable (i.e., considering the stationary case), is effected by the classical Crum-Darboux transformation, which in turn is based on factorizing the second-order Lax (one-dimensional Schrödinger) operator

$$L = -\frac{d^2}{dx^2} + V \tag{1.1}$$

for the KdV hierarchy into a product of first-order differential expressions plus a shift,

$$L = AA^+ + z_0, \quad A = \frac{d}{dx} + \phi, \quad A^+ = -\frac{d}{dx} + \phi, \quad V = \phi^2 + \phi_x + z_0. \tag{1.2}$$

Assuming $V$ to satisfy one of the stationary KdV equations and reversing the order of the two factors $A$ and $A^+$ produces a new Lax operator $\widetilde{L}$,

$$\widetilde{L} = A^+A + z_0 = -\frac{d^2}{dx^2} + \widetilde{V}, \quad \widetilde{V} = \phi^2 - \phi_x + z_0, \tag{1.3}$$

whose potential $\widetilde{V}$ is a new solution of one of the equations in the stationary KdV hierarchy. In short, the transformation

$$V \mapsto \widetilde{V} \tag{1.4}$$

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represents a Darboux transformation, or equivalently, an auto-Bäcklund transformation of the stationary KdV hierarchy. Incidentally, $\pm\phi$ in (1.2), (1.3) represent solutions of one of the stationary equations of the mKdV hierarchy and hence

$$V \mapsto \phi \mapsto -\phi \mapsto \widetilde{V} \tag{1.5}$$

represents a Bäcklund transformation from the (stationary) KdV to the mKdV hierarchy ($V \mapsto \phi$) as well as auto-Bäcklund transformations for the KdV ($V \mapsto \widetilde{V}$) and mKdV hierarchy ($\phi \mapsto -\phi$). However, while $\phi$ and $-\phi$ satisfy the identical equation(s) within the stationary mKdV hierarchy, and hence

$$D = \begin{pmatrix} 0 & A^+ \\ A & 0 \end{pmatrix} \mapsto \widetilde{D} = \begin{pmatrix} 0 & -A \\ -A^+ & 0 \end{pmatrix} \tag{1.6}$$

represents an isospectral deformation of $D$, $V$ and $\widetilde{V}$ in general do not satisfy the same stationary equation of the KdV hierarchy, that is,

$$L \mapsto \widetilde{L}, \tag{1.7}$$

in general, is not an isospectral deformation of $L$. More precisely, each solution $V$ of (one of) the $n$th stationary KdV equations is associated with a (possibly singular) hyperelliptic curve $\mathcal{K}_n$ of the type

$$\mathcal{K}_n : y^2 = \prod_{m=0}^{2n} (z - E_m), \quad \{E_m\}_{m=0,\dots,2n} \subset \mathbb{C}. \tag{1.8}$$

Similarly, $\widetilde{V}$ corresponds to a curve $\widetilde{\mathcal{K}}_{\tilde{n}}$ of the type

$$\widetilde{\mathcal{K}}_{\tilde{n}} : y^2 = \prod_{m=0}^{2\tilde{n}} (z - \widetilde{E}_m), \quad \{\widetilde{E}_m\}_{m=0,\dots,2\tilde{n}} \subset \mathbb{C} \tag{1.9}$$

and hence $V$ and $\widetilde{V}$ (resp. $L$ and $\widetilde{L}$) are isospectral if and only if $\mathcal{K}_n = \widetilde{\mathcal{K}}_{\tilde{n}}$ (i.e., $\{E_m\}_{m=0,\dots,2n} = \{\widetilde{E}_m\}_{m=0,\dots,2\tilde{n}}$, $n = \tilde{n}$).

The principal aim of the paper is to re-examine the relation between $\mathcal{K}_n$ and $\widetilde{\mathcal{K}}_{\tilde{n}}$, depending on various choices of the function $\phi$ in (1.2), and to provide a complete, yet elementary solution of this problem. Historically, the first attempts to link $\mathcal{K}_n$ and $\widetilde{\mathcal{K}}_{\tilde{n}}$ were made by Drach [14], [15], [16]. The first solution of this problem was obtained by Ehlers and Knörrer in 1982 on the basis of purely algebro-geometric techniques. An elementary but quite elaborate approach was recently developed by Ohmiya [52] (based on two additional papers [51], [53]). Our proof of Theorem 2.3 appears to be the only elementary and short one at this point.

It seems worthwhile pointing out that the curves (1.8) and (1.9) may of course be singular, that is, some (or even all) of the $E_m$'s may coincide. In fact, the class of rational KdV solutions as discussed by Adler and Moser [2] (see also [50], [53]) arises in precisely this manner (with all $E_m = 0$). Similarly, the class of soliton solutions and more generally, solitons relative to an algebro-geometric background potential, as described, for instance, in [11], [12], [17], [33], [35], [36], [43], [47], [48], App. A, [55] results in $n$ pairs of coinciding $E_m$'s, that is, $\{E_m\}_{m=0,\dots,2n} = \{E_0, E_1 = E_2, \dots, E_{2n-1} = E_{2n}\}$ for an appropriate enumeration of the $E_m$'s.

Finally, a quick description of the content of this paper. Section 2 is devoted to the KdV case and starts by summarizing the basic formalism for the KdV hierarchy and its algebro-geometric stationary solutions. Subsequently, we review Darboux transformations $V \mapsto \widetilde{V}$ and finally derive the precise connection between $\mathcal{K}_n$ and

$\widetilde{\mathcal{K}}_{\tilde{n}}$ in Theorem 2.3. Section 3 then presents the analogous results for the AKNS hierarchy. Besides of being an important hierarchy of soliton evolution equations, the AKNS hierarchy allows us to study hyperelliptic curves without a branch point at infinity as opposed to the KdV case, which necessarily leads to hyperelliptic curves branched at infinity. The corresponding Theorem 3.3, detailing the connection between $\mathcal{K}_n$ and $\widetilde{\mathcal{K}}_{\tilde{n}}$ in the AKNS case, to the best of our knowledge, appears to be without precedent.

## 2. The stationary KdV hierarchy

In this section we study Bäcklund transformations of the KdV hierarchy. Our principal aim is to prove a result by Ehlers and Knörrer [19] on hyperelliptic curves and Darboux transformations by entirely elementary means and at the same time offer a more detailed treatment (cf. Theorem 2.3).

We start by introducing a polynomial recursion formalism following Al'ber [3], [4] (see also [13], Ch. 12, [26], [27]) presented in detail in [32], [36] (see also [29], [30], [37]).

Suppose $V\colon \mathbb{C} \mapsto \mathbb{C}_\infty$, with $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, is meromorphic and consider the Schrödinger operator

$$L = -\frac{d^2}{dx^2} + V(x), \quad x \in \mathbb{C}. \tag{2.1}$$

Introducing $\{f_j\}_{j\in\mathbb{N}_0}$, with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, recursively by

$$f_0 = 1, \quad f_{j,x} = -\frac{1}{4} f_{j-1,xxx} + V f_{j-1,x} + \frac{1}{2} V_x f_{j-1}, \quad j \in \mathbb{N}, \tag{2.2}$$

one finds explicitly,

$$\begin{aligned} &f_0 = 1, \\ &f_1 = \tfrac{1}{2} V + c_1, \\ &f_2 = -\tfrac{1}{8} V_{xx} + \tfrac{3}{8} V^2 + c_1 \tfrac{1}{2} V + c_2, \quad \text{etc.}, \end{aligned} \tag{2.3}$$

where $\{c_\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{C}$ denote integration constants. The $f_j$ are well-known to be differential polynomials in $V$ (see, e.g., [20]). Given $V$ and $f_j$ one defines differential expressions $P_{2n+1}$ of order $2n+1$,

$$P_{2n+1} = \sum_{j=0}^{n} \Big( f_{n-j}(x) \frac{d}{dx} - \frac{1}{2} f_{n-j,x}(x) \Big) L^j, \quad n \in \mathbb{N}_0, \tag{2.4}$$

and one verifies

$$[P_{2n+1}, L] = 2 f_{n+1,x}, \quad n \in \mathbb{N}_0, \tag{2.5}$$

with $[\,\cdot\,,\,\cdot\,]$ the commutator. The stationary KdV hierarchy is then defined in terms of the stationary Lax relations

$$[P_{2n+1}, L] = 2 f_{n+1,x} = 0, \quad n \in \mathbb{N}_0. \tag{2.6}$$

Explicitly, one finds

$$\begin{aligned} &n = 0: \quad V_x = 0, \\ &n = 1: \quad \tfrac{1}{4} V_{xxx} - \tfrac{3}{2} V V_x - c_1 V_x = 0, \quad \text{etc.} \end{aligned} \tag{2.7}$$

$V(x)$ is called an *algebro-geometric* KdV *potential* if it satisfies one (and hence infinitely many) of the equations in the stationary KdV hierarchy (2.6).

Introducing the fundamental polynomial $F_n(z,x)$ of degree $n$ in $z$,

$$F_n(z,x)=\sum_{j=0}^{n} f_{n-j}(x)z^j, \tag{2.8}$$

equations (2.2) and (2.6), that is, $f_{n+1,x}(x)=0$, $x\in\mathbb{C}$, imply

$$F_{n,xxx}-4(V-z)F_{n,x}-2V_xF_n=0. \tag{2.9}$$

Multiplying (2.9) by $F_n$ and integrating yields

$$\frac{1}{2}F_{n,xx}(z,x)F_n(z,x)-\frac{1}{4}F_{n,x}(z,x)^2-(V(x)-z)F_n(z,x)^2=R_{2n+1}(z), \tag{2.10}$$

where the integration constant $R_{2n+1}(z)$ is a monic polynomial in $z$ of degree $2n+1$, and hence of the form

$$R_{2n+1}(z)=\prod_{m=0}^{2n}(z-E_m), \quad \{E_m\}_{m=0,\dots,2n}\subset\mathbb{C}. \tag{2.11}$$

Introducing the algebraic eigenspace,

$$\ker(L-z)=\{\psi\colon\mathbb{C}\mapsto\mathbb{C}_\infty \text{ meromorphic} \mid (L-z)\psi=0\}, \quad z\in\mathbb{C} \tag{2.12}$$

one verifies

$$P_{2n+1}\big|_{\ker(L-z)}=\bigg(F_n(z,x)\frac{d}{dx}-\frac{1}{2}F_{n,x}(z,x)\bigg)\bigg|_{\ker(L-z)}. \tag{2.13}$$

Moreover, a celebrated result by Burchnall–Chaundy [6], [7], [8] (see also [25], [54], [55], [57], [59]) then yields

$$P_{2n+1}^2+R_{2n+1}(L)=0. \tag{2.14}$$

Equation (2.14) naturally leads to the hyperelliptic curve $\dot{\mathcal{K}}_n$ defined by

$$\dot{\mathcal{K}}_n\colon \mathcal{F}_n(z,y)=y^2-R_{2n+1}(z)=0, \quad R_{2n+1}(z)=\prod_{m=0}^{2n}(z-E_m). \tag{2.15}$$

The one-point compactification of $\dot{\mathcal{K}}_n$ by joining $P_\infty$, the point at infinity, is then denoted by $\mathcal{K}_n$. A general point $P\in\mathcal{K}_n\backslash\{P_\infty\}$ will be denoted by $P=(z,y)$, where

$$\mathcal{F}_n(z,y)=0. \tag{2.16}$$

Moreover, we define the involution $*$ on $\mathcal{K}_n$ by

$$*\colon\mathcal{K}_n\mapsto\mathcal{K}_n, \quad P=(z,y)\mapsto P^*=(z,-y), \quad P_\infty^*=P_\infty. \tag{2.17}$$

Introducing the polynomial $H_{n+1}$ of degree $n+1$ in $z$,

$$H_{n+1}(z,x)=\frac{1}{2}F_{n,xx}(z,x)+(z-V(x))F_n(z,x), \tag{2.18}$$

(2.10) implies

$$R_{2n+1}(z)+\frac{1}{4}F_{n,x}(z,x)^2=F_n(z,x)H_{n+1}(z,x) \tag{2.19}$$

and we may define the following fundamental meromorphic function $\phi(P,x)$ on $\mathcal{K}_n$,

$$\phi(P,x)=\frac{iy(P)+\frac{1}{2}F_{n,x}(z,x)}{F_n(z,x)} \tag{2.20a}$$

$$= \frac{-H_{n+1}(z,x)}{iy(P) - \frac{1}{2}F_{n,x}(z,x)}, \quad P = (z,y) \in \mathcal{K}_n, \ x \in \mathbb{C}, \tag{2.20b}$$

where $y(P)$ denotes the meromorphic function on $\mathcal{K}_n$ obtained upon solving $y^2 = R_{2n+1}(z)$ with $P = (z,y)$.

Introducing the stationary Baker–Akhiezer function $\psi(P,x,x_0)$ on $\mathcal{K}_n \backslash \{P_\infty\}$ by

$$\psi(P,x,x_0) = \exp\left(\int_{x_0}^{x} dx' \, \phi(P,x')\right), \quad P = (z,y) \in \mathcal{K}_n \backslash \{P_\infty\}, \ x, x_0 \in \mathbb{C}, \tag{2.21}$$

choosing a smooth non-selfintersecting path from $x_0$ to $x$ which avoids singularities of $\phi(P,x)$, one easily verifies from (2.9), (2.10), (2.13), (2.20), and (2.21) the following result.

**Lemma 2.1.** *(see, e.g.,* [30], [32]*) Suppose $f_{n+1,x}(x) = 0$ and let $P = (z,y) \in \mathcal{K}_n \backslash \{P_\infty\}$, $x, x_0 \in \mathbb{C}$. Then $\phi(P,x)$ satisfies the Riccati-type equation*

$$\phi_x(P,x) + \phi(P,x)^2 = V(x) - z, \tag{2.22}$$

*and*

$$\phi(P,x)\phi(P^*,x) = H_{n+1}(z,x)/F_n(z,x), \tag{2.23}$$

$$\phi(P,x) + \phi(P^*,x) = F_{n,x}(z,x)/F_n(z,x), \tag{2.24}$$

$$\phi(P,x) - \phi(P^*,x) = 2iy(P)/F_n(z,x), \tag{2.25}$$

$$\phi(P,x) = iz^{-n}y(P) + O(|z|^{-1/2}) \text{ as } P = (z,y) \to P_\infty. \tag{2.26}$$

*$\psi(P,x,x_0)$ satisfies*

$$(L - z(P))\psi(P,\,\cdot\,,x_0) = 0, \quad (P_{n+1} - iy(P))\psi(P,\,\cdot\,,x_0) = 0, \tag{2.27}$$

*and*

$$\psi(P,x,x_0)\psi(P^*,x,x_0) = F_n(z,x)/F_n(z,x_0), \tag{2.28}$$

$$\psi_x(P,x,x_0)\psi_x(P^*,x,x_0) = H_{n+1}(z,x)/F_n(z,x_0), \tag{2.29}$$

$$\psi(P,x,x_0)\psi_x(P^*,x,x_0) + \psi(P^*,x,x_0)\psi_x(P,x,x_0) = F_{n,x}(z,x)/F_n(z,x_0), \tag{2.30}$$

$$W(\psi(P,\,\cdot\,,x_0), \psi(P^*,\,\cdot\,,x_0)) = -2iy(P)/F_n(z,x_0), \tag{2.31}$$

$$\psi(P,x,x_0) = \exp\big(iz^{-n}y(P)(x - x_0)\big)(1 + O(|z|^{-1/2})) \text{ as } P = (z,y) \to P_\infty. \tag{2.32}$$

(Here $W(f,g)(x) = f(x)g'(x) - f'(x)g(x)$ denotes the Wronskian of $f$ and $g$.)

In addition, we introduce the formal diagonal Green's function $G(P,x,x)$ associated with the differential expression $L$ by

$$G(P,x,x) = \frac{\psi(P,x,x_0)\psi(P^*,x,x_0)}{W(\psi(P,\,\cdot\,,x_0), \psi(P^*,\,\cdot\,,x_0))} \tag{2.33a}$$

$$= \frac{iF_n(z,x)}{2y(P)}, \quad P = (z,y) \in \mathcal{K}_n \backslash \{P_\infty\}, \ x \in \mathbb{C}, \tag{2.33b}$$

using $\phi = \psi_x/\psi$, (2.25), and (2.28). Equations (2.10) and (2.33b) then yield the universal equation

$$-2G''(P,x,x)G(P,x,x) + G'(P,x,x)^2 + 4(V(x) - z)G(P,x,x)^2 = 1, \quad P = (z,y) \in \mathcal{K}_n \backslash \{P_\infty\}. \tag{2.34}$$

These observations complete our review of the stationary KdV hierarchy. The time-dependent KdV hierarchy could readily be defined at this point but since it is not essential for our purpose we resist the temptation to do so (see, e.g., [30], [32] for the time-dependent formalism).

Next we turn to the Darboux transformations in connection with $L = -d^2/dx^2 + V$. Define

$$\psi(P,x,x_0,\sigma) = \begin{cases} \frac{1}{2}(1+\sigma)\psi(P,x,x_0) + \frac{1}{2}(1-\sigma)\psi(P^*,x,x_0) & \text{for } \sigma \in \mathbb{C}, \\ \psi(P,x,x_0) - \psi(P^*,x,x_0) & \text{for } \sigma = \infty, \end{cases}$$
$$P \in \mathcal{K}_n \backslash \{P_\infty\}, \tag{2.35}$$

pick $Q_0 = (z_0, y_0) \in \mathcal{K}_n \backslash \{P_\infty\}$, and introduce the differential expressions

$$A_\sigma(Q_0) = \frac{d}{dx} + \phi(Q_0,x,\sigma), \quad A_\sigma^+(Q_0) = -\frac{d}{dx} + \phi(Q_0,x,\sigma), \quad \sigma \in \mathbb{C}_\infty, \tag{2.36}$$

where

$$\phi(P,x,\sigma) = \psi_x(P,x,x_0,\sigma)/\psi(P,x,x_0,\sigma), \quad P \in \mathcal{K}_n \backslash \{P_\infty\}, \sigma \in \mathbb{C}_\infty. \tag{2.37}$$

One verifies (cf. (2.22))

$$L = A_\sigma(Q_0)A_\sigma^+(Q_0) + z_0 = -\frac{d^2}{dx^2} + V, \tag{2.38}$$

with

$$V(x) = \phi(Q_0,x,\sigma)^2 + \phi_x(Q_0,x,\sigma) + z_0, \tag{2.39}$$

independent of the choice of $\sigma \in \mathbb{C}_\infty$. Interchanging the order of the differential expressions $A_\sigma(Q_0)$ and $A_\sigma^+(Q_0)$ in (2.38) then yields

$$\widetilde{L}_\sigma(Q_0) = A_\sigma^+(Q_0)A_\sigma(Q_0) + z_0 = -\frac{d^2}{dx^2} + \widetilde{V}_\sigma(x,Q_0), \tag{2.40}$$

with

$$\begin{aligned} \widetilde{V}_\sigma(x,Q_0) &= \phi(Q_0,x,\sigma)^2 - \phi_x(Q_0,x,\sigma) + z_0 \\ &= V(x) - 2(\ln(\psi(Q_0,x,x_0,\sigma)))_{xx}, \quad \sigma \in \mathbb{C}_\infty. \end{aligned} \tag{2.41}$$

The transformation

$$V(x) \mapsto \widetilde{V}_\sigma(x,Q_0), \quad Q_0 \in \mathcal{K}_n \backslash \{P_\infty\}, \sigma \in \mathbb{C}_\infty \tag{2.42}$$

is usually called the Darboux transformation (also Crum-Darboux transformation or single commutation method) and goes back at least to Jacobi [40] and Darboux [10]. While we only aim at its properties from an algebraic point of view, its analytic properties in connection with spectral deformations (isospectral and non-isospectral ones) have received enormous attention in the context of spectral theory (especially, regarding the insertion of eigenvalues into spectral gaps), inverse spectral theory, and Bäcklund tranformations for the (time-dependent) KdV hierarchy. A complete bibliography in this context being impossible, we just refer to [1], [9], [11], [12], [18], Ch. 4, [19], [21], [23], [24], [28], [33], [34], [35], [36], [44], [45], [46], [56], [58] and the extensive literature therein. From a historical point of view it is very interesting to note that Drach [14], [15], [16] in his 1919 studies of Darboux transformations (being a student of Darboux) not only introduced a set of nonlinear differential equations for $V$, which today can be identified with the stationary KdV hierarchy, but also studied the effect of Darboux transformations on the underlying hyperelliptic curve. As a consequence, he seems to have been the first to explicitly establish the connection between integrable systems and spectral theory. For modern treatments of this connection see, for instance, [5], Ch. 3 and [30].

Before analyzing the Darboux tranformation (2.42) in some detail, we briefly return to the formal diagonal Green's function $G(P,x,x)$ in (2.33). In view of the possibility of linear combinations as displayed in the definition of $\psi(P,x,x_0,\sigma)$ in (2.35), and our lack of natural boundary conditions (resp. $L^2$-conditions) in connection with the (possibly singular) differential expression $L$, our definition of $G(P,x,x)$ in (2.33) appears to be highly arbitrary. However, the following considerations will show that our choice in (2.33) is the unique one that yields a bounded Green's function as $P \to P_\infty$ (for $x \in \mathbb{C}\backslash\{x_0\}$ fixed). In fact,

$$G(P,x,x) = \frac{iz^n}{2y(P)} + O(|z|^{-1}) \text{ as } P=(z,y) \to P_\infty. \tag{2.43}$$

Introducing

$$H(P,x,x,x_0,\sigma_1,\sigma_2) = \frac{\psi(P,x,x_0,\sigma_1)\psi(P,x,x_0,\sigma_2)}{W(\psi(P,\,\cdot\,,x_0,\sigma_1),\psi(P,\,\cdot\,,x_0,\sigma_2))}, \quad \sigma_1,\sigma_2 \in \mathbb{C}_\infty,\ \sigma_1 \neq \sigma_2, \tag{2.44}$$

one infers from Lemma 2.1, (2.32), and (2.35)

$$\begin{aligned} H(P,x,x,&x_0,\sigma_1,\sigma_2) \\ &= \frac{i(1+\sigma_1)(1+\sigma_2)z^n}{4(\sigma_1-\sigma_2)y(P)} \exp(2iz^{-n}y(P)(x-x_0))(1+O(|z|^{-1/2})) \\ &\quad + \frac{i(1-\sigma_1)(1-\sigma_2)z^n}{4(\sigma_1-\sigma_2)y(P)} \exp(-2iz^{-n}y(P)(x-x_0))(1+O(|z|^{-1/2})) \\ &\quad + \frac{i(1-\sigma_1\sigma_2)z^n}{2(\sigma_1-\sigma_2)y(P)} + O(|z|^{-1/2}) \text{ as } P=(z,y)\to P_\infty \end{aligned} \tag{2.45}$$

for $\sigma_1,\sigma_2 \in \mathbb{C}$, $\sigma_1 \neq \sigma_2$, and similarly if $\sigma_1 = \infty$, $\sigma_2 \in \mathbb{C}$, or $\sigma_1 \in \mathbb{C}$, $\sigma_2 = \infty$. Thus, $H(P,x,x,x_0,\sigma_1,\sigma_2)$ is uniformly bounded for $P$ in a neighborhood of $P_\infty$ and $x \in \mathbb{C}\backslash\{x_0\}$ fixed, if and only if

$$\sigma_1 = -\sigma_2 \in \{-1,1\}. \tag{2.46}$$

Since $\psi(P,x,x_0,-\sigma) = \psi(P^*,x,x_0,\sigma)$, this shows that our choice for $G(P,x,x)$ in (2.32) yields the unique diagonal Green's function of $L$ which is bounded for $P$ in a neighborhood of $P_\infty$ for fixed $x \in \mathbb{C}\backslash\{x_0\}$. In addition, $H(P,x,x,x_0,\sigma_1,\sigma_2)$ is independent of $x_0$ if and only if (2.46) holds.

Next, assuming that $\psi \in \ker(L-z)$,

$$L\psi(z) = z\psi(z), \tag{2.47}$$

one infers $A_\sigma^+(Q_0)\psi(z) \in \ker(\widetilde{L}_\sigma(Q_0)-z)$,

$$\widetilde{L}_\sigma(Q_0)(A_\sigma^+(Q_0)\psi(z)) = zA_\sigma^+(Q_0)\psi(z), \tag{2.48}$$

and

$$W(A_\sigma^+(Q_0)\psi_1(z), A_\sigma^+(Q_0)\psi_2(z)) = (z-z_0)W(\psi_1(z),\psi_2(z)), \quad \psi_1(z),\psi_2(z) \in \ker(L-z). \tag{2.49}$$

Since

$$(A_\sigma^+(Q_0)\psi(P,\,\cdot\,,x_0))(x) = \big(\phi(Q_0,x,\sigma) - \phi(P,x)\big)\psi(P,x,x_0),\\ P \in \mathcal{K}_n\backslash\{Q_0,P_\infty\}, \quad (2.50)$$

we define

$$\begin{aligned}\tilde\psi_\sigma(P,x,x_0,Q_0) &= (A_\sigma^+(Q_0)\psi(P,\,\cdot\,,x_0))(x)\\ &= \big(\phi(Q_0,x,\sigma) - \phi(P,x)\big)\psi(P,x,x_0),\\ &\qquad P \in \mathcal{K}_n\backslash\{Q_0,P_\infty\},\ \sigma\in\mathbb{C}_\infty.\end{aligned} \quad (2.51)$$

Then

$$(\widetilde L_\sigma(Q_0)\tilde\psi_\sigma(P,\,\cdot\,,x_0,Q_0))(x) = z\tilde\psi_\sigma(P,x,x_0,Q_0),\\ P=(z,y)\in\mathcal{K}_n\backslash\{Q_0,P_\infty\}, \quad (2.52)$$

and we define in analogy to (2.33a) the diagonal Green's function $\widetilde G_\sigma(P,x,x,Q_0)$ of $\widetilde L_\sigma(Q_0)$ by

$$\widetilde G_\sigma(P,x,x,Q_0) = \frac{\tilde\psi_\sigma(P,x,x_0,Q_0)\tilde\psi_\sigma(P^*,x,x_0,Q_0)}{W(\tilde\psi_\sigma(P,\,\cdot\,,x_0,Q_0),\tilde\psi_\sigma(P^*,\,\cdot\,,x_0,Q_0))},\\ P=(z,y)\in\mathcal{K}_n\backslash\{Q_0,P_\infty\}. \quad (2.53)$$

**Lemma 2.2.** *Assume $f_{n+1,x}(x)=0$ and let $Q_0=(z_0,y_0)\in\mathcal{K}_n\backslash\{P_\infty\}$, $P=(z,y)\in\mathcal{K}_n\backslash\{Q_0,P_\infty\}$, $\sigma\in\mathbb{C}_\infty$. Then the diagonal Green's function $\widetilde G_\sigma(P,x,x,Q_0)$ in* (2.34) *explicitly reads*

$$\begin{aligned}\widetilde G_\sigma(P,x,x,Q_0) &= \frac{H_{n+1}(z,x)+\phi(Q_0,x,\sigma)^2F_n(z,x)-\phi(Q_0,x,\sigma)F_{n,x}(z,x)}{-2i(z-z_0)y(P)} && (2.54a)\\ &= \frac{(\phi(P,x)-\phi(Q_0,x,\sigma))(\phi(P^*,x)-\phi(Q_0,x,\sigma))F_n(z,x)}{-2i(z-z_0)y(P)} && (2.54b)\\ &= \frac{i\widetilde F_{\sigma,\tilde n}(z,x)}{2\tilde y(P)}, && (2.54c)\end{aligned}$$

*where $\tilde y(P)$ denotes the meromorphic solution on $\dot{\widetilde{\mathcal{K}}}_{\sigma,\tilde n}(Q_0)$ obtained upon solving $y^2=\widetilde R_{\sigma,2\tilde n+1}(z)$, $P=(z,y)$ for some polynomial $\widetilde R_{\sigma,2\tilde n+1}(z)$ of degree $2\tilde n+1\in\mathbb{N}_0$ (cf.* (2.15) *and* (2.58)*) and $\widetilde F_{\sigma,\tilde n}(z,x)$ denotes a polynomial with respect to $z$ of degree $\tilde n$, with $0\le\tilde n\le n+1$. In particular, the Darboux transformation* (2.42)*, $V(x)\mapsto\widetilde V_\sigma(x,Q_0)$ maps the class of algebro-geometric KdV potentials into itself.*

*Proof.* (2.54a) and (2.54b) follow upon use of $\phi(P,x)=\psi_x(P,x,x_0)/\psi(P,x,x_0)$, (2.23), (2.24), (2.28)–(2.31), and (2.49). Since the numerator in (2.54a) is a polynomial in $z$, and

$$\widetilde G_\sigma(P,x,x,Q_0) = \frac{iz^n}{2y(P)} + O(|z|^{-1}) \text{ as } P=(z,y)\to P_\infty \quad (2.55)$$

again by (2.54a), one concludes (2.54c) and $0\le\tilde n\le n+1$. By inspection, $\widetilde F_{\sigma,\tilde n}(z,x)$ satisfies equation (2.10) with $V(x)$ replaced by $\widetilde V_\sigma(x,Q_0)$, $n$ by $\tilde n$, and $R_{2n+1}(z)$ by $\widetilde R_{\sigma,2\tilde n+1}(z)$. As a consequence, $V(x)$ being an algebro-geometric KdV potential implies that $\widetilde V_\sigma(x,Q_0)$ is one as well. ∎

The following theorem, the principal result of this section, will clarify the dependence of $\tilde n = \tilde n(n, Q_0, \sigma)$ on its variables. This result was originally derived using an entirely different algebro-geometric approach by Ehlers and Knörrer [19].

**Theorem 2.3.** *Suppose $f_{n+1,x}(x) = 0$ and let $Q_0 = (z_0, y_0) \in \mathcal{K}_n \backslash \{P_\infty\}$, $n \in \mathbb{N}_0$, $\sigma \in \mathbb{C}_\infty$, and $\tilde n = \tilde n(n, Q_0, \sigma)$ as in (2.54c). Then*

$$\tilde n(n, Q_0, \sigma) = \begin{cases} n+1 & \text{for } \sigma \in \mathbb{C}_\infty \backslash \{-1,1\} \text{ and } y_0 \neq 0, \\ n+1 & \text{for } \sigma = \infty \text{ and } y_0 = 0, \\ n & \text{for } \sigma \in \{-1,1\} \text{ and } y_0 \neq 0, \\ n & \text{for } \sigma \in \mathbb{C},\ y_0 = 0, \text{ and } R_{2n+1,z}(z_0) \neq 0, \\ n-1 & \text{for } \sigma \in \mathbb{C},\ y_0 = 0, \text{ and } R_{2n+1,z}(z_0) = 0, n \in \mathbb{N}, \end{cases} \tag{2.56}$$

*and hence the hyperelliptic curve $\dot{\widetilde{\mathcal{K}}}_{\sigma,\tilde n}(Q_0)$ associated with $\widetilde V_\sigma(x, Q_0)$ is of the type*

$$\dot{\widetilde{\mathcal{K}}}_{\sigma,\tilde n}(Q_0)\colon \widetilde{\mathcal{F}}_{\sigma,\tilde n}(z,y,Q_0) = y^2 - \widetilde R_{\sigma,2\tilde n+1}(z, Q_0) = 0, \tag{2.57}$$

*with*

$$\begin{aligned} &\widetilde R_{\sigma,2\tilde n+1}(z, Q_0) \\ &= \begin{cases} (z-z_0)^2 R_{2n+1}(z) & \text{for } \sigma \in \mathbb{C}_\infty \backslash \{-1,1\} \text{ and } y_0 \neq 0, \\ (z-z_0)^2 R_{2n+1}(z) & \text{for } \sigma = \infty \text{ and } y_0 = 0, \\ R_{2n+1}(z) & \text{for } \sigma \in \{-1,1\} \text{ and } y_0 \neq 0, \\ R_{2n+1}(z) & \text{for } \sigma \in \mathbb{C},\ y_0 = 0, \text{ and } R_{2n+1,z}(z_0) \neq 0, \\ (z-z_0)^{-2} R_{2n+1}(z) & \text{for } \sigma \in \mathbb{C},\ y_0 = 0, \text{ and } R_{2n+1,z}(z_0) = 0, n \in \mathbb{N}. \end{cases} \end{aligned} \tag{2.58}$$

*Here*

$$R_{2n+1}(z) = \prod_{m=0}^{2n} (z - E_m). \tag{2.59}$$

*Proof.* Our starting point will be (2.54b) and a careful case distinction taking into account whether or not $Q_0$ is a branch point, and distinguishing the cases $\sigma \in \mathbb{C}\backslash\{-1,1\}$, $\sigma \in \{-1,1\}$, and $\sigma = \infty$.

Case (i). $\sigma \in \mathbb{C}_\infty \backslash \{-1,1\}$ and $y_0 \neq 0$: One computes from (2.35) and (2.37),

$$\phi(Q_0, x, \sigma) = \begin{cases} \frac{(1+\sigma)\psi_x(Q_0,x,x_0) + (1-\sigma)\psi_x(Q_0^*,x,x_0)}{(1+\sigma)\psi(Q_0,x,x_0) + (1-\sigma)\psi(Q_0^*,x,x_0)} & \text{for } \sigma \in \mathbb{C}\backslash\{-1,1\}, \\ \frac{\psi_x(Q_0,x,x_0) - \psi_x(Q_0^*,x,x_0)}{\psi(Q_0,x,x_0) - \psi(Q_0^*,x,x_0)} & \text{for } \sigma = \infty, \end{cases} \tag{2.60}$$

and upon comparison with $\phi(Q_0, x) \neq \phi(Q_0^*, x)$,

$$\phi(Q_0, x) = \frac{\psi_x(Q_0, x, x_0)}{\psi(Q_0, x, x_0)}, \quad \phi(Q_0^*, x) = \frac{\psi_x(Q_0^*, x, x_0)}{\psi(Q_0^*, x, x_0)}, \tag{2.61}$$

one concludes that no cancellations can occur in (2.54b), proving $\tilde n(n, Q_0, \sigma) = n+1$ and the first statement in (2.58).

Case (ii). $\sigma = \infty$ and $y_0 = 0$: Combining (2.20a), (2.21), (2.25), and (2.35) one computes

$$\begin{aligned} &\phi(Q_0, x, \infty) = \lim_{P \to Q_0} \phi(P, x, \infty) \\ &= \lim_{P \to Q_0} \left( \frac{\phi(P,x) \exp\left(\int_{x_0}^x dx'\, \phi(P,x')\right) - \phi(P^*,x) \exp\left(\int_{x_0}^x dx'\, \phi(P^*,x')\right)}{\exp\left(\int_{x_0}^x dx'\, \phi(P,x')\right) - \exp\left(\int_{x_0}^x dx'\, \phi(P^*,x')\right)} \right) \end{aligned}$$

$$= \phi(Q_0, x)$$

$$+ \lim_{P\to Q_0} \left( \frac{\phi(P,x) - \phi(P^*,x)}{\exp\big(\int_{x_0}^x dx'\, \phi(P,x')\big) - \exp\big(\int_{x_0}^x dx'\, \phi(P^*,x')\big)} \times \right.$$

$$\left. \times \exp\left(\int_{x_0}^x dx'\, \phi(P^*,x')\right)\right)$$

$$= \phi(Q_0,x) + \exp\left(\int_{x_0}^x dx'\, \phi(Q_0,x')\right) \times$$

$$\times \lim_{P\to Q_0} \left( \frac{2iy(P)/F_n(z,x)}{\exp\big(iy(P)\int_{x_0}^x \frac{dx'}{F_n(z,x')}\big) - \exp\big(-iy(P)\int_{x_0}^x \frac{dx'}{F_n(z,x')}\big)} \times \right.$$

$$\left. \times \exp\left(-\frac{1}{2}\int_{x_0}^x dx'\, \frac{F_{n,x}(z,x')}{F_n(z,x')}\right)\right)$$

$$= \phi(Q_0,x)$$

$$+ \psi(Q_0,x,x_0)\frac{1}{F_n(z_0,x)\psi(Q_0,x,x_0)} \lim_{P\to Q_0}\left(\frac{2iy(P)}{2iy(P)\int_{x_0}^x \frac{dx'}{F_n(z,x')} + O(y(P)^2)}\right)$$

$$= \phi(Q_0,x) + \left(F_n(z_0,x)\int_{x_0}^x \frac{dx'}{F_n(z,x')}\right)^{-1}, \quad x\in\mathbb{C}\backslash\{x_0\}, \tag{2.62}$$

using $\lim_{P\to Q_0} y(P) = y(Q_0) = y_0 = 0$. From

$$\phi(Q_0,x) = \frac{1}{2}\frac{F_{n,x}(z_0,x)}{F_n(z_0,x)} \tag{2.63}$$

one concludes again that no cancellations can occur in (2.54b). Thus $\tilde{n}(n,Q_0,\infty) = n+1$ and the second statement in (2.58) holds.

The remainder of the proof requires a more refined argument, the basis of which will be derived next. First, replacing $V(x)$ by $V(x) - z_0$, we may assume without loss of generality that $z_0 = 0$ in the following. Moreover, from this point on we exclude the trivial case $V(x) = 0$, $x\in\mathbb{C}$ (the case $V(x) = E_0$ will be studied in Example 2.7 below). Writing

$$y(z)^2 = R_{2n+1}(z) \underset{z\to 0}{=} y_0^2 + \tilde{y}_1 z + \tilde{y}_2 z^2 + O(z^3), \tag{2.64}$$

a comparison of the powers $z^0$ and $z^1$ in (2.10) yields

$$2f_{n,xx}f_n = f_{n,x}^2 + 4Vf_n^2 + 4y_0^2 \tag{2.65}$$

and

$$f_{n-1,xx}f_n + f_{n,xx}f_{n-1} - f_{n,x}f_{n-1,x} - 4Vf_nf_{n-1} + 2Vf_n^2 - 2\tilde{y}_1 = 0. \tag{2.66}$$

Inserting (2.65) into (2.66), a little algebra proves the basic identity

$$f_n^2(f_{n-1}/f_n)_{xx} + f_{n,x}f_n(f_{n-1}/f_n)_x + 2Vf_n^2 + 4y_0^2(f_{n-1}/f_n) - 2\tilde{y}_1 = 0. \tag{2.67}$$

Case (iii). $\sigma\in\{-1,1\}$ and $y_0\neq 0$: Then (2.35) yields

$$\phi(Q_0,x,1) = \phi(Q_0,x), \quad \phi(Q_0,x,-1) = \phi(Q_0^*,x), \tag{2.68}$$

with $\phi(Q_0,x)\neq\phi(Q_0^*,x)$ since $y_0\neq 0$. In this case there is a cancellation in (2.54b). For instance, choosing $\sigma = 1$ one computes from (2.8) and (2.20a),

$$\phi(P,x) - \phi(Q_0,x,1) = \phi(P,x) - \phi(Q_0,x)$$

$$\begin{aligned}&\underset{P\to Q_0}{=} \frac{i(y(P)-y_0)}{f_n(x)} - iy_0\frac{f_{n-1}(x)}{f_n(x)^2}z + \frac{1}{2}\bigg(\frac{f_{n-1}(x)}{f_n(x)}\bigg)_x z + O(z^2)\\ &\underset{P\to Q_0}{=} c_1(x)z + O(z^2)\end{aligned} \tag{2.69}$$

since

$$y(P)-y_0 \underset{P\to Q_0}{=} y_1 z + O(z^2), \quad \tilde y_1 = 2y_0y_1. \tag{2.70}$$

It remains to show that $c_1(x)$ does not vanish identically in $x\in\mathbb{C}$. Arguing by contradiction we assume

$$0 = c_1(x) = \frac{iy_1}{f_n(x)} - \frac{iy_0}{f_n(x)}\frac{f_{n-1}(x)}{f_n(x)} + \frac{1}{2}\bigg(\frac{f_{n-1}(x)}{f_n(x)}\bigg)_x, \quad x\in\mathbb{C}. \tag{2.71}$$

Differentiating (2.71) with respect to $x$ and inserting the ensuing expression for $(f_{n-1}/f_n)_{xx}$ and the one for $(f_{n-1}/f_n)_x$ from (2.71) into (2.67) then results in the contradiction

$$0 = 2V(x)f_n(x)^2, \quad x\in\mathbb{C}. \tag{2.72}$$

Moreover, since

$$\phi(P^*,x) - \phi(Q_0,x,x,1) = \phi(P^*,x)-\phi(Q_0,x) \underset{P\to Q_0}{=} -\frac{2iy_0}{f_n(x)} + O(z), \tag{2.73}$$

one concludes that precisely one factor of $z$ cancels in (2.54b). Hence $\tilde n(n,Q_0,1)=n$ and the third relation in (2.58) holds. The case $\sigma=-1$ in treated analogously.

Case (iv). $\sigma\in\mathbb{C}$, $y_0=0$, and $R_{2n+1,z}(0)\neq 0$: Taking into account that $\phi(Q_0,x,\sigma)=\phi(Q_0,x)$ (using (2.37) and $Q_0=Q_0^*$) is independent of $\sigma\in\mathbb{C}$, (2.8) and (2.20a) yield

$$\big(\phi(P,x)-\phi(Q_0,x)\big)\big(\phi(P^*,x)-\phi(Q_0,x)\big) \underset{P\to Q_0}{=} \frac{y_1^2 z}{f_n(x)^2} + O(z^2) \tag{2.74}$$

since

$$y(P) \underset{P\to Q_0}{=} y_1 z^{1/2} + O(z^{3/2}), \quad y_1 = \bigg(\prod_{E_m\neq 0} E_m\bigg)^{1/2}. \tag{2.75}$$

Thus we infer again that precisely one factor of $z$ cancels in (2.54b). Hence $\tilde n(n,Q_0,\sigma)=n$ and the fourth relation in (2.58) is proved.

Case (v). $\sigma\in\mathbb{C}$, $y_0=\tilde y_1=0$, and $\tilde y_2\neq 0$ (cf. (2.64)): One calculates as in (2.74),

$$\begin{aligned}&\big(\phi(P,x)-\phi(Q_0,x)\big)\big(\phi(P^*,x)-\phi(Q_0,x)\big)\\ &\quad\underset{P\to Q_0}{=} \bigg(\frac{y_1^2}{f_n(x)^2} + \frac{1}{4}\bigg(\bigg(\frac{f_{n-1}(x)}{f_n(x)}\bigg)_x\bigg)^2\bigg)z^2 + O(z^3)\\ &\quad\underset{P\to Q_0}{=} c_2(x)z^2 + +O(z^3)\end{aligned} \tag{2.76}$$

since

$$y(P) \underset{P\to Q_0}{=} y_1 z + O(z^2), \quad y_1 = \bigg(\prod_{E_m\neq 0} E_m\bigg)^{1/2}. \tag{2.77}$$

Next we show that $c_2(x)$ does not vanish identically in $x\in\mathbb{C}$. Arguing again by contradiction we suppose that

$$0=c_2(x)=\frac{y_1^2}{f_n(x)^2}+\frac{1}{4}\left(\left(\frac{f_{n-1}(x)}{f_n(x)}\right)_x\right)^2, \quad x\in\mathbb{C}. \tag{2.78}$$

Thus

$$\left(\frac{f_{n-1}}{f_n}\right)_x=\frac{C}{f_n} \tag{2.79}$$

for some constant $C\in\mathbb{C}$. Insertion of (2.79) and its $x$-derivative into (2.67) then again yields the contradiction

$$0=2V(x)f_n(x)^2, \quad x\in\mathbb{C}. \tag{2.80}$$

Hence $\tilde n(n,Q_0,\sigma)=n-1$ and the last relation in (2.58) holds in this case.

Case (vi). $\sigma\in\mathbb{C}$, $y_0=\tilde y_1=\tilde y_2=0$ (cf. (2.64)): As in (2.76) one obtains

$$\big(\phi(P,x)-\phi(Q_0,x)\big)\big(\phi(P^*,x)-\phi(Q_0,x)\big)\underset{P\to Q_0}{=}\frac{1}{4}\left(\left(\frac{f_{n-1}(x)}{f_n(x)}\right)_x\right)^2z^2+O(z^3) \tag{2.81}$$

since

$$y(P)\underset{P\to Q_0}{=}O(z^{3/2}). \tag{2.82}$$

The remainder of the proof of case (vi) is now a special case of case (v) (with $y_1=C=0$) and one concludes again that $\tilde n(n,Q_0,\sigma)=n-1$. □

We can summarize the previous theorem in the following table.

| | | $\sigma\in\mathbb{C}\backslash\{-1,1\}$ | $\sigma\in\{-1,1\}$ | $\sigma=\infty$ |
|---|---|---|---|---|
| $y_0\neq 0$ | | $n+1$ | $n$ | $n+1$ |
| $y_0=0$ | $\tilde y_1\neq 0$ | $n$ | | |
| | $\tilde y_1=0$ | $n-1$ | | |

**Table 2.4.** The table shows the value of the arithmetic genus $\tilde n$ associated with the Darboux transformation. Here $R_{2n+1}(z)=y_0^2+\tilde y_1(z-z_0)+O((z-z_0)^2)$ as $z\to z_0$.

These results show, in particular, that Darboux transformations do not change the local structure of the original curve $y^2=R_{2n+1}(z)$, except, of course, near the point $Q_0$.

*Remark* 2.5. Theorem 2.3 was first derived by purely algebro-geometric means by Ehlers and Knörrer [19] in 1982. An elementary but rather lengthy derivation of Theorem 2.3 (focusing on the case when $\tilde n(n,\sigma)=n-1$) was provided by Ohmiya [52] in 1997 (based on two other papers [51], [53]). The current proof seems to be the only elementary and short one available at this point. As mentioned in the paragraph following (2.42), Drach [14], [15], [16] appears to have been the first to study particular aspects (the case $\tilde n=n+1$) of Theorem 2.3 around 1919. Moreover, it seems worthwhile to point out that the case $\sigma=\infty$ and $y_0=0$, which leads to $\tilde n(n,Q_0,\sigma)=n+1$, necessarily constructs an algebro-geometric KdV potential $\widetilde V_\infty(x,Q_0)$ singular at $x=x_0$ (cf. (2.62)). In fact, the class of rational algebro-geometric solutions constructed by Adler and Moser [2] arises exactly in this manner.

*Remark* 2.6. The results described in Theorem 2.3 are not confined to hyperelliptic curves $\mathcal{K}_n$ of finite (arithmetic) genus $n$. In fact, upon shifting the emphasis from $F_n(z,x)$ to $G(P,x,x)$, the results in Theorem 2.3 extend to certain classes of transcendental hyperelliptic curves of infinite (arithmetic) genus $\mathcal{K}_\infty$ (including those associated with periodic potentials $V$). More details will be presented elsewhere.

We conclude with the following elementary illustration.

**Example 2.7.** The case $n = 0$.

$$\begin{aligned}
y(P)^2 &= R_1(z) = z - E_0,\\
F_0(z,x) &= 1, \quad H_1(z,x) = z - E_0, \quad V(x) = E_0,\\
\phi(P,x) &= iy(P), \quad \psi(P,x,x_0) = \exp(iy(P)(x-x_0)),\\
G(P,x,x) &= \frac{i}{2y(P)},\\
\phi(P,x,\sigma) &= \begin{cases} iy(P)\frac{(1+\sigma)\exp(iy(P)(x-x_0))-(1-\sigma)\exp(-iy(P)(x-x_0))}{(1+\sigma)\exp(iy(P)(x-x_0))+(1-\sigma)\exp(-iy(P)(x-x_0))}, & \sigma\in\mathbb{C},\\ iy(P)\frac{\exp(iy(P)(x-x_0))-\exp(-iy(P)(x-x_0))}{\exp(iy(P)(x-x_0))+\exp(-iy(P)(x-x_0))}, & \sigma=\infty,\end{cases}\\
\phi((E_0,0),x,\sigma) &= \begin{cases} 0, & \sigma\in\mathbb{C},\\ (x-x_0)^{-1}, & \sigma=\infty.\end{cases}
\end{aligned} \tag{2.83}$$

More generally, the case $V(x) = E_0$ can be associated with any curve $y(P)^2 = R_{2n+1}(z)$ since $f_{n,x}(x) = 0$, $n \in \mathbb{N}_0$ in this special case.

## 3. The stationary AKNS hierarchy

This section is devoted to Darboux (gauge) transformations for the AKNS hierarchy. In particular, we derive the KdV analogs of Section 2 and hence determine the effect of gauge tranformations on hyperelliptic AKNS curves in the spirit of our approach to the KdV result of Ehlers and Knörrer [19].

We start by introducing a polynomial recursion for the AKNS hierarchy following the derivation in [31]. Suppose $p, q\colon \mathbb{C} \mapsto \mathbb{C}_\infty$ are meromorphic and introduce the Dirac-type differential expression

$$D = \begin{pmatrix} \frac{d}{dx} & -q \\ p & -\frac{d}{dx} \end{pmatrix}, \quad x \in \mathbb{C}. \tag{3.1}$$

Introducing $\{f_j(x)\}_{j\in\mathbb{N}_0}$, $\{g_j(x)\}_{j\in\mathbb{N}_0}$, and $\{h_j(x)\}_{j\in\mathbb{N}_0}$ recursively by

$$\begin{aligned}
f_0(x) &= -iq(x), \quad g_0(x) = 1, \quad h_0(x) = ip(x),\\
f_j(x) &= \frac{i}{2}f_{j-1,x}(x) - iq(x)g_j(x), \quad j\in\mathbb{N},\\
g_{j,x}(x) &= p(x)f_{j-1}(x) + q(x)h_{j-1}(x), \quad j\in\mathbb{N},\\
h_j(x) &= -\frac{i}{2}h_{j-1,x}(x) + ip(x)g_j(x), \quad j\in\mathbb{N}.
\end{aligned} \tag{3.2}$$

Explicitly, one computes

$$\begin{aligned}
f_0 &= -iq, \quad f_1 = \tfrac{1}{2}q_x + c_1(-iq),\\
f_2 &= \tfrac{i}{4}q_{xx} - \tfrac{i}{2}pq^2 + c_1\tfrac{1}{2}q_x + c_2(-iq),
\end{aligned}$$

$$\begin{aligned}
&g_0 = 1, \quad g_1 = c_1, \quad g_2 = \tfrac{1}{2}pq + c_2, \\
&g_3 = -\tfrac{i}{4}(p_x q - p q_x) + c_1 \tfrac{1}{2}pq + c_3, \\
&h_0 = ip, \quad h_1 = \tfrac{1}{2}p_x + c_1 ip, \\
&h_2 = -\tfrac{i}{4}p_{xx} + \tfrac{i}{2}p^2 q + c_1 \tfrac{1}{2}p_x + c_2 ip, \quad \text{etc.},
\end{aligned} \tag{3.3}$$

where $\{c_j\}_{j\in\mathbb{N}_0} \subset \mathbb{C}$ are integration constants. The coefficients $f_j$, $g_j$, $h_j$ are well-known to be differential polynomials in $p$ and $q$ (see, e.g., [30]). Given $p$, $q$ and $f_j$, $g_j$, and $h_j$ one defines the matrix-valued differential expression of order $n+1$,

$$E_{n+1} = i\sum_{j=0}^{n+1} \begin{pmatrix} -g_{n+1-j} & f_{n-j} \\ -h_{n-j} & g_{n+1-j} \end{pmatrix} D^j, \quad n \in \mathbb{N}_0, \quad f_{-1} = h_{-1} = 0, \tag{3.4}$$

and verifies

$$[E_{n+1}, D] = \begin{pmatrix} 0 & -2if_{n+1} \\ 2ih_{n+1} & 0 \end{pmatrix}, \quad n \in \mathbb{N}_0. \tag{3.5}$$

The stationary AKNS hierarchy is then defined by the stationary Lax relations

$$[E_{n+1}, D] = 0, \quad n \in \mathbb{N}, \text{ that is, } f_{n+1} = h_{n+1} = 0, \quad n \in \mathbb{N}. \tag{3.6}$$

Explicitly, one finds

$$\begin{aligned}
n = 0: &\quad \begin{pmatrix} -p_x + c_1(-2ip) \\ -q_x + c_1(2iq) \end{pmatrix} = 0, \\
n = 1: &\quad \begin{pmatrix} \tfrac{i}{2}p_{xx} - ip^2 q + c_1(-p_x) + c_2(-2ip) \\ -\tfrac{i}{2}q_{xx} + ipq^2 + c_1(-q_x) + c_2(2iq) \end{pmatrix} = 0, \text{ etc.}
\end{aligned} \tag{3.7}$$

$(p(x), q(x))$ are called *algebro-geometric* AKNS *potentials* if they satisfy one (and hence infinitely many) of the equations of the stationary AKNS hierarchy (3.6).

Introducing the polynomials $F_n(z,x)$, $G_n(z,x)$, and $H_n(z,x)$ with respect to $z$,

$$\begin{aligned}
F_n(z,x) &= \sum_{j=0}^{n} f_{n-j}(x) z^j, \\
G_{n+1}(z,x) &= \sum_{j=0}^{n+1} g_{n+1-j}(x) z^j, \\
H_n(z,x) &= \sum_{j=0}^{n} h_{n-j}(x) z^j,
\end{aligned} \tag{3.8}$$

equations (3.2) and (3.6), that is, $f_{n+1}(x) = h_{n+1}(x) = 0$, $x \in \mathbb{C}$, imply

$$F_{n,x} = -2izF_n + 2qG_{n+1}, \quad G_{n+1,x} = pF_n + qH_n, \quad H_{n,x} = 2izH_n + 2pG_{n+1}. \tag{3.9}$$

Equations (3.9) yield

$$G_{n+1}(z,x)^2 - F_n(z,x)H_n(z,x) = R_{2n+2}(z), \tag{3.10}$$

where $R_{2n+2}(z)$ is a monic polynomial in $z$ of degree $2n+2$ and hence of the form

$$R_{2n+2}(z) = \prod_{m=0}^{2n+1} (z - E_m), \quad \{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{C}. \tag{3.11}$$

Moreover, (3.9) and (3.10) imply

$$
\begin{aligned}
&F_n(z,x)F_{n,xx}(z,x) - \frac{q_x(x)}{q(x)}F_n(z,x)F_{n,x}(z,x) - \frac{1}{2}F_{n,x}(z,x)^2 \\
&+ \bigg(2z^2 - 2iz\frac{q_x(x)}{q(x)} - 2p(x)q(x)\bigg)F_n(z,x)^2 = -2q(x)^2R_{2n+2}(z), \qquad (3.12)\\
&H_n(z,x)H_{n,xx}(z,x) - \frac{p_x(x)}{p(x)}H_n(z,x)H_{n,x}(z,x) - \frac{1}{2}H_{n,x}(z,x)^2 \\
&+ \bigg(2z^2 + 2iz\frac{p_x(x)}{p(x)} - 2p(x)q(x)\bigg)H_n(z,x)^2 = -2p(x)^2R_{2n+2}(z). \qquad (3.13)
\end{aligned}
$$

Introducing the algebraic eigenspace

$$
\ker(D-z) = \bigg\{\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} : \mathbb{C} \mapsto (\mathbb{C}_\infty)^2 \text{ meromorphic} \mid (D-z)\Psi = 0\bigg\}, \quad z \in \mathbb{C}, \tag{3.14}
$$

one verifies

$$
E_{n+1}\Big|_{\ker(D-z)} = i\begin{pmatrix}-G_{n+1}(z,x) & F_n(z,x)\\ -H_n(z,x) & G_{n+1}(z,x)\end{pmatrix}\Big|_{\ker(D-z)}. \tag{3.15}
$$

Moreover, the analog of the Burchnall–Chaundy polynomial for the KdV case in Section 2 now reads

$$
E_{n+1}^2 + R_{2n+2}(D) = 0. \tag{3.16}
$$

Equation (3.16) naturally leads to the hyperelliptic curve $\dot{\mathcal{K}}_n$ defined by

$$
\dot{\mathcal{K}}_n\colon \mathcal{F}_n(z,y) = y^2 - R_{2n+2}(z) = 0, \quad R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z-E_m), \ z \in \mathbb{C}. \tag{3.17}
$$

The compactification of $\dot{\mathcal{K}}_n$, by joining $\{P_{\infty_-}, P_{\infty_+}\}$, the points at infinity, is then denoted $\mathcal{K}_n$. As in Section 2 we denote points $P \in \mathcal{K}_n\backslash\{P_{\infty_-}, P_{\infty_+}\}$ by $P = (z,y)$, where $\mathcal{F}_n(z,y) = 0$. Moreover the involution $*$ on $\mathcal{K}_n$ is defined by

$$
*\colon \mathcal{K}_n \mapsto \mathcal{K}_n, \quad P = (z,y) \mapsto P^* = (z,-y), \quad P^*_{\infty_\mp} = P_{\infty_\pm}. \tag{3.18}
$$

Because of (3.10) we may define the following fundamental meromorphic function $\phi(P,x)$ on $\mathcal{K}_n$

$$
\begin{aligned}
\phi(P,x) &= \frac{y(P) + G_{n+1}(z,x)}{F_n(z,x)} && (3.19a)\\
&= \frac{-H_n(z,x)}{y(P) - G_{n+1}(z,x)}, \quad P = (z,y) \in \mathcal{K}_n, \ x \in \mathbb{C}, && (3.19b)
\end{aligned}
$$

where $y(P)$ denotes the meromorphic function on $\mathcal{K}_n$ obtained upon solving $y^2 = R_{2n+2}(z)$ denoting $P = (z,y)$. The associated Baker–Akhiezer vector $\Psi(P,x,x_0)$ on $\mathcal{K}_n\backslash\{P_{\infty_-}, P_{\infty_+}\}$ is then defined by

$$
\begin{aligned}
&\psi_1(P,x,x_0) = \exp\bigg(\int_{x_0}^x dx'\,\big(-iz + q(x')\phi(P,x')\big)\bigg),\\
&\psi_2(P,x,x_0) = \phi(P,x)\psi_1(P,x,x_0),\\
&\Psi(P,x,x_0) = \begin{pmatrix}\psi_1(P,x,x_0)\\ \psi_2(P,x,x_0)\end{pmatrix}, \quad P = (z,y) \in \mathcal{K}_n\backslash\{P_{\infty_-}, P_{\infty_+}\}, \ x, x_0 \in \mathbb{C}, \qquad (3.20)
\end{aligned}
$$

choosing a small non-selfintersecting path from $x_0$ to $x$ avoiding singularities of $q(x)$ and $\phi(P,x)$. The analog of Lemma 2.1 then reads as follows.

**Lemma 3.1.** *(see, e.g., [30], [31]) Suppose $f_{n+1} = h_{n+1} = 0$ and let $P = (z,y) \in \mathcal{K}_n \backslash \{P_{\infty_-}, P_{\infty_+}\}$, $x, x_0 \in \mathbb{C}$. Then $\phi(P,x)$ satisfies the Riccati-type equation*

$$\phi_x(P,x) + q(x)\phi(P,x)^2 - 2iz\phi(P,x) = p(x), \tag{3.21}$$

*and*

$$\phi(P,x)\phi(P^*,x) = H_n(z,x)/F_n(z,x), \tag{3.22}$$

$$\phi(P,x) + \phi(P^*,x) = 2G_{n+1}(z,x)/F_n(z,x), \tag{3.23}$$

$$\phi(P,x) - \phi(P^*,x) = 2y(P)/F_n(z,x). \tag{3.24}$$

*Moreover, if $p(x)q(x) \neq 0$,*

$$\phi(P,x) = O(z^{\mp 1}) \text{ as } P = (z,x) \to P_{\infty_\pm}. \tag{3.25}$$

*$\Psi(P,x,x_0)$ satisfies*

$$(D - z(P))\Psi(P,\,\cdot\,,x_0) = 0, \quad (E_{n+1} - iy(P))\Psi(P,\,\cdot\,,x_0) = 0, \tag{3.26}$$

*and*

$$\psi_1(P,x,x_0)\psi_1(P^*,x,x_0) = F_n(z,x)/F_n(z,x_0), \tag{3.27}$$

$$\psi_2(P,x,x_0)\psi_2(P^*,x,x_0) = H_n(z,x)/F_n(z,x_0), \tag{3.28}$$

$$\begin{aligned}\psi_1(P,x,x_0)\psi_2(P^*,x,x_0) &+ \psi_1(P^*,x,x_0)\psi_2(P,x,x_0) \\ &= 2G_{n+1}(z,x)/F_n(z,x_0),\end{aligned} \tag{3.29}$$

$$W(\Psi(P,\,\cdot\,,x_0), \Psi(P^*,\,\cdot\,,x_0)) = -2y(P)/F_n(z,x_0). \tag{3.30}$$

*Moreover, if $p(x)q(x) \neq 0$, $q(x_0) \neq 0$, then*

$$\begin{aligned}&\psi_1(P,x,x_0) = \exp(z^{-n}y(P)(x-x_0))(1 + O(z^{-1})),\\ &\psi_2(P,x,x_0) = \exp(z^{-n}y(P)(x-x_0))O(z^{\mp 1}) \text{ as } P = (z,y) \to P_{\infty_\pm}.\end{aligned} \tag{3.31}$$

(Here $W(F,G) = f_1 g_2 - f_2 g_1$ denotes the Wronskian of $F = \binom{f_1}{f_2}$ and $G = \binom{g_1}{g_2}$.)

Since the diagonal elements $G_{\ell,\ell}(P,x,x')$, $\ell = 1,2$ of the $2\times 2$ Green's matrix $G(P,x,x')$, $x \neq x'$ associated with $D$ are discontinuous as $x \to x'$ in contrast to the off-diagonal elements $G_{\ell,\ell'}(P,x,x')$, $\ell \neq \ell'$, $\ell,\ell' = 1,2$, we consider

$$G_{1,2}(P,x,x) = -i\frac{\psi_1(P,x,x_0)\psi_1(P^*,x,x_0)}{W(\Psi(P,\,\cdot\,,x_0),\Psi(P^*,\,\cdot\,,x_0))} \tag{3.32a}$$

$$= \frac{iF_n(z,x)}{2y(P)}, \quad P = (z,y) \in \mathcal{K}_n \backslash \{P_{\infty_-}, P_{\infty_+}\},\ x \in \mathbb{C}, \tag{3.32b}$$

using $\phi = \psi_2/\psi_1$, (3.24), and (3.27). Equations (3.12) and (3.32b) then yield the universal equation

$$\begin{aligned}&2G''_{1,2}(P,x,x)G_{1,2}(P,x,x) - 2\frac{q_x(x)}{q(x)}G'_{1,2}(P,x,x)G_{1,2}(P,x,x)\\ &\quad - G'_{1,2}(P,x,x)^2 + 2\bigg(2z^2 - 2iz\frac{q_x(x)}{q(x)} - 2p(x)q(x)\bigg)G_{1,2}(P,x,x)^2 = q(x)^2,\\ &\qquad P = (z,y) \in \mathcal{K}_n \backslash \{P_{\infty_-}, P_{\infty_+}\}.\end{aligned} \tag{3.33}$$

Similarly, we find

$$G_{2,1}(P,x,x) = -i\frac{\psi_2(P,x,x_0)\psi_2(P^*,x,x_0)}{W(\Psi(P,\,\cdot\,,x_0),\Psi(P^*,\,\cdot\,,x_0))} \tag{3.34a}$$

$$= \frac{iH_n(z,x)}{2y(P)}, \quad P=(z,y)\in\mathcal{K}_n\backslash\{P_{\infty_-},P_{\infty_+}\},\ x\in\mathbb{C}, \tag{3.34b}$$

and

$$\begin{aligned} &G''_{2,1}(P,x,x)G_{2,1}(P,x,x) - 2\frac{p_x(x)}{p(x)}G'_{2,1}(P,x,x)G_{2,1}(P,x,x) \\ &\quad - G'_{2,1}(P,x,x)^2 + 2\bigg(2z^2+2iz\frac{p_x(x)}{p(x)} - 2p(x)q(x)\bigg)G_{2,1}(P,x,x)^2 = p(x)^2, \\ &\qquad P=(z,y)\in\mathcal{K}_n\backslash\{P_{\infty_-},P_{\infty_+}\},\ x\in\mathbb{C}. \end{aligned} \tag{3.35}$$

This completes our review of the stationary AKNS hierarchy. The corresponding time-dependent AKNS hierarchy can now readily be introduced, see, for instance, [30], [31].

Next we turn to gauge (i.e., Darboux-type) transformations in connection with $D = i\left(\begin{smallmatrix} d/dx & -q \\ p & -d/dx\end{smallmatrix}\right)$. Introducing

$$U(z,x) = \begin{pmatrix} -iz & -q(x) \\ p(x) & iz \end{pmatrix}, \tag{3.36}$$

the equation

$$D\Psi(z,x) = z\Psi(z,x), \quad \Psi(z,x) = \begin{pmatrix} \psi_1(z,x) \\ \psi_2(z,x)\end{pmatrix} \tag{3.37}$$

is equivalent to

$$\Psi_x(z,x) = U(z,x)\Psi(z,x). \tag{3.38}$$

The formal gauge transformation

$$\Psi(z,x) \mapsto \widetilde\Psi(z,x) = \Gamma(z,x)\Psi(z,x), \tag{3.39}$$

$$\begin{aligned} U(z,x)) \mapsto \widetilde U(z,x) &= \begin{pmatrix} -iz & -\tilde q(x) \\ \tilde p(x) & iz\end{pmatrix} \\ &= \Gamma(z,x)U(z,x)\Gamma(z,x)^{-1} + \Gamma_x(z,x)\Gamma(z,x)^{-1}, \end{aligned} \tag{3.40}$$

with $\Gamma(z,x)$ a $2\times 2$ matrix to be chosen later, then implies

$$\widetilde\Psi_x(z,x) = \widetilde U(z,x)\widetilde\Psi(z,x). \tag{3.41}$$

Hence,

$$\widetilde D\widetilde\Psi(z,x) = z\widetilde\Psi(z,x), \quad \widetilde\Psi(z,x) = \begin{pmatrix} \tilde\psi_1(z,x) \\ \tilde\psi_2(z,x)\end{pmatrix}, \tag{3.42}$$

with

$$\widetilde D = i\begin{pmatrix} d/dx & -\tilde q \\ \tilde p & -d/dx\end{pmatrix}. \tag{3.43}$$

Next, introduce

$$\Psi(P,x,x_0,\sigma) = \begin{cases} \frac12(1+\sigma)\Psi(P,x,x_0) + \frac12(1-\sigma)\Psi(P^*,x,x_0) & \text{for } \sigma\in\mathbb{C}, \\ \Psi(P,x,x_0) - \Psi(P^*,x,x_0) & \text{for } \sigma=\infty, \end{cases}$$

$$P=(z,y)\in\mathcal{K}_n\backslash\{P_{\infty_-},P_{\infty_+}\}, \quad (3.44)$$

pick $Q_0=(z_0,y_0)\in\mathcal{K}_n\backslash\{P_{\infty_-},P_{\infty_+}\}$, and define

$$\Gamma(Q_0,x,\sigma)=\begin{pmatrix} z-z_0-\frac{i}{2}q(x)\phi(Q_0,x,\sigma) & \frac{i}{2}q(x) \\ \frac{i}{2}\phi(Q_0,x,\sigma) & -\frac{i}{2}\end{pmatrix}. \quad (3.45)$$

Here $\Psi(P,x,x_0)$ is defined in (3.19) and

$$\phi(P,x,\sigma)=\psi_2(P,x,x_0,\sigma)/\psi_1(P,x,x_0,\sigma), \quad (3.46)$$
$$P=(z,y)\in\mathcal{K}_n\backslash\{P_{\infty_-},P_{\infty_+}\},\ \sigma\in\mathbb{C}_\infty.$$

According to (3.40) one then obtains

$$\widetilde{D}_\sigma(Q_0)=i\begin{pmatrix} d/dx & -\tilde{q}_\sigma(Q_0) \\ \tilde{p}_\sigma(Q_0) & -d/dx\end{pmatrix}, \quad \sigma\in\mathbb{C}_\infty, \quad (3.47a)$$
$$\tilde{p}_\sigma(x,Q_0)=\phi(Q_0,x,\sigma), \quad (3.47b)$$
$$\tilde{q}_\sigma(x,Q_0)=-2iz_0q(x)-q_x(x)+\phi(Q_0,x,\sigma)q(x)^2, \quad (3.47c)$$
$$Q_0=(z_0,y_0)\in\mathcal{K}_n\backslash\{P_{\infty_-},P_{\infty_+}\},\ \sigma\in\mathbb{C}_\infty, \quad (3.47d)$$

utilizing the fact that $\phi(P,x,\sigma)$ satisfies the Riccati-type equation (3.21) for all $\sigma\in\mathbb{C}_\infty$. The gauge transformation (or equivalently, Darboux transformation)

$$(p(x),q(x))\mapsto(\tilde{p}_\sigma(x),\tilde{q}_\sigma(x)) \quad (3.48)$$

can be inferred from the results in [41] (with a bit of additional work). Adding solitons (i.e., inserting eigenvalues into the spectrum of $D$) and its effect on the Baker-Akhiezer vector $\Psi$ has also been studied in [22], [24].

We also mention that as in Section 2 (cf. (2.43)–(2.44)), our choice of the off-diagonal elements of the Green's matrix of $D$ in (3.32) yields the only bounded Green's matrix for $P$ near $P_{\infty_\pm}$ and generic $x\in\mathbb{C}$ when comparing with more general linear combinations $\Psi(P,x,x_0,\sigma)$ in (3.44) as opposed to $\Psi(P,x,x_0)$ or $\Psi(P^*,x,x_0)$. Introducing

$$\widetilde{\Psi}_\sigma(P,x,x_0,Q_0)=\Gamma(Q_0,x,\sigma)\Psi(P,x,x_0),$$
$$P=(z,y)\in\mathcal{K}_n\backslash\{P_{\infty_-},P_{\infty_+}\},\ \sigma\in\mathbb{C}_\infty, \quad (3.49)$$

where $\Psi(P,\,\cdot\,,x_0)\in\ker(D-z)$ and $\Gamma(Q_0,x,\sigma)$ is defined in (3.45), one infers

$$\widetilde{D}_\sigma(Q_0)\widetilde{\Psi}_\sigma(P,x,x_0,Q_0)=z\widetilde{\Psi}_\sigma(P,x,x_0,Q_0). \quad (3.50)$$

Moreover,

$$\begin{aligned} &W(\widetilde{\Psi}_{\sigma,1}(P,\,\cdot\,,x_0,Q_0),\widetilde{\Psi}_{\sigma,2}(P,\,\cdot\,,x_0,Q_0)) \\ &\quad=-\frac{i}{2}(z-z_0)W(\Psi_1(P,\,\cdot\,,x_0),\Psi_2(P,\,\cdot\,,x_0)), \end{aligned} \quad (3.51)$$

where

$$\widetilde{\Psi}_{\sigma,j}(P,x,x_0,Q_0)=\Gamma(Q_0,x,\sigma)\Psi_j(P,x,x_0),$$
$$\Psi_j(P,\,\cdot\,,x_0)\in\ker(D-z),\ j=1,2. \quad (3.52)$$

Given these facts we define in analogy to (3.32a) and (3.34a) the off-diagonal Green's matrix elements associated with $\widetilde{D}_\sigma(Q_0)$ by

$$\widetilde{G}_{\sigma,1,2}(P,x,x,Q_0)=-i\frac{\tilde{\psi}_{\sigma,1}(P,x,x_0,Q_0)\tilde{\psi}_{\sigma,1}(P^*,x,x_0,Q_0)}{W(\widetilde{\Psi}_\sigma(P,\,\cdot\,,x_0,Q_0),\widetilde{\Psi}_\sigma(P^*,\,\cdot\,,x_0,Q_0))}, \quad (3.53)$$

$$\widetilde{G}_{\sigma,2,1}(P,x,x,Q_0) = -i\frac{\tilde{\psi}_{\sigma,2}(P,x,x_0,Q_0)\tilde{\psi}_{\sigma,2}(P^*,x,x_0,Q_0)}{W(\widetilde{\Psi}_\sigma(P,\,\cdot\,,x_0,Q_0),\widetilde{\Psi}_\sigma(P^*,\,\cdot\,,x_0,Q_0))}, \tag{3.54}$$
$$P=(z,y)\in\mathcal{K}_n\backslash\{Q_0,P_{\infty_-},P_{\infty_+}\},\ \sigma\in\mathbb{C}_\infty.$$

**Lemma 3.2.** *Assume $f_{n+1}=h_{n+1}=0$ and let $Q_0=(z_0,y_0)\in\mathcal{K}_n\backslash\{P_{\infty_-},P_{\infty_+}\}$, $P=(z,y)\in\mathcal{K}_n\backslash\{Q_0,P_{\infty_-},P_{\infty_+}\}$, $\sigma\in\mathbb{C}_\infty$. Then the off-diagonal elements of the Green's matrix of $\widetilde{D}_\sigma(Q_0)$ read*

$$\begin{aligned}
&\widetilde{G}_{\sigma,1,2}(P,x,x,Q_0)\\
&\quad= i\bigg(\Big(i(z-z_0)q(x)+\frac{1}{2}q(x)^2\phi(Q_0,x,\sigma)\Big)G_{n+1}(z,x)-\frac{1}{4}q(x)^2H_n(z,x)\\
&\qquad+\Big((z-z_0)^2-i(z-z_0)q(x)\phi(Q_0,x,\sigma)-\frac{1}{4}q(x)^2\phi(Q_0,x,\sigma)^2\Big)F_n(z,x)\bigg)\times\\
&\qquad\quad\times\big(-(z-z_0)y(P)\big)^{-1} &\text{(3.55a)}\\
&\quad= i\bigg((z-z_0)+\frac{i}{2}q(x)\big(\phi(P,x)-\phi(Q_0,x,\sigma)\big)\bigg)\times\\
&\qquad\times\bigg((z-z_0)+\frac{i}{2}q(x)\big(\phi(P^*,x)-\phi(Q_0,x,\sigma)\big)\bigg)F_n(z,x)\big(-(z-z_0)y(P)\big)^{-1} &\text{(3.55b)}\\
&\quad=\frac{i\widetilde{F}_{\sigma,\tilde{n}}(z,x)}{2\tilde{y}(P)} &\text{(3.55c)}
\end{aligned}$$

*and*

$$\begin{aligned}
&\widetilde{G}_{\sigma,2,1}(P,x,x,Q_0)\\
&\quad=\frac{i}{4}\big(\phi(Q_0,x,\sigma)^2F_n(z,x)-2\phi(Q_0,x,\sigma)G_{n+1}(z,x)+H_n(z,x)\big)\times\\
&\qquad\times\big((z-z_0)y(P)\big)^{-1} &\text{(3.56a)}\\
&\quad=\frac{i}{4}\big(\phi(P,x)-\phi(Q_0,x,\sigma)\big)\big(\phi(P^*,x)-\phi(Q_0,x,\sigma)\big)F_n(z,x)\big((z-z_0)y(P)\big)^{-1} &\text{(3.56b)}\\
&\quad=\frac{i\widetilde{H}_{\sigma,\tilde{n}}(z,x)}{2\tilde{y}(P)}, &\text{(3.56c)}
\end{aligned}$$

*where $\tilde{y}(P)$ denotes the meromorphic solution on $\dot{\widetilde{\mathcal{K}}}_{\sigma,\tilde{n}}(Q_0)$ obtained upon solving $y^2=\widetilde{R}_{\sigma,2\tilde{n}+2}(z)$, $P=(z,y)$ for some polynomial $\widetilde{R}_{\sigma,2\tilde{n}+2}(z)$ of degree $2\tilde{n}+2$ and $\widetilde{F}_{\sigma,\tilde{n}}(z,x)$ and $\widetilde{H}_{\sigma,\tilde{n}}(z,x)$ denote polynomials in $z$ of degree $\tilde{n}$, with $0\leq\tilde{n}\leq n+1$. In particular, the Darboux transformation (3.48), $(p(x),q(x))\mapsto(\tilde{p}_\sigma(x),\tilde{q}_\sigma(x))$ maps the class of algebro-geometric AKNS potentials into itself.*

*Proof.* We present the argument for $\widetilde{G}_{\sigma,1,2}$ only, the case $\widetilde{G}_{\sigma,2,1}$ follows similarly. As in Lemma 3.2, $\phi(P,x)=\psi_2(P,x,x_0)/\psi_1(P,x,x_0)$, (3.22), (3.23), (3.27)–(3.30), and (3.51) imply equations (3.55a) and (3.55b). Since the numerator in (3.55a) is a polynomial in $z$, and

$$\widetilde{G}_{\sigma,1,2}(P,x,x,Q_0)=\frac{\tilde{q}(x)z^n}{2y(P)}+O(|z|^{-2})\text{ as }P=(z,y)\to P_{\infty_\pm} \tag{3.57}$$

by (3.55a), one infers (3.55c) and $0 \leq \tilde{n} \leq n+1$. Again one verifies that $\widetilde{F}_{\sigma,\tilde{n}}(z,x)$ satisfies equation (3.12) with $(p(x),q(x))$ replaced by $(\tilde{p}_\sigma(x),\tilde{q}_\sigma(x))$, $n$ by $\tilde{n}$, and $R_{2n+2}(z)$ by $\widetilde{R}_{\sigma,2\tilde{n}+2}(z)$, proving that the Darboux transformation (3.48) leaves the class of algebro-geometric AKNS potentials invariant. □

The following theorem, the principal result of this section and in complete analogy to Theorem 2.3, will clarify the dependence of $\tilde{n}=\tilde{n}(n,Q_0,\sigma)$ on its variables.

**Theorem 3.3.** *Suppose $f_{n+1}=h_{n+1}=0$, let $Q_0=(z_0,y_0)\in\mathcal{K}_n\backslash\{P_{\infty_-},P_{\infty_+}\}$, $n\in\mathbb{N}_0$, $\sigma\in\mathbb{C}_\infty$, and $\tilde{n}=\tilde{n}(n,Q_0,\sigma)$ as in (3.55c). Then*

$$\tilde{n}(n,Q_0,\sigma)=\begin{cases} n+1 & \text{for } \sigma\in\mathbb{C}_\infty\backslash\{-1,1\} \text{ and } y_0\neq 0,\\ n+1 & \text{for } \sigma=\infty \text{ and } y_0=0,\\ n & \text{for } \sigma\in\{-1,1\} \text{ and } y_0\neq 0,\\ n & \text{for } \sigma\in\mathbb{C},\ y_0=0, \text{ and } R_{2n+2,z}(z_0)\neq 0,\\ n-1 & \text{for } \sigma\in\mathbb{C},\ y_0=0, \text{ and } R_{2n+2,z}(z_0)=0, n\in\mathbb{N},\end{cases} \tag{3.58}$$

*and hence the hyperelliptic curve $\dot{\widetilde{\mathcal{K}}}_{\sigma,\tilde{n}}(Q_0)$ associated with $(\tilde{p}_\sigma(x,Q_0),\tilde{q}_\sigma(x,Q_0))$ is of the type*

$$\dot{\widetilde{\mathcal{K}}}_{\sigma,\tilde{n}}\colon \widetilde{\mathcal{F}}_{\sigma,\tilde{n}}(z,y,Q_0)=y^2-\widetilde{R}_{\sigma,2\tilde{n}+2}(z,Q_0)=0, \tag{3.59}$$

*with*

$$\widetilde{R}_{\sigma,2\tilde{n}+2}(z,Q_0) = \begin{cases} (z-z_0)^2R_{2n+2}(z) & \text{for } \sigma\in\mathbb{C}_\infty\backslash\{-1,1\} \text{ and } y_0\neq 0,\\ (z-z_0)^2R_{2n+2}(z) & \text{for } \sigma=\infty \text{ and } y_0=0,\\ R_{2n+2}(z) & \text{for } \sigma\in\{-1,1\} \text{ and } y_0\neq 0,\\ R_{2n+2}(z) & \text{for } \sigma\in\mathbb{C},\ y_0=0, \text{ and } R_{2n+2,z}(z_0)\neq 0,\\ (z-z_0)^{-2}R_{2n+2}(z) & \text{for } \sigma\in\mathbb{C},\ y_0=0, \text{ and } R_{2n+2,z}(z_0)=0, n\in\mathbb{N}.\end{cases} \tag{3.60}$$

*Here*

$$R_{2n+2}(z)=\prod_{m=0}^{2n+1}(z-E_m). \tag{3.61}$$

*Proof.* The following arguments closely parallel those in the proof of Theorem 2.3. Again our starting point will be (3.55b) and (3.56b) and a careful case distinction between $\sigma\in\mathbb{C}_\infty\backslash\{-1,1\}$, $\sigma\in\{-1,1\}$, $\sigma=\infty$, and whether or not $Q_0$ is a branch point.

Case (i). $\sigma\in\mathbb{C}_\infty\backslash\{-1,1\}$ and $y_0\neq 0$: One calculates using (3.44) and (3.46),

$$\phi(Q_0,x,\sigma)=\begin{cases} \frac{(1+\sigma)\psi_2(Q_0,x,x_0)+(1-\sigma)\psi_2(Q_0^*,x,x_0)}{(1+\sigma)\psi_1(Q_0,x,x_0)+(1-\sigma)\psi_1(Q_0^*,x,x_0)} & \text{for } \sigma\in\mathbb{C}\backslash\{-1,1\},\\ \frac{\psi_2(Q_0,x,x_0)-\psi_2(Q_0^*,x,x_0)}{\psi_1(Q_0,x,x_0)-\psi_1(Q_0^*,x,x_0)} & \text{for } \sigma=\infty,\end{cases} \tag{3.62}$$

and since

$$\phi(Q_0,x)=\frac{\psi_1(Q_0,x,x_0)}{\psi_2(Q_0,x,x_0)}\neq\phi(Q_0^*,x)=\frac{\psi_1(Q_0^*,x,x_0)}{\psi_2(Q_0^*,x,x_0)}, \tag{3.63}$$

one concludes that no cancellations can occur in (3.55b) or (3.56b) and hence $\tilde{n}(n,Q_0,\sigma)=n+1$.

Case (ii). $\sigma = \infty$ and $y_0 = 0$: Then (3.19a), (3.20), (3.24) and (3.44) imply

$$
\begin{aligned}
\phi(Q_0, x, \infty) &= \lim_{P \to Q_0} \phi(P, x, \infty) \\
&= \lim_{P \to Q_0} \bigg( \bigg( \phi(P,x) \exp\bigg( \int_{x_0}^{x} dx' \big( -iz + q(x')\phi(P,x') \big) \bigg) \\
&\qquad - \phi(P^*,x) \exp\bigg( \int_{x_0}^{x} dx' \big( -iz + q(x')\phi(P^*,x') \big) \bigg) \bigg) \times \\
&\qquad \times \bigg( \exp\bigg( \int_{x_0}^{x} dx' \big( -iz + q(x')\phi(P,x') \big) \bigg) \\
&\qquad - \exp\bigg( \int_{x_0}^{x} dx' \big( -iz + q(x')\phi(P^*,x') \big) \bigg) \bigg)^{-1} \bigg) \\
&= \phi(Q_0,x) \\
&\quad + \lim_{P \to Q_0} \bigg( \frac{\phi(P,x) - \phi(P^*,x)}{\exp\big( \int_{x_0}^{x} dx'\, q(x')\phi(P,x') \big) - \exp\big( \int_{x_0}^{x} dx'\, q(x')\phi(P^*,x') \big)} \times \\
&\qquad \times \exp\bigg( \int_{x_0}^{x} dx'\, q(x')\phi(P^*,x') \bigg) \bigg) \\
&= \phi(Q_0,x) + \exp\bigg( \int_{x_0}^{x} dx'\, q(x')\phi(Q_0,x') \bigg) \times \\
&\quad \times \lim_{P \to Q_0} \bigg( \bigg( \bigg( \exp\bigg( y(P) \int_{x_0}^{x} dx' \frac{q(x')}{F_n(z,x')} \bigg) - \exp\bigg( -y(P) \int_{x_0}^{x} dx' \frac{q(x')}{F_n(z,x')} \bigg) \bigg) \times \\
&\qquad \times \exp\bigg( \int_{x_0}^{x} dx' \frac{q(x')G_{n+1}(z,x')}{F_n(z,x')} \bigg) \bigg)^{-1} \frac{2y(P)}{F_n(z,x)} \bigg) \\
&= \phi(Q_0,x) + \bigg( F_n(z_0,x) \int_{x_0}^{x} dx' \frac{q(x')}{F_n(z,x')} \bigg)^{-1}, \quad x \in \mathbb{C}\backslash\{x_0\},
\end{aligned} \tag{3.64}
$$

using $\lim_{P \to Q_0} y(P) = y(Q_0) = y_0 = 0$. Since by (3.9)

$$
\phi(Q_0, x) = \frac{G_{n+1,x}(z_0,x)}{F_n(z_0,x)} = \frac{1}{q(x)} \bigg( \frac{F_{n,x}(z_0,x)}{2F_n(z_0,x)} + iz_0 \bigg), \tag{3.65}
$$

one again concludes that no cancellation occurs in (3.55b) or (3.56b), and hence $\tilde{n}(n, Q_0, \infty) = n + 1$.

The rest of the proof relies on some additional arguments to be discussed next. First, we will assume without loss of generality that $z_0 = 0$. This can be achieved by noticing that

$$
D = i \begin{pmatrix} d/dx & -q(x) \\ p(x) & -d/dx \end{pmatrix} \text{ and } D_a = i \begin{pmatrix} d/dx & -q(x)e^{-2ax} \\ p(x)e^{2ax} & -d/dx \end{pmatrix} \tag{3.66}
$$

are related by

$$
UD_aU^{-1} = D - iaI, \quad U = \begin{pmatrix} e^{2ax} & 0 \\ 0 & e^{-2ax} \end{pmatrix}. \tag{3.67}
$$

Moreover, we will exclude the trivial case where $p(x)$ and $q(x)$ are proportional to $e^{-2ic_1x}$ and $e^{2ic_1x}$, $x \in \mathbb{C}$, respectively (cf. Example 3.4 below). We write

$$
y(z)^2 = R_{2n+2}(z) \underset{z \to 0}{=} y_0^2 + \tilde{y}_1 z + \tilde{y}_2 z^2 + O(z^3). \tag{3.68}
$$

Case (iii). $\sigma\in\{-1,1\}$ and $y_0\neq 0$: Then (3.44) yields

$$\phi(Q_0,x,1)=\phi(Q_0,x),\quad \phi(Q_0,x,-1)=\phi(Q_0^*,x), \tag{3.69}$$

with $\phi(Q_0,x)\neq\phi(Q_0^*,x)$ since $y_0\neq 0$. In this case there is a cancellation in (3.55b) and (3.56b). Choosing $\sigma=1$ one computes from (3.8) and (3.19a),

$$\phi(P^*,x)-\phi(Q_0,x,1)=\phi(P^*,x)-\phi(Q_0,x)\underset{P\to Q_0}{=}-\frac{2y_0}{f_n(x)}+O(z). \tag{3.70}$$

Furthermore,

$$\begin{aligned}
&\phi(P,x)-\phi(Q_0,x,1)=\phi(P,x)-\phi(Q_0,x)\\
&\underset{P\to Q_0}{=}\frac{y(P)-y_0}{f_n(x)}-\bigg(y_0\frac{f_{n-1}(x)}{f_n(x)^2}-\frac{g_n(x)}{f_n(x)}+\frac{g_{n+1}(x)f_{n-1}(x)}{f_n(x)^2}\bigg)z+O(z^2)\\
&\underset{P\to Q_0}{=}\bigg(\frac{y_1}{f_n(x)}-\frac{y_0f_{n-1}(x)}{f_n(x)^2}+\frac{g_n(x)}{f_n(x)}-\frac{g_{n+1}(x)f_{n-1}(x)}{f_n(x)^2}\bigg)z+O(z^2)\\
&\underset{P\to Q_0}{=}c_1(x)z+O(z^2)
\end{aligned} \tag{3.71}$$

since

$$y(P)-y_0\underset{P\to Q_0}{=}y_1z+O(z^2),\quad \tilde y_1=2y_0y_1. \tag{3.72}$$

Similarly, we find

$$z+\frac{i}{2}q(x)\big(\phi(P,x)-\phi(Q_0,x,1)\big)\underset{P\to Q_0}{=}\bigg(1+\frac{i}{2}q(x)c_1(x)\bigg)z+O(z^2). \tag{3.73}$$

It remains to show that $c_1$ does not vanish identically. We assume temporarily that $\widetilde G_{\sigma,1,2}$ and $\widetilde G_{\sigma,2,1}$ have cancellations of the same order as $z\to 0$. Arguing by contradiction we suppose that $c_1$ vanishes identically. But (3.71) and (3.73) then show that $\widetilde G_{\sigma,1,2}$ and $\widetilde G_{\sigma,2,1}$ would have cancellations of different order, which is a contradiction. We conclude that precisely one factor of $z$ cancels in (3.55b) and (3.56b), and hence $\tilde n(n,Q_0,1)=n$. The case $\sigma=-1$ is treated analogously. It remains to show that $\widetilde G_{\sigma,1,2}$ and $\widetilde G_{\sigma,2,1}$ necessarily have cancellations of the same order as $z\to 0$. A comparison of $\widetilde D_\sigma=i\left(\begin{smallmatrix}d/dx & -\tilde q_\sigma\\ \tilde p_\sigma & -d/dx\end{smallmatrix}\right)$ and its formal adjoint $\widetilde D_\sigma^*=i\left(\begin{smallmatrix}d/dx & -\overline{\tilde p_\sigma}\\ \overline{\tilde q_\sigma} & -d/dx\end{smallmatrix}\right)$ yields the replacement of $(\tilde p_\sigma,\tilde q_\sigma)$ by $(\overline{\tilde q_\sigma},\overline{\tilde p_\sigma})$ and hence the corresponding replacements of $(\widetilde F_{\sigma,\tilde n}(z,x),\widetilde G_{\sigma,\tilde n+1}(z,x),\widetilde H_{\sigma,\tilde n}(z,x))$ by $\Big(\overline{\widetilde F_{\sigma,\tilde n}(\overline z,x)},\overline{\widetilde G_{\sigma,\tilde n+1}(\overline z,x)},\overline{\widetilde H_{\sigma,\tilde n}(\overline z,x)}\Big)$ (cf. (3.2) and (3.8)) and $\widetilde R_{\sigma,2\tilde n+2}(z)$ by $\overline{\widetilde R_{\sigma,2\tilde n+2}(\overline z)}$ (cf. the notation employed in Lemma 3.2). This fact has two consequences: Firstly, from relation (3.10) we infer that the corresponding algebraic curves associated with $\widetilde D_\sigma$ and $\widetilde D_\sigma^*$ have complex conjugate branch points, that is, if $\{\widetilde E_{\sigma,m}\}_{m=0,\dots,2\tilde n+1}$ corresponds to $\widetilde D_\sigma$, then $\Big\{\overline{\widetilde E_{\sigma,m}}\Big\}_{m=0,\dots,2\tilde n+1}$ corresponds to $\widetilde D_\sigma^*$, where $\widetilde R_{\sigma,2\tilde n+2}(z)=\prod_{m=0}^{2\tilde n+1}(z-\widetilde E_{\sigma,m})$. Secondly, we infer

$$\widetilde G_{\sigma,2,1}(P,x,x)=\overline{\widetilde G^*_{\sigma,2,1}(\overline P,x,x)}, \tag{3.74}$$

where $P(z,y)$, $\overline P=(\overline z,\overline y)$, and $\widetilde G^*_\sigma(P,x,x)$ denotes the Green's matrix associated with $\widetilde D^*_\sigma$. This shows that any cancellations in $\widetilde G_{\sigma,1,2}$ and $\widetilde G_{\sigma,2,1}$ as $z\to 0$ are necessarily of identical order.

Case (iv). $\sigma \in \mathbb{C}$, $y_0 = 0$, and $R_{2n+1,z}(0) \neq 0$: Using $\phi(Q_0, x, \sigma) = \phi(Q_0, x)$ for all $\sigma \in \mathbb{C}$, (3.8) and (3.19a) yield

$$\begin{aligned} &\Big(z + \frac{i}{2}q(x)\big(\phi(P,x) - \phi(Q_0,x)\big)\Big)\Big(z + \frac{i}{2}q(x)\big(\phi(P^*,x) - \phi(Q_0,x)\big)\Big) \\ &\underset{P\to Q_0}{=} \frac{q(x)^2 y_1^2}{4f_n(x)^2} z + O(z^2) \end{aligned} \tag{3.75}$$

since

$$y(P) \underset{P\to Q_0}{=} y_1 z^{1/2} + O(z^{3/2}), \quad y_1 = \Big(\prod_{E_m \neq 0} E_m\Big)^{1/2}. \tag{3.76}$$

Thus again precisely one factor of $z$ cancels in (3.55b) (similarly, one factor cancels in (3.56b)), and hence $\tilde{n}(n, Q_0, \sigma) = n$.

Case (v). $\sigma \in \mathbb{C}$, $y_0 = \tilde{y}_1 = 0$, and $\tilde{y}_2 \neq 0$ (cf. (3.68)): One computes as in (3.75),

$$\begin{aligned} &\Big(z + \frac{i}{2}q(x)\big(\phi(P,x) - \phi(Q_0,x)\big)\Big)\Big(z + \frac{i}{2}q(x)\big(\phi(P^*,x) - \phi(Q_0,x)\big)\Big) \\ &\underset{P\to Q_0}{=} \bigg(\frac{q(x)^2 y_1^2}{4f_n(x)^2} + \Big(1 + \frac{i}{2}q(x)\Big(\frac{g_n(x)}{f_n(x)} - \frac{g_{n+1}(x)f_{n-1}(x)}{f_n(x)^2}\Big)\Big)^2\bigg) z^2 + O(z^3) \\ &\underset{P\to Q_0}{=} c_2(x) z^2 + O(z^3) \end{aligned} \tag{3.77}$$

since

$$y(P) \underset{P\to Q_0}{=} y_1 z + O(z^2), \quad y_1 = \Big(\prod_{E_m \neq 0} E_m\Big)^{1/2}. \tag{3.78}$$

Similarly, we find

$$\begin{aligned} &\big(\phi(P,x) - \phi(Q_0,x)\big)\big(\phi(P^*,x) - \phi(Q_0,x)\big) \\ &\underset{P\to Q_0}{=} -\bigg(\frac{y_1^2}{f_n(x)^2} - \Big(\frac{g_n(x)}{f_n(x)} - \frac{g_{n+1}(x)f_{n-1}(x)}{f_n(x)^2}\Big)^2\bigg) z^2 + O(z^3) \\ &\underset{P\to Q_0}{=} c_3(x) z^2 + O(z^3). \end{aligned} \tag{3.79}$$

We see that both $c_2(x)$ and $c_3(x)$ cannot vanish simultaneously, and hence precisely a factor $z^2$ cancels in (3.55b) and (3.56b). Thus $\tilde{n}(n, Q_0, \sigma) = n - 1$.

Case (vi). $\sigma \in \mathbb{C}$, $y_0 = \tilde{y}_1 = \tilde{y}_2 = 0$ (cf. (3.68)): In analogy to (3.77) and (3.79) one obtains

$$\begin{aligned} &\Big(z + \frac{i}{2}q(x)\big(\phi(P,x) - \phi(Q_0,x)\big)\Big)\Big(z + \frac{i}{2}q(x)\big(\phi(P^*,x) - \phi(Q_0,x)\big)\Big) \\ &\underset{P\to Q_0}{=} \Big(1 + \frac{i}{2}q(x)\Big(\frac{g_n(x)}{f_n(x)} - \frac{g_{n+1}f_{n-1}}{f_n^2}\Big)\Big)^2 z^2 + O(z^3) \end{aligned} \tag{3.80}$$

and

$$\begin{aligned} &\big(\phi(P,x) - \phi(Q_0,x)\big)\big(\phi(P^*,x) - \phi(Q_0,x)\big) \\ &\underset{P\to Q_0}{=} \Big(\frac{g_n(x)}{f_n(x)} - \frac{g_{n+1}f_{n-1}}{f_n^2}\Big)^2 z^2 + O(z^3) \end{aligned} \tag{3.81}$$

respectively, since

$$y(P) \underset{P \to Q_0}{=} O(z^{3/2}). \tag{3.82}$$

Thus this case subordinates to case (v) resulting again in $\tilde{n}(n, Q_0, \sigma) = n - 1$. $\square$

While Lemma 3.2 has first been noted in [39] (see also [38]), Theorem 3.3 appears to be new. We also emphasize that Table 2.4 and Remark 2.6 apply in the present AKNS context.

We conclude this section with the elementary genus zero example.

**Example 3.4.** The case $n = 0$.

$$\begin{aligned}
y(P)^2 &= R_2(z) = (z - E_0)(z - E_1), \\
c_1 &= -(E_0 + E_1)/2, \\
p(x) &= p(x_0)e^{-2ic_1(x-x_0)}, \quad q(x) = q(x_0)e^{2ic_1(x-x_0)}, \\
p(x)q(x) &= (E_0 - E_1)^2/4, \\
F_0(z,x) &= -iq(x), \quad G_1(z,x) = z + c_1, \quad H_0(z,x) = ip(x), \\
\phi(P,x) &= \frac{y(P) + z + c_1}{-iq(x)} = \frac{ip(x)}{y(P) - z - c_1}, \\
\psi_1 &= e^{i(y(P)+c_1)(x-x_0)}, \quad \psi_2 = \frac{y(P) + z + c_1}{-iq(x_0)} e^{i(y(P)-c_1)(x-x_0)},
\end{aligned}$$

$$\phi(P,x,\sigma) = \frac{i}{q(x)} \times$$

$$\times \begin{cases} \frac{(1+\sigma)(y(P)+z+c_1)\exp(iy(P)(x-x_0)) + (1-\sigma)(-y(P)+z+c_1)\exp(-iy(P)(x-x_0))}{(1+\sigma)\exp(iy(P)(x-x_0)) + (1-\sigma)\exp(-iy(P)(x-x_0))}, & \sigma \in \mathbb{C}, \\ \frac{(y(P)+z+c_1)\exp(iy(P)(x-x_0)) + (y(P)-z-c_1)\exp(-iy(P)(x-x_0))}{\exp(iy(P)(x-x_0)) + \exp(-iy(P)(x-x_0))}, & \sigma = \infty, \end{cases}$$

$$\phi((E_j,0),x,\sigma) = \frac{1}{2q(x)} \begin{cases} i(E_j + c_1), & \sigma \in \mathbb{C}, \\ i(E_j + c_1) + (x - x_0)^{-1}, & \sigma = \infty, \end{cases} \quad j = 0, 1. \tag{3.83}$$

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