SEMI-CLASSICAL COUNTING FUNCTION WITH APPLICATION TO QUANTUM CURRENT

BY
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Abstract. By adapting methods developed in [SV] to the semi-classical situation we obtain a new two-term asymptotic formula for the counting function of eigenvalues of h pseudodifferential operators in the limit as $h \to 0$. Recent results on clustering of eigenvalues obtained in [PP2] follow in Corollary and, as an application, we consider the semi-classical behaviour of the quantum current studied recently in [F].

1. Introduction and Main Results

1.1 Main result.

Let $A(h)$ denote an h admissible pseudodifferential operator (or h P.D.O. for short) as defined in [HR2] and denote the Weyl symbol of $A(h)$ by the formal, asymptotic series $a(h) = \sum_{j\geq 0} h^j a_j$. (Here the series must be understood in the sense explained in [HR2].) From here on we assume that the symbol of $A(h)$ satisfies the following standard conditions.

(H_1) $a(h)$ is real valued for all $h \in (0, h_0]$.
(H_2) There exists a constant $\gamma_0 \in \mathbb{R}$ such that $a_0(x,\xi) \geq \gamma_0$ for all $(x,\xi) \in \mathbb{R}^{2n}$.
(H_3) For some fixed $\gamma_1 \in \mathbb{R}$, there exist constants $C_0 \geq 0$ and $N_0 > 0$ such that

$$a_0(x,\xi) - \gamma_1 \leq C_0(a_0(y,\eta) - \gamma_1)(1 + |x-y|^2 + |\xi-\eta|^2)^{\frac{N_0}{2}},$$

for all $(x,\xi), (y,\eta) \in \mathbb{R}^{2n}$.
(H_4) With the same fixed γ_1 and for each $j \geq 0$ and each pair of multi-indices α, β, there exists a constant $C_{\alpha,\beta,j} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a_j\right| \leq C_{\alpha,\beta,j}(a_0 - \gamma_1),$$

for all $(x,\xi) \in \mathbb{R}^{2n}$.
(H_5) There exists a $\lambda_0 \in \mathbb{R}$ such that $a_0^{-1}((-\infty,\lambda_0))$ is non-empty and compact.

Under these hypotheses, from [HR2] and [HR1], there exists an $h_1 \in (0, h_0)$ such that $A(h)$ is essentially self-adjoint for all $h \in (0, h_1]$ when considered as an operator in $L^2(\mathbb{R}^n)$. Furthermore, for all $\lambda < \lambda_0$ and all $h \in (0, h_1]$ the sets $\mathrm{Spec}(A(h)) \cap (-\infty, \lambda]$ consist of finite sequences of discrete, finite multiplicity eigenvalues.

Author supported by European Union TMR grant FMRX-960001.

An h P.D.O. with symbol $\sum_{j\geq 0} h^j p_j$ is said to have compactly supported symbol if $p_j \in C_c^\infty(\mathbb{R}^{2n})$ for all j. The aim of this paper is to study the asymptotics of the function

$$N^{P,Q}(\lambda; h) = \sum_{\lambda_j(h)\leq\lambda} (P(h)\phi_j(h), Q(h)\phi_j(h)) \tag{1.1}$$

as $h \to 0$. Here $\lambda < \lambda_0$ denotes some fixed, regular value of a_0, $P(h)$ and $Q(h)$ denote two h P.D.O. with compactly supported symbols and $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbb{R}^n)$. The $\lambda_j(h)$ and $\phi_j(h)$ denote the eigenvalues and corresponding orthonormalised eigenfunctions of $A(h)$ and the sum is counted according to eigenvalue multiplicity.

The asymptotic behaviour of $N^{P,Q}$ is closely related to the periodic trajectories of the Hamiltonian flow generated by the principal symbol a_0. We denote by $\Phi^t_{a_0} = \exp(tH_{a_0})$ the Hamiltonian flow generated by a_0 and, for a point $(y,\eta) \in \mathbb{R}^{2n}$, we define $(x^*(t,y,\eta), \xi^*(t,y,\eta)) = \Phi^t_{a_0}(y,\eta)$. Defining

$$\Sigma_\lambda = \{(x,\xi) \in \mathbb{R}^{2n} \mid a_0(x,\xi) = \lambda\}, \tag{1.2}$$

if $(y,\eta) \in \Sigma_\lambda$ then $\Phi^t_{a_0}(y,\eta) \in \Sigma_\lambda$ for all t. Furthermore, if λ is a regular value of a_0 (ie. $\nabla a_0 \neq 0$ on Σ_λ), then Σ_λ is a smooth manifold.

A point $(y,\eta) \in \Sigma_\lambda$ is said to be periodic if there exists a constant $T > 0$ such that $\Phi^T_{a_0}(y,\eta) = (y,\eta)$ and we denote by $\Pi_\lambda \subseteq \Sigma_\lambda$ the set of all such points.

We denote by $T(\nu)$ the primitive (positive) period of a point $\nu \in \Pi_\lambda$ and define

$$\zeta(\nu) = \{\Phi^t_{a_0}(\nu) \mid 0 \leq t \leq T(\nu)\}. \tag{1.3}$$

Then the classical action, $s(\nu)$, at a point $\nu \in \Pi_\lambda$ is defined as the integral of the canonical 1-form ξdx along $\zeta(\nu)$. Explicitly,

$$s(\nu) = \int_{\zeta(\nu)} \xi dx = \int_0^{T(\nu)} \xi^*(t) \cdot \dot{x}^*(t) dt. \tag{1.4}$$

Also for a point $\nu \in \Pi_\lambda$ we define the quantity

$$q(\nu) = \int_0^{T(\nu)} a_{\text{sub}}(x^*(t), \xi^*(t)) dt - \frac{\pi}{2} m(\zeta) \tag{1.5}$$

where $m(\zeta) \in \mathbb{Z}_4$ denotes the Maslov index associated with the closed trajectory $\zeta = \zeta(\nu)$.

Analogously to [PP2], we define the oscillating function

$$Q^{P,Q}_\lambda(r;h) = (2\pi)^{-n} \int_{\Pi_\lambda} \{\pi - rT(\nu) - h^{-1}s(\nu) + q(\nu)\}_{2\pi} \frac{\mathrm{Re}(p_0\overline{q}_0)}{T(\nu)} d\nu_\lambda \tag{1.6}$$

where $r \in \mathbb{R}$ denotes a real parameter, $d\nu_\lambda$ denotes the Liouville measure on Σ_λ, defined $d\nu_\lambda = \frac{dS}{|\nabla a_0|}$, dS being the induced Lebesgue measure on Σ_λ, and $\{\cdot\}_{2\pi}$ denotes the residuum modulo 2π defined

$$\{\mu\}_{2\pi} = \mu - 2\pi k, \tag{1.7}$$

$k \in \mathbb{Z}$ being the integer such that $-\pi < \{\mu\}_{2\pi} \leq \pi$. Analogously to [SV], we define the coefficients

$$
\begin{aligned}
C_0^{P,Q}(\lambda) &= (2\pi)^{-n} \int_{a_0 \leq \lambda} \mathrm{Re}(p_0 \overline{q}_0) dx d\xi \\
C_1^{P,Q}(\lambda) &= (2\pi)^{-n} \int_{a_0 \leq \lambda} \left[\mathrm{Re}(p_0\, \overline{q_{\mathrm{sub}}} + p_{\mathrm{sub}} \overline{q}_0) - \frac{\mathrm{Im}\,(\{p_0, \overline{q_0}\})}{2} \right] dx d\xi \\
&\quad - (2\pi)^{-n} \int_{\Sigma_\lambda} \left[\frac{\mathrm{Im}\,(\{p_0, a_0\}\overline{q_0} + p_0\{a_0, \overline{q_0}\})}{2} + \mathrm{Re}(p_0 \overline{q}_0) a_{\mathrm{sub}} \right] d\nu_\lambda \\
C_2^{P,Q}(\lambda) &= (2\pi)^{-n} \int_{\Sigma_\lambda} \mathrm{Re}(p_0 \overline{q}_0) d\nu_\lambda
\end{aligned}
\tag{1.8}
$$

where $\{\cdot, \cdot\}$ denotes the usual Poison bracket defined $\{f, g\} = f_\xi \cdot g_x - f_x \cdot g_\xi$ for any differentiable f and g. Sometimes we will write $Q_\lambda^P = Q_\lambda^{P,P}$ and $C_j^P = C_j^{P,P}$. We have the following Theorem and remark that it, and all the results which follow, are valid for the imaginary part of $N^{P,Q}$ by substitution of iQ for Q.

Theorem 1.1. *Let $A(h)$ denote an h P.D.O. and suppose that the symbol of $A(h)$ satisfies hypotheses (H_1) to (H_5). Let $P(h)$ and $Q(h)$ denote h P.D.O. with compactly supported symbols and suppose that $\lambda < \lambda_0$ is such that $\nabla a_0 \neq 0$ on Σ_λ. Then for any $r_0 > 0$ there exist $h_0 > 0$ and $\epsilon_0 > 0$ such that*

$$
\begin{aligned}
&\frac{h^{1-n}}{4} \left[Q_\lambda^{P+Q}(r - \epsilon) - Q_\lambda^{P-Q}(r + \epsilon) - 4\epsilon C_2^{P,Q}(\lambda) \right] - o_\epsilon(h^{1-n}) \\
&\quad \leq \mathrm{Re}\left[N^{P,Q}(\lambda + rh; h) \right] - h^{-n} C_0^{P,Q}(\lambda) - h^{1-n} \left[C_1^{P,Q}(\lambda) + r C_2^{P,Q}(\lambda) \right] \\
&\quad \leq \frac{h^{1-n}}{4} \left[Q_\lambda^{P+Q}(r + \epsilon) - Q_\lambda^{P-Q}(r - \epsilon) + 4\epsilon C_2^{P,Q}(\lambda) \right] + o_\epsilon(h^{1-n})
\end{aligned}
$$

for all $h \in (0, h_0]$, $|r| \leq r_0$ and $0 < \epsilon \leq \epsilon_0$. In general $o_\epsilon(h^{1-n})$ depends upon ϵ but is uniform with respect to $|r| \leq r_0$. That is, for each fixed ϵ, $h^{n-1} o_\epsilon(h^{1-n}) \to 0$ as $h \to 0$ uniformly for all $|r| \leq r_0$.

Theorem 1.1 is similar to Theorem 1.1 in [PP2]. However, there are two differences which will be required for application to the quantum current later on.

Firstly, we have included the h P.D.O. $P(h)$ and $Q(h)$ in Theorem 1.1. Secondly, the coefficients $C_0^{P,Q}$ and $C_1^{P,Q}$ do not appear in Theorem 1.1 in [PP2]. Theorem 2.2 below is the crucial result, required to obtain $C_0^{P,Q}$ and $C_1^{P,Q}$, which will be proved in sections 2.2 and 2.3.

We can exploit the similarities between Theorem 1.1 and Theorem 1.1 in [PP2] to obtain analogous conditions for two-term asymptotics or clustering.

If the functions Q_λ^{P+Q} and Q_λ^{P-Q} are uniformly continuous for r in some interval we obtain a two-term asymptotic formula for Re $\left[N^{P,Q}\right]$. On the other hand, discontinuities of the functions Q_λ^{P+Q} and Q_λ^{P-Q} may cause so-called clustering (as discussed in [PP2]) to occur. The following two Corollaries are analogous to Corollaries 1.2 and 1.3 in [PP2]. Both results follow from Theorem 1.1 by almost identical arguments to those in the proofs of Corollaries 1.2 and 1.3 in [PP2].

Corollary 1.2. *Suppose that the functions Q_λ^{P+Q} and Q_λ^{P-Q} are uniformly continuous for $r \in [r_1, r_2]$ and all $h \in (0, h_0]$. Let R_1, R_2 be such that $r_1 < R_1 < R_2 < r_2$. Then there exists an $h_0 > 0$ such that*

$$\begin{aligned}&\mathrm{Re}\left[N^{P,Q}(\lambda + rh; h)\right] \\ &\qquad = h^{-n} C_0^{P,Q}(\lambda) + h^{1-n}\left[C_1^{P,Q}(\lambda) + rC_2^{P,Q}(\lambda) + Q_\lambda^{P,Q}(r; h)\right] + o(h^{1-n})\end{aligned}$$

for all $r \in [R_1, R_2]$ and $h \in (0, h_0]$.

Corollary 1.3. *Suppose there exist a positive measure set $\Pi_1 \subseteq \Pi_\lambda$, an integer $p \in \mathbb{Z}$ and a bounded, independent of ν function $r(h)$ (bounded for all $h \in (0, h_0]$) such that*

$$r(h)T(\nu) + h^{-1}s(\nu) - q(\nu) = 2\pi p \tag{1.9}$$

for all $\nu \in \Pi_1$ and $h \in (0, h_0]$. Suppose also that $\int_{\Pi_1} \frac{|p_0+q_0|^2}{T(\nu)} d\nu_\lambda > \int_{\Pi_\lambda} \frac{|p_0-q_0|^2}{T(\nu)} d\nu_\lambda$. Then for any $\delta > 0$ there exists a $c_0 > 0$ such that

$$\begin{aligned}&(2\pi)^{n-1} \liminf_{h\to 0} h^{n-1} \mathrm{Re}\left[N^{P,Q}(\lambda + r(h)h + ch; h) - N^{P,Q}(\lambda + r(h)h - ch; h)\right] \\ &\qquad\qquad \geq \int_{\Pi_1} \frac{|p_0+q_0|^2}{T(\nu)} d\nu_\lambda - \int_{\Pi_\lambda} \frac{|p_0-q_0|^2}{T(\nu)} d\nu_\lambda - \delta\end{aligned}$$

for all $c \in (0, c_0]$ and we have clustering of $N^{P,Q}$ in the sense of [PP2].

In the case when only Q_λ^{P-Q} is uniformly continuous (for example if $p_0 = q_0$) we have the following Corollary, the proof of which is elementary.

Corollary 1.4. *Suppose that Q_λ^{P-Q} is uniformly continuous for $r \in [r_1, r_2]$ and all $h \in (0, h_0]$. Let R_1, R_2 be as in Corollary 1.2. Then there exists an $h_0 > 0$ and $\epsilon_0 > 0$ such that*

$$\begin{aligned}&h^{1-n} Q_\lambda^{P,Q}(r - \epsilon) + \frac{\epsilon}{2(2\pi)^n} h^{1-n} \int_{\Pi_\lambda} |p_0 - q_0|^2 d\nu_\lambda - \epsilon h^{1-n} C_2^{P,Q}(\lambda) - o_\epsilon(h^{1-n}) \\ &\leq \mathrm{Re}\left[N^{P,Q}(\lambda + rh; h)\right] - h^{-n} C_0^{P,Q}(\lambda) - h^{1-n}\left[C_1^{P,Q}(\lambda) + rC_2^{P,Q}(\lambda)\right] \\ &\leq h^{1-n} Q_\lambda^{P,Q}(r + \epsilon) - \frac{\epsilon}{2(2\pi)^n} h^{1-n} \int_{\Pi_\lambda} |p_0 - q_0|^2 d\nu_\lambda + \epsilon h^{1-n} C_2^{P,Q}(\lambda) + o_\epsilon(h^{1-n})\end{aligned}$$

for all $h \in (0, h_0]$, $r \in [R_1, R_2]$ and $0 < \epsilon \leq \epsilon_0$.

We note that in Corollary 1.3 we may replace $P(h)$ by $-P(h)$ and obtain an analogous result for so-called *negative* clustering. Making a similar substitution in Corollary 1.4, we obtain an analogous result there if Q_λ^{P+Q} is uniformly continuous.

1.2 Quantum current.

Let $A_\epsilon(h) = (-ih\nabla - b - \epsilon a)^2 + V$ where $V(x)$ and $a(x), b(x)$ denote a scalar and two vector valued, smooth functions and $\epsilon \in \mathbb{R}$. We assume that b and V are such that $A(h) := A_0(h)$ satisfies hypotheses (H_1) to (H_5) for some λ_0. The operator

$A_\epsilon(h)$ has principal symbol $a_{0,\epsilon} = |\xi - b - \epsilon a|^2 + V$ and all other Weyl symbol terms zero.

In [F] the author considers the asymptotics (for fixed $\lambda < \lambda_0$ and as $h \to 0$) of the object defined formally,

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \operatorname{Tr}\ \left[(A_\epsilon(h) - \lambda I)\chi_{(-\infty,\lambda]}\left(A_\epsilon(h)\right)\right]. \tag{1.10}$$

Here $\chi_{(-\infty,\lambda]}(\mu)$ denotes the function equal to 1 for $\mu \leq \lambda$ and 0 for $\mu > \lambda$. According to [F] if $A(h)$ satisfies hypotheses (H_1) to (H_5) then the object in (1.10) is well defined and is said to be the *quantum current*.

Defining the operator $B(h) = a \cdot (-ih\nabla - b)$ with principal symbol $b_0 = a \cdot (\xi - b)$ and sub-principal symbol $b_{\text{sub}} = \frac{i}{2}\nabla \cdot a$, according to [F], if $A(h)$ satisfies hypotheses (H_1) to (H_5) then the quantum current in (1.10) is equal to the real part of the function $N^{B,I}$ for $A(h)$. That is, the quantum current is equal to,

$$\operatorname{Re}\left[N^{B,I}(\lambda; h)\right] = \sum_{\lambda_j(h)\leq\lambda} \operatorname{Re}\left(B(h)\phi_j(h), \phi_j(h)\right), \tag{1.11}$$

where, as before, $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbb{R}^n)$, the $\lambda_j(h)$ and $\phi_j(h)$ denote eigenvalues and corresponding orthonormalised eigenfunctions of $A(h)$ and the sum is counted according to eigenvalue multiplicity.

In the case when $a(x)$ is the gradient of some function (eg. when $a(x) = Ax$, with A a symmetric matrix) the operator $A_\epsilon(h)$ is gauge equivalent to $A(h)$ and clearly, from (1.10), the quantum current is identically zero. From here on, therefore, we consider the case when $A_\epsilon(h)$ is not gauge equivalent to $A(h)$.

We also observe that, from (1.11), the quantum current is gauge invariant (with respect to b). Thus, when $b(x)$ is the gradient of some function, the quantum current of the operator $A(h)$ is equal to the quantum current of the Schrödinger operator $-h^2\Delta + V$. In this case the eigenfunctions of $A(h)$ may be assumed real valued and, according to (1.11), the quantum current is identically zero. Henceforth we shall consider the non-trivial case when $A(h)$ is not gauge equivalent to $-h^2\Delta + V$.

From [HR2] and [B] for example, if $f \in C_c^\infty(\mathbb{R})$ then $f(A(h))$ and $B(h)f(A(h))$ are both h P.D.O. with compactly supported symbols. Choosing $f \in C_c^\infty(\mathbb{R})$ such that $f = 1$ in a neighbourhood of the set $[\gamma_0, \lambda]$, we have

$$N^{B,I}(\lambda; h) = N^{B(h)f(A(h)),f(A(h))}(\lambda; h), \tag{1.12}$$

for small enough h and hence, we can apply Theorem 1.1 to obtain results about the quantum current.

Observing that, by a change of variables, $\int_{a_0\leq\lambda} b_0 dx d\xi = 0$ and $\int_{\Sigma_\lambda} b_0 d\nu_\lambda = 0$ and hence the $C_j^{B,I} = 0$, Theorem 1.1 gives the following.

Corollary 1.5. *Let $A(h)$ and $B(h)$ be as above and let $\lambda < \lambda_0$ be a regular value of a_0. Then for any $r_0 > 0$ there exist $h_0 > 0$ and $\epsilon_0 > 0$ such that*

$$\begin{aligned}
&\frac{h^{1-n}}{4}\left[Q_\lambda^{B+I}(r-\epsilon) - Q_\lambda^{B-I}(r+\epsilon)\right] - o_\epsilon(h^{1-n}) \\
&\qquad \leq \operatorname{Re}\left[N^{B,I}(\lambda + rh; h)\right] \\
&\qquad\qquad \leq \frac{h^{1-n}}{4}\left[Q_\lambda^{B+I}(r+\epsilon) - Q_\lambda^{B-I}(r-\epsilon)\right] + o_\epsilon(h^{1-n})
\end{aligned} \tag{1.13}$$

and
(1.14)

$$\frac{h^{1-n}}{4}\left[Q_\lambda^{B+iI}(r-\epsilon)-Q_\lambda^{B-iI}(r+\epsilon)\right]-o_\epsilon(h^{1-n})$$
$$\le \operatorname{Im}\left[N^{B,I}(\lambda+rh;h)\right]+h^{1-n}(2\pi)^{-n}\left[\int_{a_0\le\lambda} ib_{\rm sub}\,dx\,d\xi-\int_{\Sigma_\lambda}\frac{\{b_0,a_0\}}{2}d\nu_\lambda\right]$$
$$\le \frac{h^{1-n}}{4}\left[Q_\lambda^{B+iI}(r+\epsilon)-Q_\lambda^{B-iI}(r-\epsilon)\right]+o_\epsilon(h^{1-n})$$

for all $h\in(0,h_0]$*,* $|r|\le r_0$ *and* $0<\epsilon\le\epsilon_0$*.*

Corollaries 1.2 and 1.3 give sufficient conditions for the existence of a two-term asymptotic formula for $\mathrm{Re}[N^{B,I}]$ or for when clustering occurs and we remark that $\mathrm{Re}[N^{B,I}]=o(h^{1-n})$ in the trivial case when the set Π_λ has zero measure. From here on we consider the case when Π_λ has positive measure.

The quantum current of the operator $A(h)$ is closely related to that of the operator $\tilde A(h)=(-ih\nabla+b)^2+V$. If $\phi_j(h)$ is an eigenfunction of $A(h)$ with corresponding eigenvalue $\lambda_j(h)$, then $\overline{\phi_j}(h)$ is an eigenfunction of $\tilde A(h)$ with the same eigenvalue. Defining the operator $\tilde B(h)=a\cdot(-ih\nabla+b)$ then $(B(h)\phi_j(h),\phi_j(h))=-(\tilde B(h)\overline{\phi_j}(h),\overline{\phi_j}(h))$ and hence,

$$\mathrm{Re}\ [\tilde N^{\tilde B,I}(\lambda;h)]=-\ \mathrm{Re}\ [N^{B,I}(\lambda;h)]. \tag{1.15}$$

Thus, the quantum current of $\tilde A(h)$ is equal to the negation of the quantum current of $A(h)$.

For example, if Q_λ^{B+I} and Q_λ^{B-I} satisfy the hypotheses of Corollary 1.2 then we have two-term asymptotics for the quantum current of $A(h)$ and $\tilde A(h)$ and, moreover,

$$\begin{aligned}\mathrm{Re}\ [\tilde N^{\tilde B,I}(\lambda+rh;h)]&=-\mathrm{Re}\ \left[N^{B,I}(\lambda+rh;h)\right]\\&=-h^{1-n}Q_\lambda^{B,I}(r;h)+o(h^{1-n}).\end{aligned} \tag{1.16}$$

Further, if the hypotheses of Corollary 1.3 are satisfied for $A(h)$ with $p_0=b_0$ and $q_0=1$ then we obtain clustering of the quantum current of $A(h)$ and negative clustering of the quantum current of $\tilde A(h)$.

We define the map $R_b:\mathbb{R}^{2n}\mapsto\mathbb{R}^{2n}$ by $R_b:(x,\xi)\mapsto(x,2b(x)-\xi)$ and define the set $\Pi_2\subseteq\Pi_\lambda$ to be the smallest set such that $R_b(\Pi_\lambda\setminus\Pi_2)=\Pi_\lambda\setminus\Pi_2$ and $(T\circ R_b)(\nu)=T(\nu)$, $(s\circ R_b)(\nu)=s(\nu)$ and $(q\circ R_b)(\nu)=q(\nu)$ for all $\nu\in\Pi_\lambda\setminus\Pi_2$. If the set Π_2 has zero measure then, using the change of variables $\eta\mapsto(2b(y)-\eta)$, we have

$$Q_\lambda^{B\pm I}=Q_\lambda^B+Q_\lambda^I, \tag{1.17}$$

and we note that $\Pi_2=\emptyset$ in the trivial case when $b=0$.

Applying the same method of proof as in the proof of Corollary 1.3 in [PP2] and using (1.17) we have the following result.

Theorem 1.6. *Suppose that Π_2 has zero (Liouville) measure. i) Suppose there exist a positive measure set $\Pi_1 \subseteq \Pi_\lambda$, an integer $p \in \mathbb{Z}$ and a bounded, independent of ν function $r(h)$ (bounded for all $h \in (0, h_0]$) such that (1.9) holds for all $\nu \in \Pi_1$ and all $h \in (0, h_0]$. Then for any $\delta > 0$ there exists $\epsilon_0 > 0$ such that*

$$\left[Q_\lambda^{B+I}(r(h)+\epsilon; h) - Q_\lambda^{B-I}(r(h)-\epsilon; h)\right] \geq (2\pi)^{1-n} \int_{\Pi_1} \frac{(|b_0|^2+1)}{T(\nu)} d\nu_\lambda - \delta \tag{1.18}$$

for all $h \in (0, h_0]$ and $\epsilon \in (0, \epsilon_0]$. ii) Suppose there exists a positive measure set $\Pi_1 \subseteq \Pi_\lambda$ on which s and q are constant (for example if Π_1 is connected). Then, defining $h_k = \frac{s}{2\pi k + q}$, $k \in \mathbb{Z}$, for any $\delta > 0$ there exists an $\epsilon_0 > 0$ such that

$$\left[Q_\lambda^{B+I}(\epsilon; h_k) - Q_\lambda^{B-I}(-\epsilon; h_k)\right] \geq (2\pi)^{1-n} \int_{\Pi_1} \frac{(|b_0|^2+1)}{T(\nu)} d\nu_\lambda - \delta \tag{1.19}$$

for all $\epsilon \in (0, \epsilon_0]$.

It has been suggested that, in certain cases, (i.e. there may exist $\lambda \in \mathbb{R}$ and functions a, b and V such that) the $O(h^{1-n})$ term of the quantum current may be non-zero and oscillating, in some sense, as $h \to 0$. We add to this conjecture that, in certain cases there may exist a bounded energy shift function $r(h)$ such that the $O(h^{1-n})$ term of the quantum current, evaluated at $(\lambda + r(h)h)$, may be oscillating as $h \to 0$.

We note that Theorem 1.6, and Corollary 1.5 do not disprove this conjecture. For if the hypotheses of Theorem 1.6 are satisfied, and the energy shift function $r(h)$ is chosen in the way described, then the bounds on either side of (1.13) are (symmetrically) bounded away from zero. The quantum current is thus not necessarily $o(h^{1-n})$ and the highest order term may be oscillating.

We briefly consider an example in which the hypotheses of Theorem 1.6 are satisfied. (In the trivial case when $b = 0$ and $V = |x|^2$, elementary calculations show that the hypotheses of Theorem 1.6 are satisfied.) Let $V = Kx \cdot x$ where K denotes a real, symmetric, positive definite matrix and let $b(x) = Bx$ with B an anti-symmetric matrix. We define Λ to be the diagonal matrix with entries $\Lambda_{jk} = \delta_{jk} k_j$, where the $\{k_j^2\}$ denote the (positive) eigenvalues of K and $k_j > 0$ for all j.

Following [MU] there exist an invertible matrix T and a unitary matrix U such that

$$\frac{d}{dt}\left(UT\Phi_{a_0}^t\right) = 2\begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix} UT\Phi_{a_0}^t. \tag{1.20}$$

Elementary calculations then show that, similarly to the case when $V = Kx \cdot x$ and $b = 0$, the flow is totally periodic if and only if $\frac{k_j}{k_1} \in \mathbb{Q}$ for all j. On the other hand, $\Pi_\lambda = \emptyset$ if and only if there exists a $j \geq 2$ such that $\frac{k_j}{k_1} \notin \mathbb{Q}$.

In the periodic case we define $\{p_j\}$ and $\{q_j\}$ to be positive integers such that $\frac{k_j}{k_1} = \frac{p_j}{q_j}$. Then $T = \frac{\pi}{k_1}$ l.c.m. $\left(\frac{q_j}{(p_j, q_j)}\right)$, $s = \lambda T$ and $q = \{\mathrm{Tr}(\sqrt{K})T + \pi\}_{2\pi} - \pi$ for all $\nu \in \Pi_\lambda = \Sigma_\lambda$. (Here (p_j, q_j) denotes the greatest common divisor of p_j and q_j, l.c.m. denotes least common multiple and $\mathrm{Tr}(\sqrt{K})$ denotes the usual matrix trace of the square root of K.)

In general, when the flow is totally periodic with constant period and Σ_λ is connected, then $\Pi_2 = \emptyset$ and, in Theorem 1.6, we can take $\Pi_1 = \Pi_\lambda$ and

$$r(h) = \left(\{q - h^{-1}s\}_{2\pi} + 2\pi p\right) T^{-1} \tag{1.21}$$

for any $p \in \mathbb{Z}$ or $h_k = \frac{s}{2\pi k + q}$, $k \in \mathbb{Z}$.

More evidence to support the conjecture described above comes from the next Theorem.

Theorem 1.7. *Suppose that there exist $\rho, h_0 > 0$ and a bounded function $r(h)$, (bounded for all $h \in (0, h_0]$) such that the set*

$$\Pi_{\rho,h_0} = \bigcup_{h\in(0,h_0]} \{\nu \in \Pi_\lambda |\inf_{k\in\mathbb{Z}} \left|r(h)T(\nu) + h^{-1}s(\nu) - q(\nu) - 2\pi k\right| < \rho\} \tag{1.22}$$

has zero (Liouville) measure. (Note that we require $\rho < \pi$ if we are to have Π_λ of positive measure.) Then for any $\delta > 0$, there exists $\epsilon_0 > 0$ such that

$$\left|Q_\lambda^{\pm B+I}(r(h) + \epsilon) - Q_\lambda^{\pm B-I}(r(h) - \epsilon) - 4Q_\lambda^{\pm B,I}(r(h))\right| < \delta \tag{1.23}$$

for all $h \in (0, h_0]$ and $\epsilon \in (0, \epsilon_0]$. Hence, applying a similar argument to that appearing in the proof of [PP2], *Corollary 1.2, there exists an $h_1 > 0$ such that*

$$\mathrm{Re}\ \left[N^{B,I}(\lambda + r(h)h; h)\right] = h^{1-n}Q_\lambda^{B,I}(r(h); h) + o(h^{1-n}) \tag{1.24}$$

for all $h \in (0, h_1]$.

Proof. As Π_{ρ,h_0} has zero measure it follows that

$$2\pi k - \pi + \rho \leq (\pi - r(h)T(\nu) - h^{-1}s(\nu) + q(\nu)) \leq 2\pi k + \pi - \rho$$

for all $h \in (0, h_0]$, $k \in \mathbb{Z}$ and almost all $\nu \in \Pi_\lambda$.

For any $\delta_1 > 0$, there exsists a $T_0 > 0$ such that the set $X_{\delta_1} = \{\nu \in \Pi_\lambda | T(\nu) > T_0\}$ has Liouville measure, denoted $\mu(X_{\delta_1})$, less than δ_1, ie. $\mu(X_{\delta_1}) < \delta_1$.

Using these two facts, for each $\delta_1 > 0$, there exists $\epsilon_0 > 0$ ($\epsilon_0 < \frac{\rho}{2T_0}$ is sufficient) such that

$$2\pi k - \pi + \frac{\rho}{2} \leq (\pi - (r(h) \pm \epsilon)T(\nu) - h^{-1}s(\nu) + q(\nu)) \leq 2\pi k + \pi - \frac{\rho}{2} \tag{1.25}$$

for all $h \in (0, h_0]$, $\epsilon \in (0, \epsilon_0]$, $k \in \mathbb{Z}$ and almost all $\nu \in (\Pi_\lambda \setminus X_{\delta_1})$.

There exists $T_1 > 0$ such that $T(\nu) \geq T_1$ for all $\nu \in \Pi_\lambda$ and hence, using (1.25), there exist constants $M_1, M_2 \geq 0$ such that the left hand side of (1.23) is less than or equal to

$$\frac{M_1\pi}{T_1}\mu(X_{\delta_1}) + M_2\epsilon < \frac{M_1\pi\delta_1}{T_1} + M_2\epsilon$$

for all $h \in (0, h_0]$ and $\epsilon \in (0, \epsilon_0]$, where we have used the facts that $\{\cdot\}_{2\pi} \leq \pi$, and $|\pm b_0 + 1|^2 - |\pm b_0 - 1|^2 = \pm 4b_0$. Choosing δ_1 and ϵ_0 small enough completes the proof of (1.23). $\square$

Theorem 1.7 gives sufficient conditions which ensure that the quantum current is $O(h^{1-n})$ and that the highest order term is oscillatory. Indeed, suppose there exist λ, a, b and V such that the flow $\Phi^t_{a_0}$ has constant period T on Π_λ, s and q constant almost everywhere on Π_λ (eg. if Π_λ is connected), and $\int_{\Pi_\lambda} b_0 d\nu_\lambda \neq 0$. Then choosing, for example,

$$r(h) = \left(\{q - h^{-1}s\}_{2\pi} + 2\pi p + \delta\right) T^{-1}, \tag{1.26}$$

with any $p \in \mathbb{Z}$ and $\delta \in (0, \pi)$, we have $\Pi_{\rho,h_0} = \emptyset$ for all $0 < \rho < \delta$ and all $h_0 > 0$ and thus,

$$\operatorname{Re}\left[N^{B,I}(\lambda + r(h)h; h)\right] = h^{1-n}\frac{(\delta - \pi)}{T}\int_{\Pi_\lambda} b_0 d\nu_\lambda + o(h^{1-n}).$$

To obtain an oscillatory top order term we could choose

$$r(h) = \left(\{q - h^{-1}s\}_{2\pi} + 2\pi p + \pi + \frac{1}{2}\{2h^{-1}\}_{2\pi}\right) T^{-1}, \tag{1.27}$$

with any $p \in \mathbb{Z}$, so that $\Pi_{\rho,h_0} = \emptyset$ for all $0 < \rho < \frac{\pi}{2}$ and all $h_0 > 0$ and thus,

$$\operatorname{Re}\left[N^{B,I}(\lambda + r(h)h; h)\right] = \frac{h^{1-n}}{2T}\{2h^{-1}\}_{2\pi}\int_{\Pi_\lambda} b_0 d\nu_\lambda + o(h^{1-n}).$$

Unfortunately, an example in which T, s and q are all constant on Π_λ and $\int_{\Pi_\lambda} b_0 d\nu_\lambda \neq 0$ has not yet been found. In the case when T, s and q are constant on Π_λ, such an example can only exist if $\Sigma_\lambda \setminus \Pi_\lambda$ is of positive measure, (if $\Sigma_\lambda \setminus \Pi_\lambda$ has zero measure then $\int_{\Pi_\lambda} b_0 d\nu_\lambda = 0$) and this rules out the example considered above.

Indeed we conclude from Theorem 1.7 that, in the case when $V = Kx \cdot x$, K a real, symmetric, positive definite matrix, $b = Bx$ with B an anti-symetric matrix and the flow is totally periodic, if we choose $r(h)$ such that Π_{ρ,h_0} has zero measure for some $\rho, h_0 > 0$ (eg. if we choose $r(h)$ as in (1.26) or (1.27)) then we have $\operatorname{Re}\left[N^{B,I}(\lambda + r(h)h; h)\right] = o(h^{1-n})$.

2. Proof of Theorem 1.1

2.1 Polarisation, Tauberian result and improved trace formula.
Firstly we comment that, due to the polarisation formula

$$4N^{P,Q} = N^{P+Q} - N^{P-Q} + iN^{P+iQ} - iN^{P-iQ} \tag{2.1}$$

appearing in [SV], the statement in Theorem 1.1 follows from the special case when $Q(h) = P(h)$. Henceforth we need only consider this particular case.

Throughout what follows we let $\rho, \gamma \in \mathcal{S}(\mathbb{R})$ denote Schwartz functions satisfying the following conditions: i) $\hat{\rho}, \hat{\gamma} \in C_c^\infty(\mathbb{R})$, ii) $\rho, \gamma > 0$, iii) $\hat{\rho}$ and $\hat{\gamma}$ are both even, iv) $\hat{\rho}(0) = 1, \hat{\rho}'(0) = 0$ and v) $\hat{\gamma}(0) = 0$, where $\hat{\rho}$ denotes Fourier transform and $\hat{\rho}'$ denotes differentiation.

For any $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 < \lambda_2$ we denote by $\mathcal{F}_+(\lambda_1, \lambda_2)$ the space of functions $\sigma_h(\lambda)$ such that, for each $h \in (0, h_0]$, $\sigma_h(\lambda)$ is a real-valued, non-negative, non-decreasing function of λ, $\sigma_h(\lambda) = 0$ for all $\lambda \le \lambda_1$ and $\sigma_h(\lambda) = \sigma_h(\lambda_2)$ for all $\lambda \ge \lambda_2$.

For any $\delta \ne 0$ we define the dilation of a function f as $f_\delta(\lambda) = \frac{1}{\delta} f(\frac{\lambda}{\delta})$ and applying the method of proof appearing in [SV], appendix B (see [B], chapter 5 for details) we have the following Tauberian type result.

Theorem 2.1. *Let $\sigma_{jh} \in \mathcal{F}_+(\lambda_1, \lambda_2)$, $j = 1, 2$. Suppose that there exists a function ρ satisfying the conditions above such that*

$$\frac{d}{d\lambda}(\sigma_{jh} * \rho_h)(\lambda) = O_j(h^{-n}), \qquad j = 1, 2$$

where the O_j are uniform with respect to $\lambda \in \mathbb{R}$. With this ρ, suppose that for all $r_0 > 0$ and all functions γ satisfying the conditions above,

$$\begin{aligned}(\sigma_{1h} * \rho_h)(\lambda + rh) &= (\sigma_{2h} * \rho_h)(\lambda + rh) + o(h^{1-n}) \\ \frac{d}{d\lambda}(\sigma_{1h} * \gamma_h)(\lambda + rh) &= \frac{d}{d\lambda}(\sigma_{2h} * \gamma_h)(\lambda + rh) + o(h^{-n})\end{aligned}$$

where the o are independent of $|r| \le r_0$. (Of course the O_j and o may depend on ρ and γ.) Then for any $\epsilon > 0$, all $|r| \le r_0$ and all $h \in (0, h_0]$,

$$\sigma_{2h}(\lambda + (r - \epsilon)h) - o_\epsilon(h^{1-n}) \le \sigma_{1h}(\lambda + rh) \le \sigma_{2h}(\lambda + (r + \epsilon)h) + o_\epsilon(h^{1-n})$$

where the o_ϵ is independent of $|r| \le r_0$ but may depend on ϵ.

With a view to applying the above Theorem we define the functions

$$\begin{aligned}\sigma_{1h}(\lambda) &= \sum_{\lambda_j(h) \le \lambda} \|P(h)\phi_j(h)\|^2 \\ \sigma_{2h}(\lambda + rh) &= h^{-n} C_0^P(\lambda) + h^{1-n} C_1^P(\lambda) + r h^{1-n} C_2^P(\lambda) + h^{1-n} Q_\lambda^P(r; h).\end{aligned} \tag{2.2}$$

Let $f, g \in C_c^\infty(\mathbb{R})$ denote real functions such that $f + g = 1$ in a neighbourhood of the set $[\gamma_0, \lambda]$, supp $g \subseteq [\lambda - \delta, \lambda + \delta]$ some $\delta > 0$, $g = 1$ in a neighbourhood of the point λ and supp $f \subseteq (-\infty, \lambda]$. Then defining $\tilde{\sigma}_{1h}$ and $\tilde{\sigma}_{2h}$ to be the same functions in (2.2) but with $P(h)$ replaced by $P(h)(f + g)(A(h))$ we have

$$\begin{aligned}\tilde{\sigma}_{1h}(\lambda) = \mathrm{Tr}\ &[f(A)P^*Pf(A)] + \\ &\sum_{\lambda_j(h) \le \lambda} \left[2\mathrm{Re}\,(Pf(A)\phi_j(h), Pg(A)\phi_j(h)) + \|Pg(A)\phi_j(h)\|^2\right]\end{aligned} \tag{2.3}$$

and $\tilde{\sigma}_{jh}(\mu) = \sigma_{jh}(\mu)$ for all μ in a neighbourhood of the set $(-\infty, \lambda]$.

In section 2.3 we will prove the following improved trace formula:

Theorem 2.2. *Let $P(h)$ and $Q(h)$ denote arbitrary h P.D.O. with compactly supported symbols. If ρ satisfies the assumptions above and has* supp $\hat{\rho}$ *contained in a sufficiently small neighbourhood of the origin then*
(2.4)
$$\begin{aligned}
&(2\pi h)^{-1}\mathrm{Re}\int_{-\infty}^{\lambda}\int \hat{\rho}(t)e^{ih^{-1}t(s+rh)}\mathrm{Tr}\ \left[P(h)e^{-ih^{-1}tA(h)}Q(h)^*\right]dtds\\
&= h^{-n}C_0^{P,Q}(\lambda)+h^{1-n}(2\pi)^{-n}\int_{a_0\le\lambda}\mathrm{Re}\left((PQ^*)_{\mathrm{sub}}\right)dxd\xi + rh^{1-n}C_2^{P,Q}(\lambda)\\
&- h^{1-n}(2\pi)^{-n}\int_{\Sigma_\lambda}\left[\frac{\mathrm{Im}\left(\{p_0,a_0\}\overline{q_0}+p_0\{a_0,\overline{q_0}\}\right)}{2}+\mathrm{Re}(p_0\overline{q}_0)a_{\mathrm{sub}}\right]d\nu_\lambda + o(h^{1-n})
\end{aligned}$$
where the $o(h^{1-n})$ is independent of r in any bounded interval and $(P^*Q)_{\mathrm{sub}}$ denotes the sub-principal symbol of the composition $P(h)^*Q(h)$.

Remark. Theorem 2.2 will be proved in an analagous way to the proof of the corresponding classical (large λ) result, see [SV] chapter 3 and Theorem 4.2.4. However, in the large λ setting, the proof in [SV] relies upon the fact that the principal symbols are homogeneous with respect to the ξ variables, the homogeneity allowing one, essentially, to change to polar coordinates and exploit some cancellations.

In the semi-classical (non-homogeneous) case we employ a slightly different argument and use an additional technique (of varying the phase functions involved in our constructions) to establish (2.4). A draw back is that this more complicated method only permits us to obtain the $o(h^{1-n})$ error in (2.4) whereas, in view of [SV] Theorem 4.2.4., one might expect to be able to obtain a better $O(h^{2-n})$ remainder.

Using the results of [HR2] to estimate the first term in (2.3), an explicit calculation and Theorem 2.2 imply that, provided supp $\hat{\rho}$ and δ are small enough,

$$(\tilde{\sigma}_{1h}*\rho_h)(\lambda+rh)=(\tilde{\sigma}_{2h}*\rho_h)(\lambda+rh)+o(h^{1-n}) \tag{2.5}$$

where the $o(h^{1-n})$ is independent of r in any bounded interval. Further, from Theorem 2.2 and (2.5), $\frac{d}{d\lambda}(\tilde{\sigma}_{jh}*\rho_h)=O_j(h^{-n})$, $j=1,2$, where the O_j are uniform in λ and, it is thus clear that, the $\tilde{\sigma}_{jh}\in\mathcal{F}_+(\gamma_0,\lambda+\delta)$.

Theorem 5.1 in [PP2] may be generalised to the case when the h P.D.O. $g(x,hD_x)$ and $g(x,hD_x)^*$, in the notation used there, are replaced by arbitrary h P.D.O. with compactly supported symbols. In this more general setting, [PP2] Theorem 5.1 gives
(2.6)
$$\begin{aligned}
&(2\pi h)^{-1}\int e^{ih^{-1}(\lambda+rh)}\hat{\gamma}(t)\mathrm{Tr}\ \left[P(h)e^{-ih^{-1}tA(h)}Q(h)^*\right]dt\\
&\qquad= (2\pi h)^{-n}\sum_{k\in\mathbb{Z}\setminus\{0\}}\int_{\Pi_\lambda}\hat{\gamma}(kT)e^{ik(h^{-1}s+rT-q)}p_0\overline{q_0}d\nu_\lambda+o(h^{-n})
\end{aligned}$$
where the $o(h^{-n})$ is independent of r in any bounded interval but may depend on γ.

Taking account of (2.3) and (2.6), another explicit calculation shows that, for all functions γ satisfying the assumptions above and provided δ is sufficiently small,
$$\frac{d}{d\lambda}(\tilde{\sigma}_{1h}*\gamma_h)(\lambda+rh)=\frac{d}{d\lambda}(\tilde{\sigma}_{2h}*\gamma_h)(\lambda+rh)+o(h^{-n})$$

where the $o(h^{-n})$ is independent of r in any bounded interval but may depend upon γ.

We have thus established that the $\tilde{\sigma}_{jh}$ satisfy the hypotheses of Theorem 2.1, and Theorem 1.1 follows by an application of Theorem 2.1.

2.2 Schrödinger equation and h F.I.O. with complex phase functions.

As a first step in proving Theorem 2.2 we approximate, in some sense, the operator $P(h)e^{-ih^{-1}tA(h)}Q(h)^*$ by a so-called h Fourier integral operator, or h F.I.O. for short. To achieve this, we adapt complex valued phase function methods developed in [LSV] to the semi-classical setting.

1. h F.I.O.

We consider a particular class of h F.I.O. with complex valued phase functions. Let $a \in C^\infty(\mathbb{R}^{2n})$ denote an arbitrary function. A function $\phi \in C^\infty(\mathbb{R}^{3n+1})$ is said to be a phase function (associated with a) if there exists a $\mu > 0$ such that

$$\phi(t,x,y,\eta) = \int_0^t \xi^*(s)\cdot \dot{x}^*(s)ds - ta(x^*,\xi^*) + \xi^*\cdot(x-x^*) + \frac{i\mu}{2}|x-x^*|^2$$

and the set of all such functions will be denoted $\Psi(a)$. (Here we recall that $x^* = x^*(t,y,\eta)$, with the same for ξ^*, and $(x^*,\xi^*) = \Phi^t_{a_0}(y,\eta)$.)

A function $u \in C^\infty(\mathbb{R}^{3n+1})$ is said to be a symbol if u has compact support in the y and η variables and the function $x \mapsto u(t,x,y,\eta)$ and all its derivatives have polynomially bounded growth for all t, y and η.

Definition. An operator valued map $U(t;h) : \mathbb{R}\times(0,h_0] \to \mathcal{L}(L^2(\mathbb{R}^n))$ is said to be an h F.I.O. if for any $T > 0$, there exist a phase function ϕ, a sequence of symbols $\{u_j\}$ and an integer $N_0 \geq 0$ such that, denoting by $\mathcal{V}_N(t;h)$ the operator with integral kernel

$$k(t,x,y;h) = (2\pi h)^{-n}\int \sum_{0\leq j\leq N} h^j u_j(t,x,y,\eta)e^{ih^{-1}\phi(t,x,y,\eta)}d\eta,$$

we have $\max_{t\in[-T,T]}\|U(t;h) - \mathcal{V}_N(t;h)\| = O(h^{N+1})$ as $h \to 0$ for all $N \geq N_0$. The function ϕ is said to be a phase function of $U(t;h)$ and the (formal) series

$$u \sim \sum_{j\geq 0} h^j u_j$$

is said to be a symbol of $U(t;h)$.

Clearly, as defined here, any particular h F.I.O. may have more than one phase and more than one symbol. Defining the matrix $Z(t,y,\eta,\mu) = \xi^*_\eta - i\mu x^*_\eta$ where x^*_η denotes the matrix of derivatives with entries $(x^*_\eta)_{jk} = \frac{\partial x^*_k}{\partial \eta_j}$ we have the following Lemma, the proof of which appears in [LS], section 4.

Lemma 2.3. *The matrix Z has non-zero deteminant for all t, y, η and $\mu > 0$.*

We define the differential operators

$$D_1 = \sum_{j,k} \frac{1}{j!}\left[\left(\frac{\partial}{\partial \eta_k}(Z^{-1})_{jk}\right)\frac{\partial}{\partial x_j} + (Z^{-1})_{jk}\frac{\partial^2}{\partial x_j \partial \eta_k} + (Z^{-1}x^*_\eta)_{jk}\frac{\partial^2}{\partial x_j \partial x_k}\right]$$
$$D_2 = -\sum_{j,k}\frac{1}{j!k!}(Z^{-1}x^*_\eta)_{jk}\frac{\partial^2}{\partial x_j \partial x_k}.$$

Then using Taylor's formula to expand symbols in polynomials in $(x - x^*)$, the identity $(x - x^*)e^{ih^{-1}\phi} = -ihZ^{-1}\nabla_\eta e^{ih^{-1}\phi}$, integration by parts and an inductive argument, the following Theorem may be proved (for more details of the technical but conceptually simple proof we refer to [B]).

Theorem 2.4. *Let $U(t; h)$ denote an h F.I.O. with phase ϕ and symbol $\sum_{j\geq 0} h^j u_j$. Then with the same phase ϕ, $U(t; h)$ has an x independent symbol $\sum_{j\geq 0} h^j \tilde{u}_j$. The $\tilde{u}_j$ are given by*

$$\tilde{u}_j(t, y, \eta) = \sum_{\substack{0\leq l\leq j \\ (j-l)\leq k\leq 2(j-l)}} D_{k,j-l}u_l\big|_{x=x^*}$$

where the $D_{k,j}$ are differential operators such that, in particular,

$$\begin{aligned} &D_{0,0} = \text{the identity operator} \\ &D_{k,k} = (i)^k D_1^k \qquad \text{for all } k \geq 1 \\ &D_{k,\frac{k}{2}} = (i)^{\frac{k}{2}} 1.3\ldots(k-1)D_2^{\frac{k}{2}} \qquad \text{for all even } k \geq 2 \end{aligned} \tag{2.7}$$

and if $\nu(x, \eta) \in C^\infty$ is an arbitrary function with a zero of order m in x at $x = x^*$ (i.e. $\partial_x^\alpha|_{x=x^*}\, \nu = 0$ for all $|\alpha| \leq m$) then

$$\sum_{j\leq k\leq 2j} D_{k,j}\nu\big|_{x=x^*} = 0 \tag{2.8}$$

for all $0 \leq j \leq \left[\frac{m}{2}\right]$ where $[\cdot]$ denotes integer part.

2. Composition of h P.D.O. and h F.I.O.

Next we consider composition of h P.D.O. and h F.I.O. We consider generalised Gauss transforms similar to those studied in [Hö], section 18.4. Let C denote the matrix

$$C = \begin{pmatrix} 0 & C_1 \\ C_1 & i\mu C_2 \end{pmatrix} \tag{2.9}$$

where C_1, C_2 denote $n \times n$, real, symmetric, positive definite matrices and $\mu > 0$. Let $\tilde{B}(x, \xi)$ denote the $2n$ vector $\tilde{B} = (0, B(x, \xi))$ where B is an n vector such that the matrix of derivatives B_x is independent of x and $B(x_0, \xi_0) = 0$ at some point $(x_0, \xi_0) \in \mathbb{R}^{2n}$.

Writing $A(s,\zeta) = \frac{1}{2}C(s,\zeta)\cdot(s,\zeta)$ and denoting $D_x = -i\partial_x$ (the same for D_ξ), the operator $e^{i(hA(D_x,D_\xi)+\tilde{B}(x,\xi)\cdot(D_x,D_\xi))}$ may be defined, for all $h > 0$, on Schwartz functions via the Fourier transform. Explicitly,

$$\begin{aligned} &e^{i(hA(D_x,D_\xi)+\tilde{B}(x,\xi)\cdot(D_x,D_\xi))}a(x,\xi) \\ &\qquad = (2\pi)^{-2n}\int e^{i(hA(s,\zeta)+(\tilde{B}(x,\xi)+(x,\xi))\cdot(s,\zeta))}\hat{a}(s,\zeta)dsd\zeta, \end{aligned} \tag{2.10}$$

$a \in \mathcal{S}(\mathbb{R}^{2n})$, the conditions on $\tilde{B}$ and C ensuring that $e^{i(hA(s,\zeta)+\tilde{B}(x,\xi)\cdot(s,\zeta))}\hat{a}(s,\zeta)$ is Schwartz (i.e. the function and all its derivatives are rapidly decreasing at infinity) in s and ζ for all $h > 0$ and for all x, ξ.

We now extend the definition of the operator in (2.10) to the case when $a \in S(m)$ where m is any σ temperate function and $S(m)$ is the space of functions $S(m, g_0)$ defined in [Hö], section 18.4 in the special case when $g_0 = |dx|^2 + |d\xi|^2$ is the Euclidean metric on $\mathbb{R}^{2n}$.

We define the constant $C_n = (2\pi)^{-n}\det((-iC)^{-1})^{\frac{1}{2}}$, where the square root is understood in the sense of [Hö], section 3.4 for example, and define the function $\psi(x,\xi;s,\zeta) = \frac{i\mu}{2}Ds\cdot s - C_1^{-1}(\zeta + B(x,\xi))\cdot s$ where $D = C_1^{-1}C_2C_1^{-1}$. Then we define (2.11)

$$e^{i(hA(D_x,D_\xi)+\tilde{B}(x,\xi)\cdot(D_x,D_\xi))}a(x,\xi) = h^{-n}C_n\int e^{ih^{-1}\psi(x,\xi;s,\zeta)}a(x-s,\xi-\zeta)dsd\zeta$$

for all a such that the convolution in (2.10) is finite. We note that when a is Schwartz the Fourier transform implies that (2.10) and (2.11) are equal.

Theorem 2.5. *Suppose that A and $\tilde{B}$ are as defined above. Then for all multi-indices $\alpha \geq 0$, the linear functional*

$$\left[\partial_x^\alpha e^{i(hA(D_x,D_\xi)+\tilde{B}(x,\xi)\cdot(D_x,D_\xi))}\ \cdot\ \right]\Big|_{(x,\xi)=(x_0,\xi_0)} : \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathbb{C} \tag{2.12}$$

is continuous : $S(m) \longrightarrow \mathbb{C}$ and is equal to $O(1)$ as $h \to 0$ for each $a \in S(m)$.

Proof. Let $a \in S(m)$ and denote by g_α the functional in (2.12) applied to a and denote $\psi_0 = \psi(x_0,\xi_0;s,\zeta)$. From (2.11), modulo constants, g_α is equal to the sum for multi-indices $\beta \leq \alpha$ of terms

$$h^{-|\beta|-n}\int(-iB_xC_1^{-1}s)^\beta e^{ih^{-1}\psi_0}(\partial_x^{\alpha-\beta}a)(x_0-s,\xi_0-\zeta)dsd\zeta$$

each of which, using intgeration by parts, is equal to

$$h^{-n}\int e^{ih^{-1}\psi_0}(\partial_x^{\alpha-\beta}(B_x\partial_\xi)^\beta a)(x_0-s,\xi_0-\zeta)dsd\zeta. \tag{2.13}$$

To handle the ζ integration in (2.13) we employ a method similar to that appearing in the proof of proposition 7.6 in [DS]. First we split the ζ integration in (2.13) by introducing the functions $\chi(h^{-\frac{1}{2}}\zeta)$ and $1-\chi(h^{-\frac{1}{2}}\zeta)$ where $\chi \in C_c^\infty(\mathbb{R}^n)$ is equal to 1 near 0.

The first integral is easy to estimate. Defining $\tilde{\psi} = -C_1^{-1}\zeta \cdot s$ we use the identity $h\frac{\tilde{\psi}_s}{|\tilde{\psi}_s|^2} \cdot D_s e^{ih^{-1}\tilde{\psi}} = e^{ih^{-1}\tilde{\psi}}$ and integration by parts to estimate the second integral. Then changing integration variables $\zeta \mapsto h^{\frac{1}{2}}\tilde{\zeta}$ and $s \mapsto h^{\frac{1}{2}}\tilde{s}$ we obtain that the second integral is, in absolute value, less than or equal to a constant times the integral $d\tilde{s}d\tilde{\zeta}$ of:

(2.14)
$$\left| \left(D_{\tilde{s}} \cdot \frac{\tilde{\psi}_s}{|\tilde{\psi}_s|^2} \right)^N \left\{ e^{-\frac{\mu}{2} D\tilde{s}\cdot\tilde{s}} (\partial_x^{\alpha-\beta}(B_x\partial_\xi)^\beta a)(x_0 - h^{\frac{1}{2}}\tilde{s}, \xi_0 - h^{\frac{1}{2}}\tilde{\zeta}) \right\} (1-\chi(\tilde{\zeta})) \right|.$$

Finally using the fact that $a \in S(m)$ there exist constants $C_{\alpha,\beta}, N_0 > 0$ such that the object in (2.14) is less than or equal to

$$C_{\alpha,\beta}\langle\tilde{s}\rangle^N \langle\tilde{\zeta}\rangle^{-N} \langle h^{\frac{1}{2}}\tilde{s}, h^{\frac{1}{2}}\tilde{\zeta}\rangle^{N_0} e^{-\frac{\mu}{2}D\tilde{s}\cdot\tilde{s}} m(x_0,\xi_0)$$

and choosing N large enough to ensure integrability of the above object with respect to $\tilde{\zeta}$ completes the proof. (Here we are using the standard notation $\langle\tilde{s},\tilde{\zeta}\rangle = (1 + |\tilde{s}|^2 + |\tilde{\zeta}|^2)^{\frac{1}{2}}$ and $\langle\tilde{s}\rangle = \langle\tilde{s},0\rangle$.) $\square$

Theorems 2.4 and 2.5 yield the following Corollary.

Corollary 2.6. *Let $A(h)$ denote an arbitrary, admissible h P.D.O. (see* [HR2], *(2.3) définition) with symbol $a(x,\xi;h)$ and let $U(t;h)$ denote an h F.I.O. with phase ϕ and finite, x independent symbol $u = \sum_{j\le N} h^j u_j$. Then the compostion $A(h)U(t;h)$ is an h F.I.O with the same phase ϕ and symbol given by the formal expression*

$$e^{\frac{h}{2}(-\mu D_\xi + iD_x)\cdot D_\xi} e^{-\mu(x-x^*)\cdot D_\xi} a(x,\xi;h)\Big|_{\xi=\xi^*} u(t,y,\eta;h) \tag{2.15}$$

where (2.15) can be made sense of by applying Theorems 2.4 and 2.5.

Proof. Denoting by $\{a_j\}$ the terms in the symbol of $A(h)$ we consider the composition $a_j^w(x,hD_x)U_k(t;h)$ where U_k has phase ϕ and symbol u_k.

Applying Theorem 2.5 with $C_1 = C_2 = I$ and $B = -\mu(x-x^*)$, for all integer $l \ge 0$,

$$\sum_{|\alpha|=l} \frac{1}{\alpha!} \left[\partial_x^\alpha e^{\frac{h}{2}(-\mu D_\xi + iD_x)\cdot D_\xi} e^{-\mu(x-x^*)\cdot D_\xi} a_j\right]\Big|_{(x,\xi)=(x^*,\xi^*)} (x-x^*)^\alpha u_k \tag{2.16}$$

is an h F.I.O. symbol uniformly with respect to $h \in (0,h_0]$.

In view of Theorem 2.4 and (2.8), the h F.I.O. with phase ϕ and symbol in (2.16) is equal to the h F.I.O. with the same phase ϕ and symbol $\sum_{m\ge[\frac{l+1}{2}]} h^m \nu_m$ where $\{\nu_m\}$ denotes some sequence of symbols. Applying Taylor's Theorem we have thus established that the operator with integral kernel given by the formal expression (understood in the sense demonstrated here in the proof)

$$(2\pi h)^{-n} \int e^{\frac{h}{2}(-\mu D_\xi + iD_x)\cdot D_\xi} e^{-\mu(x-x^*)\cdot D_\xi} a_j\Big|_{\xi=\xi^*} u_k e^{ih^{-1}\phi} d\eta \tag{2.17}$$

is an h F.I.O.

In the case when $a_j \in \mathcal{S}(\mathbb{R}^{2n})$, according to Fourier's inversion formula, the compostition $a_j^w U_k$ has kernel equal to (2.17). Hence in the general case we have established that $a_j^w U_k$ is an h F.I.O. and, noting that the finiteness of the symbol u ensures $\mathcal{L}(L^2(\mathbb{R}^n))$ boundedness of any remainder operators, we conclude that $A(h)U(t;h)$ is an h F.I.O. $\square$

Remark 2.7. In view of the proof of Corollary 2.6, to calculate the first N terms in the symbol of $A(h)U(t;h)$ it suffices to consider the first $2N$ terms in the Taylor polynomial (in $(x-x^*)$) of (2.15) and then use Theorem 2.4.

3. Initial conditions.

The operator $e^{-\frac{ih}{2}(D_x \cdot D_\xi + i\mu D_\xi^2)}$ may be defined on Schwartz functions analagously to (2.10). An application of Fourier's inversion formula then gives the following:

Theorem 2.8. *Suppose that $U(t;h)$ is an h F.I.O. with x independent symbol $u(t,y,\eta;h)$ and $P(h)$ is an h P.D.O. with compactly supported symbol $p(x,\xi;h)$. If $u(0) = e^{-\frac{ih}{2}(D_x \cdot D_\xi + i\mu D_\xi^2)} p\Big|_{(x,\xi)=(y,\eta)}$ then $U(0;h) = P(h)$.*

4. Schrödinger equation.

Let $A(h)$ denote an arbitrary admissible h P.D.O. which is essentially self-adjoint in $L^2(\mathbb{R}^n)$ for all $h \in (0,h_0]$, some $h_0 > 0$ (i.e. $A(h)$ satisfies hypotheses (H_1) to (H_4) above). Let $P(h)$ denote an h P.D.O. with compactly supported symbol.

Theorem 2.9. *The operator $U_P(t;h) = e^{-ih^{-1}tA(h)}P(h)$ is an h F.I.O. For any phase $\phi \in \Psi(a_0)$, there exists an (x independent) symbol $\sum_{j\geq 0} h^j u_j$ where the u_j can be written*

$$u_j = (\det Z)^{\frac{1}{2}} e^{-i\int_0^t a_{\mathrm{sub}}(x^*,\xi^*)ds} \nu_j \tag{2.18}$$

with $\nu_0(t,y,\eta) = p_0(y,\eta)$ for all t and

$$\sum_{j\geq 0} h^j \nu_j(0,y,\eta) = e^{-\frac{ih}{2}(D_x \cdot D_\xi + i\mu D_\xi^2)} p\Big|_{(x,\xi)=(y,\eta)}, \tag{2.19}$$

such that U_P has this phase and symbol.

Remark. In the Theorem, as $Z(0) = I$, we interpret $(\det Z)^{\frac{1}{2}}$ as the branch of $(\det Z)^{\frac{1}{2}}$ which is continuous for all t and equal to 1 when $t = 0$. Also the u_j and ν_j may depend on the choice of $\phi \in \Psi(a_0)$. This dependence is evident for u_0 if we recall that the matrix Z itself depends on μ, but is omitted from notations.

Proof. Denote by U_N an h F.I.O. with any phase $\phi \in \Psi(a_0)$ and symbol $\sum_{j\leq N} h^j u_j$, the u_j satisfying (2.18) and (2.19). Observing that $\dot\phi$ is polynomial in $(x-x^*)$, differentiating under the integral sign and applying Corollary 2.6, $(-ih\frac{\partial}{\partial t} + A(h))U_N$ is an h F.I.O. with phase ϕ and symbol $\sum_{j\geq 0} h^j \tilde{u}_j$.

By a straight forward calculation, for any smooth function $b(x,\eta)$,

$$\left\{ i\left[(D_x+i\mu D_\xi)\cdot D_\xi e^{-\mu(x-x^*)\cdot D_\xi}a_0\right]\Big|_{(x,\xi)=(x^*,\xi^*)} + \mathrm{Tr}\ (Z^{-1}\dot{Z})\right\} b(x^*,\eta)$$
$$= 2\left\{ (D_1+D_2)\left[b\left(\dot{\phi}+e^{-\mu(x-x^*)\cdot D_\xi}a_0\Big|_{\xi=\xi^*}\right)\right]\right\}\Big|_{x=x^*}$$

where we have used (2.7) and (2.8). Using this fact, (2.7), (2.8), the fact that $\phi\in\Psi(a_0)$ and taking account of remark 2.7 we find $\tilde{u}_0=0$, $\tilde{u}_1=-i\dot{u}_0+\left[a_{\mathrm{sub}}(x^*,\xi^*)+\frac{i}{2}\mathrm{Tr}\ (Z^{-1}\dot{Z})\right]u_0$ and that there exist differential operators $\{L_{j,k}\}$ such that

$$\tilde{u}_j=-i\dot{u}_{j-1}+\left[a_{\mathrm{sub}}(x^*,\xi^*)+\frac{i}{2}\mathrm{Tr}\ (Z^{-1}\dot{Z})\right]u_{j-1}+\sum_{k\le j-2}L_{j,k}u_k$$

for all $2\le j\le N+1$. Differentiating and using Liouville's formula to re-write $\frac{\frac{d}{dt}\det Z}{\det Z}=\mathrm{Tr}\ (Z^{-1}\dot{Z})$ we find $\tilde{u}_1=0$ and

$$\tilde{u}_j=(\det Z)^{\frac{1}{2}}e^{-i\int_0^t a_{\mathrm{sub}}(x^*,\xi^*)ds}\dot{\nu}_{j-1}+\sum_{k\le j-2}L_{j,k}u_k$$

for $2\le j\le N+1$.

The ν_j satisfy (2.19), however, we may choose the ν_j in such a way that $\tilde{u}_j=0$ for all $2\le j\le N+1$. We have thus established that

$$\max_{t\in[-T,T]}\left\|\left(-ih\frac{\partial}{\partial t}+A(h)\right)(U_N-U_P)\right\|_{\mathcal{L}(L^2(\mathbb{R}^n))}=O(h^{N+2})$$

and from (2.19) and Theorem 2.8

$$\|U_N(0;h)-P(h)\|_{\mathcal{L}(L^2(\mathbb{R}^n))}=O(h^{N+1}).$$

The proof is thus completed by an application of the Duhamel formula:

$$R(t;h)=\int_0^t e^{ih^{-1}(s-t)A(h)}\left[\left(\frac{\partial}{\partial s}+ih^{-1}A(h)\right)R(s;h)\right]ds+e^{-ih^{-1}tA(h)}R(0;h).$$

□

Let $Q(h)$ denote another h P.D.O. with compactly supported symbol. Then from Theorem 2.9 and Corollary 2.6, $U_{P,Q}(t;h)=P(h)e^{-ih^{-1}tA(h)}Q(h)^*$ is an h F.I.O. and for any phase $\phi\in\Psi(a_0)$ there exists a symbol $\sum_{j\ge 0}h^ju_j$, with

$$u_0=(\det Z)^{\frac{1}{2}}e^{-i\int_0^t a_{\mathrm{sub}}(x^*,\xi^*)ds}p_0(x^*,\xi^*)\overline{q_0}(y,\eta)$$
$$\sum_{j\ge 0}h^ju_j(0)=e^{-\frac{ih}{2}(D_x\cdot D_\xi+i\mu D_\xi^2)}(p\sharp\overline{q})\Big|_{(x,\xi)=(y,\eta)} \tag{2.20}$$

(where $p\sharp\overline{q}$ denotes the Weyl symbol of the composition $P(h)Q(h)^*$) such that $U_{P,Q}$ has this phase and symbol. We denote $L_{(s)}(h)$ the h P.D.O. with symbol $L(x,\xi)=\langle\xi\rangle^s$. Then applying Theorem 2.9 and Corollary 2.6 to $L_{(s)}(h)^*U_{P,Q}L_{(s)}(h)$ for $s>\frac{n}{2}$, we obtain:

Corollary 2.10. *For any $T \geq 0$ and integer $N \geq 0$,*

$$\sup_{x,y\in\mathbb{R}^n} \max_{t\in[-T,T]} |u_{P,Q}(t,x,y;h) - u_N(t,x,y;h)| = O(h^{N+1})$$

as $h \to 0$ where, respectively, $u_{P,Q}$ and u_N denote the integral kernels of $U_{P,Q}$ and the h F.I.O. with phase ϕ and symbol $\sum_{j\leq N} h^j u_j$, where ϕ and $\{u_j\}$ are those in the preceding paragraph.

2.3 Proof of Theorem 2.2.

In this section we use a stationary phase argument to prove Theorem 2.2. Let ϕ and u denote an arbitrary phase function and x independent symbol and let $\rho \in \mathcal{S}(\mathbb{R})$ satisfy the assumptions above. We are going to consider integrals of the form

$$(2\pi h)^{-1} \int_{-\infty}^{\lambda} \int \hat{\rho}(t) e^{irt} e^{ih^{-1}\psi} u(t,y,\eta)\, dy d\eta dt ds \tag{2.21}$$

where $\psi = st + \phi(t,y,y,\eta)$.

We define $\tau(y,\eta) = a_0(y,\eta)$ and recall that throughout we assume $\nabla a_0 \neq 0$ on Σ_λ. If u has y,η support contained in a sufficiently small neighbourhood of Σ_λ then we can choose functions $\{\omega_j\}_{j=1}^{2n-1}$, each $\omega_j : \mathbb{R}^{2n} \to \mathbb{R}$, such that (τ,ω) is a diffeormorphism from $\mathrm{supp}_{y,\eta} u$ onto the set $U \times V$ (where $U = \tau(\ \mathrm{supp}\ _{y,\eta} u) \subseteq \mathbb{R}$ and $V = \omega(\ \mathrm{supp}\ _{y,\eta} u) \subseteq \mathbb{R}^{2n-1}$) and the matrix of derivatives $\frac{\partial(y,\eta)}{\partial(\tau,\omega)}$ has determinant of absolute value $|\nabla a_0|^{-1}$ when evaluated at $\tau = \lambda$.

As $\nabla a_0 \neq 0$ on Σ_λ, if u has y,η support contained in a sufficiently small neighbourhood of Σ_λ and $\hat{\rho}$ has support contained in a sufficiently small neighbourhood of 0 then we have $\Phi_{a_0}^t(y,\eta) = (y,\eta)$ if and only if $t = 0$ for all $(t,y,\eta) \in \mathrm{supp}\ (\hat{\rho}u)$.

From here on we assume that $\mathrm{supp}\ (\hat{\rho}u)$ is contained in a sufficiently small neighbourhood of $\{0\}\times\Sigma_\lambda$ such that the statements of the preceding two paragraphs are true.

Differentiating we have

$$\begin{aligned} \psi_t &= s - \tau + (\dot{\xi}^* - i\mu\dot{x}^*)\cdot(y - x^*) \\ \psi_\tau &= (\xi^*_\tau - i\mu x^*_\tau)\cdot(y-x^*) + y_\tau \cdot (\xi^* - \eta + i\mu(y - x^*)) \end{aligned} \tag{2.22}$$

and thus, ψ is real-valued and stationary (with respect to t and τ) on $\mathrm{supp}\ (\hat{\rho}u)$ if and only if $(t,\tau) = (0,s)$. Although ψ is complex-valued, as this unique stationary point is independent of the ω variables, we may apply a (real) stationary phase argument for the (t,τ) integration in (2.21), see [Hö] Theorem 7.7.6 for example.

Elementary differentiations establish that

$$\begin{aligned} \partial_\tau^k \psi(0,\tau) &= 0 \text{ for all } k \geq 0 \\ \partial_\tau^k \dot{\psi}(0,\tau) &= 0 \text{ for all } k \geq 2 \\ (\partial_\tau \dot{\psi})(0,s) &= -1. \end{aligned} \tag{2.23}$$

We denote the hessian matrix of second derivatives of ψ with respect to t and τ evaluated at the statinary point by ψ'' and, according to (2.23), ψ'' has determinant of absolute value 1. Furthermore, $\mathrm{Re}(\psi'')$ has zero signature, $\mathrm{Im}(\psi'') \geq 0$

and, in matrix supremum norm, $\mathrm{Im}(\psi'') \to 0$ as $\mu \to 0$. Thus, the square root $(\det(-i\psi''))^{-\frac{1}{2}}$, as defined in [Hö] section 3.4, is equal to $e^{i\theta(\mu)}$, θ some function $\theta : \mathbb{R}_+ \longrightarrow [0, 2\pi)$ such that $\theta(\mu) \to 0$ as $\mu \to 0$.

We define the function $g(t,\tau) = \psi(t,\tau) - \frac{1}{2}\psi''(t,\tau - s)\cdot(t,\tau - s)$, and denote by D the differential operator $D = 2\partial_t\partial_\tau - \psi_{tt}\partial_\tau^2$. Then for any suitable f, using the fact that g has a zero of order 6 at the stationary point and using (2.23), $D^3(fg^2) = 0$ when evaluated at $(0,s)$. Furthermore, using (2.23), $D^2(fg)\big|_{(0,s)} = 8(\partial_\tau\partial_t^2\psi)(\partial_\tau f)\big|_{(0,s)} + 4\,(\partial_\tau^2\partial_t^2\psi)f\big|_{(0,s)}$. Hence, defining the differential operator

$$Lf = -i\left[(\partial_\tau\partial_t f) + \frac{1}{2}\left[\partial_\tau^2(\psi_{tt}f)\right]\right]\Bigg|_{(t,\tau)=(0,s)}, \tag{2.24}$$

and changing variables $(y,\eta) \mapsto (\tau,\omega)$, [Hö] Theorem 7.7.6 implies that the integral in (2.21) is equal to

(2.25)

$$e^{i\theta(\mu)}\int_{a_0\le\lambda} u(0)dyd\eta + he^{i\theta(\mu)}\int_{-\infty}^{\lambda}\int_V L\left(\hat\rho e^{irt}u\left|\det\frac{\partial(y,\eta)}{\partial(\tau,\omega)}\right|\right)d\omega ds + O_\mu(h^2).$$

For any suitable f, according to how the $\{\omega_j\}$ were chosen,

(2.26)

$$\int_{-\infty}^{\lambda}\int_V \frac{\partial}{\partial\tau}\bigg|_{\tau=s}\left(f\left|\det\frac{\partial(y,\eta)}{\partial(\tau,\omega)}\right|\right)d\omega ds = \int_V f|_{\tau=\lambda}\frac{d\omega}{|\nabla a_0|} = \int_{\Sigma_\lambda} f|_{\tau=\lambda}\,d\nu_\lambda.$$

We can apply (2.25) in the case when $\phi \in \Psi(a_0)$ is a phase and u is replaced by $(2\pi h)^{-n}(u_0 + hu_1)$, the u_j being the symbol terms (for this particular ϕ) of the h F.I.O. $P(h)e^{-ih^{-1}tA(h)}Q(h)^*$, see the end of the previous section. (To be precise, we must apply (2.25) in the case when the u_j are the symbol terms of the h F.I.O. $P(h)f(A(h))e^{-ih^{-1}tA(h)}Q(h)^*$, with $f \in C_c^\infty(\mathbb{R})$ having $f(\lambda) = 1$ and support contained in a sufficiently small neighbourhood of λ. This is to ensure that $\hat\rho(u_0 + hu_1)$ has support contained in a sufficiently small neighbourhood of $\{0\}\times\Sigma_\lambda$.)

As $\theta(\mu) \to 0$ as $\mu \to 0$, we can choose $\mu_0 > 0$ such that $\cos(\theta(\mu)) \neq 0$ for all $0 \le \mu < \mu_0$. Differentiating the explicit formula for u_0 given in (2.20), using the formula for $u_1(0)$ given in (2.20), denoting the left-hand side of (2.4) by $I_{P,Q}$, using

(2.25), (2.24) and (2.26), there exists a constant $C^{P,Q}(\lambda)$ such that,

(2.27)

$$\begin{aligned}\frac{I_{P,Q}}{\cos(\theta(\mu))} &= h^{-n}C_0^{P,Q}(\lambda) + h^{1-n}(2\pi)^{-n}\int_{a_0\le\lambda}\mathrm{Re}\left((PQ^*)_{\mathrm{sub}}\right)dxd\xi \\ &- h^{1-n}(2\pi)^{-n}\int_{\Sigma_\lambda}\left[\frac{\mathrm{Im}\left(\{p_0,a_0\}\overline{q_0}+p_0\{a_0,\overline{q_0}\}\right)}{2}+\mathrm{Re}(p_0\overline{q}_0)a_{\mathrm{sub}}\right]d\nu_\lambda \\ &+ h^{1-n}(2\pi)^{-n}\int_{a_0\le\lambda}\frac{1}{2}\mathrm{Im}\left[\left(D_x\cdot D_\xi+i\mu D_\xi^2\right)(p_0\overline{q_0})\right]dxd\xi + rh^{1-n}C_2^{P,Q}(\lambda) \\ &+ \frac{h^{1-n}}{2}(2\pi)^{-n}\int_{-\infty}^{\lambda}\int_V \mathrm{Im}\left[\frac{\partial^2}{\partial\tau^2}\bigg|_{\tau=s}\left(\psi_{tt}(0)p_0\overline{q}_0\left|\det\frac{\partial(y,\eta)}{\partial(\tau,\omega)}\right|\right)\right]d\omega ds \\ &- h^{1-n}(2\pi)^{-n}\int_{\Sigma_\lambda}\frac{1}{2}\mathrm{Im}\left(\mathrm{Tr}\ \left[(a_0)_{x\xi}+i\mu(a_0)_{\xi\xi}\right]p_0\overline{q}_0\right)d\nu_\lambda \\ &+ h^{-n}\tan(\theta(\mu))\left[C_0^{iP,Q}(\lambda)+hC^{P,Q}(\lambda)\right]+O_\mu(h^{2-n}),\end{aligned}$$

for all $0\le\mu<\mu_0$ and all r in any bounded interval.

Recall that $\hat{\rho}$ is even. A change of variables therefore implies that $I_{P,Q}=I_{Q,P}$. Using this fact in (2.27) and the fact that $\mathrm{Im}\psi_{tt}(0)=\mu|\dot{x}^*(0)|^2$, there exists another constant $C(\lambda)$ such that

$$\begin{aligned}\frac{I_{P,Q}}{\cos(\theta(\mu))} &= h^{-n}C_0^{P,Q}(\lambda) + h^{1-n}(2\pi)^{-n}\int_{a_0\le\lambda}\mathrm{Re}\left((PQ^*)_{\mathrm{sub}}\right)dxd\xi + \mu h^{1-n}C(\lambda) \\ &\quad - h^{1-n}(2\pi)^{-n}\int_{\Sigma_\lambda}\left[\frac{\mathrm{Im}\left(\{p_0,a_0\}\overline{q_0}+p_0\{a_0,\overline{q_0}\}\right)}{2}+\mathrm{Re}(p_0\overline{q}_0)a_{\mathrm{sub}}\right]d\nu_\lambda \\ &\quad + rh^{1-n}C_2^{P,Q}(\lambda)+\frac{h^{1-n}}{2}\tan(\theta(\mu))\left[C^{P,Q}(\lambda)+C^{Q,P}(\lambda)\right]+O_\mu(h^{2-n}),\end{aligned}$$

where we have used the fact that $C_0^{iP,Q}(\lambda)=-C_0^{iQ,P}(\lambda)$. Letting $\mu\to 0$, $\theta(\mu)\to 0$ and we can replace $\frac{h^{1-n}}{2}\tan(\theta(\mu))\left[C^{P,Q}(\lambda)+C^{Q,P}(\lambda)\right]+\mu h^{1-n}C(\lambda)+O_\mu(h^{2-n})$ by $o(h^{1-n})$ which completes the proof. □

Acknowledgements

The author would like to thank Bernard Helffer for useful comments and the suggestion of working on this problem. Section 2.2 constitutes an extension of part of the work contained in the author's PhD thesis and the author would therefore like to thank Yuri Safarov for useful suggestions there. The author is also grateful to Vesselin Petkov for pointing out the 'correct' definition of absolute periodicity and Søren Fournais for useful comments.

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