On the essential spectrum of two dimensional periodic magnetic Schrödinger operators

H. D. Cornean *

26.08.99

Abstract

For two dimensional Schrödinger operators with a nonzero constant magnetic field perturbed by an infinite number of periodically disposed, long range magnetic and electric wells, it is proven that when the inter-well distance (R) grows to infinity, the essential spectrum near the eigenvalues of the "one well Hamiltonian" is located in mini-bands whose width shrink faster than any exponential with R. This should be compared with our previous result [4], which stated that in the case of compactly supported wells, the mini-bands shrink Gaussian like with R.

1 Introduction

In this paper we continue the study (begun in [3] and [4]) of the spectral properties of two dimensional magnetic Schrödinger operators. In [3] we considered the "one well problem" i.e.

$$H = (\mathbf{p} - \mathbf{a}_0 - \mathbf{a})^2 + V, \tag{1.1}$$

where $\mathbf{a}_0$ corresponds to a nonzero constant magnetic field $\mathbf{B}_0$, the magnetic perturbation $\mathbf{B}'(\mathbf{x}) = \mathrm{curl}\,\mathbf{a}(\mathbf{x})$ is long range but bounded in the sense that:

$$b \equiv max\{||D^\alpha B'||_\infty, |\alpha| \leq 1\} < \infty, \tag{1.2}$$

and the scalar perturbation $V = V_1 + V_2$ obeys:

$$V_1 \in \mathbf{L}^2(\mathbf{R}^2), \qquad V_2 \in \mathbf{L}^\infty(\mathbf{R}^2). \tag{1.3}$$

It is known that if both the magnetic and the scalar perturbations are vanishing at infinity, then (see [6], [13]):

$$\sigma_{Landau}(B_0) := \{(2n+1)B_0 \mid n = 0, 1, ...\} = \sigma_{ess}(H). \tag{1.4}$$

*Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 70700 Bucharest, Romania; e-mail cornean@barutu.fizica.unibuc.ro

While in [3] we studied the exponential decay at infinity of the eigenfunctions of H, in [4] we dealt with the multiple well case. The reasoning was that the Landau spectrum suffers a radical change and one is expecting to find essential spectrum and gaps in between the Landau levels. The fact that the "one well" perturbations were taken compactly supported allowed us to conclude that the Hamiltonian with an infinite number of wells (not necessarily disposed in a periodic lattice) has (for a sufficiently large inter well distance) essential spectrum near the discrete eigenvalues of H, and is located in mini-bands whose width presents a *Gaussian* decay with the inter well distance.

In this paper we intend to study the case in which the "one well" perturbations are no longer compactly supported, but periodic. We will show that (see Theorem 2.1 for the precise statement) the essential spectrum which appears near a discrete eigenvalue of H looks like a mini-band too, and its width is shrinking (at least) *faster than any exponential.* This together with an improved resolvent estimate for the "one well" Hamiltonian (see Lemma 2.1) is what we add new to the already known results. We want to stress though that the magnetic perturbation considered here is the magnetic field which is not always a very regular perturbation and one has to be a little bit more careful with it. Nevertheless, from the technical point of view, our work follows rather closely the geometric perturbation theory developed in [1].

Let us comment a few previous results which are connected to ours.

In [7] the case of pure magnetic field Hamiltonians is considered. Take $H_a(\lambda) = (-\imath\nabla - \lambda\mathbf{a})^2$ with $\lambda \in \mathbf{R}$, $\mathbf{a} \in \mathcal{C}^1\left(\mathbf{R}^d, \mathbf{R}^d\right)$, $d \geq 2$ and $\mathbf{B} = d\mathbf{a}$ periodic with respect to $\mathbf{Z}^d$. Let $M_B = \{\mathbf{x}|\mathbf{B}(\mathbf{x}) = 0\}$ and $M_a = \{\mathbf{x}|\mathbf{a}(\mathbf{x}) = 0\}$. Under the condition that $|M_B \backslash M_a| = 0$, it is shown that $H_a(\lambda)$ converges in the norm resolvent sense as $\lambda \to \infty$ to the Dirichlet Laplacian on M, therefore the spectrum of the pure magnetic operator will have band structure.

Starting from these results, Nakamura [14] studied the existence of open gaps for operators of the following type:

$$H_a(\lambda) = (-\imath\nabla - \lambda\mathbf{a})^2 + V,$$

where V is bounded. He also considered that the electric potential and the magnetic field are periodic with respect to a lattice $\Gamma \subset \mathbf{Z}^d$, with bounded fundamental domain Ω_Γ. Let $H_a^D(\lambda)$ be the restriction of $H_a(\lambda)$ to Ω_Γ with Dirichlet boundary conditions imposed on its frontier. Denote with $\Sigma(\lambda)$ the spectrum of $H_a^D(\lambda)$. If the magnetic field is different from zero on $\partial\Omega_\Gamma$, it is proven that $\sigma(H_a(\lambda))$ is situated near $\Sigma(\lambda)$ up to an error of order $\mathcal{O}(\exp(-\alpha\sqrt{\lambda}))$, for some positive α.

If in these two papers "tight-binding limit" means that the magnetic field outside the wells is very large, in [5] (as in our case) the large quantity is the distance between wells.

More precisely, Daumer [5] considered the two dimensional case $d = 2$, with a periodic lattice $\Gamma = \mathbf{Z}^2$ and a constant magnetic field B. The Hamiltonian is given by:

$$H_R^B = (\mathbf{p}-\mathbf{a})^2 + V_R,$$

where $\mathbf{a}$ is the transversal gauge which gives B and

$$V_R = \sum_{\gamma\in\Gamma} V(\cdot - R\gamma), \quad R > 0,$$

where V is a smooth potential obeying the following conditions:

$$\forall \mathbf{x} \in \mathbf{R}^2,\ \forall \alpha \in \mathbf{N}^2,\ |D^\alpha V|(\mathbf{x}) \leq \mathrm{C}_\alpha (1+|\mathbf{x}|)^{-\sigma-|\alpha|}, \sigma > 2.$$

One of the many problems considered there is the comparison between the negative spectrum of H_R^B and the negative spectrum of the "one well" operator $H = -\Delta + V$ for B small enough and R sufficiently large. It is shown that the spectral problem can be reduced to that of a perturbed Harper type pseudo-differential operator which can be handled with the theory developed by B. Helffer and J. Sjöstrand in [8], [9],..., [12].

Amongst other things, it is proven that if $\lambda < 0$ is a not degenerate eigenvalue of H, then in the limit $(R,B) \to (\infty, 0)$, H_R^B has a nonempty spectral band near λ included in $[\lambda - o(1), \lambda + o(1)]$.

The major difference between Daumer's work and ours is that the constant magnetic field is treated there as a perturbation to H, which restricts the spectral study to the negative part of the spectrum. In our setting, it is the constant magnetic field which is responsible for the enhanced localisation of the mini-bands. Moreover, because our one well Hamiltonian (see (2.9)) obeys (1.4), it could have positive eigenvalues which generate mini-bands at their turn and in fact we make no difference between the negative and the positive regions.

2 Preliminaries and the results

As already said, we shall consider only the two dimensional case (i.e. the particle is confined in the plane $x_3 = 0$ and the magnetic field is orthogonal to that plane). Let $B(\mathbf{x}) \in C^1(\mathbf{R}^2)$. We shall use the following family of vector potentials corresponding to $B(\mathbf{x})$:

$$\mathbf{a}(\mathbf{x},\mathbf{x}') = \int_0^1 ds\, s\, \mathbf{B}(\mathbf{x}' + s(\mathbf{x}-\mathbf{x}')) \wedge (\mathbf{x}-\mathbf{x}'). \tag{2.1}$$

For $\mathbf{x}' = 0$, this is nothing but the usual transversal gauge (see e.g. [17]):

$$\mathbf{a}(\mathbf{x},0) \equiv \mathbf{a}(\mathbf{x}) = \int_0^1 ds\, s\, \mathbf{B}(s\,\mathbf{x}) \wedge \mathbf{x}. \tag{2.2}$$

If we define (see [5] for details):

$$\varphi(\mathbf{x},\mathbf{x}') = -\left(\int_0^1 dt \int_0^1 ds\, s\, \mathbf{B}(s\,t\,(\mathbf{x}-\mathbf{x}') + s\,\mathbf{x}')\right) \cdot (\mathbf{x} \wedge \mathbf{x}'), \tag{2.3}$$

then one can see that:

$$\nabla_{\mathbf{x}}\varphi(\mathbf{x},\mathbf{x}') = \mathbf{a}(\mathbf{x}) - \mathbf{a}(\mathbf{x},\mathbf{x}'). \tag{2.4}$$

If $B(\mathbf{x}) = B_0$ is constant, then:

$$\begin{aligned}\varphi_0(\mathbf{x},\mathbf{x}') &= -\frac{1}{2}B_0\,(x_1\,x_2' - x_1'\,x_2)\\ \mathbf{a}_0(\mathbf{x},\mathbf{x}') &= \frac{1}{2}\mathbf{B}_0 \wedge (\mathbf{x}-\mathbf{x}').\end{aligned} \tag{2.5}$$

The Hamiltonian of a particle in the presence of a magnetic field and a scalar potential V is (in the transversal gauge):

$$\begin{aligned} H &= (\mathbf{p}-\mathbf{a}(\mathbf{x}))^2 + V(\mathbf{x}),\\ \mathbf{p} &= \left(-\imath\frac{\partial}{\partial x_1}, -\imath\frac{\partial}{\partial x_2}\right),\\ \mathbf{a}(\mathbf{x}) &= \left(-x_2\int_0^1 ds\; s\,B(s\,\mathbf{x}), x_1\int_0^1 ds\; s\,B(s\,\mathbf{x})\right). \end{aligned} \tag{2.6}$$

In order to give our first result (used in the estimation on the width of the mini-bands which appear in the multiple well case, but also interesting in itself), we need a few auxiliary notations.

Let $B_0 > 0$, $B'(\mathbf{x}) \in \mathcal{C}^1(\mathbf{R}^2)$ and $B(\mathbf{x}) = B_0 + B'(\mathbf{x})$. Let $V_1 \in \mathbf{L}^2(\mathbf{R}^2)$, $V_2 \in \mathbf{L}^\infty(\mathbf{R}^2)$ and $V = V_1 + V_2$.

Define $\mathbf{a}(\mathbf{x}) = \int_0^1 ds\, s\,\mathbf{B}(s\mathbf{x}) \wedge \mathbf{x}$; then $H = (\mathbf{p}-\mathbf{a})^2 + V$ is essentially self adjoint on $\mathcal{C}_0^\infty(\mathbf{R}^2)$.

Define $\mathcal{H}_\mu := \mathbf{L}^2\left(\mathbf{R}^2, dx\ \exp\left(2\mu\langle\cdot\rangle\right)\right)$, where $\langle\cdot\rangle = \left(1+|\cdot|^2\right)^{\frac{1}{2}}$ and $\mu \geq 0$.

Then our first result is contained in the following lemma:

Lemma 2.1 *Let z be a complex number such that* $\operatorname{dist}\{z,\sigma(H)\} = d > 0$ *and let $\chi_n(\mathbf{x})$, $n \in \mathbf{N}$ be the characteristic function of $\{|\mathbf{x}| \geq n\}$.*

Define $b_n = \max\left(||\chi_n \mathbf{D}^\alpha B'||_\infty, |\alpha| \leq 1\right)$ and $v_n = ||\chi_n V_1||_2 + ||\chi_n V_2||_\infty$, obeying the condition $\lim_{n\to\infty}(b_n + v_n) = 0$. Take $\mathbf{n} \in \mathbf{R}^2$, $|\mathbf{n}| = 1$; then for every positive number μ the following operators:

$$A_1(z) = (H-z)^{-1} \tag{2.7}$$

and

$$A_2(z) = \left[\mathbf{n}\cdot(\mathbf{p}-\mathbf{a})\,(H-z)^{-1}\right] \tag{2.8}$$

belong to $\mathcal{B}(\mathcal{H}_\mu)$ and their norms are uniform in $0 < d_0 \leq d \leq d_1 < \infty$.

From now on, a single well will consist from a magnetic field $B(\mathbf{x}) = B_0 + B'(\mathbf{x})$ and a scalar potential V, obeying the following condition (in order to insure the existence of (2.10)):

$$|D^\alpha B'|(\cdot) + |V|(\cdot) \leq \text{const}\ \langle\cdot\rangle^{-\beta},\ \beta > 2,\ |\alpha| \leq 1. \tag{2.9}$$

If $\mathbf{a}$ gives $B = B_0 + B'$, then the one well Hamiltonian will be $H = (\mathbf{p} - \mathbf{a})^2 + V$.

Denote with $\Gamma = \mathbf{Z}^2$ and define a dilated, rectangular lattice $\Gamma_R = \{R\gamma | \gamma \in \Gamma,\ R > 0\}$. The periodic potentials will be:

$$V_{\Gamma_R}(\cdot) = \sum_{\gamma \in \Gamma} V(\cdot - R\gamma), \quad B_{\Gamma_R}(\cdot) = B_0 + \sum_{\gamma \in \Gamma} B'(\cdot - R\gamma). \tag{2.10}$$

If $\mathbf{a}_{\Gamma_R}$ stands for the (transversal) gauge which gives B_{Γ_R}, then our Hamiltonian will be $H_{\Gamma_R} = (\mathbf{p} - \mathbf{a}_{\Gamma_R})^2 + V_{\Gamma_R}$.

Let $c < d$ such that $[c, d] \cap \sigma_{Landau}(B_0) = \emptyset$ (see (1.4)) and let $\sigma(H) \cap (c, d) = \{E_1\}$, where E_1 is a discrete eigenvalue of H (for simplicity we're considering it not degenerate; see Remark 4 at the end of the proofs for a short discussion on the degenerate case).

Then the existence of highly localised essential spectrum near E_1 is stated in the following theorem:

Theorem 2.1 *Fix $\mu > 0$. Then there exists a real function $E_1(R)$ obeying $\lim_{R \to \infty} E_1(R) = E_1$, and $R(\mu) > 0$ such that, for every $R > R(\mu)$ one has:*
i) $\sigma(H_{\Gamma_R}) \cap [c, d] \subset [E_1(R) - 2\exp(-\mu R), E_1(R) + 2\exp(-\mu R)]$
and
ii) $\dim \mathrm{Ran} P_{\Gamma_R}([c, d]) = \infty$ *(i.e. the band is not empty).*

3 The proofs

3.1 An improved resolvent estimate

Before starting the proof of Lemma 2.1, let us remark the following proposition:

Proposition 3.1 *Let K be a compact in $\mathbf{C}$ and $z \in K$. Let $A_1(z)$ and $A_2(z)$ defined as in (2.7) and (2.8). Then there exists a constant $\mathcal{C}(K)$ such that:*

$$||A_2(z)|| \leq \mathcal{C}(K)\,(1 + ||A_1(z)||)\,.$$

Proof:

In the argument we are presenting here, appear the same basic ideas as in the proof of Theorem B.6.5. from [16] (which treats atomic Hamiltonians without magnetic fields and with polynomial weights). Nevertheless, we decided to give a rather detailed proof in order to have a better control on the bounds (see the reasoning for this in Remark 2 below the proof of Lemma 2.1).

In order to work with bounded weights only, we shall consider that $f_m(\mathbf{x})$ is a sequence of some smooth and bounded functions such that $f_m(\mathbf{x}) = \langle \mathbf{x} \rangle$ if $|\mathbf{x}| \leq m/2$ and $\sup_m \max\{|\nabla f_m|, |\Delta f_m|\} < 2$.

Because $b_0 + v_0 < \infty$, there exists a $\lambda < 0$ such that:

$$\lambda < -1 - (\inf \sigma(H))\,, \tag{3.1}$$

and therefore $R(\lambda) = (H - \lambda)^{-1} > 0$.

Denote with $R^0(\lambda) = \left[(\mathbf{p} - \mathbf{a})^2 - \lambda\right]^{-1}$; then for $-\lambda$ large enough, one has the following relations:

$$|| \left(1 + (R^0(\lambda))^{\frac{1}{2}} V (R^0(\lambda))^{\frac{1}{2}}\right)^{-1} ||_{\mathcal{B}(\mathcal{H}_0)} < 1/2 \tag{3.2}$$

and

$$R(\lambda) = R^0(\lambda)^{\frac{1}{2}} \left(1 + (R^0(\lambda))^{\frac{1}{2}} V (R^0(\lambda))^{\frac{1}{2}}\right)^{-1} R^0(\lambda)^{\frac{1}{2}}. \tag{3.3}$$

Under the same condition on $-\lambda$, it is known from the usual Combes-Thomas theory that $R(\lambda) \in \mathcal{B}(\mathcal{H}_\mu)$:

$$|| \exp(\mu\langle\cdot\rangle) R(\lambda) \exp(-\mu\langle\cdot\rangle) ||_{\mathcal{B}(\mathcal{H}_0)} < \infty. \tag{3.4}$$

Then $A_2(z)$ can be rewritten (as an operator in $\mathcal{B}(\mathcal{H}_0)$) in the following form:

$$\exp(\mu\langle\cdot\rangle) \left[\mathbf{n} \cdot (\mathbf{p} - \mathbf{a})(R(\lambda) + (z - \lambda) R(\lambda) R(z))\right] \exp(-\mu\langle\cdot\rangle), \tag{3.5}$$

so it would be sufficient proving that $A_2(\lambda)$ is bounded.

Let $\phi, \psi \in C_0^\infty\left(\mathbf{R}^2\right)$ with their support included in $|\mathbf{x}| < M < \infty$. If one manages to prove that:

$$\begin{aligned} & |\langle \exp(\mu\langle\cdot\rangle)\phi, [\mathbf{n} \cdot (\mathbf{p} - \mathbf{a})]\, R(\lambda)\psi\rangle| \leq \\ \leq\ & \text{const}\ (1 + ||A_1(\lambda)||)\ ||\phi||_{\mathcal{H}_0} ||\psi||_{\mathcal{H}_\mu}, \end{aligned} \tag{3.6}$$

then via a density argument the proof would be finished.

Let us put the scalar product from (3.6) in a more convenient form. Provided $M < m/2$, one has:

$$\begin{aligned} & \langle \exp(\mu\langle\cdot\rangle)\phi, [\mathbf{n} \cdot (\mathbf{p} - \mathbf{a})]\, R(\lambda)\psi\rangle_{\mathcal{H}_0} = \\ =\ & \langle [\mathbf{n} \cdot (\mathbf{p} - \mathbf{a})] \exp(\mu f_m(\cdot))\phi, R(\lambda)\psi\rangle_{\mathcal{H}_0} = \\ =\ & \langle [\mathbf{n} \cdot (\mathbf{p} - \mathbf{a})]\phi, \exp(\mu f_m(\cdot)) R(\lambda)\psi\rangle + \\ +\ & \imath\, \mu\, \langle [\mathbf{n} \cdot (\nabla f_m(\cdot))]\phi, \exp(\mu f_m(\cdot)) R(\lambda)\psi\rangle. \end{aligned} \tag{3.7}$$

The second term of the last equality is easy to control; we will treat here only the first one.

Commuting $\exp(\mu f_m(\cdot))$ with $R(\lambda)$, one obtains:

$$\begin{aligned} & \exp(\mu f_m(\cdot)) R(\lambda)\psi = \\ =\ & R(\lambda) \exp(\mu\langle\cdot\rangle)\psi + R(\lambda)[H, \exp(\mu f_m(\cdot))] R(\lambda)\psi = \\ =\ & R(\lambda) \exp(\mu\langle\cdot\rangle)\psi + R(\lambda)(\mu^2 |(\nabla f_m(\cdot))|^2 + \mu\, \Delta f_m(\cdot)) \times \\ \times\ & \exp(\mu f_m(\cdot)) R(\lambda)\psi - 2\, \imath\, \mu\, R(\lambda)[(\mathbf{p} - \mathbf{a}) \cdot (\nabla f_m(\cdot))] \times \\ \times\ & \exp(\mu f_m(\cdot)) R(\lambda)\psi. \end{aligned} \tag{3.8}$$

With the help of the following two inequalities:

$$||\mathbf{n}\cdot(\mathbf{p}-\mathbf{a})(R^0(\lambda))^{\frac{1}{2}}||_{\mathcal{B}(\mathcal{H}_0)} < \infty, \tag{3.9}$$

and

$$\sup_m ||(R^0(\lambda))^{\frac{1}{2}}(\mathbf{p}-\mathbf{a})\cdot(\nabla f_m)||_{\mathcal{B}(\mathcal{H}_0)} < \infty, \tag{3.10}$$

after using (3.3) and taking the supremum over m in (3.8), one obtains (3.6).□

In the proof of Lemma 2.1, a central role will be played by the following lemma (proved in [3] in a slightly different form as Corollary 3.1), which controls the "asymptotic Hamiltonians":

Lemma 3.1 *Let z be a complex number, with* $\mathrm{dist}(z, \sigma_L(B_0)) = d > 0$. *Define:*

$$B_n(\mathbf{x}) = B_0 + B_n'(\mathbf{x}), \quad V_n = V_{1,n} + V_{2,n}$$

and

$$b_n = \max(\sup_{\mathbf{x}\in\mathbf{R}^2} |D^\alpha B_n'(\mathbf{x})|,\ |\alpha| \leq 1);\ v_n = ||V_{1,n}||_2 + ||V_{2,n}||_\infty,$$

with

$$\lim_{n\to\infty}(b_n + v_n) = 0.$$

Define:

$$H_n = (\mathbf{p}-\mathbf{a}_n)^2 + V_n.$$

Fix $\mu > 0$. Then there exists $N(\mu, d) > 0$ such that, for any $n \geq N(\mu, d)$, one has $z \notin \sigma(H_n)$ and

$$||\exp(\mu <\cdot>)\,(H_n - z)^{-1}\exp(-\mu <\cdot>)||_{\mathcal{B}(\mathcal{H}_0)} \leq \mathrm{C}(\mu, d). \square \tag{3.11}$$

Proof of Lemma 2.1:

The main idea of the proof consists in constructing an "almost inverse" operator for $H - z$, using the fact that after "gauging" away some magnetic phases and going sufficiently far from the origin, the particle "feels" only smaller and smaller magnetic and scalar perturbations. In that region, its dynamics should be very close to that of a particle which is placed in a *globally* small perturbation, for which we have the informations given in Lemma 3.1. The "almost inverse" operator will be constructed via a "cut and paste" technique which takes into account that for long range magnetic fields the gauge transformations we need can be performed only locally. Firstly, we need a quadratic partition of unity constituted from five cut-off functions with convex supports and of the following type:

$$g \in C^\infty(\mathbf{R}^2)\,,\ 0 \leq g(\mathbf{x}) \leq 1. \tag{3.12}$$

The origin (where the perturbations are centred) is covered by a compactly supported function:

$$g_0(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad |x_i| \leq \frac{1}{2},\ i \in \{1,2\} \\ 0 & \text{if} \quad |x_1| \geq \frac{3}{2} \text{ or } |x_2| \geq \frac{3}{2} \end{cases} \tag{3.13}$$

The other four are not compactly supported and are covering the rest of the plane. The right hand side is covered by g_1 which is equal to one if $x_1 \geq \frac{3}{2}$ and $|x_2| \leq \frac{1}{2}$, and equal to zero if $x_1 \leq \frac{1}{2}$ or $|x_2| \geq \frac{3}{2}$. The upper side is covered by g_2 which is equal to one if $x_2 \geq \frac{3}{2}$ and equal to zero if $x_2 \leq \frac{1}{2}$. The left and down sides are covered by $g_3(\mathbf{x}) := g_1(-\mathbf{x})$ and $g_4(\mathbf{x}) := g_2(-\mathbf{x})$. The last condition imposed on these functions is:

$$\sum_{j=0}^{4} g_j^2(\mathbf{x}) = 1. \tag{3.14}$$

In order to define the asymptotic potentials, let us choose another four functions $\tilde{g}_j$, $j \in \{1, \ldots, 4\}$ such that the origin is away from their supports and $\tilde{g}_j g_j = g_j$.

Let us now "dilate" the partition:

$$g_{j,n}(\mathbf{x}) = g_j\left(\frac{\mathbf{x}}{n}\right); \quad \tilde{g}_{j,n}(\mathbf{x}) = \tilde{g}_j\left(\frac{\mathbf{x}}{n}\right); \quad n \in \{1, 2, \ldots\}. \tag{3.15}$$

From these definitions one can see that all the derivatives of $g_{j,n}$ converge uniformly to zero when n goes to infinity.

Let us define the asymptotic potentials:

$$\begin{aligned} B_{j,n} &= B_0 + \tilde{g}_{j,n} B', \quad V_{j,n} = \tilde{g}_{j,n} V, \\ \mathbf{a}_{j,n}(\mathbf{x}) &= \int_0^1 ds\ s\ \mathbf{B}_{j,n}(s\mathbf{x}) \wedge \mathbf{x}, \quad j \in \{1, \ldots 4\}. \end{aligned} \tag{3.16}$$

Take now four points situated in the support of the functions which are far from the origin. They will be useful in order to write down the needed local gauge transformations:

$$\{\mathbf{x}^j\}_{j \in \{1,\ldots,4\}}, \quad \mathbf{x}^j \in \text{supp } g_{j,n}.$$

From (2.4) and (2.3), one can see that:

$$\begin{aligned} \mathbf{a}(\mathbf{x}) &= \mathbf{a}(\mathbf{x}, \mathbf{x}^j) + \nabla_{\mathbf{x}} \varphi(\mathbf{x}, \mathbf{x}^j) \quad \text{and} \\ \mathbf{a}_{j,n}(\mathbf{x}) &= \mathbf{a}_{j,n}(\mathbf{x}, \mathbf{x}^j) + \nabla_{\mathbf{x}} \varphi_{j,n}(\mathbf{x}, \mathbf{x}^j), \quad j \in \{1, \ldots, 4\}, \end{aligned} \tag{3.17}$$

and, for $\mathbf{x} \in \text{supp } g_{j,n}$, one can easily show that:

$$\mathbf{a}(\mathbf{x}, \mathbf{x}^j) = \mathbf{a}_{j,n}(\mathbf{x}, \mathbf{x}^j).$$

With the help of the following definition:

$$\delta\varphi_{j,n}(\mathbf{x},\mathbf{x}^j) = \varphi(\mathbf{x},\mathbf{x}^j) - \varphi_{j,n}(\mathbf{x},\mathbf{x}^j),$$

one can see (at least at a formal level) how the local gauge transformations are acting:

$$(\mathbf{p}-\mathbf{a})\exp(\imath\,\delta\varphi_{j,n}(\cdot,\mathbf{x}^j))g_{j,n} = \exp(\imath\,\delta\varphi_{j,n}(\cdot,\mathbf{x}^j))(\mathbf{p}-\mathbf{a}_{j,n})g_{j,n}. \tag{3.18}$$

At this point we are able to define the "asymptotic Hamiltonians" (they have *globally* small perturbations, uniformly in n; see the conditions imposed on B' and V):

$$H_{j,n} = (\mathbf{p}-\mathbf{a}_{j,n})^2 + V_{j,n}, \quad j \in \{1,\dots,4\}.$$

Take z as stated. For n_0 large enough and $n \geq n_0$, Lemma 3.1 implies that:

$$\mathrm{dist}(z,\sigma(H_{j,n})) \geq \frac{d_0}{2}. \tag{3.19}$$

The operator whose role is to approximate the resolvent reads as:

$$\begin{aligned} S(z) \;&=\; g_{0,n}(H-z)^{-1}g_{0,n} + \qquad (3.20)\\ &+ \sum_{j=1}^{4}\exp(\imath\,\delta\varphi_{j,n}(\cdot,\mathbf{x}^j))g_{j,n}((H_{j,n}-z)^{-1}g_{j,n}\exp(-\imath\,\delta\varphi_{j,n}(\cdot,\mathbf{x}^j)). \end{aligned}$$

The essential self-adjointness of H and $\{H_{j,n}\}$ on the same core $C_0^\infty(\mathbf{R}^2)$ and the use of (3.18) enable us to conclude that

$$(H-z)S(z) = 1 + T_0 + T_1, \tag{3.21}$$

where T_0 and T_1 are given by:

$$T_0 = [H, g_{0,n}](H-z)^{-1}g_{0,n}, \tag{3.22}$$

$$T_1 = \sum_{j=1}^{4}\exp(\imath\,\delta\varphi_{j,n})[H_{j,n},g_{j,n}]((H_{j,n}-z)^{-1}g_{j,n}\exp(-\imath\,\delta\varphi_{j,n}), \tag{3.23}$$

and $[H_{j,n},g_{j,n}] = -2\imath(\nabla g_{j,n})\cdot(\mathbf{p}-\mathbf{a}_{j,n}) - (\Delta g_{j,n})$.

Fix $\mu > 0$. Proposition 3.1, Lemma 3.1, the compact support of $g_{0,n}$ and the fact that the derivatives of $g_{j,n}$ behave at least like $1/n$, assure the existence of $n(\mu,d)$ such that, for $n = n(\mu,d)+1$, one has:

$$\begin{aligned} &||\exp(\mu<\cdot>)T_1\exp(-\mu<\cdot>)|| + ||T_0|| \leq \frac{1}{4},\\ &||\exp(\mu<\cdot>)T_0|| < \infty,\\ &||\exp(\mu<\cdot>)S(z)\exp(-\mu<\cdot>)|| \leq \mathrm{C}(\mu,d). \end{aligned} \tag{3.24}$$

Then we can perform the following computations:

$$\begin{aligned} & (1+T_1+T_0)^{-1} = (1+T_1)^{-1}\left[1+T_0(1+T_1)^{-1}\right]^{-1} = \\ = \; & (1+T_1)^{-1}\left\{1-T_0(1+T_1)^{-1}\left[1+T_0(1+T_1)^{-1}\right]^{-1}\right\}. \end{aligned} \tag{3.25}$$

If z has a small imaginary part, then we know that $(H-z)^{-1}$ exists and from (3.21) it should be given by the following relation:

$$\begin{aligned} (H-z)^{-1} &= S(z)(1+T_1+T_0)^{-1} = \\ &= S(z)(1+T_1)^{-1} - \\ &- S(z)(1+T_1)^{-1}T_0(1+T_1)^{-1}\left[1+T_0(1+T_1)^{-1}\right]^{-1}. \end{aligned} \tag{3.26}$$

Because the right hand side of (3.26) is bounded for all z obeying $0 < d_0 \leq d \leq d_1 < \infty$, it follows through analytic continuation that (3.26) is true on the real axis too. Using now (3.24) in (3.26), the proof is finished.□

Remark 1:

For $\alpha \in \mathbf{R}$, define the following analytic family of operators:

$$H(\alpha) = \exp(\imath\alpha < \cdot >)H\exp(-\imath\alpha < \cdot >).$$

Lemma 2.1 implies in particular that, if $z \in \rho(H)$, then $(H(\alpha)-z)^{-1}$ admits an entire extension to the whole complex plane and $\rho(H) \subset \rho(H(\alpha))$, therefore the essential spectrum of $H(\alpha)$ is not moving towards its discrete eigenvalues (from the usual analytic perturbation theory we know that its discrete spectrum is independent of α). Moreover, the projector $P(\alpha)$ (written with the Riesz formula) associated with a discrete eigenvalue of $H(\alpha)$ admits an entire extension too, which (via O'Connor's Lemma) implies that the discrete eigenfunctions of H decay faster than any exponential (this result was obtained in a simpler way in [3]).

Remark 2:

Suppose that one has a family of scalar and magnetic perturbations V_k and B'_k such that (see the formulation of Lemma 2.1) $\lim_{n\to\infty}\{\sup_k(b_{n,k}) + \sup_k(v_{n,k})\} = 0$, and $\inf_k\{\mathrm{dist}(z,\sigma(H_k))\} \geq d > 0$. Then it can be seen that the choices for λ in (3.1) and (3.2), the bounds in (3.4), (3.2), (3.3), (3.9) and (3.10), the choice of $N(\mu,d)$ and the bound in (3.11) can be made uniform in k, and so are the bounds on the norms which appear in Lemma 2.1.

3.2 Periodic wells in two dimensions

Proof of Theorem 2.1:

i). The strategy in finding the spectrum location of the "whole" Hamiltonian H_{Γ_R} in the given interval $[c,d]$ consists in constructing its resolvent $(H_{\Gamma_R}-z)^{-1}$ for all $z \in [c,d]$ situated outside of an exponentially small strip centred in

$E_1(R)$, where $E_1(R)$ will turn out to be an eigenvalue of an "almost one well" Hamiltonian H_R. So one has to make the following two steps:

a) Guess H_R and $E_1(R)$,

and

b) construct a bounded operator $S_R(z)$ such that, as soon as $R \geq R(\mu)$ one can write:

$$(H_{\Gamma_R} - z)S_R(z) = 1 + T_R(z), \tag{3.27}$$

$$||T_R(z)|| < \frac{\exp(-\mu R)}{|z - E_1(R)|} + 1/2. \tag{3.28}$$

Let us introduce some cut-off functions which will help to write down the needed operators. Let $j \in \{1, 2, 3\}$, and let g_j be of the same kind as in (3.12) and with convex supports included in $|\mathbf{x}| \leq 7/8$. Choose $g_1(\mathbf{x}) = 1$ if $|\mathbf{x}| \leq 1/4$ and equal to zero if $|\mathbf{x}| \geq 3/4$. Take then g_2 and g_3 such that $g_2 g_3 = g_2$ and $g_1 g_2 = g_1$. If $\mathrm{supp}(\partial g_2)$ denotes the support of at least one derivative of g_2, then we ask that:

$$\delta = \mathrm{dist}(\mathrm{supp}(\partial g_2), \mathrm{supp} g_1) > 0. \tag{3.29}$$

Define then $g_{j,\gamma,R} = g_j((\cdot - R\gamma)/R)$. The final requirement will be:

$$\sum_{\gamma \in \Gamma} g_{1,\gamma,R}(\mathbf{x}) = 1. \tag{3.30}$$

Remark 3:

It is very important to see that, for any $R > 0$ and any γ, the function $g_{2,\gamma,R}$ has at most 8 neighbours whose supports are not disjoint from $\mathrm{supp} g_{2,\gamma,R}$. Moreover,

$$\mathrm{dist}(\mathrm{supp}(\partial g_{2,\gamma,R}), \mathrm{supp} g_{1,\gamma,R}) = \delta R, \qquad g_{2,\gamma,R}\, g_{1,\gamma,R} = g_{1,\gamma,R}. \tag{3.31}$$

Let us introduce the "almost one well potentials":

$$\begin{aligned} B_R(\mathbf{x}) &= B_0 + \left[\sum_{\gamma \in \Gamma} B'(\mathbf{x} - R\gamma)\right] g_{3,0,R}(\mathbf{x}) = \\ &= B(\mathbf{x}) + (1 - g_{3,0,R}(\mathbf{x}))B'(\mathbf{x}) + \left[\sum_{\gamma \neq 0} B'(\mathbf{x} - R\gamma)\right] g_{3,0,R}(\mathbf{x}), \end{aligned} \tag{3.32}$$

$$\begin{aligned} V_R(\mathbf{x}) &= V_{\Gamma_R}(\mathbf{x}) g_{3,0,R}(\mathbf{x}) = \\ &= V(\mathbf{x}) + (1 - g_{3,0,R}(\mathbf{x}))V(\mathbf{x}) + \left[\sum_{\gamma \neq 0} V(\mathbf{x} - R\gamma)\right] g_{3,0,R}(\mathbf{x}). \end{aligned} \tag{3.33}$$

If $\mathbf{a}_R$ is the transversal gauge corresponding to $B_R(\mathbf{x})$, then the "almost one well" Hamiltonian will be $H_R = (\mathbf{p} - \mathbf{a}_R)^2 + V_R$.

From the above definitions and using (2.9), one can see that:

$$\lim_{R\to\infty} \{ \sup_{\mathbf{x}\in\mathbf{R}^2} [|D^\alpha(B - B_R)|(\mathbf{x}) + |V - V_R|(\mathbf{x})] \} = 0. \tag{3.34}$$

The stability of spectrum for such Hamiltonians (see [15] for a more general discussion) implies that for $R \geq R_0$, in $[c, d]$ lives only one not degenerated eigenvalue of H_R (denoted with $E_1(R)$) which converges to E_1 when R goes to infinity.

At this point we have to establish the link between H_R and H_{Γ_R}. Denote with $t_{\gamma R}$ the usual translation $((t_{\gamma R} f)(\mathbf{x}) = f(\mathbf{x} - R\gamma))$; from (2.1) and (2.10) one obtains that $\mathbf{a}_{\Gamma_R}(\mathbf{x}, \gamma R) = \mathbf{a}_{\Gamma_R}(\mathbf{x} - \gamma R)$ and

$$\mathbf{a}_{\Gamma_R}(\mathbf{x}) g_{2,0,R} = \mathbf{a}_R(\mathbf{x}) g_{2,0,R}. \tag{3.35}$$

One can write down a gauge phase φ_γ such that:

$$H_{\Gamma_R} \exp(\imath\varphi_\gamma) g_{2,\gamma,R} t_{\gamma R} = \exp(\imath\varphi_\gamma) t_{\gamma R} H_R g_{2,0,R}. \tag{3.36}$$

The resolvent's approximation will then be:

$$S_R(z) = \sum_{\gamma\in\Gamma} g_{2,\gamma,R} \exp(\imath\varphi_\gamma) t_{\gamma R} (H_R - z)^{-1} t_{-\gamma R} \exp(-\imath\varphi_\gamma) g_{1,\gamma,R}, \tag{3.37}$$

where the infinite sum should be understood as a strong limit. Using (3.31) (see Remark 3) and the fact that $\{g_{1,\gamma,R}\}$ is a locally finite partition of unity, one obtains:

$$||S_R(z)|| \leq \text{const}|E_1(R) - z|^{-1}, \tag{3.38}$$

where the above constant does not depend on R. After a similar reasoning with that one made in obtaining (3.21), one has the following two equalities:

$$(H_{\Gamma_R} - z) S_R(z) = 1 + T_R(z), \tag{3.39}$$
$$T_R(z) = \sum_{\gamma\in\Gamma} \exp(\imath\varphi_\gamma) t_{\gamma R} [H_R, g_{2,0,R}] (H_R - z)^{-1} g_{1,0,R} t_{-\gamma R} \exp(-\imath\varphi_\gamma).$$

The operator $T_R(z)$ is also bounded:

$$||T_R(z)|| \leq \text{const}||[H_R, g_{2,0,R}] (H_R - z)^{-1} g_{1,0,R}||, \tag{3.40}$$

where $[H_R, g_{2,0,R}] = -2\imath(\nabla g_{2,0,R}) \cdot (\mathbf{p} - \mathbf{a}_R) - \Delta g_{2,0,R}$. Because z is situated in the interval $[c, d]$ which contains only E_1, one can find a contour $\mathcal{C}$ in the complex plane which surrounds E_1 such that $\text{dist}\{\sigma(H), \mathcal{C}\} \geq d > 0$; due to the spectrum stability of H_R, for $R > R_0$ one has $\mathrm{dist}\{\sigma(H_R), \mathcal{C}\} \geq d/2$. Riesz formula gives at once that:

$$\begin{aligned} P_R &= -\frac{1}{2\pi\imath}\int_{\mathcal{C}} d\xi (H_R - \xi)^{-1}, \\ (1-P_R)(H_R - z)^{-1} &= \frac{1}{2\pi\imath}\int_{\mathcal{C}} d\xi \frac{1}{\xi - z}(H_R - \xi)^{-1}, \\ (H_R - z)^{-1} &= \frac{1}{E_1(R) - z} P_R + (1-P_R)(H_R - z)^{-1}. \end{aligned} \tag{3.41}$$

Take (see (3.29) for the definition of δ) $\tilde{\mu} > 2\mu/\delta$. From (3.31), Lemma 2.1, Remark 2 and (3.41) one can write:

$$\begin{aligned} \|T_R(z)\| &\leq \mathrm{const}\|[H_R, g_{2,0,R}]\,(H_R - z)^{-1}\, g_{1,0,R}\| \leq \\ &\leq \frac{\mathrm{const}}{|E_1(R) - z|}\exp\left(-\tilde{\mu}\delta R\right) + 1/2, \end{aligned} \tag{3.42}$$

where the above constant depends on everything but R; take now $R(\mu)$ such that $\mathrm{const}\exp\left(-\tilde{\mu}\delta R(\mu)/2\right) < 1$; then for $R \geq R(\mu)$ and $|E_1(R) - z| \geq 2\exp\left(-\mu R\right)$ one finally obtains $\|T_R(z)\| < 1$ and the proof of i) is finished.

ii). First of all, because H_{Γ_R} commutes with the magnetic translations it can not have discrete spectrum, therefore the only thing we should check is that its spectral projection corresponding to $[c,d]$ is not zero. For $R \geq R(\mu)$, one has:

$$P_{\Gamma_R}([c,d]) = -\frac{1}{2\pi\imath}\int_{\mathcal{C}} d\xi (H_{\Gamma_R} - \xi)^{-1}, \tag{3.43}$$

and using the expression of $S_R(z)$, one obtains (R kept sufficiently large):

$$\|P_{\Gamma_R}([c,d]) - \sum_{\gamma\in\Gamma} g_{2,\gamma,R}\exp\left(\imath\varphi_\gamma\right) t_{\gamma R} P_R t_{-\gamma R}\exp\left(-\imath\varphi_\gamma\right) g_{1,\gamma,R}\| < 1/2. \tag{3.44}$$

Here P_R stands for the one dimensional projector of H_R which corresponds to $E_1(R)$. Denote with $\tilde{P}(R)$ the sum which appears in (3.44):

$$\tilde{P}(R) = g_{2,0,R} P_R g_{1,0,R} + \sum_{\gamma\neq 0} g_{2,\gamma,R}\exp\left(\imath\varphi_\gamma\right) t_{\gamma R} P_R t_{-\gamma R}\exp\left(-\imath\varphi_\gamma\right) g_{1,\gamma,R}, \tag{3.45}$$

where we used the fact that $\varphi_0 = 0$ (see (2.3)). Denote also with $\psi_1(R)$ the normalised eigenfunction of H_R. If P (ψ_1) denotes the projector (eigenvector) of H corresponding to E_1, then we know that

$$\lim_{R\to\infty}\|P_R - P\| = 0. \tag{3.46}$$

From now on, ϵ will mean a generic, infinitesimally small positive quantity. For $R_0 > 0$, denote with χ_0 the characteristic function of $\{|\mathbf{x}| \geq R_0\}$; provided R_0

sufficiently large, one has:

$$\sup_{R\geq R_0} ||P_R - P|| \leq \epsilon. \tag{3.47}$$

If $U(R)$ denotes the Nagy unitary which intertwines P_R and P, then $\psi_1(R) = U(R)\psi_1$ and $\sup_{R\geq R_0} ||U(R)-1|| \leq \epsilon$. Due to the exponential localisation of ψ_1 near the origin, one has $||\chi_0\psi_1|| \leq \epsilon$, therefore:

$$\langle\psi_1(R),(1-\chi_0)\psi_1(R)\rangle \geq 1-\epsilon.$$

Moreover, $(1-\chi_0)\psi_1(R)$ is compactly supported. It follows that:

$$\psi_1(R) = (\langle\psi_1(R),(1-\chi_0)\psi_1(R)\rangle)^{-1}P_R(1-\chi_0)\psi_1(R). \tag{3.48}$$

Therefore:

$$\sup_{R\geq R(\mu)} ||\exp(\mu\langle\cdot\rangle)\psi_1(R)|| \leq \mathrm{C}(\mu), \tag{3.49}$$

which means that $\psi_1(R)$ has an uniform exponential localisation in $R \geq R(\mu)$. Then $||\tilde{P}(R)\psi_1(R)|| > 1/2$ for sufficiently large R. This and (3.44) finally imply that $||P_{\Gamma_R}([c,d])\psi_1(R)|| > 0$ and the proof is finished.□

Remark 4:

If E_1 is degenerated of order n, (say $n=2$), then we will show that the above argument can be repeated in order to obtain that for $R \geq R(\mu)$ and $k \in \{1,2\}$ one has:

$$\sup_{R\geq R(\mu)} ||\exp(\mu\langle\cdot\rangle)\psi_k(R)|| \leq \mathrm{C}(\mu). \tag{3.50}$$

First of all, $\psi_{1,2}(R)$ are normalised linear combinations of $U(R)\psi_{1,2}$, therefore:

$$\sup_{R\geq R_0} ||\chi_0\psi_{1,2}(R)|| \leq \epsilon. \tag{3.51}$$

The second step is to express $\psi_{1,2}(R)$ with the help of $P_R(1-\chi_0)\psi_{1,2}(R)$. The relation between them is given by:

$$\begin{aligned} P_R(1-\chi_0)\psi_{1,2}(R) &= \psi_1(R)\langle\psi_1(R),(1-\chi_0)\psi_{1,2}(R)\rangle + \\ &+ \psi_2(R)\langle\psi_2(R),(1-\chi_0)\psi_{1,2}(R)\rangle. \end{aligned} \tag{3.52}$$

Using (3.51), one can invert the above matrix and (3.50) follows easily.

Then H_R has two eigenvalues $E_1(R)$ and $E_2(R)$ near E_1; looking at (3.41), one realizes that the only thing that changes is:

$$P_R(H_R - z)^{-1} = \frac{|\psi_1(R)\rangle\langle\psi_1(R)|}{E_1(R)-z} + \frac{|\psi_2(R)\rangle\langle\psi_2(R)|}{E_2(R)-z}. \tag{3.53}$$

Using (3.50), one concludes that the spectrum of H_{Γ_R} in that region consists from two mini-bands (of the same type as in i)) localised near $E_1(R)$ and $E_2(R)$.

Acknowledgements.

This paper was conceived during a visit to Erwin Schrodinger Institute in Vienna at the invitation of Professor T. Hoffmann-Ostenhof. The financial support is hereby gratefully acknowledged.

We also thank the referee for valuable suggestions and for drawing our attention toward Daumer's work [5].

References

[1] Briet, P., Combes, J. M., Duclos, P.: Spectral stability under tunneling. Commun. Math. Phys. **126** (1989), 133.

[2] Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: Schrödinger operators with application to quantum mechanics and global geometry, Berlin, Heidelberg, New York, Springer-Verlag, 1987.

[3] Cornean, H.D., Nenciu, G.: On eigenfunction decay of two dimensional magnetic Schrödinger operators. Commun. Math. Phys. **192** (1998), 671.

[4] Cornean, H.D., Nenciu, G.: Two dimensional magnetic Schrödinger operators: width of mini-bands in the tight binding approximation. Preprint E.S.I. n^o 608, Vienna, (1998). To appear in Ann. I.H.P. (Physique Théorique)

[5] Daumer, F.: Equation de Schrödinger avec champ éléctrique périodique et champ magnétique constant dans l'approximation du tight-binding. Comm. in P.D.E. **18** n^o 5, 6 (1993), 1021.

[6] Helffer, B.: On spectral theory for Schrödinger operators with magnetic potentials. Advanced Studies in Pure Mathematics. **23** (1994), 113.

[7] Hempel, R., Herbst I.: Strong magnetic fieds, Dirichlet boundaries and spectral gaps. Preprint E.S.I. n^o 74, Vienna, (1994).

[8] Helffer, B., Sjöstrand, J.: Multiple wells in the semi-classical limit **I**. Comm. in P.D.E. **9** (1984), 337.

[9] Helffer, B., Sjöstrand, J.: Puits multiples en limite semi-classique **II**. Interaction moléculaire. Symétries. Perturbations. Ann. I.H.P. **42** (1985), 127.

[10] Helffer, B., Sjöstrand, J.: Multiple wells in the semi-classical limit **III**. Math. Nachr. **124** (1985), 263.

[11] Helffer, B., Sjöstrand, J.: Effet tunnel pour l'équation de Schrödinger avec champ magnétique. Ann. l'école de Pise. **14** (1987), 625.

[12] Helffer, B., Sjöstrand, J.: Equation de Schrödinger avec champ magnétique et équation de Harper, Lect. notes in Physics, n^o 34, Springer Verlag, 118-197, 1988

[13] Iwatsuka, A.: The essential spectrum of two dimensional Schrödinger operators with perturbed magnetic fields. J. Math. Kyoto Univ. **23** (1983), 475.

[14] Nakamura, S.: Band spectrum for Schrödinger operators with strong periodic magnetic fields. Operator Theory: Advances and Applications **78**, (1995), 261.

[15] Nenciu, G.: Dynamics of band electrons in electric and magnetic fields: rigorous justification of the effective Hamiltonians. Rev. Mod. Phys. **63** (1991), 91.

[16] Simon, B.: Schrödinger semigroups. Bull. Amer. Math. Soc. (N. S.) **7** (1982), 447.

[17] Thaller, B.: The Dirac equation, Berlin, Heidelberg, New York, Springer-Verlag, 1992