Limiting Absorption Principle at Critical Values for the Dirac Operator

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Abstract

We prove estimates for the resolvent $(H_0 - z)^{-1}$ of the Dirac operator $H_0 = \boldsymbol{\alpha} \cdot \boldsymbol{P} + m\beta$, valid even for z close to the critical points $\pm m$. In particular, it is shown that the operator $(1 + |x|^2)^{-1/2}$ is globally H_0-smooth. As a by-product, the absence of the singular spectrum as well as the existence and unitarity of the wave operators are obtained for a class of perturbations of $H = H_0 + V$.

1 Introduction

We shall work in the Hilbert space $\mathscr{H} := L^2(\mathbb{R}^3, \mathbb{C}^4)$ with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. It will be freely identified to $\mathcal{H} \otimes \mathbb{C}^4$ where we set $\mathcal{H} := L^2(\mathbb{R}^3)$. Then the free Dirac operator

$$H_0 = \boldsymbol{\alpha} \cdot \boldsymbol{P} + m\beta \tag{1.1}$$

describes (in suitable units) a relativistic electron with mass $m > 0$. Here the dot means the usual scalar product in $\mathbb{R}^3$, $\boldsymbol{P} := -i\boldsymbol{\nabla}$ is the momentum observable and $\beta := \alpha_0$, α_1, α_2, α_3 are the (hermitian) Dirac matrices satisfying the anti-commutation relations: $\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}$. H_0 is a self-adjoint operator in $\mathscr{H}$ with domain $\mathscr{H}^1 := \mathcal{H}^1 \otimes \mathbb{C}^4$, where $\mathcal{H}^1$ is the usual Sobolev space of order one in $\mathbb{R}^3$. The spectrum of H_0 is purely absolutely continuous and equal to $(-\infty, -m] \cup [m, \infty)$. We refer to [Th] for a comprehensive discussion of the Dirac operator.

In connection with the spectral analysis of H_0 and large classes of perturbations, extensive use was made of the limiting absorption principle (LAP). Let us state a rough version, only for the free operator. For two Hilbert spaces $\mathscr{X}$ and $\mathscr{Y}$, denote by $\mathbb{B}(\mathscr{X}, \mathscr{Y})$ the Banach space of linear, continuous operators : $\mathscr{X} \to \mathscr{Y}$; if $\mathscr{X} = \mathscr{Y} \equiv \mathscr{H}$ we simply set $\mathbb{B}(\mathscr{H})$ and use the symbol $\| \cdot \|$ also for the norm in $\mathbb{B}(\mathscr{H})$. It is known that one cannot expect to have estimates uniform in $|\mathfrak{Im} z|$ for $(H_0 - z)^{-1}$ in $\mathbb{B}(\mathscr{H})$ if $\mathfrak{Re} z$ belongs to $(-\infty, -m] \cup [m, \infty)$. But weaker estimates are in fact available if one avoids the points $\pm m$.

Let us set $\langle x \rangle = \sqrt{1 + |x|^2}$ for any $x \in \mathbb{R}^3$, Q_j for the operator of multiplication with x_j and let $\mathcal{S}'$ be the space of tempered distributions in $\mathbb{R}^3$, dual of the Schwartz space $\mathcal{S}$. Then $\mathcal{H}_s := \{\varphi \in \mathcal{S}' \mid \|\varphi\|_{\mathcal{H}_s} := \|\langle Q \rangle^s \varphi\|_{\mathcal{H}} < \infty\}$ is the weighted Lebesgue space of order $s \in \mathbb{R}$ and we set $\mathscr{H}_s := \mathcal{H}_s \otimes \mathbb{C}^4$ (Hilbert tensor product).

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The LAP states that for any $s > \frac{1}{2}$ and any compact subset J of $(-\infty, -m) \cup (m, \infty)$ there is a constant C such that for any $\lambda \in J$ and $\mu > 0$:

$$\|(H_0 - \lambda \mp i\mu)^{-1}\|_{\mathbb{B}(\mathscr{H}_s, \mathscr{H}_{-s})} \le C. \tag{1.2}$$

The fact that (1.2) fails for $\lambda = \pm m$ and $s - \frac{1}{2}$ small is one of the motivations of the present work.

The points $\pm m$ also play a special role in the spectral theory of perturbations $H = H_0 + V$. The main feature is that for large classes of potentials V (that roughly are sums of a short-range and a long-range part), $\lambda = \pm m$ are the only possible accumulation points of the point spectrum of H and, moreover, any eigenvalue that differs of $\pm m$ has finite multiplicity. We also know that the singular continuous spectrum is absent. These results (see e.g. [BMP]) are closely related to (1.2) as well as to the fact that it fails for $\lambda = \pm m$.

The break-down of the resolvent estimates near some special points is a well-known phenomenon. Let us consider the case of the (positive) Laplace operator $\Delta := |P|^2 = -\sum_{j=1}^n \partial_j^2$ acting in $L^2(\mathbb{R}^n)$. Then one has for any $s > \frac{1}{2}$:

$$\|(\Delta - \lambda \mp i\mu)^{-1}\|_{\mathbb{B}(\mathcal{H}_s(\mathbb{R}^n), \mathcal{H}_{-s}(\mathbb{R}^n))} \le C, \tag{1.3}$$

uniformly in $\mu > 0$ and $\lambda \in J \subset (0, \infty)$, J compact. For an improved version of the estimate (1.3) we refer to [Ag]. But one cannot take $\lambda = 0$ in (1.3); the operators $(\Delta \mp i\mu)^{-1} : \mathcal{H}_{\frac{1}{2}+\varepsilon}(\mathbb{R}^n) \mapsto \mathcal{H}_{-\frac{1}{2}-\varepsilon}(\mathbb{R}^n)$ are not bounded uniformly in $\mu > 0$ if $\varepsilon > 0$ is small. This is connected to the fact that $\Delta = h(P)$ with $h(x) = |x|^2$ and 0 is a critical value for h. Actually, the Laplace operator is just a particular case. The operator $h(P)$ may be defined in $L^2(\mathbb{R}^n)$ for large classes of functions $h : \mathbb{R}^n \mapsto \mathbb{R}$. Under certain conditions on h, the LAP can also be proved for $h(P)$ if λ belongs to regions away from the critical set $\mathbf{Cr}(h) := h\left[(\nabla h)^{-1}(\{0\})\right]$. The auxiliary Hilbert space $\mathcal{H}_s(\mathbb{R}^n)$ (with $s > \frac{1}{2}$) is playing the same role as above.

In the last period, the problem of obtaining the LAP for the operator $h(P)$ in a global form, i.e. also at the critical points, has become one of special interest in the literature; as a short (and subjective) selection we refer to [KY], [Si] for the laplacian and to [B-A], [B-AK], [CS], [KRS], [MP] for the general case. The main new fact brought by the global estimates is that one is forced to use a smaller space in the above estimates, namely one has (roughly) to replace $\mathcal{H}_{\frac{1}{2}+\varepsilon}(\mathbb{R}^n)$ by $\mathcal{H}_1(\mathbb{R}^n)$.

The main purpose of the present paper is to obtain similar results, namely a *global* LAP for the Dirac operator. The set $\{-m, m\}$ may be viewed as a critical set because m is the only critical value of the function $h_m : \mathbb{R}^3 \mapsto \mathbb{R}$ with $h_m(x) := \sqrt{|x|^2 + m^2}$ and $H_0 = \boldsymbol{\alpha} \cdot \boldsymbol{P} + m\beta$ is unitarily equivalent to the operator $[h_m(\boldsymbol{P}) \otimes 1_{\mathbb{C}^2}] \oplus [-h_m(\boldsymbol{P}) \otimes 1_{\mathbb{C}^2}]$. Unfortunately, the unitary transformation is complicated (see [Th]) hence a direct treatment of H_0 is preferable.

Let us describe the method of proof. We recall first that one of the strongest tools in proving the LAP for a self-adjoint operator H is the method of E. Mourre (see [Mo], [ABG]). It relies mainly on the so-called Mourre estimate:

$$E_H(J) i[H, A] E_H(J) \ge a E_H(J), \tag{1.4}$$

where E_H is the spectral measure of H, J is a suitable real interval, $a > 0$ and A is a self-adjoint operator. One often says that A is *conjugated* to H on the interval J. Under some extra technical assumptions one gets a LAP for H with respect to the operator norm in $\mathbb{B}(\mathscr{X}, \mathscr{X}')$. A simple but rough option is to take $\mathscr{X}$ as the domain of A endowed with the graph norm and $\mathscr{X}'$ the dual of $\mathscr{X}$. Unfortunately, if $H = h(P)$ the best choice for A is essentially

$$A = \frac{1}{2}\{(\nabla h)(P) \cdot Q + Q \cdot (\nabla h)(P)\},$$

and (1.4) cannot be obtained if the closure of J contains critical points of the function h. The strict positivity fails. In order to overcome this obstruction an approach called the "*method of the weakly conjugate operator*" was developed in [BKM] and [BM], in which the LAP can be obtained from the weaker positivity assumption $[iH, A] > 0$ and some extra technical conditions. This may be seen as an extension of the well-known Kato-Putnam theory (see [RS2]). It has already been applied to different types of partial differential operators with anisotropic coefficients, leading to results on the absence of the singular spectrum in various situations in which Mourre's approach did not work. The task of the present work is to apply this method to Dirac operators.

2 Main results

Let H_0 be the self-adjoint operator defined by (1.1). Recall that the operator $T \in \mathbb{B}(\mathscr{H})$ is called H_0-*smooth* if there is a finite constant C such that $\int_{\mathbb{R}} \|Te^{-itH_0}\psi\|^2 dt \le C\|\psi\|^2$ for any $\psi \in \mathscr{H}$.

Theorem 2.1 (i) *There exists $C > 0$ such that for any $\lambda \in \mathbb{R}$ and $\mu > 0$:*

$$\|(H_0 - \lambda \mp i\mu)^{-1}\|_{\mathbb{B}(\mathscr{H}_1, \mathscr{H}_{-1})} \le C.$$

(ii) *The operator $\langle \boldsymbol{Q} \rangle^{-1}$ is H_0-smooth.*

Corollary 2.2 *Let $V : \mathbb{R}^3 \to \mathbb{B}(\mathbb{C}^4)$ be a Borel, hermitian valued function such that for all $x \in \mathbb{R}^3$, $\|V(x)\|_{\mathbb{B}(\mathbb{C}^4)} \le \langle x \rangle^{-2}$. Then there exists a constant $\Gamma > 0$ such that for all $\gamma \in (-\Gamma, \Gamma)$ the wave operators for the couple H_0, $H_\gamma := H_0 + \gamma V$*

$$\Omega^{\pm} = \operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{itH_\gamma} \mathrm{e}^{-itH_0}$$

exist and are unitary. In particular, H_γ has purely absolutely continuous spectrum.

The corollary follows easily by combining (i) of Theorem 2.1 with standard smoothness results (see e.g. [RS4]). Notice that it states much more than just the absence of the point spectrum (which might be a known fact for perturbations as above). As for the statement (ii) in Theorem 2.1, it follows immediately (see [RS4]) from (i). We thus consider (i) as being the main result of the present paper and we devote the Section 3 to its proof.

As explained in the Introduction, the first problem one has to solve is to find a "weakly conjugate operator" A satisfying $[iH_0, A] > 0$. Our choice is

(2.1) $$A = \frac{1}{2}(H_0^{-1} \boldsymbol{P} \cdot \boldsymbol{Q} + \boldsymbol{Q} \cdot \boldsymbol{P} H_0^{-1}).$$

Easy computations performed on $\mathscr{S}$ show that the above commutator equals $\Delta(\Delta + m^2)^{-1} \equiv B$. Then B extends to a bounded operator in $\mathscr{H}$ which satisfies $\langle \psi, B\psi \rangle > 0$ for any $\psi \in \mathscr{H} \setminus \{0\}$. This is the precise meaning to give to "weak positivity". In [BG] and [BMP] another operator A was used in order to get the Mourre estimate for the perturbed hamiltonian $H = H_0 + V$, under suitable conditions on V. It does not fit to our purposes. But one *may* use (2.1) in order to perform Mourre theory in a rather simplified manner. We also send to [GM] in which a systematic way to treat a large class of generalizations of the Dirac operator perturbed by highly singular potentials is given. Note however that this does not prove the LAP at critical points and give only the absence of the singular continuous spectrum and some qualitative information on the point spectrum. Let us mention that one cannot use directly the abstract results of [BKM] and [BM]. Indeed, in these papers some technical difficulties are solved by working intensively with the evolution group $\{W(t) = e^{itA} \mid t \in \mathbb{R}\}$. This is not convenient in our case, the group generated by (2.1) being too involved. In exchange, although both H_0 and A are unbounded operators, the commutators B and $i[B, A]$ are bounded and this helps us to overcome the technical difficulties. On the other hand, in an abstract setting a supplementary space $\mathscr{A}$ appears in the LAP. But its rigorous definition is based also on the evolution group $\{W(t) \mid t \in \mathbb{R}\}$. Moreover, in applications the space $\mathscr{A}$ is too involved and somewhat artificial. Our direct proof puts instead in evidence the simple and familiar space $\mathscr{H}_1 = \mathcal{H}_1 \otimes \mathbb{C}^4$.

3 Proof of Theorem 2.1

Our goal is to prove the estimate

$$|\langle \psi, (H_0 - \lambda \mp i\mu)^{-1}\psi\rangle| \leq C\|\psi\|^2_{\mathscr{H}_1}, \tag{3.1}$$

where the constant C is independent of $\psi \in \mathscr{H}_1$, $\lambda \in \mathbb{R}$ and $\mu > 0$. As stated in the Introduction, the strategy is to find a self-adjoint operator A having positive commutator with H_0 and fulfilling some extra technical conditions. We define A on $\mathscr{S} := \mathcal{S} \otimes \mathbb{C}^4$ (the completion in $\|\cdot\|_{\mathscr{H}}$ of the algebraical tensor product $\mathcal{S} \odot \mathbb{C}^4$) by the formula (2.1). Notice that $H_0^2 = |\boldsymbol{P}|^2 + m^2 = \Delta + m^2$, hence $H_0^{-1} = H_0(\Delta + m^2)^{-1}$. Thus

$$A = \frac{1}{2}\left[H_0(|\boldsymbol{P}|^2 + m^2)^{-1}\boldsymbol{P}\cdot\boldsymbol{Q} + \boldsymbol{Q}\cdot\boldsymbol{P}(|\boldsymbol{P}|^2 + m^2)^{-1}H_0\right]$$

leaves $\mathscr{S}$ invariant. The operator A may also be set in the non-symmetric forms

$$A = H_0^{-1}\boldsymbol{P}\cdot\boldsymbol{Q} + \frac{3i}{2}H_0^{-1} - \frac{i}{2}H_0^{-1}\boldsymbol{\alpha}\cdot\boldsymbol{P}H_0^{-1} = H_0^{-1}\boldsymbol{P}\cdot\boldsymbol{Q} + iH_0^{-1} + \frac{1}{2}miH_0^{-1}\beta H_0^{-1} \tag{3.2}$$

if one takes into account the commutation formulae

$$[Q_j, P_k] = i\delta_{jk} \text{ and } [Q_j, H_0^{-1}] = -H_0^{-1}[Q_j, H_0]H_0^{-1} = -iH_0^{-1}\alpha_j H_0^{-1},\ j = 1, 2, 3.$$

Lemma 3.1 (i) *A is an essentially self-adjoint operator.*
(ii) *The domain of the (self-adjoint) closure of A contains $\mathscr{H}_1$.*

Proof: Obviously, A is symmetric. According to the commutator theorem of Nelson (see [RS2] Theorem X.37, with $N \equiv \langle\boldsymbol{Q}\rangle \geq 1$) one needs to prove for any $\psi \in \mathscr{S}$:

$$\|A\psi\| \leq C\,\|\langle\boldsymbol{Q}\rangle\psi\| \quad \text{and} \quad |\langle A\psi, \langle\boldsymbol{Q}\rangle\psi\rangle - \langle\langle\boldsymbol{Q}\rangle\psi, A\psi\rangle| \leq C\,\langle\psi, \langle\boldsymbol{Q}\rangle\psi\rangle\,.$$

Since the operators H_0^{-1} and $H_0^{-1}P_j$ are bounded (for each j), the first inequality follows from (3.2). The second one is an immediate consequence of the equality

$$i[A, \langle\boldsymbol{Q}\rangle] = H_0^{-1}\frac{|\boldsymbol{Q}|^2}{\langle\boldsymbol{Q}\rangle} + \frac{|\boldsymbol{Q}|^2}{\langle\boldsymbol{Q}\rangle} - H_0^{-1}\frac{\boldsymbol{\alpha}\cdot\boldsymbol{Q}}{\langle\boldsymbol{Q}\rangle}H_0^{-1}\boldsymbol{P}\cdot\boldsymbol{Q} - \boldsymbol{Q}\cdot\boldsymbol{P}H_0^{-1}\frac{\boldsymbol{\alpha}\cdot\boldsymbol{Q}}{\langle\boldsymbol{Q}\rangle}H_0^{-1},$$

obtained by a direct commutator calculus performed on the invariant domain $\mathscr{S}$. ■

In Section 2 we introduced the notation B for the bounded operator $\Delta(\Delta+m^2)^{-1}=1-m^2(\Delta+m^2)^{-1}$. As a consequence of the Lemma 3.1, a standard computation performed on vectors $\varphi_1,\ \varphi_2\in\mathscr{H}_1\cap\mathscr{H}^1$ gives

$$\langle\varphi_1,B\varphi_2\rangle=\langle iH_0\varphi_1,A\varphi_2\rangle-\langle iA\varphi_1,H_0\varphi_2\rangle. \tag{3.3}$$

This allows us to interpret B as the commutator $[iH_0,A]$ on $\mathscr{H}_1\cap\mathscr{H}^1$. Since it will play an important rôle in our proof, we shall point out some of its properties. If one sets $B^{1/2}=|\boldsymbol{P}|(\boldsymbol{P}^2+m^2)^{-1/2}\in\mathbb{B}(\mathscr{H})$ we may see $B^{1/2}$ as an operator $b(\boldsymbol{P})$ defined via the functional calculus by the Borel function $b:\mathbb{R}^3\to\mathbb{R}$, $b(\boldsymbol{\kappa})=|\boldsymbol{\kappa}|(\boldsymbol{\kappa}^2+m^2)^{-1/2}$. The left inverse $B^{-1/2}$ of $B^{1/2}$ satisfies the property $B^{-1/2}\in\mathbb{B}(\mathscr{H}_1,\mathscr{H})$, i.e. for all $\psi\in\mathscr{H}_1$

$$\|B^{-1/2}\psi\|\le C\|\psi\|_{\mathscr{H}_1}. \tag{3.4}$$

Indeed, since $b(\boldsymbol{\kappa})\le 1-m|\boldsymbol{\kappa}|^{-1}$ it will be enough to show that $|\boldsymbol{P}|^{-1}\in\mathbb{B}(\mathscr{H}_1,\mathscr{H})$. But this is a direct consequence of the Hardy inequality $\||\boldsymbol{P}|^{-1}\psi\|_{\mathscr{H}}\le\frac{1}{4}\|\psi\|_{\mathscr{H}_1}$, valid for each $\psi\in\mathscr{H}_1$. Also, remark that $B^{-1/2}A$ is a well defined linear operator : $\mathscr{S}\to\mathscr{H}$. We stress that it extends to an element of $\mathbb{B}(\mathscr{H}_1,\mathscr{H})$ i.e. there is some $C>0$ such that for any $\psi\in\mathscr{H}_1$

$$\|B^{-1/2}A\psi\|\le C\|\psi\|_{\mathscr{H}_1}. \tag{3.5}$$

This clearly follows from (3.4) and (3.2) if one takes into account that $B^{-1/2}=|H_0|\,|\boldsymbol{P}|^{-1}$ on $\mathscr{S}$. Moreover, an easy computation performed on $\mathscr{S}$ yields

$$\begin{aligned}[iB,A]=i[1-m^2H_0^{-2},A] &= m^2H_0^{-1}([iH_0,A]H_0^{-1}+H_0^{-1}[iH_0,A])H_0^{-1}\\ &= m^2H_0^{-1}(\Delta H_0^{-3}+H_0^{-3}\Delta)H_0^{-1}=2m^2\Delta H_0^{-5}.\end{aligned}$$

Actually, $[iB,A]$ extends to an element of $\mathbb{B}(\mathscr{H})$, and one checks by continuity that for all $\varphi_1,\ \varphi_2\in\mathscr{H}_1\cap\mathscr{H}^1$,

$$\langle\varphi_1,2m^2\Delta H_0^{-5}\varphi_2\rangle=\langle iB\varphi_1,A\varphi_2\rangle-\langle iA\varphi_1,B\varphi_2\rangle. \tag{3.6}$$

It is also clear that $B^{-1/2}\Delta H_0^{-5}B^{-1/2}$ extends to a bounded operator in $\mathscr{H}$.

We start now the main part of the proof of the Theorem 2.1 by defining for each $\lambda\in\mathbb{R}$ and $\mu>0$ the family of operators $T_\varepsilon^\pm(\lambda,\mu):=H_0-\lambda\mp i\mu\mp i\varepsilon B$, indexed by $\varepsilon\in[0,1)$. Remark first that they are (linear) homeomorphisms : $\mathscr{H}^1\to\mathscr{H}$. Indeed, for $\mu>0$ and $\varphi\in\mathscr{H}^1$, the estimate

$$\|T_\varepsilon^\pm\varphi\|^2=\|(H_0-\lambda\mp i\varepsilon B)\varphi\|^2+\mu^2\|\varphi\|^2+2\mathfrak{Re}\langle(H_0-\lambda\mp i\varepsilon B)\varphi,\mp i\mu\varphi\rangle\ge\mu^2\|\varphi\|^2 \tag{3.7}$$

based on the weak positivity of B, shows that $T_\varepsilon^\pm\equiv T_\varepsilon^\pm(\lambda,\mu)$ are injective. Since $T_\varepsilon^\pm=(T_\varepsilon^\mp)^*$ their ranges are dense. For each $\varepsilon<1$ they are closed operators and their surjectivity follows easily. We may thus conclude by the open mapping theorem.

Set now $G_\varepsilon^\pm\equiv G_\varepsilon^\pm(\lambda,\mu):=(T_\varepsilon^\pm(\lambda,\mu))^{-1}$. They are operators of $\mathbb{B}(\mathscr{H},\mathscr{H}^1)\subset\mathbb{B}(\mathscr{H})$ which verify $G_\varepsilon^\pm=(G_\varepsilon^\mp)^*$. For a vector ψ arbitrarily fixed in $\mathscr{H}$ we shall systematically use the notation $F_\varepsilon^\pm=\langle\psi,G_\varepsilon^\pm(\lambda,\mu)\psi\rangle$. The function $\varepsilon\mapsto F_\varepsilon^\pm$ is continuous on $[0,1)$ and in particular in $\varepsilon=0$ as it is shown by

$$\begin{aligned}|F_\varepsilon^\pm-F_0^\pm| &= |\langle\psi,G_\varepsilon^\pm(\lambda,\mu)\psi\rangle-\langle\psi,(H_0-\lambda\mp i\mu)^{-1}\psi\rangle|\le|\langle\psi,G_\varepsilon^\pm(\lambda,\mu)i\varepsilon B(H_0-\lambda\mp i\mu)^{-1}\psi\rangle|\\ &\le\varepsilon\|B\|\,\|G_\varepsilon^\pm(\lambda,\mu)\|\,\|(H_0-\lambda\mp i\mu)^{-1}\|\,\|\psi\|^2\le\frac{\varepsilon}{\mu^2}\|\psi\|^2,\end{aligned}$$

which is an easy consequence of the second resolvent identity and of (3.7). Hence $\lim_{\varepsilon\to 0} F_\varepsilon = F_0$ exists, and the formula (3.1) to be proved may be rewritten as

$$|F_0^{\pm}| \le C\|\psi\|^2_{\mathscr{H}_1}. \tag{3.8}$$

In order to show (3.8) we need detailed information about the family of resolvents $\{G_\varepsilon^{\pm}\}_\varepsilon$. An important feature is that it leaves $\mathscr{H}_1$ invariant:

$$G_\varepsilon^{\pm}\mathscr{H}_1 \subset \mathscr{H}_1. \tag{3.9}$$

For, let us assume that $\psi \in \mathscr{H}_1$ i.e. the application $\mathbb{R}^3 \ni \boldsymbol{x} \mapsto \mathrm{e}^{i\boldsymbol{x}\cdot\boldsymbol{Q}}\psi \in \mathscr{H}$ is C^1. Then

$$\begin{aligned} \mathrm{e}^{i\boldsymbol{x}\cdot\boldsymbol{Q}} G_\varepsilon^{\pm}\psi &= (\mathrm{e}^{i\boldsymbol{x}\cdot\boldsymbol{Q}} H_0 \mathrm{e}^{-i\boldsymbol{x}\cdot\boldsymbol{Q}} - \lambda \mp i\mu \mp i\varepsilon \mathrm{e}^{i\boldsymbol{x}\cdot\boldsymbol{Q}} B \mathrm{e}^{-i\boldsymbol{x}\cdot\boldsymbol{Q}})^{-1}\mathrm{e}^{i\boldsymbol{x}\cdot\boldsymbol{Q}}\psi \\ &= \left(H_0 - \boldsymbol{\alpha}\cdot\boldsymbol{x} - \lambda \mp i\mu \mp i\varepsilon\frac{(\boldsymbol{P}-\boldsymbol{x})^2}{(\boldsymbol{P}-\boldsymbol{x})^2+m^2}\right)^{-1}\mathrm{e}^{i\boldsymbol{x}\cdot\boldsymbol{Q}}\psi \end{aligned}$$

is a C^1 function of $\boldsymbol{x}$ in the topology of $\mathscr{H}$, because

$$\mathbb{R}^3 \ni \boldsymbol{x} \mapsto \left(H_0 - \boldsymbol{\alpha}\cdot\boldsymbol{x} - \lambda \mp i\mu \mp i\varepsilon(\boldsymbol{P}-\boldsymbol{x})^2\left[(\boldsymbol{P}-\boldsymbol{x})^2+m^2\right]^{-1}\right)^{-1} \in \mathbb{B}(\mathscr{H})$$

is C^1 in the operator norm.

Technical estimates on $G_\varepsilon^{\pm}$ are gathered below.

Lemma 3.2 (i) $\|B^{1/2}G_\varepsilon^{\pm}\psi\|^2 \le \varepsilon^{-1}|F_\varepsilon^{\pm}|$ *for each* $\psi \in \mathscr{H}$.
(ii) $\|B^{1/2}G_\varepsilon^{\pm}\psi\| \le C\varepsilon^{-1}\|\psi\|_{\mathscr{H}_1}$ *for each* $\psi \in \mathscr{H}_1$ *(C is some positive constant).*

Proof: Let $\psi \in \mathscr{H}$. Then (i) follows from

$$|F_\varepsilon^{\pm}| \ge \mp\mathfrak{Im}\langle\psi, G_\varepsilon^{\pm}\psi\rangle = \mp\mathfrak{Im}\langle(H_0-\lambda\mp i\mu\mp i\varepsilon B)G_\varepsilon^{\pm}\psi, G_\varepsilon^{\pm}\psi\rangle = \mu\|G_\varepsilon^{\pm}\psi\|^2 + \varepsilon\|B^{1/2}G_\varepsilon^{\pm}\psi\|^2.$$

For (ii), let $\psi \in \mathscr{H}_1$. As a consequence of (3.4),

$$|F_\varepsilon^{\pm}| = |\langle\psi, G_\varepsilon^{\pm}\psi\rangle| \le \|B^{-1/2}\psi\|\,\|B^{1/2}G_\varepsilon^{\pm}\psi\| \le C\|\psi\|_{\mathscr{H}_1}\,\|B^{1/2}G_\varepsilon^{\pm}\psi\|.$$

Thus one may use (i) in order to find the estimate of (ii). ■

We are now ready for the last step of the proof. It relies mainly on a differential inequality in terms of F_ε. We fix a $\psi \in \mathscr{H}_1$ and denote for simplicity $G_\varepsilon \equiv G_\varepsilon^{+}$ and $F_\varepsilon \equiv F_\varepsilon^{+}$. The proof for the case $G_\varepsilon \equiv G_\varepsilon^{-}$ follows analogously. Notice first that the resolvent identity gives $F'_\varepsilon \equiv \frac{d}{d\varepsilon}F_\varepsilon = i\langle G_\varepsilon^*\psi, BG_\varepsilon\psi\rangle$. If one puts $\varphi_1 = G_\varepsilon^*\psi$ and $\varphi_2 = G_\varepsilon\psi$, (3.9), (3.3) and (ii) of Lemma 3.1 yield

$$\begin{aligned} F'_\varepsilon &= \langle AG_\varepsilon^*\psi, H_0G_\varepsilon\psi\rangle - \langle H_0G_\varepsilon^*\psi, AG_\varepsilon\psi\rangle \\ &= \langle G_\varepsilon^*\psi, A\psi\rangle - \langle A\psi, G_\varepsilon\psi\rangle + i\varepsilon(\langle BG_\varepsilon^*\psi, AG_\varepsilon\psi\rangle - \langle AG_\varepsilon^*\psi, BG_\varepsilon\psi\rangle) \\ &= \langle G_\varepsilon^*\psi, A\psi\rangle - \langle A\psi, G_\varepsilon\psi\rangle - 2m^2\varepsilon\langle G_\varepsilon^*\psi, \Delta H_0^{-5}G_\varepsilon\psi\rangle. \end{aligned}$$

For the last equality we used (3.6). Further, (3.5) and Lemma 3.2 furnish the estimate

$$\begin{aligned} |F'_\varepsilon| &\le \|B^{-1/2}A\psi\|\,(\|B^{1/2}G_\varepsilon^*\psi\| + \|B^{1/2}G_\varepsilon\psi\|) + C\varepsilon\|B^{1/2}G_\varepsilon^*\psi\|\,\|B^{1/2}G_\varepsilon\psi\| \\ &\le \tilde{C}\varepsilon^{-1/2}|F_\varepsilon|^{1/2}\|\psi\|_{\mathscr{H}_1}. \end{aligned}$$

After integration on $[\rho, \rho_0] \subset (0,1)$ one gets

$$|F_\rho| \leq |F_{\rho_0}| + C\|\psi\|_{\mathscr{H}_1} \int_\rho^{\rho_0} \varepsilon^{-1/2} |F_\varepsilon|^{1/2} \, \mathrm{d}\varepsilon$$

which may be treated with the help of a Gronwall type lemma. Indeed, if one takes $\theta = 1/2$ in Lemma 7.A.1 of [ABG] we may find a positive constant C such that

$$|F_\rho| \leq C \left(|F_{\rho_0}|^{1/2} + \|\psi\|_{\mathscr{H}_1} (\sqrt{\rho_0} - \sqrt{\rho}) \right)^2 .$$

On the other hand, (3.4) and (ii) of Lemma 3.2 ensure the existence of a constant $C > 0$ such that

$$|F_{\rho_0}|^{1/2} \equiv |\langle \psi, G_{\rho_0} \psi \rangle|^{1/2} \leq \|B^{-1/2}\psi\|^{1/2} \, \|B^{1/2} G_{\rho_0} \psi\|^{1/2} \leq C \|\psi\|_{\mathscr{H}_1} .$$

If one replaces the above estimate in the previous one and let $\rho \to 0$ we obtain (3.8), which finishes the proof of the Theorem 2.1.

Acknowledgements. The second author is indebted to the Swiss National Science Foundation for financial support.

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