Absence of transport under the slowly varying potential in disordered systems

Fumihiko Nakano * and Masahiro Kaminaga †

Abstract

In the tight-binding random Hamiltonian on $\mathbf{Z}^d$, we consider the charge transport induced by the electric potential which varies sufficiently slowly in time, and prove that it is almost surely equal to zero at high disorder. In order to compute the charge transport, we adopt the adiabatic approximation and prove a weak form of adiabatic theorem while there is no spectral gap at the Fermi energy.

KEY WORDS: charge transport, Anderson localization, adiabatic theorem.

1 INTRODUCTION

About four decades ago, Anderson(4) discussed that a certain disorder may cause materials to have insulating property. And recently, there is much progress toward the mathematical understanding of this phenomenon (e.g., 1, 2, 3, 10, 11, 20, 22, etc.) such as derivation of the exponential decay of the eigenfunctions. In this paper, we consider the charge transport raised by the time-dependent flux(6,13), and prove that this quantity is zero when the external field varies slowly in time.

Our model is the standard, tight-binding, random Hamiltonian on $l^2(\mathbf{Z}^d)$:

$$(H_\omega \varphi)(x) := \sum_{|x-y|=1} \varphi(y) + \lambda V_\omega(x)\varphi(x), \quad \varphi \in l^2(\mathbf{Z}^d). \tag{1.1}$$

$\lambda > 0$ is the coupling constant which represents the strength of disorder potential. $\{V_\omega(x)\}_{x\in\mathbf{Z}^d}$ is the independent, identically distributed random variables

*present address: Department of Physics, Princeton University, Princeton, NJ08544, USA, On leave of absence from: Mathematical Institute, Tohoku University, Sendai, 980-77, Japan.

†Multimedia Systems Research Department, Hitachi Ltd., Central Research Laboratory, 1-280, Higashi-Koigakubo, Kokubunji-shi, Tokyo, 185-8601, Japan.

on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that the probability distribution of $V_\omega(0)$ has the density $r(v)dv$, where $r \in L^p(\mathbf{R})$, $1 < p \leq +\infty$, and $r(v)$ is compactly supported. It follows that, for $a \in \mathbf{Z}^d$, there is a corresponding measure preserving map $T^a \,:\, \Omega \to \Omega$, such that

$$U(a)H_\omega U(a)^* = H_{T^a\omega}, \tag{1.2}$$

where $U(a)$ is the translation operator on $l^2(\mathbf{Z}^d)$: $(U(a)\varphi)(x) := \varphi(x-a)$. $\mathbf{P}$ is ergodic with respect to T, and $\{H_\omega\}_{\omega\in\Omega}$ is the ergodic family of bounded self-adjoint operators. Let $V(t)$ be a multiplication operator on $l^2(\mathbf{Z}^d)$: $(V(t)\varphi)(x) := \exp[i\int_0^t g(x^{(1)},s)ds]\varphi(x)$, where $g(x^{(1)},s) : \mathbf{Z}\times[0,1] \to \mathbf{R}$ is bounded and periodic in $x^{(1)}$ with period $L \in \mathbf{N}$. $x^{(1)}$ is the first component of $x \in \mathbf{Z}^d$. We define the time-dependent random Hamiltonian:

$$H_\omega(t) := V(t)H_\omega V^*(t), \quad t \in [0,1],$$

or equivalently,

$$(H_\omega(t)\varphi)(x) := \sum_{|x-y|=1} \exp\Big[i\int_0^t \big(g(x^{(1)},s) - g(y^{(1)},s)\big)ds\Big]\varphi(y) \\ +\lambda V_\omega(x)\varphi(x), \quad \varphi \in l^2(\mathbf{Z}^d). \tag{1.3}$$

$H_\omega(t)$ describes non-interacting electrons in the random potential $\lambda V_\omega(x)$ under the time-dependent electric field potential $g(x^{(1)},t)$ [9, Chapter 7]. This formulation was introduced by Avron-Seiler-Yaffe and Klein-Seiler [6, 13] to study the quantum Hall effect.

We shall prepare some notations which are necessary to define the charge transport. Let $U_\omega(t)$ be the unitary evolution operator of $H_\omega(t)$ which satisfies:

$$i\frac{d}{dt}U_\omega(t)\varphi = H_\omega(t)U_\omega(t)\varphi, \quad t \in (0,1), \\ U_\omega(0)\varphi = \varphi, \quad \varphi \in l^2(\mathbf{Z}^d). \tag{1.4}$$

Let $\varepsilon_F \in \mathbf{R}$ be the Fermi energy which can be arbitrary in this paper, and $P_\omega(t) := \chi_{(-\infty,\,\varepsilon_F]}(H_\omega(t))$ be the corresponding Fermi projection, where χ_C is the characteristic function of the set $C \subset \mathbf{R}$. For simplicity, we write $P_\omega := P_\omega(0)$. For an operator A on $l^2(\mathbf{Z}^d)$, we define $\mathcal{T}(A)$ as the trace per unit volume:

$$\mathcal{T}(A) := \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|}\mathrm{trace}(\chi_\Lambda A \chi_\Lambda), \tag{1.5}$$

as long as it exists, where $\Lambda := \{x \in \mathbf{Z}^d : |x^{(i)}| \le M, i = 1, \cdots, d\} \subset \mathbf{Z}^d$, $M \in \mathbf{N}$. In what follows, we shall define the quantity which we call the charge transport induced by slowly varying potential. This is based on what was considered in (refs. 6, 13). Firstly, we consider the charge transport from $t = 0$ to $t = 1$, which is defined as the thermal average of the current operator: $J_\omega(t) := U_\omega^*(t) i[H_\omega(t), x] U_\omega(t)$ at zero temperature, in the grand canonical ensemble:

$$j_\omega := \int_0^1 dt\ \mathcal{T}(U_\omega^*(t) i[H_\omega(t),\ x] U_\omega(t) P_\omega), \tag{1.6}$$

where $[A, B] := AB - BA$ is the commutator which is defined by those integral kernels (this will be mentioned in Remark after Lemma 2.1). We will confirm that j_ω is well-defined in Lemma 2.4. Secondly, we consider the limit under which the electric potential varies slowly in time. In order to do this, we rescale the time $s := t/\tau$ ($\tau > 0$) and replace t by s. When s varies from $s = 0$ to $s = 1$, the real time t goes from $t = 0$ to $t = \tau$, and the electric potential becomes $g(x^{(1)}, s)/\tau$. The corresponding time evolution operator is the solution to:

$$\begin{aligned} i\frac{d}{ds} U_{\omega,\tau}(s)\varphi &= \tau H_\omega(s) U_{\omega,\tau}(s)\varphi, \quad s \in (0, 1), \\ U_{\omega,\tau}(0)\varphi &= \varphi, \quad \varphi \in l^2(\mathbf{Z}^d). \end{aligned} \tag{1.7}$$

The charge transport from $t = 0$ to $t = \tau$ becomes:

$$j_{\omega,\tau} := \int_0^1 ds\ \mathcal{T}\left(U_{\omega,\tau}^*(s) i\tau[H_\omega(s),\ x] U_{\omega,\tau}(s) P_\omega\right). \tag{1.8}$$

We define $\tau \to \infty$ limit of j_ω as the charge transport induced by slowly varying electric potential.

$$\sigma_\omega := \lim_{\tau\to\infty} j_{\omega,\tau}. \tag{1.9}$$

Remarks.

(1) A typical example of $g(x^{(1)}, s)$ is the bounded periodic potential which is modulated in time: $g(x^{(1)}, s) = h(x^{(1)}/L - t)$, where $h(x)$ is a function of period 1. It was first introduced by Thouless and Niu[14, 15, 16, 21] in their study of quantized adiabatic charge transport. Niu[14] showed that, in some special choices of $h(x)$ and $V(x) \equiv 0$, there are nonzero transport (that is, $\sigma \neq 0$). Therefore, it seems to be plausible to take $g(x^{(1)}, s)$ as an electric potential to study the transport properties.

(2) Since the electric potential becomes $g(x^{(1)}, s)/\tau$ under the above scaling, $\tau \to \infty$ limit is equivalent to the small electric field limit. Thus, σ_ω can be regarded as the electrical conductivity under the bounded and time-dependent electric potential $g(x^{(1)}, s)$.

(3) There have been some definitions of the electrical conductivity in the study of quantum Hall effect. The definition of the charge transport in (refs. 6, 13) is almost the same as that of ours except the following two points: (a) usual trace in (refs. 6, 13) is replaced by the trace per unit volume here. This replacement seems to be important to consider Anderson localization, (b) taking average with respect to flux is not necessary here.

The definition of the electrical conductivity in (ref. 7, 19) is also similar to that of ours. The difference is: (a) they considered time independent Hamiltonian with constant electric field and dissipative term, (b) they adopted Abel limit to compute the conductivity explicitly, while we will use adiabatic approximation here.

The main result of this paper is, at high disorder, the charge transport defined in (1.9) indeed vanishes almost surely.

Theorem 1.1.
For sufficiently large $\lambda > 0$, $\sigma_\omega = 0$, for $\mathbf{P}$-a.e. ω.

Remarks.

(1) More precisely, λ is taken large such that $\lambda > \lambda_0(s) := 2\|r\|_p^{1/\sigma}(\frac{2\sigma}{\sigma - s}d)^{1/s}$, for some $0 < s < \sigma$, where $\|\cdot\|_p$ is the $L^p(\mathbf{R})$ norm, and $\sigma := 1 - \frac{1}{p}$ ($\sigma = 1$, if $p = +\infty$)[2]. On the other hand, our proof does not apply to the weak disorder case (that is, λ is small) even if ϵ_F lies in the localized states, because we can not control the tunneling to the delocalized states.

(2) The main tool we use to prove Theorem 1.1 is the exponential decay estimates of the fractional moment of Green's function[1,2,3]. Thus, the above assertion also holds to many other Hamiltonians, such as that with long range hopping terms, or that with periodic background potential[2].

(3) It is desirable to consider the constant electric field. However, if we take $g(x^{(1)}, s) := x^{(1)}$, then (a) the proof of Lemma 2.2 in section 2 fails, (b) the adiabatic Hamiltonian defined in (1.10) would be unbounded and analysis of it

will not be easy.

(4) Many people might think that our result is almost trivial because spectrum of H_ω is all pure point and corresponding eigenfunctions decay exponentially. Our opposition to this opinion is: (a) they do not consider any external fields, (b) We only know that each eigenfunction decays exponentially at each rate. However, from the statistical mechanics point of view, we have to consider infinitely many states at the same time at least near the Fermi energy. And, del Rio, et. al.[10] suggest that the eigenfunctions of H_ω do not localize "uniformly", (c) the eigenvalues of H_ω distribute densely in the spectrum[20] which implies there is no spectral gap, and there are infinitely many eigenvalues near the Fermi energy. Since there might be infinitely small excitation beyond the Fermi energy, our result seems not to be trivial.

(5) By using Kubo formula, vanishing conductivity follows immediately from the exponential decay estimates of Green's function [2, 11, 19]. Thus, our result can be regarded as another presentation of the vanishment of the electrical conductivity except the electric field must be bounded.

In fact, we expect that the excitation beyond the Fermi energy can be negligible under the weak electric field. In another words, we consider the adiabatic approximation of $j_{\omega,\tau}$ to prove Theorem 1.1. To this end, we consider the adiabatic evolution operator $U^A_{\omega,\tau}(s)$ which satisfies:

$$\begin{aligned} i\frac{d}{ds}U^A_{\omega,\tau}(s)\varphi &= \tau H^A_\omega(s)U^A_{\omega,\tau}(s)\varphi, \quad s\in(0,1),\\ U^A_{\omega,\tau}(0)\varphi &= \varphi, \quad \varphi\in l^2(\mathbf{Z}^d), \end{aligned} \tag{1.10}$$

where $H^A_\omega(s) := H_\omega(s) + \frac{i}{\tau}[\dot{P}_\omega(s), P_\omega(s)]$ (which is bounded and self-adjoint on $l^2(\mathbf{Z}^d)$). $\dot{P}_\omega(s)$ is the derivative of $P_\omega(s)$ in the operator norm topology. In fact, $\dot{P}_\omega(s) = iV(s)[g(x,s), P_\omega]V^*(s)$. Since $U^A_{\omega,\tau}(s)$ is expected to approach to $U_{\omega,\tau}(s)$ for sufficiently large τ [6, 12, 13], we consider the adiabatic approximation of $j_{\omega,\tau}$ as:

$$j^A_{\omega,\tau} := \int_0^1 ds\, \mathcal{T}(U^{A*}_{\omega,\tau}(s)i\tau[H^A_\omega(s), x]U^A_{\omega,\tau}(s)P_\omega). \tag{1.11}$$

Our theorem is the corollary of following two propositions.

Proposition 1.2.

For sufficiently large $\lambda > 0$, $j_{\omega,\tau} = j^A_{\omega,\tau} + o(1)$, as $\tau \to \infty$, for **P***-a.e. ω.*

Proposition 1.3.
For sufficiently large $\lambda > 0$, $j^A_{\omega,\tau} = 0$, for **P***-a.e. ω.*

The rest of this paper is organized as follows. In section 2, we study some properties of $\mathcal{T}$, derive rapid decay of $U_{\omega,\tau}(s)$ and $U^A_{\omega,\tau}(s)$, and prove the well-definedness of $j_{\omega,\tau}$ and $j^A_{\omega,\tau}$. In section 3, we prove Proposition 1.2 and 1.3. Proposition 1.3 follows easily from the intertwining property of $U^A_{\omega,\tau}(s)$[6] and the time reversal invariance of H_ω. The proof of Proposition 1.2 needs a sort of adiabatic theorem. Adiabatic theorem is proved in (refs. 6, 12, 13) in an abstract setting when ϵ_F lies in the spectral gap. We follow their argument to prove our adiabatic theorem. Then we will have an additional term which originates from the fact that there is no spectral gap in our case. We estimate this term by exponential decay of the fractional moment of Green's function [1, 2, 3].

2 PRELIMINARIES

For an operator A_ω on $l^2(\mathbf{Z}^d)$, we write $< x|\ A_\omega\ |y >:= (\delta_x,\ A_\omega \delta_y)$, x, $y \in \mathbf{Z}^d$, where $(\cdot\ ,\ \cdot)$ is the inner product on $l^2(\mathbf{Z}^d)$, and $\delta_x(z) \in l^2(\mathbf{Z}^d)$ is defined as: $\delta_x(z) = 1$ (if $z = x$), $\delta_x(z) = 0$ (otherwise). We call $< x|\ A_\omega\ |y >$ the integral kernel of A_ω. We say an operator A_ω satisfies covariance relation ((CR) in short), if A_ω obeys

$$U(a) A_\omega U^*(a) = A_{T^a \omega}, \tag{2.1}$$

whenever $a \in \{x \in \mathbf{Z}^d : x^{(1)} = kL,\ k \in \mathbf{Z}\}$. L was defined in section 1 to be the period of $g(x^{(1)}, s)$. We introduce some classes of families of operators on $l^2(\mathbf{Z}^d)$:

$$\begin{aligned}
\mathcal{S}_\alpha := \{\ & A_\omega \ \text{ an operator on } l^2(\mathbf{Z}^d) : \\
& (1)\ \ A_\omega \text{ satisfies (CR)}, \\
& (2)\ \ \forall\beta > 0,\ \exists C_{\alpha\beta} > 0 \text{ s.t. } \mathbf{E}|< x|\ A_\omega\ |y >|^\alpha \le C_{\alpha\beta} < x - y >^{-\beta}\},
\end{aligned}$$

$$\mathcal{S} := \cup_{\alpha>1}\, \mathcal{S}_\alpha, \quad \tilde{\mathcal{S}} := \cap_{\alpha>1} \mathcal{S}_\alpha,$$

where $< x >:= (1+|x|^2)^{1/2}$. $\mathbf{E}$ stands for the expectation value with respect to $\mathbf{P}$. We set $\|A\|_{\mathcal{S}_\alpha,\beta} := (\sup_{x,y\in\mathbf{Z}^d} < x-y >^\beta \mathbf{E}| < x|\ A_\omega\ |y > |^\alpha)^{1/\alpha}$.

Lemma 2.1.

The followings hold for $\mathbf{P}-a.e.\omega$.

(1) Let $A_i \in \mathcal{S}_{\alpha_i}$, $i = 1, \cdots, n$ such that $\sum_{i=1}^n \alpha_i^{-1} = 1$. Then, for $\beta > 1$,

$$|\mathcal{T}(\prod_{i=1}^n A_i)| \le C_n \prod_{i=1}^n \|A_i\|_{\mathcal{S}_{\alpha_i},\beta\alpha_i}, \tag{2.2}$$

where $C_n > 0$ is an universal constant.

(2) Let $A_i \in \mathcal{S}_{\alpha_i}$, $i = 1, 2$, such that $\sum_{i=1}^2 \alpha_i^{-1} = 1$. Then,

$$\mathcal{T}(A_1A_2) = \mathcal{T}(A_2A_1). \tag{2.3}$$

Remark.

$A_\omega \in \mathcal{S}_\alpha$ is not necessarily bounded on $l^2(\mathbf{Z}^d)$ for fixed $\omega \in \Omega$. In this case, we define the product of two operators A, $B \in \mathcal{S}_\alpha$ by their integral kernels:

$$< x|\ AB\ |y >:= \sum_{z\in\mathbf{Z}^d} < x|\ A\ |z >< z|\ B\ |y >, \tag{2.4}$$

whenever it exists and finite. In all cases in this paper, we can prove that for all x, $y \in \mathbf{Z}^d$, the above expression converges absolutely and $< x|\ AB\ |y >$ is finite for $\mathbf{P}$-a.e. ω. Therefore, $A_\omega \in \mathcal{S}_\alpha$ always has a meaning as a function on $\Omega \times \mathbf{Z}^d \times \mathbf{Z}^d$.

proof.

(1) By definition,

$$\begin{aligned} < x|\ \prod_{i=1}^n A_i\ |x > = \sum_{z_1,z_2,\cdots,z_{n-1}\in\mathbf{Z}^d} & < x|\ A_1\ |z_1 >< z_1|\ A_2\ |z_2 > \\ & \cdots < z_{n-2}|\ A_{n-1}\ |z_{n-1} >< z_{n-1}|\ A_n\ |x > \end{aligned}$$

We take expectation on both sides and use Fubini's Theorem and Hölder's inequality with respect to $\mathbf{P}$:

$$\mathbf{E}| < x|\prod_{i=1}^n A_i\ |x > |$$

$$\leq \mathbf{E} \sum_{z_1,z_2,\cdots,z_{n-1}\in\mathbf{Z}^d} |<x|\, A_1\, |z_1>||<z_1|\, A_2\, |z_2>|$$
$$\cdots|<z_{n-2}|\, A_{n-1}\, |z_{n-1}>||<z_{n-1}|\, A_n\, |x>|$$
$$\leq \sum_{z_1,z_2,\cdots,z_{n-1}\in\mathbf{Z}^d} (\mathbf{E}|<x|\, A_1\, |z_1>|^{\alpha_1})^{1/\alpha_1}(\mathbf{E}|<z_1|\, A_2\, |z_2>|^{\alpha_2})^{1/\alpha_2}$$
$$\cdots(\mathbf{E}|<z_{n-2}|\, A_{n-1}\, |z_{n-1}>|^{\alpha_{n-1}})^{1/\alpha_{n-1}}(\mathbf{E}|<z_{n-1}|\, A_n\, |x>|^{\alpha_n})^{1/\alpha_n}$$
$$\leq \sum_{z_1,z_2,\cdots,z_{n-1}\in\mathbf{Z}^d} \|A_1\|_{\mathcal{S}_{\alpha_1},\beta\alpha_1} <x-z_1>^{-\beta} \|A_2\|_{\mathcal{S}_{\alpha_2},\beta\alpha_2} <z_1-z_2>^{-\beta}$$
$$\cdots\|A_{n-1}\|_{\mathcal{S}_{\alpha_{n-1}},\beta\alpha_{n-1}} <z_{n-2}-z_{n-1}>^{-\beta} \|A_n\|_{\mathcal{S}_{\alpha_n},\beta\alpha_n} <z_{n-1}-x>^{-\beta}$$
$$\leq C_n \prod_{i=1}^{n} \|A_i\|_{\mathcal{S}_{\alpha_i},\beta\alpha_i}.$$

Thus, it suffices to show the following equation which follows from Birkhoff's ergodic theorem:

$$\mathcal{T}(\prod_{i=1}^{n} A_i) = \frac{1}{L}\sum_{x^{(1)}=1}^{L} \mathbf{E} <x|\, \prod_{i=1}^{n} A_i\, |x>, \quad a.e.\ \omega, \tag{2.5}$$

where $x^{(i)}=0$, $i=2,\cdots,d$.

(2) We write $x\in\mathbf{Z}^d$ as $x=(p,q)$, $p\in\mathbf{Z}$, $q\in\mathbf{Z}^{d-1}$. From (2.5),

$$\mathcal{T}(A_1A_2) = \frac{1}{L}\sum_{p=1}^{L}\mathbf{E}\sum_{k\in\mathbf{Z},r\in\mathbf{Z}^{d-1}}\sum_{q=1}^{L}$$
$$<p,0\,|\,A_{1,\omega}\,|\,kL+q,r><kL+q,r\,|\,A_{2,\omega}\,|\,p,0>, \tag{2.6}$$

for a.e. ω. Since $A_{1,\omega}$, $A_{2,\omega}$ satisfy (CR), we have

$$<p,0\,|\,A_{1,\omega}\,|\,kL+q,r>=<-kL+p,-r\,|\,A_{1,T^{-k\vec{L}-\vec{r}}\omega}\,|\,q,0>, \tag{2.7}$$
$$<kL+q,r\,|\,A_{2,\omega}\,|\,p,0>=<q,0\,|\,A_{2,T^{-k\vec{L}-\vec{r}}\omega}\,|-kL+p,-r>, \tag{2.8}$$

where $\vec{L}:=(L,0)\in\mathbf{Z}^d$, $\vec{r}:=(0,r)\in\mathbf{Z}^d$.

We substitute (2.7) and (2.8) into (2.6), and use Fubini's theorem (which is permitted by Lemma 2.1 (1)). Therefore,

$$\mathcal{T}\,(A_1A_2)$$
$$= \frac{1}{L}\sum_{p=1}^{L}\sum_{k\in\mathbf{Z},r\in\mathbf{Z}^{d-1}}\sum_{q=1}^{L}\mathbf{E}$$

$$
\begin{aligned}
&< q,0 \,|\, A_{2,T^{-k\vec{L}-\vec{r}}\omega} \,|\, -kL+p,-r >< -kL+p,-r \,|\, A_{1,T^{-k\vec{L}-\vec{r}}\omega} \,|\, q,0 > \\
&= \frac{1}{L}\sum_{q=1}^{L} \mathbf{E} \sum_{k\in\mathbf{Z}, r\in\mathbf{Z}^{d-1}} \sum_{p=1}^{L} < q,0 \,|\, A_{2,\omega} \,|\, kL+p,r >< kL+p,r \,|\, A_{1,\omega} \,|\, q,0 > \\
&= \mathcal{T}(A_2A_1).
\end{aligned}
$$

$\square$

The following lemma implies that electrons are still localized even under the electric field, and it is here that we have to restrict our analysis to bounded electric potential.

Lemma 2.2.

For arbitrary $\alpha > 0$, there exists a constant $C_\alpha > 0$ such that

$$
\mathbf{E}| < x|\, U_{\omega,\tau}(s)\, |y > | \leq C_\alpha < x-y >^{-\alpha}, \tag{2.9}
$$

$$
\mathbf{E}| < x|\, U^A_{\omega,\tau}(s)\, |y > | \leq C_\alpha < x-y >^{-\alpha}, \tag{2.10}
$$

for $\lambda > 0$ sufficiently large, where C_α is independent of x, $y \in \mathbf{Z}^d$, $s \in [0,1]$, and $\tau > 0$.

proof.

(1) At first, we prove (2.9), (2.10) when $d = 1$.

Let $\bar{U}_{\omega,\tau}(s)$, $\bar{U}^A_{\omega,\tau}(s)$ be the unitary evolution operators of time-dependent Schrödinger equations:

$$
\begin{aligned}
i\frac{d}{ds}\bar{U}_{\omega,\tau}(s)\varphi &= \tau\big(H_\omega + \frac{1}{\tau}g(x,s)\big)\bar{U}_{\omega,\tau}(s)\varphi, \\
\bar{U}_{\omega,\tau}(0)\varphi &= \varphi,
\end{aligned} \tag{2.11}
$$

$$
\begin{aligned}
i\frac{d}{ds}\bar{U}^A_{\omega,\tau}(s)\varphi &= \tau\Big(H_\omega + \frac{1}{\tau}g(x,s) \\
&\qquad + \frac{1}{\tau}[[P_\omega, g(x,s)], P_\omega]\Big)\bar{U}^A_{\omega,\tau}(s)\varphi, \\
\bar{U}^A_{\omega,\tau}(0)\varphi &= \varphi, \quad s \in (0,1), \quad \varphi \in l^2(\mathbf{Z}).
\end{aligned} \tag{2.12}
$$

It follows that$^{(9,\, Chapter\ 7)}$

$$
U_{\omega,\tau}(s) = V(s)\bar{U}_{\omega,\tau}(s), \quad U^A_{\omega,\tau}(s) = V(s)\bar{U}^A_{\omega,\tau}(s).
$$

Therefore (2.9), (2.10) are equivalent to the rapid decay of $\bar{U}_{\omega,\tau}(s)$, $\bar{U}^A_{\omega,\tau}(s)$ repectively. We will only show the decay of $\bar{U}^A_{\omega,\tau}(s)$ (that is, $\mathbf{E}| < x|\bar{U}^A_{\omega,\tau}(s)|y > | \leq C_\alpha < x-y >^{-\alpha}$), because the proof of the decay of $\bar{U}_{\omega,\tau}(s)$ is similar.

When $x = y$, this is obvious because $\bar{U}^A_{\omega,\tau}(s)$ is bounded on $l^2(\mathbf{Z})$, so that we will show

$$\mathbf{E}| < x| \, \bar{U}^A_{\omega,\tau}(s) \, |y > | \leq C_\alpha |x-y|^{-\alpha},$$

when $x \neq y$. $\bar{U}^A_{\omega,\tau}(s)$ can be written by Dyson expansion which converges in operator norm:

$$\bar{U}^A_{\omega,\tau}(s) = \sum_{j=0}^{\infty} T_j(s),$$

where $T_0(s) := e^{-i\tau s H_\omega}$, and for $j \geq 1$,

$$\begin{aligned}
T_j(s) := (-i)^j & \int_{0 \leq \sum_{i=1}^j s_i \leq s} ds_1 ds_2 \cdots ds_j \\
& U^0_{\omega,\tau}(s - \sum_{i=1}^j s_i) X_\omega(\sum_{i=1}^j s_i) U^0_{\omega,\tau}(s_1) X_\omega(\sum_{i=2}^j s_i) U^0_{\omega,\tau}(s_2) X_\omega(\sum_{i=3}^j s_i) \\
& \cdots X_\omega(s_{j-1} + s_j) U^0_{\omega,\tau}(s_{j-1}) X_\omega(s_j) U^0_{\omega,\tau}(s_j). \qquad (2.13)
\end{aligned}$$

In (2.13), we wrote $U^0_{\omega,\tau}(s) := e^{-i\tau s H_\omega}$, and $X_\omega(s) := g(x,s) + [[P_\omega, g(x,s)], P_\omega]$. We will show that, $\sum_{j=0}^{\infty} \mathbf{E}| < x| \, T_j(s) \, |y > |$ converges and has the rapid decay for sufficiently small $s > 0$, and that how small s should be does not depend on α. Then, the conclusion follows by the semigroup property of $\bar{U}^A_{\omega,\tau}(s)$ (we have to consider $\bar{U}^A_{\omega,\tau}(t,s)$ which is defined similarly to (2.12) except $\bar{U}^A_{\omega,\tau}(0)\varphi = \varphi$ is replaced by $\bar{U}^A_{\omega,\tau}(t,t)\varphi = \varphi$) and the following simple fact: let A_ω, B_ω be bounded operators on $l^2(\mathbf{Z})$ which satisfy ($\alpha > 2$)

$$\begin{aligned}
\mathbf{E}| < x| \, A_\omega \, |y > | \; & \leq < x - y >^{-\alpha}, \\
\mathbf{E}| < x| \, B_\omega \, |y > | \; & \leq < x - y >^{-\alpha}, \quad x, y \in \mathbf{Z},
\end{aligned}$$

Then, the operator $C_\omega := A_\omega B_\omega$ satisfies

$$\mathbf{E}| < x| \, C_\omega \, |y > | \leq C < x - y >^{-\alpha/2},$$

for some constant $C > 0$.

The integral kernel of $< x| \, T_j \, |y >$ can be written as:

$$\begin{aligned}
<x| \, T_j \, |y >= (-i)^j & \int_{0 \leq \sum_{i=1}^j s_i \leq s} ds_1 ds_2 \cdots ds_j \\
\sum_{z_1, \cdots, z_j} & < x| \, U^0_{\omega,\tau}(s - \sum_{i=1}^j s_i) X_\omega(\sum_{i=1}^j s_i) \, |z_1 >< z_1| \, U^0_{\omega,\tau}(s_1) X_\omega(\sum_{i=2}^j s_i) \, |z_2 > \\
& \cdots < z_{j-1}| \, U^0_{\omega,\tau}(s_{j-1}) X_\omega(s_j) \, |z_j >< z_j| \, U^0_{\omega,\tau}(s_j) \, |y > . \qquad (2.14)
\end{aligned}$$

In order to use Fubini's theorem, we take expectation at first and consider:

$$\begin{aligned}\tilde{T}_j := \sum_{z_1, z_2, \cdots, z_j \in \mathbf{Z}} & \mathbf{E}| < x| \; U^0_{\omega,\tau}(s - \sum_{i=1}^{j} s_i) X_\omega(\sum_{i=1}^{j} s_i) \; |z_1 > | \\ & | < z_1| \; U^0_{\omega,\tau}(s_1) X_\omega(\sum_{i=2}^{j} s_i) \; |z_2 > | \\ & \cdots | < z_{j-1}| \; U^0_{\omega,\tau}(s_{j-1}) X_\omega(s_j) \; |z_j > || < z_j| \; U^0_{\omega,\tau}(s_j) \; |y > |. \quad (2.15)\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}\tilde{T}_j \leq & \sum_{z_1, z_2, \cdots, z_j \in \mathbf{Z}} (\mathbf{E}| < x| \; U^0_{\omega,\tau}(s - \sum_{i=1}^{j} s_i) X_\omega(\sum_{i=1}^{j} s_i) \; |z_1 > |^{j+1})^{\frac{1}{j+1}} \\ & (\mathbf{E}| < z_1| \; U^0_{\omega,\tau}(s_1) X_\omega(\sum_{i=2}^{j} s_i) \; |z_2 > |^{j+1})^{\frac{1}{j+1}} \cdots \\ & (\mathbf{E}| < z_{j-1}| \; U^0_{\omega,\tau}(s_{j-1}) X_\omega(s_j) \; |z_j > |^{j+1})^{\frac{1}{j+1}} (\mathbf{E}| < z_j| \; U^0_{\omega,\tau}(s_j) \; |y > |^{j+1})^{\frac{1}{j+1}}.\end{aligned}$$

We use the results in (ref. 1, 2):

$$\mathbf{E}| < x| \; U^0_{\omega,\tau}(s) \; |y > | \leq C_1 e^{-\mu|x-y|}, \tag{2.16}$$

$$\mathbf{E}| < x| \; P_\omega \; |y > | \leq C_1 e^{-\mu|x-y|}, \tag{2.17}$$

for sufficiently large $\lambda > 0$, where constants $C_1 > 0$ and $\mu > 0$ are independent of x, $y \in \mathbf{Z}$, $s \in [0,1]$, and $\tau > 0$. We used the fact that, in this case, H_ω has only pure point spectrum [9, 20]. By (2.16), (2.17), Hölder's inequality, and the boundedness of $U^0_{\omega,\tau}(s)$ and $X_\omega(s)$, we can deduce that, when $\lambda > 0$ is sufficiently large,

$$\mathbf{E}| < x| \; U^0_{\omega,\tau}(s) X_\omega(t) \; |y > | \leq C_2 e^{-\mu|x-y|} \tag{2.18}$$

for some constant $C_2 > 0$ which is also independent of x, $y \in \mathbf{Z}$, s, $t \in [0,1]$, and $\tau > 0$. Moreover, since $U^0_{\omega,\tau}(s)$ and $X_\omega(t)$ are bounded,

$$\mathbf{E}| < x| \; U^0_{\omega,\tau}(s - \sum_{i=1}^{j} s_i) X_\omega(\sum_{i=1}^{j} s_i)| \; z_1 > |^{j+1} \leq C_3^j C_2 e^{-\mu|x-z_1|},$$

for some constant $C_3 > 0$. By estimating the other terms in the same way,

$$\begin{aligned}\tilde{T}_j \leq \sum_{z_1, z_2, \cdots, z_j \in \mathbf{Z}} & C_3^{\frac{j}{j+1}} C_2^{\frac{1}{j+1}} e^{-\frac{\mu}{j+1}|x-z_1|} C_3^{\frac{j}{j+1}} C_2^{\frac{1}{j+1}} e^{-\frac{\mu}{j+1}|z_1-z_2|} \\ & \cdots C_3^{\frac{j}{j+1}} C_2^{\frac{1}{j+1}} e^{-\frac{\mu}{j+1}|z_{j-1}-z_j|} C_1^{\frac{1}{j+1}} e^{-\frac{\mu}{j+1}|z_j-y|}.\end{aligned}$$

We notice that, for $i = 1, 2, 3$, $C_i^{\frac{1}{j+1}} \le 1$, $C_i^{\frac{j}{j+1}} \le 1$ if $C_i \le 1$, and $C_i^{\frac{1}{j+1}} \le C_i$, $C_i^{\frac{j}{j+1}} \le C_i$ if $C_i > 1$. Therefore, by taking $C_4 := \max(1, C_1, C_2, C_3)$,

$$\begin{aligned}\tilde{T}_j &\le \sum_{z_1, z_2, \cdots, z_j \in \mathbf{Z}} C_4^{2j+1} e^{-\frac{\mu}{j+1}(|x-z_1|+|z_1-z_2|+\cdots+|z_{j-1}-z_j|+|z_j-y|)} \\ &\le C_4^{2j+1} \sum_{z_1, z_2, \cdots, z_j \in \mathbf{Z}} e^{-(\frac{\mu-\epsilon}{j+1})|x-y|} e^{-\frac{\epsilon}{j+1}(|x-z_1|+|z_1-z_2|+\cdots+|z_{j-1}-z_j|+|z_j-y|)}.\end{aligned}$$

For $\epsilon > 0$ small, $\sum_{x \in \mathbf{Z}} e^{-\epsilon|x|} \le \frac{4}{\epsilon}$. We ignore $e^{-\frac{\epsilon}{j+1}|z_j - y|}$, and take the sum in the order of z_j, z_{j-1}, $\cdots$, z_2, z_1. The conclusion is

$$\begin{aligned}\tilde{T}_j &\le C_4^{2j+1} (\sum_{z \in \mathbf{Z}} e^{-\frac{\epsilon}{j+1}|z|})^j e^{-\frac{\mu-\epsilon}{j+1}|x-y|} \\ &\le C_4^{2j+1} \{\frac{4}{\epsilon}(j+1)\}^j e^{-\frac{\mu-\epsilon}{j+1}|x-y|}.\end{aligned}$$

In general, for $\mu > 0$, $\alpha > 0$, $e^{-\mu x} \le (\frac{\alpha}{\mu})^\alpha e^{-\alpha} x^{-\alpha}$. Therefore, for arbitrary $\alpha > 0$,

$$\tilde{T}_j \le C_4^{2j+1} (\frac{4}{\epsilon}(j+1))^j (\frac{j+1}{\mu-\epsilon}\alpha)^\alpha e^{-\alpha} |x-y|^{-\alpha}.$$

We return to (2.14). We interchange $\mathbf{E}$ with $\sum_{z_1, z_2, \cdots, z_j \in \mathbf{Z}}$ and compute

$$\begin{aligned}&\mathbf{E}| < x|\, T_j\, |y > | \\ &\qquad \le \int_{0 \le \sum_{i=1}^j s_i \le s} ds_1 ds_2 \cdots ds_j \; C_4^{2j+1} (\frac{4}{\epsilon}(j+1))^j (\frac{j+1}{\mu-\epsilon}\alpha)^\alpha e^{-\alpha} |x-y|^{-\alpha} \\ &\qquad \le \frac{s^j}{j!} C_4^{2j+1} (\frac{4}{\epsilon}(j+1))^j (\frac{j+1}{\mu-\epsilon}\alpha)^\alpha e^{-\alpha} |x-y|^{-\alpha}.\end{aligned}$$

By taking $C_5 := (\frac{\alpha}{\mu-\epsilon})^\alpha e^{-\alpha} C_4$, and $C_6 := 4C_4^2/\epsilon$,

$$\mathbf{E}| < x|\; T_j\; |y > | \le \frac{s^j}{(j+1)!} (j+1)^\alpha C_6^j C_5 (j+1)^{j+1} |x-y|^{-\alpha}.$$

By Stirling formula, for some constants C_7, $C_8 > 0$,

$$(C_7 n)^n \le n! \le (C_8 n)^n.$$

Therefore,

$$\begin{aligned}\mathbf{E}| < x|\; T_j\; |y > | &\le (j+1)^\alpha C_6^j s^j C_7^{-(j+1)} C_5 |x-y|^{-\alpha} \\ &\le (j+1)^\alpha (\frac{C_6}{C_7})^j \frac{C_5}{C_7} s^j |x-y|^{-\alpha}.\end{aligned}$$

Hence, for sufficiently small $s > 0$, there exists a constant $C_\alpha > 0$ such that

$$\sum_{j=1}^{\infty} \mathbf{E}| < x|\ T_j\ |y > | \leq C_\alpha |x-y|^{-\alpha}.$$

By Fubini's theorem,

$$\begin{aligned} \mathbf{E}| < x|\ \bar{U}^A_{\omega,\tau}(s)\ |y > | &= \mathbf{E}| < x|\ \sum_{j=0}^{\infty} T_j\ |y > | \\ &\leq \mathbf{E} \sum_{j=0}^{\infty} | < x|\ T_j\ |y > | \\ &= \sum_{j=0}^{\infty} \mathbf{E}| < x|\ T_j\ |y > | \\ &\leq C_\alpha |x-y|^{-\alpha}. \end{aligned}$$

(2) When $d > 1$, we write j as $j = dk + l$, $k \in \mathbf{N}$, $l = 0, 1, \cdots, d-1$, and write the sum $\sum_{j=1}^{\infty}$ as $\sum_{j=1}^{\infty} = \sum_{j \equiv l, mod\, d} \sum_{k=1}^{\infty}$. Then, it is sufficient to use the following estimates instead of (2.18):

$$\mathbf{E}| < x|\ \overbrace{(U^0X)(U^0X)\cdots(U^0X)}^{d}\ |y > | \leq Ce^{-\mu|x-y|}, \tag{2.19}$$

$$\mathbf{E}| < x|\ \overbrace{(U^0X)(U^0X)\cdots(U^0X)}^{l(<d)}(U_0XU_0X)|y > | \leq Ce^{-\mu|x-y|}, \tag{2.20}$$

(we omit the s_i-dependence). The rest is similar. □

Lemma 2.3.
For sufficiently large $\lambda > 0$, $U_{\omega,\tau}(s)$, $U^A_{\omega,\tau}(s)$ and $P_\omega(s) \in \tilde{\mathcal{S}}$.

proof.
It is straightforward to see $U_{\omega,\tau}(s)$, $U^A_{\omega,\tau}(s)$ and $P_\omega(s)$ satisfy (CR). Then, $P_\omega \in \tilde{\mathcal{S}}$ follows from (2.17) and boundedness of P_ω on $l^2(\mathbf{Z}^d)$. $U_{\omega,\tau}(s)$, $U^A_{\omega,\tau}(s) \in \tilde{\mathcal{S}}$ follows from Lemma 2.2 and their unitarity. □

Lemma 2.4.
$j_{\omega,\tau}$, $j^A_{\omega,\tau}$ is well-defined for $\mathbf{P}$-a.e. ω.

proof.

It follows easily from Lemma 2.1(1) and 2.3. □

The following lemma is not necessary to prove our theorem. However, it gives us another point of view of the charge transport.

Lemma 2.5.

$$j_\omega = \mathcal{T}\left((U_\omega^*(1)xU_\omega(1) - x)P_\omega\right), \ for\ \mathbf{P}-a.e.\ \omega. \tag{2.21}$$

Remarks.

(1) In the RHS of (2.21), $(U_\omega^*(1)xU_\omega(1) - x)$ is at first defined as a form on $C_0 := \{\varphi \in l^2(\mathbf{Z}^d) \ : \ \varphi(x) = 0$, for sufficiently large $|x|\ \}$, and then extended to an operator.

(2) We can consider $(U_\omega^*(1)xU_\omega(1) - x)$ stands for the displacement of electrons from $t = 0$ to $t = 1$. Therefore, multiplying P_ω by this operator, and taking trace per unit volume corresponds to the measurement of the displacement of electrons below the Fermi energy.

proof.

At first, we note that

$$\begin{aligned} \text{the RHS of (2.21)} &= \mathcal{T}\left(\int_0^1 dt\, U_\omega^*(t)i[H_\omega(t), x]U_\omega(t)P_\omega\right) \\ &= \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|}\text{trace}\left(\chi_\Lambda \int_0^1 dt\, U_\omega^*(t)i[H_\omega(t), x]U_\omega(t)P_\omega\chi_\Lambda\right). \end{aligned}$$

Thus, it suffices to show that we can interchange $\int_0^1 dt$ with $\lim_{|\Lambda|\to\infty}$ and $\int_0^1 dt$ with trace. By Lemma 2.1 (2), 2.3,

$$\begin{aligned} &\mathcal{T}(U_\omega^*(t)i[H_\omega(t), x]U_\omega(t)P_\omega) \\ &\qquad = \mathcal{T}\left(P_\omega U_\omega^*(t)i[H_\omega(t), x]U_\omega(t)P_\omega\right) \\ &\qquad = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|}\text{trace}\left(\chi_\Lambda P_\omega U_\omega^*(t)i[H_\omega(t), x]U_\omega(t)P_\omega\chi_\Lambda\right), \end{aligned}$$

for $\mathbf{P}$-a.e. ω. $A_\omega := U_\omega^*(t)i[H_\omega(t), x]U_\omega(t)$ is bounded on $l^2(\mathbf{Z}^d)$. Then, by decomposing A_ω into the linear combination of four unitrary operators[17, Chapter 6], we can assume A_ω is unitary (we should note it still satisfies (CR)). By using $|\mathrm{trace}(AB)| \leq \mathrm{trace}(AA^*)^{1/2}\mathrm{trace}(B^*B)^{1/2}$ (where A, B belong to Hilbert-Schmidt class),

$$\begin{aligned}
&|\frac{1}{|\Lambda|}\mathrm{trace}(\chi_\Lambda P_\omega A_\omega P_\omega \chi_\Lambda)| \\
&\quad \leq \frac{1}{|\Lambda|}\mathrm{trace}(\chi_\Lambda P_\omega A_\omega A_\omega^* P_\omega \chi_\Lambda)^{1/2}\mathrm{trace}(\chi_\Lambda P_\omega P_\omega \chi_\Lambda)^{1/2} \\
&\quad = \frac{1}{|\Lambda|}|\mathrm{trace}(\chi_\Lambda P_\omega \chi_\Lambda)|
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\frac{1}{|\Lambda|}\mathrm{trace}(\chi_\Lambda P_\omega U_\omega^*(t) i[H_\omega, x] U_\omega(t) P_\omega \chi_\Lambda)| \\
&\qquad \leq \|U_\omega^*(t) i[H_\omega, x] U_\omega(t)\|_{op} \frac{1}{|\Lambda|}\mathrm{trace}(\chi_\Lambda P_\omega \chi_\Lambda) \\
&\qquad \leq C\frac{1}{|\Lambda|}\mathrm{trace}(\chi_\Lambda P_\omega \chi_\Lambda), \qquad (2.22)
\end{aligned}$$

where the constant $C > 0$ can be taken independent of $t \in [0,1]$. $\|\cdot\|_{op}$ is the operator norm on $l^2(\mathbf{Z}^d)$. Since the RHS of (2.22) converges to $C\,\mathcal{T}(P_\omega)$ for $\mathbf{P}$-a.e. ω, the LHS of (2.22) is uniformly bounded w.r.t. $t \in [0,1]$ for $|\Lambda|$ sufficiently large. Then, by the dominated convergence theorem, we can interchange $\int_0^1 dt$ with $\lim_{|\Lambda|\to\infty}$. Next,

$$\begin{aligned}
&\int_0^1 dt\, \mathrm{trace}(\chi_\Lambda P_\omega U_\omega^*(t) i[H_\omega(t), x] U_\omega(t) P_\omega \chi_\Lambda) \\
&\qquad = \int_0^1 dt \sum_{n=1}^{N} (\varphi_n, (\chi_\Lambda P_\omega U_\omega^*(t) i[H_\omega(t), x] U_\omega(t) P_\omega \chi_\Lambda)\varphi_n),
\end{aligned}$$

where $\{\varphi_n(x)\}_{n=1}^N$ is the complete orthonormal basis of $l^2(\Lambda)$. Since $\sum_{n=1}^N$ is a finite sum, we can interchange $\int_0^1 dt$ with trace. □

3 PROOF OF THEOREM 1.1.

We defer the proof of Proposition 1.2 which is a little complicated, and prove Proposition 1.3 first.

proof of Proposition 1.3.

We decompose the integrand of (1.11) = I + II, where

$$\mathrm{I} = i\tau\mathcal{T}(U^{A*}_{\omega,\tau}(s)[H_\omega(s),x]U^{A}_{\omega,\tau}(s)P_\omega),$$
$$\mathrm{II} = -\mathcal{T}(U^{A*}_{\omega,\tau}(s)[[\dot{P}_\omega(s),P_\omega(s)],x]U^{A}_{\omega,\tau}(s)P_\omega).$$

Since $U^{A}_{\omega,\tau}(s)$ satisfies the following intertwining property[6]:

$$U^{A}_{\omega,\tau}(s)P_\omega = P_\omega(s)U^{A}_{\omega,\tau}(s), \tag{3.1}$$

We can simplify I:

$$\begin{aligned}\mathrm{I} &= i\tau\mathcal{T}(U^{A*}_{\omega,\tau}(s)[H_\omega(s),x]P_\omega(s)U^{A}_{\omega,\tau}(s))\\ &= i\tau\mathcal{T}(P_\omega(s)[H_\omega(s),x]P_\omega(s))\\ &= i\tau\mathcal{T}(V(s)P_\omega[H_\omega,x]P_\omega V^*(s))\\ &= i\tau\mathcal{T}(P_\omega[H_\omega,x]P_\omega).\end{aligned}$$

In the second and fourth equality, we used Lemma 2.1 (2). Thus, by Proposition 2 in (ref. 6), I = 0 for **P**-a.e. ω. As for II, by using intertwining property (3.1) and Lemma 2.1(2),

$$\begin{aligned}\mathrm{II} &= -\mathcal{T}(U^{A*}_{\omega,\tau}(s)[[\dot{P}_\omega(s),P_\omega(s)],x]P_\omega(s)U^{A}_{\omega,\tau}(s))\\ &= -\mathcal{T}(P_\omega(s)[[\dot{P}_\omega(s),P_\omega(s)],x]P_\omega(s))\\ &= i\mathcal{T}(P_\omega[[[P_\omega,g(x,s)],P_\omega],x]P_\omega)\\ &= i\mathcal{T}(P_\omega[[P_\omega,g(x,s)],[P_\omega,x]]P_\omega)\\ &= -2\mathrm{Im}\mathcal{T}(P_\omega[[P_\omega,g(x,s)],[P_\omega,x]]P_\omega),\end{aligned}$$

Therefore II = 0, because $< x|P_\omega|y >$ is real due to the time-reversal invariance. $\square$

We shall compare $U_{\omega,\tau}(s)$ with $U^{A}_{\omega,\tau}(s)$, and prove a kind of adiabatic theorem (Lemma 3.2) which is necessary to prove Proposition 1.2. Let $\Omega_{\omega,\tau}(s) := U^{A*}_{\omega,\tau}(s)U_{\omega,\tau}(s)$. Then, it solves the following integral equation.

$$\Omega_{\omega,\tau}(s) = I - \int_0^s dt\, U^{A*}_{\omega,\tau}(t)X_\omega(t)U^{A}_{\omega,\tau}(t)\Omega_{\omega,\tau}(t), \tag{3.2}$$

where $X_\omega(t) := [\dot{P}_\omega(t),P_\omega(t)]$ (it is different from what is defined in the proof of Lemma 2.2), and I is the identity operator. We shall estimate the second term in the RHS of (3.2), and will show that it becomes small in certain sense when τ is large, basically along the argument in (refs. 6, 13). At first we introduce

$$\tilde{X}_\omega(t) := -\frac{1}{2\pi i}\int_{\Gamma_\delta} dz R_\omega(z,t) X_\omega(t) R_\omega(z,t), \tag{3.3}$$

where $R_\omega(z,t) := (H_\omega(t) - z)^{-1}$. $\Gamma_\delta := \Gamma \setminus \tilde{\Gamma}_\delta$, where $\Gamma \subset \mathbf{C}$ is a rectangle whose vertices are located at $\epsilon_F + ia$, $-N + ia$, $-N - ia$, and $\epsilon_F - ia$ (N is taken sufficiently large such that $\inf \sigma(H_\omega) > -N$). And $\tilde{\Gamma}_\delta := \{\epsilon_F + ic : -\delta \le c \le \delta\}$, for some $\delta \in (0, a)$. Then, by direct computation, we have

$$\begin{aligned}[H_\omega(t), \tilde{X}_\omega(t)] = [X_\omega(t),\ P_\omega(t)] \\ + \frac{1}{2\pi i}[X_\omega(t),\ \int_{\tilde{\Gamma}_\delta} dz R_\omega(z,t)],\end{aligned}$$

where $\int_{\tilde{\Gamma}_\delta} dz R_\omega(z,t)$ is defined to be

$$\int_{\tilde{\Gamma}_\delta} dz R_\omega(z,t) := \mathrm{s}-\lim_{\eta \downarrow 0}\left(\int_{\epsilon_F - i\delta}^{\epsilon_F - i\eta} + \int_{\epsilon_F + i\eta}^{\epsilon_F + i\delta}\right) R_\omega(z,t) dz.$$

We used the fact [9, Chapter 9]: $\mathbf{P}\{\omega \in \Omega$: ϵ_F is an eigenvalue of $H_\omega\} = 0$. We define:

$$S_\omega(t) := \frac{1}{2\pi i}[X_\omega(t),\ T_\omega(t)], \tag{3.4}$$

$$T_\omega(t) := \int_{\tilde{\Gamma}_\delta} dz R_\omega(z,t). \tag{3.5}$$

Let $Q_\omega(t) := I - P_\omega(t)$, and $Q_\omega := Q_\omega(0)$. The second term of (3.2) multiplied by Q_ω from the left is computed as in the same way in (ref. 6, Lemma 2.5). The result is:

$$\begin{aligned} & Q_\omega \int_0^s dt\, U^{A*}_{\omega,\tau}(t) X_\omega(t) U^A_{\omega,\tau}(t) \Omega_{\omega,\tau}(t) \\ & = -\frac{i}{\tau} Q_\omega \Big\{ [U^{A*}_{\omega,\tau}(t) \tilde{X}_\omega(t) U^A_{\omega,\tau}(t) P_\omega \Omega_{\omega,\tau}(t)]^{t=s}_{t=0} \\ & - \int_0^s dt\, U^{A*}_{\omega,\tau}(t) \tilde{X}_\omega(t) U^A_{\omega,\tau}(t) P_\omega \dot{\Omega}_{\omega,\tau}(t) \\ & - \int_0^s dt\, U^{A*}_{\omega,\tau}(t) (\dot{\tilde{X}}_\omega(t) - [\dot{P}_\omega(t), \tilde{X}_\omega(t)]) U^A_{\omega,\tau}(t) P_\omega \Omega_{\omega,\tau}(t) \Big\} \\ & - Q_\omega \int_0^s dt\, U^{A*}_{\omega,\tau}(t) S_\omega(t) U^A_{\omega,\tau}(t) \Omega_{\omega,\tau}(t). \end{aligned} \tag{3.6}$$

We estimate the last term in the RHS of (3.6) which is the additional term we mentioned at the end of section 1.

Lemma 3.1.

$T_\omega(t) \in \mathcal{S}$. Moreover, for α, β which satisfy $1 \le \alpha < 2$, $\alpha - 1 < \beta < 1 - \frac{1}{p}$, there exists a constant $C_{\alpha\beta} > 0$ such that for sufficiently large $\lambda > 0$,

$$\mathbf{E}| < x|\ T_\omega(t)\ |y > |^\alpha \le C_{\alpha\beta}\delta^\beta e^{-\mu|x-y|}. \tag{3.7}$$

proof.

Let $z = \epsilon_F + i\eta$, $\eta \in (-\delta, \delta)$.

$$\begin{aligned}\mathbf{E}| < x\ \mid\ T_\omega(t)\ |y > |^\alpha &\le (2\delta)^{\alpha-1}\mathbf{E}\int_{\tilde{\Gamma}_\delta} dz| < x|\ R_\omega(z,t)\ |y > |^\alpha \\ &= (2\delta)^{\alpha-1}\mathbf{E}\int_{-\delta}^{\delta} d\eta| < x|\ R_\omega(\epsilon_F + i\eta, t)\ |y > |^\alpha \\ &\le (2\delta)^{\alpha-1}\mathbf{E}\int_{-\delta}^{\delta} d\eta\eta^{-(\alpha-\beta)}| < x|\ R_\omega(\epsilon_F + i\eta, t)\ |y > |^\beta,\end{aligned} \tag{3.8}$$

where $0 < \beta < 1 - \frac{1}{p}$. In the first inequality, we used the Jensen's inequality. If $\alpha - \beta < 1$, we use the exponential decay estimates of Green's function derived in (ref. 3):

$$\mathbf{E}| < x|\ R_\omega(\epsilon_F + i\eta, t)|\ y > |^\beta \le Ce^{-\mu|x-y|}, \tag{3.9}$$

for $0 < \beta < 1 - \frac{1}{p}$ and $\lambda > 0$ is sufficiently large. Hence we obtain

$$\text{the RHS of (3.8)} \le C\delta^\beta e^{-\mu|x-y|}.$$

□

By using above estimates, we prove $\Omega_{\omega,\tau}(s)$ is close to identity in certain sense when τ is large (adiabatic theorem). Usually, adiabatic theorem is stated as[6,12,13]

$$U_\tau(s) = U_\tau^A(s) + O(\tau^{-1}), \quad as\ \tau \to \infty,$$

in the operator norm. However, since there is no spectral gap in the cases where Anderson localization takes place, our adiabatic theorem presented below is rather weak.

Lemma 3.2.

$\Omega_{\omega,\tau}(s)$ *has the following form:*

$$\Omega_{\omega,\tau}(s)=I+\frac{1}{\tau}\left\{A_1(s)+\int_0^s dt\,A_2(t)\right\}+\int_0^s dt\,B(t), \tag{3.10}$$

$$A_1=\prod_{i=1}^{l}A_1^{(i)},\quad A_2(t)=\prod_{i=1}^{m}A_2^{(i)}(t),\quad B(t)=\prod_{i=1}^{n}B^{(i)}(t), \tag{3.11}$$

where,

(1) $A_1^{(i)}$, $A_2^{(i)}(t)\in\tilde{\mathcal{S}}$ *and* $\|A_1^{(i)}\|_{\mathcal{S}_\alpha,\beta}$, $\|A_2^{(i)}(t)\|_{\mathcal{S}_\alpha,\beta}$ *are all independent of* $\tau>0$, $t\in[0,1]$.

(2) There is an $i_0\le n$ *such that,* $B^{(i)}(t)\in\tilde{\mathcal{S}}$ *for* $i\ne i_0$, *where* $\|B^{(i)}(t)\|_{\mathcal{S}_\alpha,\beta}$ *are all independent of* $\tau>0$, $t\in[0,1]$. *And* $B^{(i_0)}(t)=T_\omega(t)$ *which is defined in (3.5) and satisfies (3.7).*

Remark.

We can not deduce the adiabatic theorem in the operator norm from (3.10). What Lemma 3.2 tells us is that

$$\lim_{\tau\to\infty}\mathbf{E}(\varphi,\big(U_{\omega,\tau}(s)-U^A_{\omega,\tau}(s)\big)\psi)=0, \tag{3.12}$$

$$\lim_{\tau\to\infty}\mathbf{E}\|(U_{\omega,\tau}(s)-U^A_{\omega,\tau}(s))\varphi\|_2^2=0, \tag{3.13}$$

for φ, $\psi\in l^2(\mathbf{Z}^d)$. $\|\cdot\|_2$ is the $l^2(\mathbf{Z}^d)$-norm. The rate of convergence is $\tau^{-\alpha}$, where $\alpha<\left(1+\frac{2}{\beta}(d+1)\right)^{-1}$, if $\lambda>\lambda_0(\beta)$, $0<\beta<1-\frac{1}{p}$. Thus (3.12), (3.13) can be regarded as a weak form of the adiabatic theorem. On the other hand, we should note that the adiabatic theorem in the operator norm is proved in (ref. 5) without gap condition in general setting, provided Range$P(s)$ is finite dimensional.

proof.

It suffices to estimate the RHS of (3.6), since we can estimate the second term of (3.2) multiplied by P_ω from the left in the same manner. We will show the first term of (3.6) is the component of A_1 in (3.10), and the second and third terms are the components of $A_2(t)$. In order to do this, we have to check the operators appearing in the first, second and third terms in (3.6) satisfy (1) in the statement of Lemma 3.2. We have only to check $\dot{\Omega}_{\omega,\tau}(t)$ and $\dot{\tilde{X}}_\omega(t)$, because for the rest, it is the direct conclusion of Lemma 2.3. By (3.2), $\dot{\Omega}_{\omega,\tau}(t)$ can be written as

$$\dot{\Omega}_{\omega,\tau}(s)=-U^{A*}_{\omega,\tau}(s)X_\omega(s)U^A_{\omega,\tau}(s)\Omega_{\omega,\tau}(s).$$

By Lemma 2.3, the RHS of this equation obviously satisfies (1) in the statement of Lemma 3.2. As for $\dot{\tilde{X}}_\omega(s)$, it has the following form,

$$\dot{\tilde{X}}_\omega(s) = -\frac{1}{2\pi i}\int_{\Gamma_\delta} dz\,(\dot{R}_\omega(z,s)X_\omega(s)R_\omega(z,s)$$
$$+R_\omega(z,s)\dot{X}_\omega(s)R_\omega(z,s) + R_\omega(z,s)X_\omega(s)\dot{R}_\omega(z,s)).$$

We note that Γ_δ does not contain the spectrum of H_ω. Thus $R_\omega(z,s)$ satisfies the exponential decay estimate which can be shown by Combes-Thomas argument [1,8] for $z \in \Gamma_\delta$:

$$| < x|\ R_\omega(z,s)\ |y > | \le Ce^{-\mu|x-y|}.$$

Then, the conclusion follows from the explicit form

$\dot{R}_\omega(z,s) = iV(s)[g(x,s), R_\omega(z,0)]V^*(s)$, $\dot{P}_\omega(s) = iV(s)[g(x,s), P_\omega]V^*(s)$, the boundedness of $R_\omega(z,0)$ and P_ω, and the Jensen's inequality.

That the last term of (3.6) satisfies (2) in the statement of Lemma 3.2 follows easily from Lemma 2.3, 3.1. □

We are at the stage of the proof of Proposition 1.2.

proof of Proposition 1.2.

We shall compare the following two quantities:

$$j_{\omega,\tau} := i\tau\int_0^1 ds\mathcal{T}(U^*_{\omega,\tau}(s)[H_\omega(s), x]U_{\omega,\tau}(s)P_\omega),$$
$$j^A_{\omega,\tau} := i\tau\int_0^1 ds\mathcal{T}(U^{A*}_{\omega,\tau}(s)[H^A_\omega(s), x]U^A_{\omega,\tau}(s)P_\omega).$$

We compute

$$j_{\omega,\tau} = i\tau\int_0^1 ds\,\mathcal{T}([U^*_{\omega,\tau}(s)H_\omega(s), x]U_{\omega,\tau}(s)P_\omega)$$
$$- i\tau\int_0^1 ds\mathcal{T}([U^*_{\omega,\tau}(s), x]H_\omega(s)U_{\omega,\tau}(s)P_\omega).$$

Since $[U^*_{\omega,\tau}(s)H_\omega(s), x]$, $[U^*_{\omega,\tau}(s), x] \in \tilde{\mathcal{S}}$, the above two terms are both well-defined by Lemma 2.1 (1). Here, we defined the product of operators by their integral kernels, since $[U^*_{\omega,\tau}(s)H_\omega(s), x]$ and $[U^*_{\omega,\tau}(s), x]$ could be unbounded (Remark after Lemma 2.1). We substitute the Schrödinger equation (1.4):

$$j_{\omega,\tau} = i\tau\int_0^1 ds\mathcal{T}([-\frac{i}{\tau}\frac{d}{ds}U^*_{\omega,\tau}(s), x]U_{\omega,\tau}(s)P_\omega)$$

$$-i\tau\int_0^1 ds\mathcal{T}([U^*_{\omega,\tau}(s),x]\frac{i}{\tau}\frac{d}{ds}U_{\omega,\tau}(s)P_\omega)$$
$$=\int_0^1 ds\mathcal{T}\left(\frac{d}{ds}([U^*_{\omega,\tau}(s),x]U_{\omega,\tau}(s)P_\omega)\right).$$

Because $\frac{d}{ds}([U^*_{\omega,\tau}(s),x]U_{\omega,\tau}(s)P_\omega)$ is the sum of two operators, and each operators are products of operators which belong to $\tilde{\mathcal{S}}$, we can interchange $\int_0^1 ds$ with $\int_\Omega d\mathbf{P}$. Therefore,

$$j_{\omega,\tau}=\mathcal{T}([U^*_{\omega,\tau}(1),x]U_{\omega,\tau}(1)P_\omega).$$

And in the same manner,

$$j^A_{\omega,\tau}=\mathcal{T}([U^{A*}_{\omega,\tau}(1),x]U^A_{\omega,\tau}(1)P_\omega).$$

To compare $j^A_{\omega,\tau}$ with $j_{\omega,\tau}$, we substitute $U^A_{\omega,\tau}(s)=U_{\omega,\tau}(s)\Omega^*_{\omega,\tau}(s)$ into the above equation.

$$\begin{aligned} j^A_{\omega,\tau} &= \mathcal{T}([\Omega_{\omega,\tau}(1)U^*_{\omega,\tau}(1),x]U_{\omega,\tau}(1)\Omega^*_{\omega,\tau}(1)P_\omega) \\ &= \mathcal{T}(\Omega_{\omega,\tau}(1)[U^*_{\omega,\tau}(1),x]U_{\omega,\tau}(1)\Omega^*_{\omega,\tau}(1)P_\omega)+\mathcal{T}([\Omega_{\omega,\tau}(1),x]\Omega^*_{\omega,\tau}(1)P_\omega) \\ &=: C+D. \end{aligned}$$

We will show that $C=j_{\omega,\tau}+o(1)$ and $D=o(1)$ as $\tau\to\infty$. This concludes the proof. By subtracting the following two equations,

$$P_\omega\Omega_{\omega,\tau}(s)-P_\omega\Omega_{\omega,\tau}(s)P_\omega=P_\omega\Omega_{\omega,\tau}(s)Q_\omega,$$
$$\Omega_{\omega,\tau}(s)P_\omega-P_\omega\Omega_{\omega,\tau}(s)P_\omega=Q_\omega\Omega_{\omega,\tau}(s)P_\omega,$$

we obtain

$$\begin{aligned} P_\omega\Omega_{\omega,\tau}(s)-\Omega_{\omega,\tau}(s)P_\omega &= P_\omega\Omega_{\omega,\tau}(s)Q_\omega-Q_\omega\Omega_{\omega,\tau}(s)P_\omega \\ &=: W(s), \\ \text{where}\quad W(s) &:= P_\omega\left[\frac{1}{\tau}\left\{A_1(s)+\int_0^s dtA_2(t)\right\}+\int_0^s dtB(t)\right]Q_\omega \\ &- Q_\omega\left[\frac{1}{\tau}\left\{A_1(s)+\int_0^s dtA_2(t)\right\}+\int_0^s dtB(t)\right]P_\omega. \end{aligned}$$

Then,

$$\mathcal{T}(\Omega_{\omega,\tau}(1)[U^*_{\omega,\tau}(1),x]U_{\omega,\tau}(1)(\Omega^*_{\omega,\tau}(1)P_\omega-P_\omega\Omega^*_{\omega,\tau}(1)))$$
$$=\frac{1}{\tau}\mathcal{T}(\Omega_{\omega,\tau}(1)[U^*_{\omega,\tau}(1),x]U_{\omega,\tau}(1)(Q_\omega A_1^*(1)P_\omega-P_\omega A_1^*(1)Q_\omega))$$

$$
\begin{aligned}
&+\frac{1}{\tau}\int_0^1 ds\mathcal{T}(\Omega_{\omega,\tau}(1)[U^*_{\omega,\tau}(1),x]U_{\omega,\tau}(1)(Q_\omega A_2^*(s)P_\omega - P_\omega A_2^*(s)Q_\omega)) \\
&+\int_0^1 ds\mathcal{T}(\Omega_{\omega,\tau}(1)[U^*_{\omega,\tau}(1),x]U_{\omega,\tau}(1)(Q_\omega B^*(s)P_\omega - P_\omega B^*(s)Q_\omega)) \\
&:= E_1 + E_2 + F. \qquad (3.14)
\end{aligned}
$$

By Lemma 2.1(1), 3.2, there exist constants $\beta > 0$, $C_\beta > 0$ such that F in (3.14) satisfies $|F| \le C_\beta \delta^\beta$. Therefore, for arbitrary small $\epsilon > 0$, we have $|F| < \epsilon$ by taking $\delta > 0$ sufficiently small. After fixing such $\delta > 0$, we let $\tau > 0$ sufficiently large and obtain $|E_1|$, $|E_2| < \epsilon$ (by using Lemma 2.1(1), 3.2 again). Hence

$$
\mathcal{T}(\Omega_{\omega,\tau}(1)[U^*_{\omega,\tau}(1),x]U_{\omega,\tau}(1)(\Omega^*_{\omega,\tau}(1)P_\omega - P_\omega\Omega^*_{\omega,\tau}(1))) = o(1),
$$

and therefore,

$$
C = \mathcal{T}(\Omega_{\omega,\tau}(1)[U^*_{\omega,\tau}(1),x]U_{\omega,\tau}(1)P_\omega\Omega^*_{\omega,\tau}(1)) + o(1),
$$

as $\tau \to \infty$. By Lemma 2.1 (2), it follows that

$$
\begin{aligned}
C &= \mathcal{T}(P_\omega\Omega^*_{\omega,\tau}(1)\Omega_{\omega,\tau}(1)[U^*_{\omega,\tau}(1),x]U_{\omega,\tau}(1)P_\omega) + o(1) \\
&= \mathcal{T}(P_\omega[U^*_{\omega,\tau}(1),x]U_{\omega,\tau}(1)P_\omega) + o(1) \\
&= j_{\omega,\tau} + o(1).
\end{aligned}
$$

It remains to show $D = o(1)$. In fact, by Lemma 3.2

$$
\begin{aligned}
D &= \mathcal{T}([\Omega_{\omega,\tau}(1),x]\Omega^*_{\omega,\tau}(1)P_\omega) \\
&= \frac{1}{\tau}\mathcal{T}([A_1(1),x]\Omega^*_{\omega,\tau}(1)P_\omega) + \frac{1}{\tau}\int_0^1 ds\mathcal{T}([A_2(s),x]\Omega^*_{\omega,\tau}(1)P_\omega) \\
&\quad + \int_0^1 ds\mathcal{T}([B(s),x]\Omega^*_{\omega,\tau}(1)P_\omega).
\end{aligned}
$$

In the same manner as in C, we obtain $D = o(1)$. $\square$

ACKNOWLEDGEMENTS

It is our pleasure to thank Michael Aizenman, Akira Iwatsuka, Tohru Koma, Shu Nakamura, and Hermann Schultz-Baldes for their useful discussions and suggestions. We would like to thank the referee for important critical comments. One of the authors (F.N.) is partially supported by the Japan Society for the Promotion of Science.

References

[1] Aizenman, M., *Localization at Weak Disorder: Some Elementary Bounds,* Rev. Math. Phys. **6**(1994), 1163-1182.

[2] Aizenman, M., Graf, G. M., *Localization Bounds for an Electron Gas,* J. Phys. A: Math. Gen. **31**(1998), 6783-6806.

[3] Aizenman, M., Molchanov, S., *Localization at Large Disorder and at Extreme Energies: An Elementary Derivation,* Commun. Math. Phys. **157**(1993), 245-278.

[4] Anderson, P. W., *Absence of diffusion in certain random lattices,* Phys. Rev. **109**(1958), 1492-1505.

[5] Avron, J. E., Elgart, A., *Adiabatic Theorem without a Gap Condition,* Commun. Math. Phys. **203**(1999), 445-463.

[6] Avron, J. E., Seiler, R., Yaffe, L. G., *Adiabatic Theorems and Applications to the Quantum Hall Effect,* Commun. Math. Phys. **110**(1987), 33-49.

[7] Bellissard, J., van Elst, A., Schultz-Baldes, H., *The Non-Commutative Geometry of The Quantum Hall Effect,* J. Math. Phys. **35(10)**(1994), 5373-5451.

[8] Combes, J. M., Thomas, L. E., *Asymptotic behaviour of eigenfunctions for multi-particle Schrödinger operators,* Commun. Math. Phys. **34**(1973), 251-270.

[9] Cycon, H., Froese, R., Kirsch, W., Simon, B., *Schrödinger Operators,* Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[10] del Rio, R., Jitomirskaya, S., Last, Y., Simon, B., *Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations, and localization,* J. Analyse Math. **69**(1996), 153-200.

[11] Fröhlich, J., Spencer, T., *Absence of diffusion in the Anderson tight binding model for large disorder or low energy,* Commun. Math. Phys. **88**(1983), 151-184.

[12] Kato, T., *On the adiabatic theorem of quantum mechanics,* J. Phys. Soc. Jpn. **5**(1950), 435-439.

[13] Klein, M., Seiler, R., *Power-Low Corrections to the Kubo Formula Vanish in Quantum Hall Systems,* Commun. Math. Phys. **128**(1990), 141-160.

[14] Niu, Q., *Quantum adiabatic particle transport,* Phys. Rev. **B34**(1986), 5093.

[15] Niu, Q., *Towards a Quantum Pump of Electric Charge,* Phys. Rev. Lett. **64**(1990), 1812.

[16] Niu, Q., and Thouless D. J., *Quantized adiabatic charge transport in the presence of disorder and many-body interaction,* J. Phys. A:Math. Gen. **17**(1984), 2453-2462.

[17] Reed, M., Simon, B., *Methods of modern mathematical physics, vol. I, Functional analysis,* Academic Press, New York, 1975.

[18] Reed, M., Simon, B., *Methods of modern mathematical physics, vol. II, Fourier analysis, self-adjointness,* Academic Press, New York, 1975.

[19] Schultz-Baldes, H., Bellissard, J., *A Kinetic Theory for Quantum Transport in Aperiodic Media*, J. Stat. Phys. **91**(1998), 991-1026.

[20] Simon, B., Wolff, T., *Singular Continuous Spectrum under Rank One Perturbations and Localization for Random Hamiltonians,* Commun. Pure. Appl. Math. **39**(1986), 75-90.

[21] Thouless, D. J., *Quantization of particle transport,* Phys. Rev. **B27**(1983), 6083.

[22] von Dreifus, H., Krein, A., *A new proof of localization in the Anderson tight binding model,* Commun. Math. Phys. **124**(1989), 285-299.