Dissipative Quasigeostrophic Motion under Temporally Almost Periodic Forcing *

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January 19, 1999

Abstract

The full nonlinear dissipative quasigeostrophic model is shown to have a unique temporally almost periodic solution when the wind forcing is temporally almost periodic under suitable constraints on the spatial square–integral of the wind forcing and the β parameter, Ekman number, viscosity and the domain size. The proof involves the pullback attractor for the associated nonautonomous dynamical system.

Key words: Quasigeostrophic fluid model, dissipative nonautonomous dynamics, almost periodic motion, pullback attractor.

Short running title: Almost Periodic Quasigeostrophic Motion

*This work was partly supported by the National Science Foundation Grant DMS-9704345 and the DFG Forschungschwerpunkt "Ergodentheorie, Analysis und effiziente Simulation dynamischer Systeme". This work was begun during a visit to the Oberwolfach Mathematical Research Institute, Germany.

1 Introduction

The barotropic quasigeostrophic (QG) flow model is derived as an approximation of the rotating shallow water equations by an asymptotic expansion in a small Rossby number. The lowest order approximation, which is also the conservation law for the 0th order potential vorticity, gives the barotropic QG equation ([16], page 234)

$$\Delta\psi_t + J(\psi, \Delta\psi) + \beta\psi_x = \nu\Delta^2\psi - r\Delta\psi + f(x, y, t), \tag{1}$$

where $\psi(x, y, t)$ is the stream function, $\beta > 0$ the meridional gradient of the Coriolis parameter, $\nu > 0$ the viscous dissipation constant, $r > 0$ the Ekman dissipation constant and $f(x, y, t)$ the wind forcing. In addition, $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplacian operator in the plane and $J(f, g) = f_x g_y - f_y g_x$ is the Jacobian operator.

Equation (1) can be rewritten in terms of the relative vorticity $\omega(x, y, t) = \Delta\psi(x, y, t)$ as

$$\omega_t + J(\psi, \omega) + \beta\psi_x = \nu\Delta\omega - r\omega + f(x, y, t)\ , \tag{2}$$

For an arbitrary bounded planar domain D with area $|D|$ and piecewise smooth boundary this equation can be supplemented by homogeneous Dirichlet boundary conditions for both ψ and $\omega = \Delta\psi$, namely

$$\psi(x, y, t) = 0, \qquad \omega(x, y, t) = 0 \quad \text{on } \partial D\ , \tag{3}$$

together with an appropriate initial condition,

$$\omega(x, y, 0) = \omega_0(x, y) \quad \text{on } D\ . \tag{4}$$

The global well–posedness (i.e. existence and uniqueness of smooth solutions) of the dissipative model (2)–(4) can be obtained similarly as in, for example, [1, 11, 19]; see also [2]. Steady wind forcing has been used in numerical simulations [5] and Duan [10] has shown the existence of time periodic quasigeostrophic response of time periodic wind forcing by means of a Leray–Schauder topological degree argument and Browder's principle. In this paper it is assumed that the wind forcing $f(x, y, t)$ is temporally almost periodic and a concept of pullback attraction [7, 15] will be used to establish the existence of a unique temporally almost periodic solution of (2)–(3) under appropriate constraints on the model parameters. The main result is

Theorem 1 *Assume that*

$$\frac{r}{2} + \frac{\pi\nu}{|D|} > \frac{1}{2}\beta\left(\frac{|D|}{\pi} + 1\right)$$

and that the wind forcing $f(x, y, t)$ is temporally almost periodic with its $L^2(D)$–norm bounded uniformly in time $t \in I\!R$ by

$$||f(\cdot, \cdot, t)|| \le \sqrt{\frac{\pi r}{|D|}}\ \left[\frac{r}{2} + \frac{\pi\nu}{|D|} - \frac{1}{2}\beta\left(\frac{|D|}{\pi} + 1\right)\right]^{\frac{3}{2}}.$$

Then the dissipative quasigeostrophic model (2)–(3) has a unique temporally almost periodic solution that exists for all time $t \in I\!R$.

The necessary terminology will be presented as required in the text and proof that follow. Some mathematical preliminaries are stated below, while dissipativity and strong contraction properties of QG flow are established in Section 2. Background material on pullback attraction for nonautonomous systems is presented in Section 3 and that for almost periodicity in Section 4, where it is applied to the QG model under consideration to complete the proof of Theorem 1.

Standard abbreviations $L^2 = L^2(D)$, $H_0^k = H_0^k(D)$, $k = 1, 2, \ldots$, are used for the common Sobolev spaces in fluid mechanics [18], with $< \cdot, \cdot >$ and $\| \cdot \|$ denoting the usual scalar product and norm, respectively, in L^2. We need the following properties and estimates (see also [11]) of the Jacobian operator $J : H_0^1 \times H_0^1 \to L^1$.

$$\int_D J(f, g)h\, dxdy = -\int_D J(f, h)g\, dxdy, \tag{5}$$

$$\int_D J(f, g)g\, dxdy = 0, \tag{6}$$

$$\left|\int_D J(f, g)\, dxdy\right| \le \|\nabla f\| \|\nabla g\| \tag{7}$$

for all f, g, $h \in H_0^1$, and

$$\left|\int_D J(\Delta f, g)\Delta h\, dxdy\right| \le \sqrt{\frac{2|D|}{\pi}} \|\Delta f\|\ \|\Delta g\|\ \|\Delta h\| \tag{8}$$

for all f, g, $h \in H_0^2$, as well as the *Poincaré inequality* [13]

$$\|g\|^2 = \int_D g^2(x,y)\,dxdy \le \frac{|D|}{\pi}\int_D |\nabla g|^2\,dxdy = \frac{|D|}{\pi}\|\nabla g\|^2 \tag{9}$$

for $g \in H_0^1$, and *Young's inequality* [13]

$$AB \le \frac{\epsilon}{2}A^2 + \frac{1}{2\epsilon}B^2, \tag{10}$$

where A, B are non-negative real numbers and $\epsilon > 0$.

2 Dynamics of dissipative QG flow

We first show that the equation (2) with boundary conditions (3) is a dissipative system in the sense ([14, 18], see also [10]) that all solutions $\omega(x,y,t)$ approach a bounded set in the phase space L^2 as time goes to infinity provided that the L^2 norm of the forcing term is uniformly bounded in time and that the system parameters satisfy the inequality of Theorem 1. Then we show that the system is strongly contracting under the restriction on the magnitude of the L^2 norm of the forcing term assumed in Theorem 1.

2.1 Dissipativity property

Define the solution operator $S_{t,t_0} : L^2 \to L^2$ by $S_{t,t_0}\omega_0 := \omega(t)$ for $t \ge t_0$, where $\omega(t)$ is the solution of the QG equations in L^2 starting at $\omega_0 \in L^2$ at time t_0. Since the the dissipative QG equations (2)–(3) are strictly parabolic, the solution operators S_{t,t_0} exist and are compact for all $t > t_0$; see, for example, [18]. In fact, the S_{t,t_0} are compact in H_0^k for all $k \ge 0$ and so, in particular, $S_{t,t_0}B$ is a compact subset of L^2 for each $t > t_0$ and every closed and bounded subset B of L^2.

Multiplying (2) by ω and integrating over D, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega\|^2 &= -\nu\|\nabla\omega\|^2 - r\|\omega\|^2 + \int_D f(x,y,t)\omega\,dxdy \\ &\quad - \int_D J(\psi,\omega)\omega\,dxdy - \beta\int_D \psi_x\omega\,dxdy.\end{aligned} \tag{11}$$

Now $\int_D J(\psi,\omega)\omega\, dxdy = 0$ by (5) and from the Young and Poincaré inequalities we have

$$\begin{aligned}\left|\beta \int_D \psi_x \omega\, dxdy\right| &\leq \frac{1}{2}\beta\left(\int_D \psi_x^2\, dxdy + \int_D \omega^2\, dxdy\right)\\ &\leq \frac{1}{2}\beta\left(\frac{|D|}{\pi}\int_D \omega^2\, dxdy + \int_D \omega^2\, dxdy\right),\end{aligned}$$

that is

$$\left|\beta \int_D \psi_x \omega\, dxdy\right| \leq \frac{1}{2}\beta\left(\frac{|D|}{\pi}+1\right)\|\omega\|^2, \tag{12}$$

and by the Poincaré inequality again we also have

$$-\nu\|\nabla\omega\|^2 \leq -\frac{\pi\nu}{|D|}\|\omega\|^2. \tag{13}$$

Now assume that the square-integral of the wind forcing $f(x,y,t)$ with respect to $(x,y)\in D$ is uniformly bounded in time, i.e.

$$\|f(\cdot,\cdot,t)\| \leq M \tag{14}$$

for some positive constant M. (This is mild assumption because a temporally almost periodic function is bounded in time, see [3] and later). Then

$$\begin{aligned}\left|\int_D f(x,y,t)\omega\, dxdy\right| &\leq \frac{1}{2r}\int_D f^2(x,y,t)\, dxdy + \frac{r}{2}\int_D \omega^2\, dxdy\\ &\leq \frac{M^2}{2r} + \frac{r}{2}\|\omega\|^2.\end{aligned}$$

Putting (12)–(15) into (11) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2 + \alpha\|\omega\|^2 \leq \frac{M^2}{2r}, \tag{15}$$

where

$$\alpha = \frac{r}{2} + \frac{\pi\nu}{|D|} - \frac{1}{2}\beta\left(\frac{|D|}{\pi}+1\right). \tag{16}$$

Then $\alpha > 0$ if we assume that

$$\frac{r}{2} + \frac{\pi\nu}{|D|} > \frac{1}{2}\beta\left(\frac{|D|}{\pi}+1\right), \tag{17}$$

which is in fact the first constraint of Theorem 1. Thus, by the Gronwall inequality, we have

$$\|\omega\|^2 \leq \|\omega_0\|^2 e^{-2\alpha t} + \frac{M^2}{2r\alpha}\left(1 - e^{-2\alpha t}\right). \tag{18}$$

Hence all solutions ω enter the closed and bounded set

$$\mathcal{B} = \left\{\omega : \ \|\omega\| \leq \frac{M}{\sqrt{2r\alpha}}\right\}$$

in finite time and stay there. The set $\mathcal{B}$ is thus an absorbing set of the system and is positively invariant in the sense that $S_{t,t_0}\mathcal{B} \subset \mathcal{B}$ for all $t \geq t_0$ and $t_0 \in I\!\!R$.

For later purposes note that the solution operator S_{t,t_0} satisfies $S_{t_0,t_0} = id.$ and $S_{t_2,t_1} \circ S_{t_1,t_0} = S_{t_2,t_0}$ for any $t_0 \leq t_1 \leq t_2$, that is $\{S_{t,t_0} \ : t \geq t_0, t_0 \in I\!\!R\}$ is a nonautonomous process or cocycle mapping. In addition, it follows from existence and uniqueness theory that $(t, t_0, \omega_0) \to S_{t,t_0}\omega_0$ is continuous. Hence, in particular, when the forcing f is independent of time there exists a global autonomous attractor defined by

$$\mathcal{A}_0 = \bigcap_{t \geq 0} S_{t,0}\mathcal{B},$$

which is a nonempty compact subset of L^2, and is invariant under the autonomous semigroup $\{S_{t,0} \ : t \geq 0\}$ in the sense that $S_{t,0}\mathcal{A}_0 = \mathcal{A}_0$ for all $t \geq 0$.

2.2 Strong contraction property

Now consider two trajectories $\omega^{(i)}$ corresponding to initial values $\omega_0^{(i)} \in \mathcal{B}$, $i = 1$ and 2. Note that these trajectories remain inside $\mathcal{B}$. Their difference $\delta\omega = \omega^{(1)} - \omega^{(2)}$ satisfies the equation

$$\delta\omega_t + J\left(\psi^{(1)}, \omega^{(1)}\right) - J\left(\psi^{(2)}, \omega^{(2)}\right) + \beta\delta\psi_x = \nu\Delta\delta\omega - r\delta\omega.$$

Similarly to the proof above it can be shown from this equation that

$$\frac{1}{2}\frac{d}{dt}\|\delta\omega\|^2 + \int_D \delta J\delta\ \omega\, dxdy + \beta\int_D \delta\psi_x\, \delta\omega\, dxdy = -\nu\|\nabla\delta\omega\|^2 - r\|\delta\omega\|^2 \tag{19}$$

where

$$\delta J(\psi, \omega) := J(\psi^{(1)}, \omega^{(1)}) - J(\psi^{(2)}, \omega^{(2)}).$$

Now from the properties (5)–(8) of the Jacobian J we have

$$\begin{aligned}
\left|\int_D \delta J \delta\omega \, dxdy\right| &= \left|\int_D \left(J\left(\psi^{(1)}, \omega^{(1)}\right) - J\left(\psi^{(2)}, \omega^{(2)}\right)\right)\left(\omega^{(1)} - \omega^{(2)}\right) dxdy\right| \\
&= \left|\int_D J\left(\psi^{(1)}, \omega^{(1)}\right)\omega^{(2)} \, dxdy + \int_D J\left(\psi^{(2)}, \omega^{(2)}\right)\omega^{(1)} \, dxdy\right| \\
&= \left|\int_D J\left(\psi^{(1)}, \omega^{(1)}\right)\omega^{(2)} \, dxdy - \int_D J\left(\psi^{(2)}, \omega^{(1)}\right)\omega^{(2)} \, dxdy\right| \\
&= \left|\int_D J\left(\psi^{(1)} - \psi^{(2)}, \omega^{(1)}\right)\left(\omega^{(1)} - \omega^{(2)}\right) dxdy\right| \\
&= \left|\int_D J\left(\delta\psi, \omega^{(1)}\right)\delta\omega \, dxdy\right| \\
&= \left|\int_D J\left(\omega^{(1)}, \delta\psi\right)\delta\omega \, dxdy\right| \\
&= \left|\int_D J\left(\Delta\psi^{(1)}, \delta\psi\right)\Delta\delta\psi \, dxdy\right| \\
&\leq \sqrt{\frac{2|D|}{\pi}}\|\Delta\psi^{(1)}\| \, \|\Delta\delta\psi\| \, \|\Delta\delta\psi\| \\
&= \sqrt{\frac{2|D|}{\pi}}\|\omega^{(1)}\| \, \|\delta\omega\|^2,
\end{aligned}$$

where in the last two steps,we have used (8) with $f = \psi^{(1)}, g = h = \delta\psi$, and the fact $\Delta\delta\psi = \delta\Delta\psi = \delta\omega$. Using this and noting that $\omega^{(1)}$ is in the positively invariant absorbing set $\mathcal{B}$ so $\|\omega^{(1)}\| \leq M/\sqrt{2r\alpha}$, we have

$$\begin{aligned}
\left|\int_D \delta J \delta\omega \, dxdy\right| &\leq \sqrt{\frac{2|D|}{\pi}}\|\omega^{(1)}\| \, \|\delta\omega\|^2 \\
&\leq \sqrt{\frac{2|D|}{\pi}}\frac{M}{\sqrt{2r\alpha}}\|\delta\omega\|^2
\end{aligned}$$

$$= \sqrt{\frac{|D|}{\pi r \alpha}} \; M\|\delta\omega\|^2. \tag{20}$$

Then from equation (19), using (20) and (12), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\delta\omega\|^2 &= -\nu\|\nabla\delta\omega\|^2 - r\|\delta\omega\|^2 - \int_D \delta J \delta\omega \, dxdy \\
&\quad -\beta \int_D \delta\psi_x \delta\omega \, dxdy \\
&\leq -\nu\|\nabla\delta\omega\|^2 - r\|\delta\omega\|^2 + \left|\int_D \delta J \delta\omega \, dxdy\right| \\
&\quad + \left|\beta \int_D \delta\psi_x \delta\omega \, dxdy\right| \\
&\leq -\frac{\pi\nu}{|D|}\|\delta\omega\|^2 - r\|\delta\omega\|^2 + \sqrt{\frac{|D|}{\pi r \alpha}} \; M\|\delta\omega\|^2 \\
&\quad + \frac{1}{2}\beta\left(\frac{|D|}{\pi} + 1\right)\|\delta\omega\|^2 \\
&\leq -\gamma\|\delta\omega\|^2
\end{aligned}$$

where

$$\gamma := r + \frac{\pi\nu}{|D|} - \frac{1}{2}\beta\left(\frac{|D|}{\pi} + 1\right) - \sqrt{\frac{|D|}{\pi r \alpha}} \; M.$$

Note that $\gamma > \alpha - \sqrt{\frac{|D|}{\pi r \alpha}} \; M$. Thus, $\gamma > 0$ if we assume that

$$\|f(\cdot,\cdot,t)\| \leq M < \sqrt{\frac{\pi r}{|D|}}\alpha^{\frac{3}{2}}, \tag{21}$$

for all $t \in I\!R$. Here α is defined in (16), so (21) holds because of the assumption on the L^2 norm of f in Theorem 1. This gives

$$\|\delta\omega(t)\|^2 \leq \|\omega_0\| e^{-2\gamma t} \to 0 \quad \text{as} \quad t \to \infty,$$

for solutions starting within the positively invariant absorbing set $\mathcal{B}$. This is the desired strong contractive condition. This means there is a unique solution $\omega^*(t)$ in $\mathcal{B}$ to which all other solutions converge. This solution $\omega^*(t)$ can be determined by the pullback convergence to be discussed in the following two Sections.

3 Nonautonomous dynamical systems

In order to show existence of temporally almost periodic solutions, we need some results from the theory of nonautonomous dynamical systems. Consider first an autonomous dynamical system on a metric space P described by a group $\theta = \{\theta_t\}_{t\in I\!R}$ of mappings of P into itself.

Let X be a complete metric space and consider a continuous mapping

$$\Phi : I\!R^+ \times P \times X \to X$$

satisfying the properties

$$\Phi(0,p,\cdot) = \mathrm{id}_X, \qquad \Phi(\tau + t, p, x) = \Phi(\tau, \theta_t p, \Phi(t,p,x))$$

for all t, $\tau \in I\!R^+$, $p \in P$ and $x \in X$. The mapping Φ is called a cocycle on X with respect to θ on P.

The appropriate concept of an attractor for a nonautonomous cocyle systems is the *pullback attractor*. In contrast to autonomous attractors it consists of a family subsets of the original state space X that are indexed by the cocycle parameter set.

Definition 1 *A family $\widehat{A} = \{A_p\}_{p\in P}$ of nonempty compact sets of X is called a* pullback attractor *of the cocycle Φ on X with respect to θ_t on P if it is Φ–invariant, i.e.*

$$\Phi(t,p,A_p) = A_{\theta_t p} \qquad \textit{for all} \quad t \in I\!R^+, p \in P,$$

and pullback attracting*, i.e.*

$$\lim_{t\to\infty} H_X^*\left(\Phi(t,\theta_{-t}p,D), A_p\right) = 0 \qquad \textit{for all} \quad D \in K(X),\ p \in P,$$

where $K(X)$ is the space of all nonempty compact subsets of the metric space (X, d_X).

Here H_X^* is the Hausdorff semi–metric between nonempty compact subsets of X, i.e. $H_X^*(A,B) := \max_{a\in A} \mathrm{dist}(a,B) = \max_{a\in A}\min_{b\in B} d_X(A,b)$ for A, $B \in K(X)$.

The following theorem combines several known results. See Crauel and Flandoli [9], Flandoli and Schmalfuß [12], and Cheban [6] as well as [7, 15] for this and various related proofs.

Theorem 2 *Let Φ be a continuous cocycle on a metric space X with respect to a group θ of continuous mappings on a metric space P. In addition, suppose that there is a nonempty compact subset B of X and that for every $D \in K(X)$ there exists a $T(D) \in I\!R^+$, which is independent of $p \in P$, such that*

$$\Phi(t, p, D) \subset B \quad \textit{for all} \quad t > T(D). \tag{22}$$

Then there exists a unique pullback attractor $\widehat{A} = \{A_p\}_{p \in P}$ of the cocycle Φ on X, where

$$A_p = \bigcap_{\tau \in I\!R^+} \overline{\bigcup_{\substack{t > \tau \\ t \in I\!R^+}} \Phi\left(t, \theta_{-t} p, B\right)}. \tag{23}$$

Moreover, the mapping $p \mapsto A_p$ is upper semicontinuous.

Moreover, in [7] it is shown that the pullback attractor consists of a single trajectory when the cocycle dynamics are in fact strongly contracting.

Theorem 3 *Suppose that the cocycle Φ in Theorem 2 is strongly contracting inside the absorbing set B. Then the pullback attractor consists of singleton valued sets, i.e. $A_p = \{a^*(p)\}$, and the mapping $p \mapsto a^*(p)$ is continuous.*

4 Almost periodicity

A function $\varphi : I\!R \to X$, where (X, d_X) is a metric space, is called *almost periodic* [3] if for every $\varepsilon > 0$ there exists a relatively dense subset M_ε of $I\!R$ such that

$$d_X\left(\varphi(t + \tau), \varphi(t)\right) < \varepsilon$$

for all $t \in I\!R$ and $\tau \in M_\varepsilon$. A subset $M \subseteq I\!R$ is called *relatively dense* in $I\!R$ if there exists a positive number $l \in I\!R$ such that for every $a \in I\!R$ the interval $[a, a + l] \bigcap I\!R$ of length l contains an element of M, i.e. $M \bigcap [a, a + l] \neq \emptyset$ for every $a \in I\!R$.

The QG solution operators S_{t,t_0} form a cocycle mapping on $X = L^2$ with parameter set $P = I\!R$, where $p = t_0$, the initial time, and $\theta_t t_0 = t_0 + t$, the left shift by time t. However, the space $P = I\!R$ is not compact here. Though more complicated, it is more useful to consider P to be the closure of the subset $\{\theta_t f, t \in I\!R\}$, i.e. the hull of f, in the metric space $L^2_{loc}\left(I\!R, L^2(D)\right)$ of locally $L^2(I\!R)$–functions $f : I\!R \to L^2(D)$ with the metric

$$d_P(f,g) := \sum_{N=1}^{\infty} 2^{-N} \min\left\{1, \sqrt{\int_{-N}^{N} \|f(t)-g(t)\|^2\, dt}\right\}$$

with θ_t defined to be the left shift operator, i.e. $\theta_t f(\cdot) := f(\cdot + t)$. By a classical result [3, 17], a function f in the above metric space is almost periodic if and only if the the hull of f is compact and minimal. Here minimal means nonempty, closed and invariant with respect to the autonomous dynamical system generated by the shift operators θ_t such that with no proper subset has these properties. The cocycle mapping is defined to be the QG solution $\omega(t)$ starting at ω_0 at time $t_0 = 0$ for a given forcing mapping $f \in P$, i.e.

$$\Phi(t, f, \omega_0) := S^f_{t,0}\, \omega_0,$$

where we have included a superscript f on S to denote the dependence on the forcing term f. (This dependence is in fact continuous). The cocycle property here follows from the fact that $S^f_{t,t_0}\omega_0 = S^{\theta_{t_0} f}_{t-t_0,0}\, \omega_0$ for all $t \geq t_0$, $t_0 \in I\!R$, $\omega_0 \in L^2$ and $f \in P$.

Theorem 4 *Let the assumptions of Theorem 1 hold. Then the dissipative QG model (2)–(3) has a unique almost periodic solution ω^* defined by*

$$\omega^*(t) := a^*\left(\theta_t f\right), \qquad t \in I\!R,$$

where $\{a^*(p)\}$ is the singleton valued pullback attractor–trajectory of the cocycle $\Phi(t, f, \omega_0)$ on $L_2(D)$, P is the hull in the metric space $L^2_{loc}(I\!R, L^2(D))$ of the almost periodic forcing term f and the θ_t are the left shift operators on P.

This is proved as follows. By Theorems 2 and 3 the pullback attractor exists, consists of singleton valued components $\{a^*(p)\}$ and the mapping $p \mapsto a^*(p)$ is continuous on P. In fact, the mapping $p \mapsto a^*(p)$ is uniformly continuous on P because P is compact subset of $L^2_{loc}(I\!R, L^2(D))$ due to the assumed almost periodicity. That is, for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\|a^*(p) - a^*(q)\| < \varepsilon$ whenever $d_P(p,q) < \delta$. Now let the point $\bar{p}$ ($= f$, the given temporal forcing function) be almost periodic and for $\delta = \delta(\varepsilon) > 0$

denote by M_δ the relatively dense subset of $I\!R$ such that $d_P(\theta_{t+\tau}\bar{p}, \theta_t\bar{p}) < \delta$ for all $\tau \in M_\delta$ and $t \in I\!R$. From this and the uniform continuity we have

$$\|a^*(\theta_{t+\tau}\bar{p}) - a^*(\theta_t\bar{p})\| < \varepsilon$$

for all $t \in I\!R$ and $\tau \in M_{\delta(\varepsilon)}$. Hence $t \mapsto \omega^*(t) := a^*(\theta_t\bar{p})$ is almost periodic. It is unique as the single-trajectory pullback attractor is the only trajectory that exists and is bounded for the entire time line.

This completes the proof of the main result, Theorem 1.

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