Herbert Spohn

Dynamics of Charged Particles and Their Radiation Field

Prof. Dr. Herbert Spohn
Zentrum Mathematik und Physik Department
Technische Universität München
D-80290 München, Germany
spohn@mathematik.tu-muenchen.de

Preface

By intention, my project has two parts. The first one covers the classical electron theory. It is essentially self–contained and will be presented in the following chapters. 75 years after the discovery of quantum mechanics, to discuss only the classical version of the theory looks somewhat obsolete, in particular since many phenomena, like the stability of atoms, the existence of spectral lines and their life time, the binding of atoms, and many others, are described only by the quantized theory. Thus it is a necessity to discuss the quantized version of the classical models studied here. This is *not* quantum electrodynamics. It is the quantum theory of electrons, stable nuclei, and photons with no pair production allowed. Well said, but the quantum part turns out to be a difficult task. There is a lot of material with the mathematical physics side in flux and very active at present. Thus it remains to be seen whether the quantum part will be ever finished. In the meantime I invite the reader to comments, criticisms, and improvements on the classical part.

In thank my collaborators, Sasha Komech and Markus Kunze, for their constant help and insistence. I am very grateful to Joel Lebowitz. He initiated my interest in tracer particle problems and I always wanted to apply these ideas to electrons coupled to the Maxwell field. I am indebted to F. Rohrlich for discussions and important hints on the literature. I acknowledge instructive discussions with A. Arai, V. Bach, D. Bambusi, G. Bauer, J. Bellissard, F. Bonetto, D. Dürr, J.-P. Eckmann, L. Erdös, J. Fröhlich, L. Galgani, G. Gallavotti, S. Goldstein, M. Hirokawa, F. Hiroshima, V. Jakšić, M. Kiessling, E. Lieb, M. Loss, D. Noja, C.-A. Pillet, M. Rauscher, L. Rey–Bellet, H.T. Yau.

I am most thankful to Mrs. L. Vásquez de Rosswag for her cheerful and tireless effort in typing.

Herbert Spohn München, August 1, 1999

Contents

0 Introduction

"Classical Electron Theory" is an attempt of building a dynamical theory of electromagnetic fields coupled to well concentrated lumps of charges, like electrons and nuclei. As the name indicates, it is a theory of classical fields in interaction with classical particles. Two limiting cases have been confirmed experimentally and are the subject of any course on electrodynamics: either the currents, i.e. the motion of the charges, or the electromagnetic fields are given with the task to predict the behavior of the remaining piece. In stark contrast, there are only few examples where one is still in the classical domain and the full power of a coupled theory is required. In fact, I only know one example, namely the motion of a single electron in a Penning trap which will be discussed in Section 7. Thus the classical electron theory is a mostly theoretical enterprise, but with this taste it has intrigued physicists for almost a century.

The motives have varied with time. The founding fathers, like Abraham and Lorentz, tried to develop a dynamic theory for the then newly discovered electron. In particular, depending on the model, they predicted its energy–momentum relation, cf. Section 3. This enterprise came to a stand still with the advent of the theory of special relativity. Based on totally disjoint arguments it required for any massive particle the relativistically covariant link between energy and momentum. The classical electron theory flourished for a second time in the early days of quantum electrodynamics. The hope was that a refined understanding of the classical theory would give a hint on how to properly quantize and how to correctly handle the infinities. The notion of mass renormalization was taken from the classical theory. But as the proper quantum theory surfaced, the classical considerations had little value. In fact, there is no classical analogue of the renormalization in quantum field theory. There the interaction with the quantized field reduces the quantum fluctuations of the electron. The, yet unproven, construction is to counterbalance by increasing the bare electron fluctuations through letting its bare mass tend to *zero* in such a way that, upon removing the ultraviolet cut–off, finite fluctuations remain which are adjusted to yield the experimental mass of the electron.

At large, the classical electron theory has a poor reputation. One has to fight with infinities, the bare mass of the electron is supposedly negative and tends to $-\infty$ in the limit of a point charge, the theory has instabilities as demonstrated by the runaway solutions, physical solutions suffer from preacceleration in contradiction to a causal description, and more. In contrast, our approach will be rather conservative with no need for such spectacular revisions of the conventional concepts. It is based on two corner stones

- a well–defined dynamical theory of extended charges in interaction with the electromagnetic field,
- a study of the effective dynamics of the charges under the condition that the external potentials vary slowly on the scale given by the size of the charge distribution. This is the *adiabatic limit*.

Of course, the familiar objects reappear in disguise with one important difference. In the adiabatic limit the ratio 'mass induced through the self–interaction with the electromagnetic field' to 'bare mass' remains finite and does not diverge as it is the case in the traditional point charge limit.

Our approach reflects the great progress which has taken place in the theory of dynamical systems. After all, a charge coupled to its radiation field is just one particular case, however with some rather special features. Perhaps the most unusual one is the appearance of a central manifold in the effective dynamics if friction through radiation is included.

A few words on the style and scope of the book are in order. First of all we systematically develop the theory, no review is intended. For a subject with a long history, such an attitude looks questionable. After all, what did the many physicists in that field contribute? To make up, we included one historical chapter, which as very often in physics is the history as viewed from our present understanding. Since there are already excellent historical studies, we hope to be excused. Further we added at the end of each chapter *Notes and References* which are intended as a guide to all the material which has been left out. As prerequisites for reading, only a basic knowledge of Maxwell's theory of electromagnetism and of Hamiltonian mechanics is required.

The reader will notice that we state several theorems and give their proof. In some cases the proof is complete. In other cases we only present the essential idea. The more technical steps can be found in the original literature and are therefore not reproduced. I am convinced that, in particularly in a subject where there is so little control through experience, one needs fixed points in terms of mathematical theorems. Over the years a lot of common knowledge has been accumulated and we made an effort to disentangle hard facts from truth by tradition.

The introduction might give the impression as if all problems are resolved and the classical electron theory is in good shape. This would be too simplistic an attitude. What I hope is to convince the reader to view our dynamical problem from a particular perspective. Once this point has been reached, there are then many loose ends. To mention only two: our comparison between the true and approximate particle dynamics could be sharper and a fully relativistic theory of an extended charge distribution exists only on a formal level. The greatest reward would be if my notes encourage further research.

1 A Charge Coupled to its Electromagnetic Field

We plan to study the dynamics of a well localized charge, like an electron or a proton, when coupled to its own electromagnetic field. The case of several particles is reserved for Chapter 9. In a first attempt, one models the particle as a point charge with some definite mass. If its world line is prescribed, then the fields are determined through the inhomogeneous Maxwell equations. On the other hand, if the electromagnetic fields are given, then the motion of the point charge is governed by the Lorentz force equation. While it then seems obvious how to marry the two equations, so to have a coupled dynamics for the charge and its electromagnetic field, ambiguities and inconsistencies arise due to the infinite electrostatic energy of the Coulomb field of the point charge. Thus one is forced to introduce a slightly smeared charge distribution, i.e. an extended charge model. Mathematically this means that the interaction between particle and field is cut–off or regularized at short distances, which seems to leave a lot of freedom. There are strong constraints however. In particular one has to satisfy local charge conservation, the theory should be of Lagrangian form, and it has to reproduce the limiting cases mentioned above. In addition, as to be expected for any decent physical model, the theory should be well–defined and empirically accurate in its domain of validity. In fact, only two models have been worked out in some detail: (i) the semi–relativistic Abraham model of a rigid charge distribution, and (ii) the Lorentz model of a relativistically covariant extended charge distribution. The aim of this chapter is to introduce both models at some length. On the way we recall some properties of the Maxwell equations for later use.

1.1 The Maxwell equations

We prescribe a charge density $\rho(\boldsymbol{x},t)$ and an associated current, $\boldsymbol{j}(\boldsymbol{x},t)$, which are linked through the conservation law

$$\partial_t\, \rho(\boldsymbol{x},t) + \nabla \cdot \boldsymbol{j}(\boldsymbol{x},t) = 0\,. \tag{1.1}$$

Of course, $\boldsymbol{x} \in \mathbb{R}^3$, the physical space, and $t \in \mathbb{R}$, the time. The Maxwell equations for the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ consist of the two evolution equations

$$\begin{aligned} c^{-1}\partial_t \boldsymbol{B}(\boldsymbol{x},t) &= -\nabla \times \boldsymbol{E}(\boldsymbol{x},t)\,, \\ c^{-1}\partial_t \boldsymbol{E}(\boldsymbol{x},t) &= \nabla \times \boldsymbol{B}(\boldsymbol{x},t) - c^{-1}\boldsymbol{j}(\boldsymbol{x},t) \end{aligned} \tag{1.2}$$

and the two constraints

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x},t) = \rho(\boldsymbol{x},t)\,, \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x},t) = 0\,. \tag{1.3}$$

We use the Heaviside–Lorentz units. The vacuum susceptibilities are $\varepsilon_0 = 1 = \mu_0$, which fixes the unit of charge. c is the speed of light. Mostly we will set $c = 1$ for convenience, thereby linking the units of space and time. If needed, one can easily reintroduce these natural constants in the convential way. At some parts we will do this without notice, so to have the dimensions right and to better keep track of the order of magnitudes.

We solve the Maxwell equations as a Cauchy problem, i.e. by prescribing the fields at time $t = 0$. If the constraints (1.3) are satisfied at $t = 0$, then by the continuity equation (1.1) they are satisfied at all times. Thus the initial data are

$$\boldsymbol{E}(\boldsymbol{x},0)\,,\ \ \boldsymbol{B}(\boldsymbol{x},0) \tag{1.4}$$

together with the constraints

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x},0) = \rho(\boldsymbol{x},0)\,, \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x},0) = 0\,. \tag{1.5}$$

The choice $t = 0$ is merely a convention. In some cases it is preferable to prescribe the fields either in the distant past or the remote future. We will only consider physical situations where the fields decay at infinity and thus have the finite energy

$$\mathcal{E} = \frac{1}{2} \int d^3x \, \big(\boldsymbol{E}(\boldsymbol{x},t)^2 + \boldsymbol{B}(\boldsymbol{x},t)^2\big) < \infty \,. \tag{1.6}$$

In a thermal state at non–zero temperature, one would be forced to consider infinite energy solutions. But this is outside the present scope.

The Maxwell equations (1.2), (1.3) are inhomogeneous wave equations and thus easy to solve. This will be done in Fourier space first, where Fourier transform is denoted by $\widehat{\ }$ and defined through

$$\widehat{f}(\boldsymbol{k}) = (2\pi)^{-n/2} \int d^n x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}) \,. \tag{1.7}$$

Then (1.2) becomes

$$\begin{aligned} \partial_t \widehat{\boldsymbol{B}}(\boldsymbol{k},t) &= -i\boldsymbol{k} \times \widehat{\boldsymbol{E}}(\boldsymbol{k},t) \,, \\ \partial_t \widehat{\boldsymbol{E}}(\boldsymbol{k},t) &= i\boldsymbol{k} \times \widehat{\boldsymbol{B}}(\boldsymbol{k},t) - \widehat{\boldsymbol{j}}(\boldsymbol{k},t) \end{aligned} \tag{1.8}$$

with the constraints

$$i\boldsymbol{k} \cdot \widehat{\boldsymbol{E}}(\boldsymbol{k},t) = \hat{\rho}(\boldsymbol{k},t) \,, \quad i\boldsymbol{k} \cdot \widehat{\boldsymbol{B}}(\boldsymbol{k},t) = 0 \tag{1.9}$$

and the conservation law

$$\partial_t \hat{\rho}(\boldsymbol{k},t) + i\boldsymbol{k} \cdot \widehat{\boldsymbol{j}}(\boldsymbol{k},t) = 0 \,. \tag{1.10}$$

To solve the inhomogeneous equations (1.8), as usual, we rely on the solution of the homogeneous equations,

$$\begin{aligned} \begin{pmatrix} \widehat{\boldsymbol{E}}_{(0)}(\boldsymbol{k},t) \\ \widehat{\boldsymbol{B}}_{(0)}(\boldsymbol{k},t) \end{pmatrix} &= \Big[\big(\cos|\boldsymbol{k}|t + (1 - \cos|\boldsymbol{k}|t) \, \frac{1}{\boldsymbol{k}^2} \, |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \big) \begin{pmatrix} \widehat{\boldsymbol{E}}(\boldsymbol{k},0) \\ \widehat{\boldsymbol{B}}(\boldsymbol{k},0) \end{pmatrix} \\ &\quad + \big(\frac{1}{|\boldsymbol{k}|} \sin|\boldsymbol{k}|t\big) i\boldsymbol{k} \times \begin{pmatrix} \widehat{\boldsymbol{B}}(\boldsymbol{k},0) \\ -\widehat{\boldsymbol{E}}(\boldsymbol{k},0) \end{pmatrix} \Big] \,. \end{aligned} \tag{1.11}$$

We insert (1.11) in the time–integrated version of (1.8). We impose the constraints and, with one partial integration and using the conservation law, arrive at

$$
\begin{aligned}
\widehat{\boldsymbol{E}}(\boldsymbol{k},t) &= (\cos|\boldsymbol{k}|t)\widehat{\boldsymbol{E}}(\boldsymbol{k},0) + (|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|t)i\boldsymbol{k}\times\widehat{\boldsymbol{B}}(\boldsymbol{k},0) \\
&\quad + \int_0^t ds\big(-(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|(t-s))i\boldsymbol{k}\hat{\rho}(\boldsymbol{k},s) - (\cos|\boldsymbol{k}|(t-s))\widehat{\boldsymbol{j}}(\boldsymbol{k},s)\big) \\
&= \widehat{\boldsymbol{E}}_{(0)}(\boldsymbol{k},t) + \widehat{\boldsymbol{E}}_{\mathrm{ret}}(\boldsymbol{k},t)\,, \qquad (1.12) \\
\widehat{\boldsymbol{B}}(\boldsymbol{k},t) &= (\cos|\boldsymbol{k}|t)\widehat{\boldsymbol{B}}(\boldsymbol{k},0) - (|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|t)i\boldsymbol{k}\times\widehat{\boldsymbol{E}}(\boldsymbol{k},0) \\
&\quad + \int_0^t ds(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|(t-s))i\boldsymbol{k}\times\widehat{\boldsymbol{j}}(\boldsymbol{k},s) \\
&= \widehat{\boldsymbol{B}}_{(0)}(\boldsymbol{k},t) + \widehat{\boldsymbol{B}}_{\mathrm{ret}}(\boldsymbol{k},t)\,. \qquad (1.13)
\end{aligned}
$$

The first terms are the initial fields propagated up to time t, while the second terms are the retarded fields. If we would solve the Maxwell equations into the past, then the retarded fields are to be replaced by the advanced fields.

Let us introduce the fundamental propagator, $G_t(\boldsymbol{x})$, of the wave equation which satisfies

$$\partial_t^2 G - \Delta G = \delta(\boldsymbol{x})\delta(t) \qquad (1.14)$$

and is defined as the Fourier transform of $(2\pi)^{-3/2}\,|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|t$. This means $G_t(\boldsymbol{x}) = (2\pi)^{-1}\,\delta(|\boldsymbol{x}|^2 - t^2)$ and for $t \geq 0$

$$G_t(\boldsymbol{x}) = \frac{1}{4\pi t}\delta(|\boldsymbol{x}| - t)\,. \qquad (1.15)$$

Then in physical space the solution (1.12), (1.13) of the inhomogeneous

Maxwell equations reads as

$$
\begin{aligned}
\boldsymbol{E}(t) &= \partial_t G_t * \boldsymbol{E}(0) + \nabla \times G_t * \boldsymbol{B}(0) - \int_0^t ds \big(\nabla G_{t-s} * \rho(s) + \partial_t G_{t-s} * \boldsymbol{j}(s)\big) \\
&= \boldsymbol{E}_{(0)}(t) + \boldsymbol{E}_{\mathrm{ret}}(t)\,, \qquad (1.16) \\
\boldsymbol{B}(t) &= \partial_t G_t * \boldsymbol{B}(0) - \nabla \times G_t * \boldsymbol{E}(0) + \int_0^t ds \nabla \times G_{t-s} * \boldsymbol{j}(s) \\
&= \boldsymbol{B}_{(0)}(t) + \boldsymbol{B}_{\mathrm{ret}}(t)\,. \qquad (1.17)
\end{aligned}
$$

Here $*$ denotes convolution, i.e. $f_1 * f_2(\boldsymbol{x}) = \int d^n y f_1(\boldsymbol{x} - \boldsymbol{y}) f_2(\boldsymbol{y})$.

The expressions (1.16), (1.17) remain meaningful when $\rho, \boldsymbol{j}$ are generated by the motion of a single point charge. Let us denote by $\boldsymbol{q}(t)$ the position and $\boldsymbol{v}(t) = \dot{\boldsymbol{q}}(t)$ the velocity of the particle carrying the charge e. Then

$$\rho(\boldsymbol{x}, t) = e\delta(\boldsymbol{x} - \boldsymbol{q}(t))\ ,\ \boldsymbol{j}(\boldsymbol{x}, t) = e\delta(\boldsymbol{x} - \boldsymbol{q}(t))\boldsymbol{v}(t)\,. \qquad (1.18)$$

It is assumed that the particle is relativistic and therefore $|\boldsymbol{v}(t)| < 1$. Inserting in (1.16), (1.17) one arrives at the Liénard–Wiechert fields. Since their derivation is handled in any textbook, we do not repeat the computation here and discuss only the result. We imagine that the world line, $t \mapsto \boldsymbol{q}(t)$, of the particle is given for all times. We prescribe the initial data for the fields at time $t = t_0$ and take the limit $t_0 \to -\infty$ in (1.16), (1.17). Then, at a fixed space–time point $(t, \boldsymbol{x})$, the contribution from the initial fields vanishes and the retarded fields become the Liénard–Wiechert fields. To define them we introduce the retarded time t_{ret}, depending on $t, \boldsymbol{x}$, as the unique solution of

$$t_{\mathrm{ret}} = t - |\boldsymbol{x} - \boldsymbol{q}(t_{\mathrm{ret}})|\,. \qquad (1.19)$$

t_{ret} is the unique time when the world line crosses the backwards light cone with apex at $(t, \boldsymbol{x})$. We also introduce the unit vector

$$\widehat{\boldsymbol{n}} = \frac{\boldsymbol{x} - \boldsymbol{q}(t_{\mathrm{ret}})}{|\boldsymbol{x} - \boldsymbol{q}(t_{\mathrm{ret}})|}\,. \qquad (1.20)$$

Then the electric field generated by a moving point charge is given by

$$\boldsymbol{E}(\boldsymbol{x},t) = (e/4\pi)\Big[\frac{(1-\boldsymbol{v}^2)(\widehat{\boldsymbol{n}}-\boldsymbol{v})}{(1-\boldsymbol{v}\cdot\widehat{\boldsymbol{n}})^3|\boldsymbol{x}-\boldsymbol{q}|^2} + \frac{\widehat{\boldsymbol{n}}\times[(\widehat{\boldsymbol{n}}-\boldsymbol{v})\times\dot{\boldsymbol{v}}]}{(1-\boldsymbol{v}\cdot\widehat{\boldsymbol{n}})^3|\boldsymbol{x}-\boldsymbol{q}|}\Big]\Big|_{t=t_{\mathrm{ret}}} \tag{1.21}$$

and the corresponding magnetic field is

$$\boldsymbol{B}(\boldsymbol{x},t) = \widehat{\boldsymbol{n}}\times\boldsymbol{E}(\boldsymbol{x},t)\,. \tag{1.22}$$

(1.21) and (1.22) are less explicit than the notation suggests, since t_{ret} depends through (1.19) on the reference point $(t,\boldsymbol{x})$ and the particle trajectory. The first contribution in (1.21) is proportional to $|\boldsymbol{x}-\boldsymbol{q}|^{-2}$ and independent of the acceleration. This is the near field which in a certain sense remains attached to the particle. The second contribution is proportional to $|\boldsymbol{x}-\boldsymbol{q}|^{-1}$ and to the acceleration. This is the far field, which carries the information on the radiation field escaping to infinity. If $\boldsymbol{q}(t)$ is smooth in t, then the Liénard–Wiechert fields are smooth functions except at $\boldsymbol{x}=\boldsymbol{q}(t)$, where they diverge as $|\boldsymbol{x}-\boldsymbol{q}(t)|^{-2}$. The corresponding potentials have a Coulomb singularity at the world line of the particle.

1.2 The Lorentz force equation

We take now the point of view that the electromagnetic fields $\boldsymbol{E},\boldsymbol{B}$ are given. The motion of a charged particle, charge e, position $\boldsymbol{q}(t)$, velocity $\boldsymbol{v}(t)$, is then governed by the Lorentz force equation

$$\frac{d}{dt}(m_0\gamma\boldsymbol{v}(t)) = e\big(\boldsymbol{E}(\boldsymbol{q}(t),t) + c^{-1}\boldsymbol{v}(t)\times\boldsymbol{B}(\boldsymbol{q}(t),t)\big)\,, \tag{1.23}$$

which as ordinary differential equation has to be supplemented by the initial conditions $\boldsymbol{q}(0),\boldsymbol{v}(0)$. Here $\gamma(\boldsymbol{v}) = 1/\sqrt{1-(\boldsymbol{v}/c)^2}$. The particle is relativistic with rest mass m_0 as measured experimentally. Once the particle is coupled to the Maxwell field, m_0 will attain a new meaning.

The $(\boldsymbol{E},\boldsymbol{B})$ fields in (1.23) are not completely arbitrary. They have to be solutions of the Maxwell equations with sources $(\rho,\boldsymbol{j})$. So to speak, we have separated all charges into a single charged particle whose motion is to be determined through (1.23) and a rest whose motion is taken to be known.

(1.23) is of Hamiltonian form. To see this we introduce the vector potentials ϕ, $\boldsymbol{A}$ such that

$$\boldsymbol{E}(\boldsymbol{x},t) = -\nabla\phi(\boldsymbol{x},t) - c^{-1}\partial_t \boldsymbol{A}(\boldsymbol{x},t)\,, \quad \boldsymbol{B}(\boldsymbol{x},t) = \nabla \times \boldsymbol{A}(\boldsymbol{x},t)\,. \tag{1.24}$$

Then the Lagrangian associated to (1.23) is

$$L(\boldsymbol{q},\dot{\boldsymbol{q}}) = -m_0 c^2 (1 - c^{-2}\dot{\boldsymbol{q}}^2)^{1/2} - e(\phi(\boldsymbol{q},t) - c^{-1}\dot{\boldsymbol{q}} \cdot \boldsymbol{A}(\boldsymbol{q},t))\,. \tag{1.25}$$

Introducing the canonical momentum

$$\boldsymbol{p} = m_0 \gamma(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \frac{e}{c}\boldsymbol{A}(\boldsymbol{q},t) \tag{1.26}$$

the Hamiltonian is given by

$$H(\boldsymbol{q},\boldsymbol{p},t) = ((c\,\boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{q},t))^2 + m_0^2 c^4)^{1/2} + e\phi(\boldsymbol{q},t)\,. \tag{1.27}$$

In particular, if the fields are time–independent, then the energy

$$\mathcal{E}(\boldsymbol{q},\boldsymbol{v}) = m_0\gamma(\boldsymbol{v}) + e\phi(\boldsymbol{q}) \tag{1.28}$$

is conserved along the solution trajectories of (1.23).

It should be remarked that in general the solutions to the Lorentz force equation will have a complicated structure even for time–independent fields. This has been amply demonstrated for particular cases. Depending on the external fields the motion ranges from regular to fully chaotic with a mixed phase space as a rule.

1.3 The coupled Maxwell–Lorentz equations

While, up to a minute fraction, sufficient for all of electrodynamics, from a more fundamental point of view it is unsatisfactory that in the Maxwell and Lorentz force equations so to speak "one half" must be prescribed and one would hope to have a coupled system of equations for the time evolution of the charged particles together with their electromagnetic field. If we restrict ourselves to a single particle, it is obvious how to proceed. From (1.2), (1.3) we have

$$\begin{aligned}\partial_t \boldsymbol{B}(\boldsymbol{x},t) &= -\nabla\times\boldsymbol{E}(\boldsymbol{x},t)\,,\\ \partial_t \boldsymbol{E}(\boldsymbol{x},t) &= \nabla\times\boldsymbol{B}(\boldsymbol{x},t) - e\delta(\boldsymbol{x}-\boldsymbol{q}(t))\boldsymbol{v}(t)\end{aligned}\tag{1.29}$$

with the constraints

$$\nabla\cdot\boldsymbol{E}(\boldsymbol{x},t) = e\delta(\boldsymbol{x}-\boldsymbol{q}(t))\,,\quad \nabla\cdot\boldsymbol{B}(\boldsymbol{x},t)=0\tag{1.30}$$

and from (1.23) we have

$$\frac{d}{dt}(m_0\gamma\boldsymbol{v}(t)) = e\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}(t)) + \boldsymbol{E}(\boldsymbol{q}(t),t) + \boldsymbol{v}(t)\times(\boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q}(t)) + \boldsymbol{B}(\boldsymbol{q}(t),t))\big)\,.\tag{1.31}$$

We added in explicitely external electromagnetic fields $\boldsymbol{E}_{\mathrm{ex}}$, $\boldsymbol{B}_{ex}$, which will play a prominent role later on. They are derived from potentials as

$$\boldsymbol{E}_{\mathrm{ex}} = -\nabla\phi_{\mathrm{ex}}\,,\quad \boldsymbol{B}_{\mathrm{ex}} = \nabla\times\boldsymbol{A}_{\mathrm{ex}}\,.\tag{1.32}$$

We assume the potentials to be time–independent for simplicity, although considerable parts of the theory to be developed will work also for time–dependent fields. As before, (1.29), (1.31) are to be solved as initial value problem. Thus $\boldsymbol{E}(\boldsymbol{x},0)$, $\boldsymbol{B}(\boldsymbol{x},0)$, $\boldsymbol{q}(0)$, $\boldsymbol{v}(0)$ are given. Note that the continuity equation is satisfied by fiat.

(1.29), (1.31) are the stationary points of a Lagrangian action, which strengthens our trust in these equations, since every microscopic classical evolution equation seems to be of that form. We introduce the electromagnetic potentials as in (1.24), (1.32). Then the action for (1.29), (1.31) reads

$$\begin{aligned}S([\boldsymbol{q},\phi,\boldsymbol{A}]) &= \int dt\big[-m_0(1-\dot{\boldsymbol{q}}(t)^2)^{1/2} - e\big(\phi_{\mathrm{ex}}(\boldsymbol{q}(t)) + \phi(\boldsymbol{q}(t),t)\\ &\qquad -\dot{\boldsymbol{q}}(t)\cdot(\boldsymbol{A}_{\mathrm{ex}}(\boldsymbol{q}(t)) + \boldsymbol{A}(\boldsymbol{q}(t),t))\big)\big] \qquad (1.33)\\ &\quad +\frac{1}{2}\int dt\int d^3x\big[(\nabla\phi(\boldsymbol{x},t)+\partial_t\boldsymbol{A}(\boldsymbol{x},t))^2 - (\nabla\times\boldsymbol{A}(\boldsymbol{x},t))^2\big]\,.\end{aligned}$$

The only difficulty is that (1.29), (1.31) make no sense mathematically. As explained, the solution to the Maxwell equations is singular at $\boldsymbol{x} = \boldsymbol{q}(t)$ and in the Lorentz force equation we are asked to evaluate the fields exactly at that point. One might be tempted to put the blame on the mathematics which refuses to handle equations as singular as (1.29), (1.31). However before such a drastic conclusion is drawn, the physics should be properly understood. The point charge carries along with it a potential which at short distances diverges as the Coulomb potential, cf. (1.21), and which therefore has the electrostatic energy

$$\frac{1}{2} \int_{\{|\boldsymbol{x}-\boldsymbol{q}(t)|\leq R\}} d^3x \boldsymbol{E}(\boldsymbol{x}, t)^2 \simeq \int_0^R dr r^2 (r^{-2})^2 = \int_0^R dr r^{-2} = \infty\,. \tag{1.34}$$

Taking literally such an object would have an infinite mass and cannot respond to external forces. It would maintain its velocity forever, which is not what is observed.

Thus we are forced to regularize the Maxwell–Lorentz equation (1.29), (1.31) at short distances.

To carry out such a program there are two in part complementary points of view. The first one, which we will *not* adopt here, regards the regularization as a mathematical device with the sole purpose to make sense out of a singular mathematical object through a suitable limiting procedure. To mention only one prominent mathematical physics example. The free scalar field, $\phi(\boldsymbol{x})$, in Euclidean quantum field theory in 1+1 dimensions fluctuates so wildly at short distances that an interaction as $\int d^2x V(\phi(\boldsymbol{x}))$ with $V(\phi) = \phi^2 + \lambda\phi^4$ cannot be properly defined. One way, not necessarily optimal, to regularize the theory is to introduce a spatial lattice with lattice spacing a. Such a lattice field theory is well defined in finite volume. One then carries out the limit spacing $a \to 0$ at a simultaneous readjustment of the interaction potential, $V(\phi) = V_a(\phi)$ and obtains a Euclidean invariant, interacting quantum field theory. Ideally the limit theory should be independent of the regularization procedure. E.g. we could start with the free scalar field in the continuum and also regularize $\phi(\boldsymbol{x})$ as $\phi * g(\boldsymbol{x})$ with g a test function concentrated at 0. Then the regularized interaction is $\int d^2x V(\phi * g(\boldsymbol{x}))$ and in the limit $g(\boldsymbol{y}) \mapsto \delta(\boldsymbol{y})$ a quantum field theory should be obtained identical to the one of the lattice regularization.

In the second point of view one argues that there is a physical cut–off coming from a more refined theory, which is then modelled in a phenomenological way. While this is a standard procedure, it is worthwhile to illustrate it in a concrete example. We consider many ($\cong 10^{23}$) He^4 atoms in a container of adjustable size and we want to compute their free energy according to the rules of statistical mechanics. The more refined theory is here non–relativistic quantum mechanics which treats the electrons and nuclei as point particles carrying a spin $\frac{1}{2}$, resp. spin 0. As far as we can tell, this model approximately covers a temperature range $T = 0\ {}^\circ K$ to $T = 10^5\ {}^\circ K$, way beyond dissociation, and a density range $\rho = 0$ to $\rho = \textit{close packing}$. Beyond that relativistic effects must be taken into account. However there is a more limited range where we can get away with a model of classical point particles interacting through an effective potential of Lennard–Jones type. Once we specify this pair potential, classical statistical mechanics makes well–defined predictions at *any* T, ρ. There is no limitation in theory. Only outside a certain range the classical model looses the correspondence to the real world. Already from the way we describe the physical cut–off, there is a fair amount of vagueness. How much error do we allow in the free energy? What about more refined properties like density correlations? An effective potential can be defined quantum mechanically, but it is temperature–dependent and never strictly pair. Despite all these imprecisions and shortcomings, the equilibrium theory of fluids relies heavily on the availability of a classical model.

In the same spirit we modify the Maxwell–Lorentz equations by introducing an extended charge distribution as a phenomenological model for the left out quantum electrodynamics. The charge distribution is stabilized by strong interactions which are outside the realm of electromagnetic forces. If the particle is an electron then on the classical level it looks like an extended charged object with a size roughly of the order of its Compton wave length, i.e. 4×10^{-11} cm. We impose the condition that the extended charge distribution has to be adjusted such that, over the range where classical electrodynamics is applicable, the coupled Maxwell–Lorentz equations correctly reproduce the empirical observations.

Such general clauses seem to leave a lot of freedom. However, charge conservation and the derivability of the equations of motion from an action severely limits the possibilities. In fact, essentially only two models of an extended charge distribution have been investigated so far.
(i) *The semi–relativistic Abraham model of a rigid charge distribution.* The δ–function in (1.29), (1.30), and (1.31) is replaced by a smooth function $\rho(\boldsymbol{x})$ which is radial and vanishes for $|\boldsymbol{x}| \geq R_\rho$. In Fourier space it means that couplings between the particle and Fourier modes with $|\boldsymbol{k}| \geq R_\rho^{-1}$ are smoothly suppressed. The Abraham model will be studied in considerable detail . While defined for all velocities $|\boldsymbol{v}(t)| < 1$, it becomes empirically incorrect at velocities close to one. Despite this draw–back we hope that the Abraham model serves as a blue–print for more realistic cut–off prescriptions.
(ii) *The relativistic Lorentz model of a deformable charge distribution.* The Abraham model violates relativistic invariance, since the choice of ρ singles out a specific reference frame. More in accord with relativity is to require that the charge distribution is radial in its momentary rest frame. We will discuss details in Section 1.5 and only remark already now that relativistic invariance forces the equations of motion to be nonlocal in time.

We emphasize that for extended charge models the diameter R_ρ of the charge distribution defines a length (and upon dividing by c also a time) scale, relative to which the approximate validity of effective theories, like the Lorentz–Dirac equation, can be addressed quantitatively. In fact, apart from the external forces, R_ρ is the only length scale available.

1.4 The Abraham model

Following Abraham, we model the charge as a spherically symmetric, rigid body to which the charge is permanently attached. To be specific the charge distribution, ρ, with total charge e is assumed to be smooth, radial, and supported in a ball of radius R_ρ, i.e.

Condition (C):

$$\rho \in C_0^\infty(\mathbb{R}^3)\,,\ \rho(\boldsymbol{x}) = \rho_r(|\boldsymbol{x}|)\,,\ \rho(\boldsymbol{x}) = 0 \text{ for } |\boldsymbol{x}| \geq R_\rho\,, \int d^3x \rho(\boldsymbol{x}) = e\,. \tag{1.35}$$

To be definite, we require $e > 0$. With the obvious sign change, also $e < 0$ is then covered.

Equivalenty, we could introduce the form factor f such that

$$\rho(\boldsymbol{x}) = e\, f(\boldsymbol{x}^2)\ , \qquad \int d^3x\, f(\boldsymbol{x}^2) = 1\,. \tag{1.36}$$

The equations of motion for the Abraham model read then

$$\begin{aligned} \partial_t \boldsymbol{B}(\boldsymbol{x},t) &= -\nabla \times \boldsymbol{E}(\boldsymbol{x},t)\,, \\ \partial_t \boldsymbol{E}(\boldsymbol{x},t) &= \nabla \times \boldsymbol{B}(\boldsymbol{x},t) - \rho(\boldsymbol{x}-\boldsymbol{q}(t))\boldsymbol{v}(t)\,, \end{aligned} \tag{1.37}$$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x},t) = \rho(\boldsymbol{x}-\boldsymbol{q}(t))\,, \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x},t) = 0\,, \tag{1.38}$$

$$\begin{aligned} \frac{d}{dt}(m_{\mathrm{b}}\gamma\boldsymbol{v}(t)) = \int d^3x\ & \rho(\boldsymbol{x}-\boldsymbol{q}(t))\big[\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{x}) + \boldsymbol{v}(t) \times \boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{x}) \\ & + \boldsymbol{E}(\boldsymbol{x},t) + \boldsymbol{v}(t) \times \boldsymbol{B}(\boldsymbol{x},t)\big]\,. \end{aligned} \tag{1.39}$$

In contrast to the Lorentz force equation, for the Abraham model we denote the mechanical mass of the particle by m_{b} to emphasize that this bare mass will differ from the observed mass of the compound object "particle plus surrounding Coulomb field". For the external potentials $\phi_{\mathrm{ex}}, \boldsymbol{A}_{\mathrm{ex}}$ we have considerable freedom. We require them to be smooth and locally bounded, including their derivates to avoid too strong local oscillations. No condition on the increase at infinity is needed, since $|\boldsymbol{v}(t)| \le 1$. However, it is convenient to have the energy (1.42) uniformly bounded from below. To keep matters simple we make the unnecessarily strong assumptions

Condition (P):

$$\phi_{\rm ex} \in C^\infty(\mathbb{R}^3)\ ,\ \ \boldsymbol{A}_{\rm ex} \in C^\infty(\mathbb{R}^3,\mathbb{R}^3),\ \ \phi_{\rm ex} \ge \overline{\phi} > -\infty\,. \tag{1.40}$$

There exists a constant C such that $|\nabla^k \phi_{\rm ex}| \le C$, $|\nabla^k \boldsymbol{A}_{\rm ex}| \le C$ componentwise for $k = 1, 2, 3$.

Physically, the Abraham model is not quite consistent. Besides the center of mass $\boldsymbol{q}(t)$ and its velocity $\boldsymbol{v}(t)$ a rigid body has also rotational degrees of freedom. As we will see in Chapter 8 even if initially non–rotating the external fields and the self–interaction necessarily induce a rotation. The translational and rotational degrees of freedom are coupled through the field. This makes the model considerably more intricate and it is an advisable strategy to understand the simplified version first.

The Abraham model is derived from the Lagrangian

$$\begin{aligned} L &= -m_{\rm b}(1-\dot{\boldsymbol{q}}^2)^{1/2} - (\phi_{\rm ex} + \phi - \dot{\boldsymbol{q}}\cdot \boldsymbol{A}_{\rm ex} - \dot{\boldsymbol{q}}\cdot\boldsymbol{A}) * \rho(\boldsymbol{q}) \\ &\quad + \frac{1}{2}\int d^3x \big[(\nabla\phi + \partial_t \boldsymbol{A})^2 - (\nabla\times\boldsymbol{A})^2\big]\,. \end{aligned} \tag{1.41}$$

Correspondingly the energy

$$\mathcal{E}(\boldsymbol{E},\boldsymbol{B},\boldsymbol{q},\boldsymbol{v}) = m_{\rm b}\gamma(\boldsymbol{v}) + e\phi_{\rm ex} * \rho\,(\boldsymbol{q}) + \frac{1}{2}\int d^3x \big(\boldsymbol{E}(\boldsymbol{x})^2 + \boldsymbol{B}(\boldsymbol{x})^2\big) \tag{1.42}$$

is conserved.

As for any dynamical system, we first have to construct a suitable phase space. The dynamical variables are $(\boldsymbol{E}(\boldsymbol{x}), \boldsymbol{B}(\boldsymbol{x}), \boldsymbol{q}, \boldsymbol{v}) = Y$ which is called a state of the system. We have $\boldsymbol{q} \in \mathbb{R}^3, \boldsymbol{v} \in \mathbb{V} = \{\boldsymbol{v} : |\boldsymbol{v}| < 1\}$. In addition the field energy (1.42) should be bounded. Thus it is natural to introduce the (real) Hilbert space

$$L^2 = L^2(\mathbb{R}^3,\mathbb{R}^3) \tag{1.43}$$

with norm $||\boldsymbol{E}|| = (\int d^3x|\boldsymbol{E}(\boldsymbol{x})|^2)^{1/2}$ and to define $\mathcal{L}$ as the set of states satisfying

$$\|Y\|_{\mathcal{L}} = \|\boldsymbol{E}\| + \|\boldsymbol{B}\| + |\boldsymbol{q}| + |\gamma(\boldsymbol{v})\boldsymbol{v}| < \infty\,. \tag{1.44}$$

In particular the field energy, $\frac{1}{2}(\|\boldsymbol{E}\|^2 + \|\boldsymbol{B}\|^2)$, is bounded. We equip $\mathcal{L}$ with the metric

$$d(Y_1, Y_2) = \|\boldsymbol{E}_1 - \boldsymbol{E}_2\| + ||\boldsymbol{B}_1 - \boldsymbol{B}_2|| + |\boldsymbol{q}_1 - \boldsymbol{q}_2| + |\gamma(\boldsymbol{v}_1)\boldsymbol{v}_1 - \gamma(\boldsymbol{v}_2)\boldsymbol{v}_2|\,. \tag{1.45}$$

In addition, one has to satisfy the constraints (1.38). Thus the phase space, $\mathcal{M}$, for the Abraham model is the nonlinear submanifold of $\mathcal{L}$ defined through

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x}) = \rho(\boldsymbol{x} - \boldsymbol{q})\,, \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x}) = 0\,. \tag{1.46}$$

$\mathcal{M}$ inherits its metric from $\mathcal{L}$.

At several occasions we will need that the system forgets its initial field data. For this purpose it is helpful to have a little bit of smoothness and some decay at infinity. Formally we introduce the "good" subset $\mathcal{M}^\sigma \subset \mathcal{M}$, $0 \le \sigma \le 1$, which consists of fields such that componentwise and outside some ball of radius R_0, $|\boldsymbol{x}| \ge R_0$, we have

$$|\boldsymbol{E}(\boldsymbol{x})| + |\boldsymbol{B}(\boldsymbol{x})| + |\boldsymbol{x}|(|\nabla \boldsymbol{E}(\boldsymbol{x})| + |\nabla \boldsymbol{B}(\boldsymbol{x})|) \le C\,|\boldsymbol{x}|^{-1-\sigma}\,. \tag{1.47}$$

The Liénard–Wiechert fields (1.21), (1.22) are in $\mathcal{M}^1$ and $\mathcal{M}^1$ is dense in $\mathcal{M}$. However $\mathcal{M}^\sigma = \emptyset$ for $\sigma > 1$, since $\int d^3x \rho(\boldsymbol{x}) = e \neq 0$.

The evolution equations (1.37) to (1.39) are of the general form

$$\frac{d}{dt} Y(t) = F(Y(t)) \tag{1.48}$$

with $Y(0) = Y^0 \in \mathcal{M}$. We have to turn to the question of the existence and uniqueness of solutions of the Abraham model (1.48).

Theorem 1.1 *Let conditions* (C) *and* (P) *hold and let* $Y^0 = (\boldsymbol{E}^0(\boldsymbol{x}), \boldsymbol{B}^0(\boldsymbol{x}), \boldsymbol{q}^0, \boldsymbol{v}^0) \in \mathcal{M}$. *Then the integrated version of Equation (1.48),*

$$Y(t) = Y^0 + \int_0^t ds\, F(Y(s))\,, \tag{1.49}$$

has a unique solution $Y(t) = (\boldsymbol{E}(\boldsymbol{x}, t), \boldsymbol{B}(\boldsymbol{x}, t), \boldsymbol{q}(t), \boldsymbol{v}(t)) \in \mathcal{M}$, *which is continuous in* t *and satisfies* $Y(0) = Y^0$. *Along the solution trajectory*

$$\mathcal{E}(Y(t)) = \mathcal{E}(Y^0)\,, \tag{1.50}$$

i.e. the energy is conserved.

For short times existence and uniqueness follows through the contraction mapping principle with constants depending only on the initial energy. For smooth initial data energy conservation is verified directly and by continuity it extends to all finite energy data. Thus we can construct iteratively the solution for all times.

We first summarize some properties of the inhomogeneous Maxwell equations. They follow directly from the Fourier and convolution representations (1.12), (1.13), resp. (1.16), (1.17).

Lemma 1.2 *In the inhomogeneous Maxwell equations (1.2), (1.3) let* $\rho(\boldsymbol{x}, t) = \rho(\boldsymbol{x} - \boldsymbol{q}(t))$, $\boldsymbol{j}(\boldsymbol{x}, t) = \rho(\boldsymbol{x} - \boldsymbol{q}(t))\boldsymbol{v}(t)$ *with* $t \mapsto \boldsymbol{q}(t), \boldsymbol{v}(t)$ *continuous. Then (1.2), (1.3) has a unique solution in* $C(\mathbb{R}, L^2 \oplus L^2)$. *The solution map* $(\boldsymbol{E}^0, \boldsymbol{B}^0) \mapsto (\boldsymbol{E}(t), \boldsymbol{B}(t))$ *depends continuously on* $\boldsymbol{q}(t), \boldsymbol{v}(t)$.

Proof of Theorem 1.1: Let us fix some $b > 0$ and choose initial data such that $\mathcal{E}(Y^0) \leq b$.
(i) There exists a unique solution $Y(t) \in C([0, \varepsilon], \mathcal{M})$ for $\varepsilon = \varepsilon(b)$ sufficiently small.

We write (1.42) in the form

$$\frac{d}{dt}(m_{\mathrm{b}}\gamma\, \boldsymbol{v}(t)) = \boldsymbol{F}_{\mathrm{ex}}(t) + \boldsymbol{F}_{(0)}(t) + \boldsymbol{F}_{\mathrm{self}}(t) \tag{1.51}$$

by inserting $\boldsymbol{E}(\boldsymbol{x},t), \boldsymbol{B}(\boldsymbol{x},t)$ from the Maxwell equations according to (1.16), (1.17). Let

$$\begin{aligned} W_t(\boldsymbol{x}) &= \int d^3k |\hat{\rho}(\boldsymbol{k})|^2 \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{1}{|\boldsymbol{k}|} \sin|\boldsymbol{k}|t \\ &= (2\pi)^3 \int d^3y \int d^3y' \rho(\boldsymbol{y})\rho(\boldsymbol{y}') \frac{1}{4\pi t} \, \delta(|\boldsymbol{y}+\boldsymbol{x}-\boldsymbol{y}'| - t) \, . \end{aligned} \tag{1.52}$$

Then

$$\boldsymbol{F}_{\mathrm{e}x}(t) = \boldsymbol{E}_{\mathrm{e}x} * \rho(t) + \boldsymbol{v}(t) \times \boldsymbol{B}_{\mathrm{e}x} * \rho(t) \, , \tag{1.53}$$

$$\begin{aligned} \boldsymbol{F}_{(0)}(t) &= \int d^3x \, \rho(\boldsymbol{x}-\boldsymbol{q}(t)) \big[\partial_t G_t * \boldsymbol{E}^0(\boldsymbol{x}) + (\nabla \times G_t) * \boldsymbol{B}^0(\boldsymbol{x}) \\ &\quad + \boldsymbol{v}(t) \times \partial_t G_t * \boldsymbol{B}^0(\boldsymbol{x}) - \boldsymbol{v}(t) \times (\nabla \times G_t * \boldsymbol{E}^0(\boldsymbol{x}))\big] \, , \end{aligned} \tag{1.54}$$

$$\begin{aligned} \boldsymbol{F}_{\mathrm{self}}(t) &= \int_0^t ds \big[-\nabla W_{t-s}(\boldsymbol{q}(t)-\boldsymbol{q}(s)) - \boldsymbol{v}(s)\partial_t W_{t-s}(\boldsymbol{q}(t)-\boldsymbol{q}(s)) \\ &\quad + \boldsymbol{v}(t) \times (\nabla \times \boldsymbol{v}(s) W_{t-s}(\boldsymbol{q}(t)-\boldsymbol{q}(s)))\big] \, . \end{aligned} \tag{1.55}$$

We now integrate both sides of (1.51) over the time interval $[0,t]$. The resulting expression is regarded as a map from the trajectory $t \mapsto \boldsymbol{q}(t), \boldsymbol{v}(t), 0 \le t \le \delta$, to the trajectory $t \mapsto \bar{\boldsymbol{q}}(t), \bar{\boldsymbol{v}}(t)$ and is defined by

$$\bar{\boldsymbol{q}}(t) = \boldsymbol{q}^0 + \int_0^t ds \, \boldsymbol{v}(s) \, , \tag{1.56}$$

$$m_{\mathrm{b}}\gamma(\bar{\boldsymbol{v}}(t))\bar{\boldsymbol{v}}(t) = m_{\mathrm{b}}\gamma(\boldsymbol{v}^0)\boldsymbol{v}^0 + \int_0^t ds \big(\boldsymbol{F}_{\mathrm{ex}}(s) + \boldsymbol{F}_{(0)}(s) + \boldsymbol{F}_{\mathrm{self}}(s)\big) \, ,$$

where $\boldsymbol{F}_{\mathrm{ex}}(s)$, $\boldsymbol{F}_{(0)}(s)$, $\boldsymbol{F}_{\mathrm{self}}(s)$ are functionals of $\boldsymbol{q}(\cdot), \boldsymbol{v}(\cdot)$ according to (1.53) to (1.55). Since $\rho, W, \phi_{\mathrm{ex}}, \boldsymbol{A}_{\mathrm{ex}}$ are smooth, this map is a contraction in $C([0,t], \mathbb{R}^3 \times \mathbb{V})$, i.e.

$$\begin{aligned} &\sup_{0\le s\le t} \big(|\bar{\boldsymbol{q}}_2(s) - \bar{\boldsymbol{q}}_1(s)| + |\bar{\boldsymbol{v}}_2(s) - \bar{\boldsymbol{v}}_1(s)|\big) \\ &\le c(t,b) \sup_{0\le s\le t} \big(|\boldsymbol{q}_1(s) - \boldsymbol{q}_2(s)| + |\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)|\big) \, , \end{aligned} \tag{1.57}$$

with a constant $c(t,b)$ depending on b and $c(t,b) < 1$ for sufficiently small t. Such a map has a unique fixed point which is the desired solution $\boldsymbol{q}(t), \boldsymbol{v}(t)$. By the Maxwell equations also $\boldsymbol{E}(\boldsymbol{x},t)$,$\boldsymbol{B}(\boldsymbol{x},t)$ is uniquely determined.
(ii) The solution map $Y^0 \mapsto Y(t)$ is continuous in $\mathcal{M}$.

This follows from Lemma 1.2 and the continuity of $\boldsymbol{q}(t), \boldsymbol{v}(t)$ in dependence on the initial data.
(iii) The energy is conserved.

We choose smooth initial fields such that $\boldsymbol{E}, \boldsymbol{B} \in C^\infty(\mathbb{R}^3)$ and

$$|\nabla^\alpha \boldsymbol{E}(\boldsymbol{x})| + |\nabla^\alpha \boldsymbol{B}(\boldsymbol{x})| \leq C(1+|\boldsymbol{x}|)^{-(2+|\alpha|)}\,. \tag{1.58}$$

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi–index with $\alpha_i = 0, 1, 2, \ldots$. This set is dense in $\mathcal{M}$. By the convolution representation (1.16), (1.17) of the solution to the Maxwell equations we have $\boldsymbol{E}(\boldsymbol{x},t), \boldsymbol{B}(\boldsymbol{x},t) \in C^1([0,\delta] \times \mathbb{R}^3)$ and $|\boldsymbol{E}(\boldsymbol{x},t)| + |\boldsymbol{B}(\boldsymbol{x},t)| \leq C(1+|\boldsymbol{x}|)^{-2}$. Also $\boldsymbol{v}(t) \in C^1([0,\delta])$. Thus we are allowed to differentiate,

$$\begin{aligned}\frac{d}{dt}\,\mathcal{E}(Y(t)) &= \gamma^3 \boldsymbol{v}\cdot\dot{\boldsymbol{v}} + \boldsymbol{v}\cdot\nabla\phi_{\mathrm{ex}} * \rho(\boldsymbol{q}) + \int d^3x (\boldsymbol{E}\cdot\partial_t\boldsymbol{E} + \boldsymbol{B}\cdot\partial_t\boldsymbol{B}) \\ &= \int d^3x \big(\boldsymbol{E}\cdot(\nabla\times\boldsymbol{B}) - \boldsymbol{B}\cdot(\nabla\times\boldsymbol{E})\big) = 0\,,\end{aligned} \tag{1.59}$$

since the fields decay and hence the surface terms vanish. Thus $\mathcal{E}(Y(t)) = \mathcal{E}(Y^0)$ for $0 \leq t \leq \delta$. By continuity this equality extends to all of $\mathcal{M}$.
(iv) The global solution exists.

¿From (iii) we know that $\mathcal{E}(Y(\delta)) = \mathcal{E}(Y^0) \leq b$. Thus we can repeat the previous argument for $\delta \leq t \leq 2\delta$, etc.. Backwards in time we still have the solution (1.16), (1.17) of the Maxwell equations, only the retarded fields have to be replaced by the advanced ones. Thereby we obtain the solution for all times. □

Theorem 1.1 ensures the existence and uniqueness of solutions for the Abraham model. For initial data $Y^0 \in \mathcal{M}$ the solution trajectory $t \to Y(t)$ lies in the phase space $\mathcal{M}$, is continuous in t, and its energy is conserved. We have thus established the basis for further investigations on the dynamics of the Abraham model.

1.5 Appendix: Long time asymptotics

For dynamical systems one of the first qualitative question is to understand whether there are general patterns governing the long time behavior. For the Abraham model the long time asymptotics is dominated through the loss of energy radiated to infinity, which is proportional to $\dot{\boldsymbol{v}}(t)^2$ according to Larmor's formula. Since the energy is bounded from below, we expect that

$$\lim_{t\to\infty} \dot{\boldsymbol{v}}(t) = 0 \tag{1.60}$$

under rather general initial conditions. In fact, one would also expect that

$$\lim_{t\to\infty} \boldsymbol{v}(t) = \boldsymbol{v}_\infty \in \mathbb{V}, \tag{1.61}$$

where $\boldsymbol{v}_\infty = 0$ for bounded motion and $\boldsymbol{v}_\infty \neq 0$ for a scattering solution.

In this section we will prove (1.60) under the extra hypothesis

Wiener Condition (W)*:*

$$\hat{\rho}(\boldsymbol{k}) > 0\,. \tag{1.62}$$

The proof follows rather closely the physical intuition and leads to an equation of convolution type which has a definite long time limit only under (W). (W) means that all modes of the charge distribution couple to electromagnetic field. According to (1.60) the Abraham model does not admit then any periodic solution. Since ρ has compact support, in general, $\hat{\rho}(\boldsymbol{k})$ may vanish for a discrete set of shell radii $|\boldsymbol{k}|$. At present, it remains as an open problem, whether periodic solution become then possible.

Let us consider a ball of radius R centered at the origin. At time t the field energy in this ball and the mechanical energy of the charge is given by

$$\mathcal{E}_R(t) = \mathcal{E}(0) - \frac{1}{2} \int\limits_{\{|\boldsymbol{x}|\geq R\}} d^3x \big(\boldsymbol{E}(\boldsymbol{x},t)^2 + \boldsymbol{B}(\boldsymbol{x},t)^2\big) \tag{1.63}$$

for R sufficiently large, using the conservation of total energy. $\mathcal{E}_R$ changes in time as

$$\frac{d}{dt}\mathcal{E}_R(t) = -R^2 \int d^2\omega\; \boldsymbol{\omega}\cdot[\boldsymbol{E}(R\boldsymbol{\omega},t)\times \boldsymbol{B}(R\boldsymbol{\omega},t)]\,, \tag{1.64}$$

where $\boldsymbol{\omega}$ is a vector on the unit sphere, $d^2\omega$ the surface measure normalized to 4π, and $\boldsymbol{E}\times\boldsymbol{B}$ the Poynting vector for the flux in energy at the surface of the ball under consideration. Since the total energy is bounded from below, we conclude that

$$\mathcal{E}_R(R) - \mathcal{E}_R(R+t) = -\int_R^{R+t} ds\, \frac{d}{ds}\,\mathcal{E}_R(s) \le C \tag{1.65}$$

with the constant $C = \mathcal{E}(0) - \overline{\phi}$ independent of R, t.

In (1.65) we first take the limit $R \to \infty$, which yields the energy radiated to infinity during the time interval $[0, t]$ through a large sphere centered at the origin. Subsequently we take the limit $t \to \infty$ to obtain the total radiated energy. To state the result we define

$$\begin{aligned} \boldsymbol{E}_\infty(\boldsymbol{\omega}, t) \quad &= \quad -\frac{1}{4\pi}\int d^3y\, \rho(\boldsymbol{y} - \boldsymbol{q}(t + \boldsymbol{\omega}\cdot\boldsymbol{y})) \\ &\qquad \big[(1 - \boldsymbol{\omega}\cdot\boldsymbol{v})^{-1}\dot{\boldsymbol{v}} + (1 - \boldsymbol{\omega}\cdot\boldsymbol{v})^{-2}(\boldsymbol{\omega}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v} - \boldsymbol{\omega})\big]|_{t+\boldsymbol{\omega}\cdot\boldsymbol{y}} \end{aligned} \tag{1.66}$$

which is a functional of the actual trajectory of the particle. Whatever its motion we must have

$$\int_0^\infty dt \int d^2\omega\, |\boldsymbol{E}_\infty(\boldsymbol{\omega}, t)|^2 \le C < \infty\,. \tag{1.67}$$

Note that the integrand in (1.67) is proportional to $\dot{\boldsymbol{v}}(t)^2$, which therefore has to decay to zero for large t.

To establish (1.67) is somewhat tedious with pieces of the argument explained in Sections 4.3 and 6.4, 6.5. One imagines that the trajectory $t \mapsto \boldsymbol{q}(t)$ is given and solves the inhomogeneous Maxwell equations according to (1.16), (1.17). If the initial fields are in $\mathcal{M}^\sigma$, $\frac{1}{2} < \sigma \le 1$, then they decay and make no contribution to (1.65) in the limit $R \to \infty$, cf. our treatment of the initial time slip in Section 4.3. Next one has to study the asymptotics of the retarded fields, which is carried out in Sections 6.4, 6.5. There ε is fixed and for our purpose we may set $\varepsilon = 1$. In addition in (6.49) the sphere is centered at $\boldsymbol{q}^\varepsilon(t)$, rather than at the origin. This means we can use (6.52, (6.53) with $\boldsymbol{q}^\varepsilon(t)$ replaced by 0 for our case.

The real task is to extract from (1.67) that the acceleration vanishes for long times.

Theorem 1.3 *For the Abraham model satisfying* (C), (P), *and the Wiener condition* (W) *let the initial data* $Y(0) = (\boldsymbol{E}^0, \boldsymbol{B}^0, \boldsymbol{q}^0, \boldsymbol{v}^0) \in \mathcal{M}^\sigma$ *with* $\frac{1}{2} < \sigma \leq 1$. *Then*

$$\lim_{t\to\infty} \dot{\boldsymbol{v}}(t) = 0\,. \tag{1.68}$$

Proof: By energy conservation $|\boldsymbol{v}(t)| \leq \overline{v} < 1$. Inserting in (1.39) and using (P) we conclude that $|\dot{\boldsymbol{v}}(t)| \leq C$. Differentiating (1.39) and using again (P) also $|\ddot{\boldsymbol{v}}(t)| \leq C$ uniformly in t. Therefore $\boldsymbol{E}_\infty(\boldsymbol{\omega}, t)$ is Lipschitz continuous jointly in $\boldsymbol{\omega}, t$. Since the energy dissipation (1.67) is bounded, this implies

$$\lim_{t\to\infty} \boldsymbol{E}_\infty(\boldsymbol{\omega}, t) = 0 \tag{1.69}$$

uniformly in $\boldsymbol{\omega}$.

We analyze the structure of the integrand in (1.66). In the retarded argument only $y_\| = \boldsymbol{\omega}\cdot\boldsymbol{y}$ appears. Therefore the integration over $\boldsymbol{y}_\perp = \boldsymbol{y} - y_\|\boldsymbol{\omega}$ can be carried out and we are left with a one–dimensional integral of convolution type. We set $\rho_a(x_3) = \int dx_1 dx_2\, \rho(\boldsymbol{x})$. Then

$$\begin{aligned}\boldsymbol{E}_\infty(\boldsymbol{\omega}, t) &= \frac{1}{4\pi}\int dy_\|\, \rho_a(y_\| - q_\|(t + y_\|)) \\ &\qquad \big[(1-\boldsymbol{\omega}\cdot\boldsymbol{v})^{-2}\boldsymbol{\omega}\times((\boldsymbol{\omega}-\boldsymbol{v})\times\dot{\boldsymbol{v}})\big]\big|_{t+y_\|} \\ &= \frac{1}{4\pi}\int ds \rho_a(t-(s-q_\|(s)))\big[(1-\boldsymbol{\omega}\cdot\boldsymbol{v})^{-2}\boldsymbol{\omega}\times((\boldsymbol{\omega}-\boldsymbol{v})\times\dot{\boldsymbol{v}})\big]\big|_s\,.\end{aligned} \tag{1.70}$$

Since $|\dot{q}_\|(s)| < 1$, we can substitute $\theta = s - q_\|(s)$ and obtain the convolution representation

$$\boldsymbol{E}_\infty(\boldsymbol{\omega}, t) = \int d\theta\, \rho_a(t-\theta)\boldsymbol{g}_{\boldsymbol{\omega}}(\theta) = \rho_a * \boldsymbol{g}_{\boldsymbol{\omega}}(t)\,, \tag{1.71}$$

where

$$\boldsymbol{g}_{\boldsymbol{\omega}}(\theta) = \frac{1}{4\pi}\left[(1-\boldsymbol{\omega}\cdot\boldsymbol{v})^{-2}\boldsymbol{\omega}\times((\boldsymbol{\omega}-\boldsymbol{v})\times\dot{\boldsymbol{v}})\right]|_{s(\theta)}\,. \tag{1.72}$$

¿From (1.69) we know that $\lim\limits_{t\to\infty}\rho_a * \boldsymbol{g}_{\boldsymbol{\omega}}(t) = 0$. If $\hat{\rho}(\boldsymbol{k}_0) = 0$ for some $\boldsymbol{k}_0$, and hence the Wiener condition would not be satisfied, then the convolution integral admits a periodic solution and no further progress seems to be possible. However with (W) and the smoothness of $\boldsymbol{g}_{\boldsymbol{\omega}}(\theta)$ already established, Pitt's extension to the Tauberian theorem of Wiener ensures us that

$$\lim_{\theta\to\infty}\boldsymbol{g}_{\boldsymbol{\omega}}(\theta) = 0\,, \tag{1.73}$$

which, since $\theta(t)\to\infty$ as $t\to\infty$, implies

$$\lim_{t\to\infty}\boldsymbol{\omega}\times((\boldsymbol{\omega}-\boldsymbol{v}(t))\times\dot{\boldsymbol{v}}(t)) = 0 \tag{1.74}$$

for every $\boldsymbol{\omega}$ in the unit sphere. Replacing $\boldsymbol{\omega}$ by $-\boldsymbol{\omega}$ and summing both expressions yields $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\dot{\boldsymbol{v}}(t))\to 0$ as $t\to\infty$. Since this is true for every $\boldsymbol{\omega}$, the claim follows. □

Next we study the comoving electromagnetic fields for large times. As used already, under our assumptions the initial fields decay as $t\to\infty$. Thus we only have to consider the true retarded fields $\boldsymbol{E}_{\mathrm{ret}}(\boldsymbol{x}+\boldsymbol{q}(t),t)$, $\boldsymbol{B}_{\mathrm{ret}}(\boldsymbol{x}+\boldsymbol{q}(t),t)$ centered at the position of the particle. Since $\dot{\boldsymbol{v}}(t)\to 0$, these fields become almost stationary. In Section 3.1 we show that there is a unique comoving field with velocity $\boldsymbol{v}$. These charge soliton fields are denoted by $\boldsymbol{E}_{\boldsymbol{v}}, \boldsymbol{B}_{\boldsymbol{v}}$ when centered at the origin, compare with (3.21), (3.22). Thus the true retarded fields $\boldsymbol{E}_{\mathrm{ret}}(\boldsymbol{x}+\boldsymbol{q}(t),t), \boldsymbol{B}_{\mathrm{ret}}(\boldsymbol{x}+\boldsymbol{q}(t),t)$ have to compared with the soliton fields $\boldsymbol{E}_{\boldsymbol{v}(t)}, \boldsymbol{B}_{\boldsymbol{v}(t)}$. For this purpose we use the representations (3.21), (3.22) for the charge soliton and (1.16), (1.17) for the retarded fields.

We insert the explicit form (1.15) of the propagator. This yields

$$\begin{aligned}
\boldsymbol{E}_{\boldsymbol{v}}(\boldsymbol{x}) &= \int d^3y\,(4\pi|\boldsymbol{x}-\boldsymbol{y}|)^{-1}\big(|\boldsymbol{x}-\boldsymbol{y}|^{-1}\rho(\boldsymbol{y}-\boldsymbol{v}|\boldsymbol{x}-\boldsymbol{y}|)\widehat{\boldsymbol{n}} \\
&\qquad +\boldsymbol{v}\cdot\nabla\rho(\boldsymbol{y}-\boldsymbol{v}|\boldsymbol{x}-\boldsymbol{y}|)(\boldsymbol{v}-\widehat{\boldsymbol{n}})\big)\,, \qquad (1.75)\\
\boldsymbol{B}_{\boldsymbol{v}}(\boldsymbol{x}) &= \int d^3y(4\pi|\boldsymbol{x}-\boldsymbol{y}|)^{-1}\widehat{\boldsymbol{n}}\times\big(-|\boldsymbol{x}-\boldsymbol{y}|^{-1}\rho(\boldsymbol{y}-|\boldsymbol{x}-\boldsymbol{y}|\boldsymbol{v})\boldsymbol{v} \\
&\qquad +\boldsymbol{v}\cdot\nabla\rho(\boldsymbol{y}-|\boldsymbol{x}-\boldsymbol{y}|\boldsymbol{v})\boldsymbol{v}\big)\,, \qquad (1.76)
\end{aligned}$$

where $\widehat{\boldsymbol{n}} = (\boldsymbol{x}-\boldsymbol{y})/|\boldsymbol{x}-\boldsymbol{y}|$. Similarly for the retarded fields

$$\begin{aligned}
\boldsymbol{E}_{\mathrm{ret}}(\boldsymbol{x}+\boldsymbol{q}(t),t) &= \int d^3y\,\big(4\pi|\boldsymbol{x}-\boldsymbol{y}|\big)^{-1}\big(|\boldsymbol{x}-\boldsymbol{y}|^{-1}\rho(\boldsymbol{y}+\boldsymbol{q}(t)-\boldsymbol{q}(\tau))\widehat{\boldsymbol{n}} \\
&\qquad +\boldsymbol{v}(\tau)\cdot\nabla\rho(\boldsymbol{y}+\boldsymbol{q}(t)-\boldsymbol{q}(\tau))(\boldsymbol{v}(\tau)-\widehat{\boldsymbol{n}}) \\
&\qquad -\rho(\boldsymbol{y}+\boldsymbol{q}(t)-\boldsymbol{q}(\tau))\dot{\boldsymbol{v}}(\tau)\big)\,, \qquad (1.77)\\
\boldsymbol{B}_{\mathrm{ret}}(\boldsymbol{x}+\boldsymbol{q}(t),t) &= \int d^3y(4\pi|\boldsymbol{x}-\boldsymbol{y}|)^{-1}\widehat{\boldsymbol{n}}\times\big(-|\boldsymbol{x}-\boldsymbol{y}|^{-1}\rho(\boldsymbol{y}+\boldsymbol{q}(t)-\boldsymbol{q}(\tau)) \\
&\qquad \boldsymbol{v}(\tau)+\boldsymbol{v}(\tau)\cdot\nabla\rho(\boldsymbol{y}+\boldsymbol{q}(t)-\boldsymbol{q}(\tau))\boldsymbol{v}(\tau) \\
&\qquad -\rho(\boldsymbol{y}+\boldsymbol{q}(t)-\boldsymbol{q}(\tau))\dot{\boldsymbol{v}}(\tau)\big)\,, \qquad (1.78)
\end{aligned}$$

where $\tau = t-|\boldsymbol{x}-\boldsymbol{y}|$ and $t \geq t_\rho = 2R_\rho/(1-\overline{v})$.

We compare the fields locally and use that $\lim_{t\to\infty}\dot{\boldsymbol{v}}(t)=0$. Then, for any fixed $R>0$,

$$\begin{aligned}
\lim_{t\to\infty}\int_{\{|\boldsymbol{x}|\leq R\}} d^3x\Big(&\big(\boldsymbol{E}(\boldsymbol{x}+\boldsymbol{q}(t),t)-\boldsymbol{E}_{\boldsymbol{v}(t)}(\boldsymbol{x})\big)^2 \\
&+\big(\boldsymbol{B}(\boldsymbol{x}+\boldsymbol{q}(t),t)-B_{v(t)}(\boldsymbol{x})\big)^2\Big) = 0\,. \qquad (1.79)
\end{aligned}$$

Armed with this information we can sketch qualitatively the long time behavior of the Abraham model. We distinguish bounded and scattering trajectories. If ϕ_{ex} and $\boldsymbol{A}_{\mathrm{ex}}$ decay sufficiently fast for large $|\boldsymbol{x}|$ and if the particle escapes into the essentially force free region, then $\lim_{t\to\infty}\boldsymbol{v}(t)=\boldsymbol{v}_\infty\neq 0$ and the fields are well approximated by the soliton fields, compare with (1.79). In particular, if there are no external fields the charge travels with some definite velocity in the long time limit. This point is discussed in more detail by a different method in Section 4.3.

If the motion is bounded,

$$|\boldsymbol{q}(t)| \leq \overline{q} \tag{1.80}$$

for all t, then $\dot{\boldsymbol{v}}(t) \to 0$ for $t \to \infty$ implies that

$$\lim_{t\to\infty} \boldsymbol{v}(t) = 0\,. \tag{1.81}$$

Inserting in the Lorentz force equation (1.39) and using that the fields become soliton–like, one infers that

$$\lim_{t\to\infty} \nabla\phi_{\mathrm{ex}} * \rho(\boldsymbol{q}(t)) = 0\,, \tag{1.82}$$

i.e. $\boldsymbol{q}(t)$ approaches the set of critical points $\mathcal{A} = \{\boldsymbol{q},\ \nabla\phi_{\mathrm{ex}} * \rho(\boldsymbol{q}) = 0\}$.

(1.81), (1.82) still leave a lot of freedom. Generically, one expects that $\boldsymbol{q}(t)$ approaches a definite limit. This is indeed the case if $\mathcal{A}$ happens to be a discrete set. By continuity of solutions in t, $\boldsymbol{q}(t)$ has to converge then to some definite $\boldsymbol{q}^* \in \mathcal{A}$. In particular if $\phi_{\mathrm{ex}} * \rho$ is strictly convex, the charge will come to rest at the minimum of $\phi_{\mathrm{ex}} * \rho$. If $\mathcal{A}$ is not discrete, the long time behavior depends on be the specific situation. E.g. let $\boldsymbol{A}_{\mathrm{ex}} = 0$ and ϕ_{ex} be strictly convex outside a ball of radius one and let $\phi_{\mathrm{ex}} = 0$ inside this ball. Each time the particle is reflected by the confining potential it looses in energy. Thus (1.81) holds, but $\boldsymbol{q}(t)$ has no limit as $t \to \infty$. A more realistic example is a constant magnetic field and $\phi_{\mathrm{ex}} = 0$. Then $\mathcal{A} = \mathbb{R}^3$, but using the sharper estimates of Section 4.3 it can be shown that the particle spirals inwards to come to rest at its center of gyration.

So far (1.80) was an assumption and it would be nice to have some sufficient criteria. It seems that the only one available is conservation of energy. Bounded motion requires then sufficiently deep local minima of ϕ_{ex}. Thus we see that bounded energy dissipation yields reasonable results for electrostatic forces but is rather weak for motion in magnetic fields, except for the general fact that the acceleration has to vanish for large t.

In general, bounded and unbounded motion coexist. The prime example is a charge bound by an infinitely heavy nucleus, which is modelled by an attractive, locally smoothened Coulomb potential. If the initial field supplies sufficient energy, then the "atom" becomes ionized and the charge will travel freely in the long time limit. Conversely a charge may loose energy through radiation and become trapped by the external potential.

1.6 The Lorentz model

To improve on the semi–relativistic Abraham model, with Lorentz it is natural to assume that the charge distribution is rigid in its own rest frame. The actual construction of such a charge distribution requires some effort. For obvious reasons we will switch now to a relativistic notation.

We fix a laboratory frame, $\mathcal{S}$, with coordinates $(t, \boldsymbol{x}) = x^\mu, \mu = 0, 1, 2, 3$, and invariant length $t^2 - \boldsymbol{x}^2 = t^2 - x_i x_i = x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu$, where we adopt the standard summation convention and use latin indices for three–vectors, greek indices for four–vectors. In $\mathcal{S}$ we prescribe the world line of a charged particle $t \mapsto z_i(t), i = 1, 2, 3$, with velocity $v_i(t) = \frac{d}{dt} z_i(t)$, $\boldsymbol{v}(t)^2 < 1$. We will also parametrize the world line through its eigentime as $s \mapsto z^\mu(s)$, $\mu = 0, 1, 2, 3$.

To construct the extended charge distribution we consider some time $t = t_0$ when the particle has velocity $v_i = v_i(t_0)$. The rest frame for the particle at time t_0 is denoted by $\mathcal{S}'$ with coordinates x'^μ. It is connected to $\mathcal{S}$ by the Lorentz transformation

$$x'^\mu = \Lambda(\boldsymbol{v})^\mu{}_\nu x^\nu \, . \tag{1.83}$$

We require the origins of $\mathcal{S}$ and $\mathcal{S}'$ to coincide and their spatial axes to be parallel to each other, i.e. $\Lambda(\boldsymbol{v})$ is a Lorentz boost and reads explicitly

$$\begin{aligned} x'_i &= x_i + (\gamma^2/(1+\gamma)) v_i v_k x_k - \gamma v_i t \, , \\ t' &= \gamma(t - v_k x_k) \, . \end{aligned} \tag{1.84}$$

The Lorentz model assumes that in $\mathcal{S}'$ the four–current j'^μ is given by

$$j'^\mu = e\big(f((x'_j - z'_j(t'))(x'_j - z'_j(t'))), 0\big)|_{t'=t'_0} \tag{1.85}$$

with the relativistic form factor f, where $ef(\boldsymbol{x}^2) = \rho_r(|\boldsymbol{x}|)$ as before. In particular, f vanishes for $|\boldsymbol{x}| \geq R_\rho$.

We have to transform j'^μ back to the laboratory frame $\mathcal{S}$. Using (1.84), (1.85) yields

$$j^\mu = e(\gamma, \gamma \boldsymbol{v}) f((x'_j - z_j(t'))(x'_j - z_j(t')))|_{t'=t'_0} \,. \tag{1.86}$$

The condition $t' = t'_0$ means $\gamma(t - v_k x_k) = \gamma(t_0 - v_k z_k(t_0))$, i.e.

$$t_0 = t - v_k(x_k - z_k(t_0)) \,. \tag{1.87}$$

Therefore

$$j^\mu(t, \boldsymbol{x}) = e(\gamma, \gamma \boldsymbol{v}) f((t - t_0)^2 - (x_j - z_j(t_0))(x_j - z_j(t_0))) \tag{1.88}$$

with $t_0 = t_0(t, \boldsymbol{x})$ defined through (1.87).

An example is shown in Figure 1. $j^\mu(t, \boldsymbol{x})$ vanishes outside a tube around the world line. This tube is fibered into cross sections corresponding to $\{t' = t'_0\}$ in the momentary rest frame. Along each cross section the charge is smeared according to the form factor f. From the figure we notice that, when the acceleration becomes too large, the cross sections overlap, which means that for given $(t, \boldsymbol{x})$ Equation (1.87) has several solutions. (1.88) seems to indicate that one should add the contribution from each solution. This however would violate charge conservation and the proper prescription is to reverse the sign of a charge element when it moves backwards in time. Taking into account multiple solutions to (1.87) and their proper sign leads to the four-current

$$j^\mu(x) = \int_{-\infty}^{\infty} ds\, v^\mu [1 - \dot{v}_\nu (x - z)^\nu] f((x - z)^2) \delta(v_\lambda (x - z)^\lambda) \tag{1.89}$$

with $x = (x^0, \ldots, x^3)$. Here $z^\mu = z^\mu(s)$ is parametrized by its eigentime s and $v^\mu(s) = \dot{z}^\mu(s) = \frac{d}{ds} z^\mu(s)$. The current (1.89) satisfies the charge conservation

$$\partial_\mu j^\mu = 0 \,. \tag{1.90}$$

There is an instructive way to rewrite the four–current by using the Thomas precession, which we recall first. The elementary observation is that the Lorentz boosts, cf. (1.84), do not form a subgroup of the Lorentz group. Let us consider the inertial frame $\mathcal{S}'$ with velocity $\boldsymbol{v}'$ relative to $\mathcal{S}$ and the inertial frame $\mathcal{S}''$ with velocity $\boldsymbol{v}''$ relative to $\mathcal{S}'$ and thus velocity $\boldsymbol{w}$, the relativistic sum of $\boldsymbol{v}$ and $\boldsymbol{v}'$, relative to $\mathcal{S}$. If we denote by $\Lambda(\boldsymbol{v})$ the Lorentz boost with velocity $\boldsymbol{v}$, then

$$\Lambda(\boldsymbol{v}')\Lambda(\boldsymbol{v}) = R(\boldsymbol{v}, \boldsymbol{v}')\Lambda(\boldsymbol{w})\,, \tag{1.91}$$

where $R(\boldsymbol{v}, \boldsymbol{v}')$ is a suitable pure spatial rotation.

To apply this observation to a relativistically rigid charge distribution we imagine that it has a body fixed frame. We want to define an inertial frame $\mathcal{K}(s)$ such that relative to this frame the body axes maintain their orientation throughout time. We agree that the time axis of $\mathcal{K}(s)$ is parallel to the four–velocity $v^\mu(s)$ and that the origin of $\mathcal{S}$ and $\mathcal{K}(s)$ coincide. Let $\Lambda(s)$ be the Lorentz transformation from $\mathcal{S}$ to $\mathcal{K}(s)$. To determine it we subdivide the time axis into little intervals of length Δs and require that $\mathcal{K}((m+1)\Delta s)$ is related to $\mathcal{K}(m\Delta s)$ by a Lorentz boost with the properly adjusted velocities, i.e. the space axes of $\mathcal{K}((m+1)\Delta s)$ and $\mathcal{K}(m\Delta s)$ are parallel to each other. Taking the limit $\Delta s \to 0$ one finds that $\Lambda(s)$ is determined through

$$\dot{\Lambda}^\mu{}_\nu = \eta^\mu{}_\lambda \Lambda^\lambda{}_\nu\,, \quad \Lambda^\mu{}_\nu(0) = \Lambda^\mu{}_\nu(\boldsymbol{v}(0))\,, \tag{1.92}$$

where

$$\eta^\mu{}_\nu = v^\mu \dot{v}_\nu - v^\nu \dot{v}_\mu\,. \tag{1.93}$$

Using this definition and noting that $v_\nu \xi^\nu \delta(v_\lambda \xi^\lambda) = 0$ the four–current (1.89) equals

$$j^\mu(x) = \int\limits_{-\infty}^{\infty} ds [v^\mu - \eta^\mu{}_\nu (x-z)^\nu] f((x-z)^2)\, \delta(v_\lambda (x-z)^\lambda)\,, \tag{1.94}$$

in which form it has a transparent physical meaning. The current is the sum of two contributions. There is a translational part proportional to v^μ as expected from the nonrelativistic limit. In addition there is a rotational component due to the Thomas precession. The minus sign is a convention. With our definition a vector time–independent in $\mathcal{K}(s)$ has components $x^\mu(s)$ in $\mathcal{S}$ and they change as

$$\dot{x}^\mu = -\eta^\mu{}_\nu x^\nu \,. \tag{1.95}$$

Before proceeding to the action for the dynamics, we should understand whether the current (1.89) conforms with the naive physical intuition. An instructive example is a uniformly accelerated charge, the so–called hyperbolic motion. We assume that the particle is accelerated along the positive 1–axis starting from rest at the origin. In the orthogonal direction the current traces out a tube of diameter $2R_\rho$ and it suffices to treat the two–dimensional space–time problem. The center, C, of the charge moves along the orbit

$$C = (t, g^{-1}(\sqrt{1+g^2t^2} - 1)),\ t \geq 0\,, \tag{1.96}$$

where $g > 0$ is the acceleration. The curves traced by the right and left ends, C_+ and C_-, are determined from (1.84) and are given in parameter form as

$$C_\pm = \big((1 \pm R_\rho g)t, g^{-1}((1+R_\rho g)\sqrt{1+g^2t^2} - 1)\big),\ t \geq 0\,. \tag{1.97}$$

The equal–time distance between the center and C_+ is $t^{-1}((R_\rho g)^2 + 2R_\rho g)/(2g^2 (1+R_\rho g))$ for large t and thus well localized. However the left end motion depends crucially on the magnitude of $R_\rho g$. If $R_\rho g < 1$, then the distance to the center is $t^{-1}((R_\rho g)^2 - 2R_\rho g)/(2g^2(1-R_\rho g))$ for large t. On the other hand for $R_\rho g > 1$, the left end moves into the past, cf. (1.97), and the current density looks strangely distorted. To have a feeling for the order of magnitudes involved we insert the classical electron radius. Then

$$g > \frac{c^2}{R_\rho} = 10^{31}\,[m/\sec^2]\,, \tag{1.98}$$

which is huge and way beyond the domain of the validity of the theory. Of course, one would hope that for reasonable initial data such accelerations can never be reached. But the mere fact that charge elements may move backwards in time is an extra difficulty.

Armed with the four–current (1.94) we can write down the relativistically covariant action, S, as a functional of the the four–potential A^μ and the particle coordinates z^μ. The action of the Lorentz model has four pieces,

$$\mathsf{S} = \mathsf{S}_0 + \mathsf{S}_\mathrm{f} + \mathsf{S}_\mathrm{int} + \mathsf{S}_\mathrm{ex}\,. \tag{1.99}$$

S_0 is the mechanical action,

$$\mathsf{S}_0 = m_\mathrm{b} \int_{s_1}^{s_2} ds\,. \tag{1.100}$$

S_f is the field action,

$$\mathsf{S}_\mathrm{f} = -\frac{1}{4} \int_\Omega d^4x F^{\mu\nu}(x) F_{\mu\nu}(x)\,, \tag{1.101}$$

where the field tensor $F^{\mu\nu}$ is defined through

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,. \tag{1.102}$$

The interaction is bilinear in the current and the field,

$$\mathsf{S}_\mathrm{int} = \int_\Omega d^4x A^\mu(x) j_\mu(x)\,, \tag{1.103}$$

and correspondingly for the interaction with the external potentials

$$\mathsf{S}_\mathrm{ex} = \int_\Omega d^4x\, A^\mu_\mathrm{ex}(x) j_\mu(x)\,. \tag{1.104}$$

The world line $z^\mu(s)$ is specified for $s_1 \le s \le s_2$. Consequently Ω is the volume in $\mathcal{S}$ bounded by the two hyperplanes $v^\mu(s_1)(x - z(s_1))_\mu = 0$ and $v^\mu(s_2)(x - z(s_2))_\mu = 0$.

The actual dynamical trajectory is a stationary point of the action S at fixed endpoints. The variation with respect to the potentials A^μ leads to the inhomogeneous Maxwell equations

$$\partial_\nu F^{\nu\mu} = j^\mu \,. \tag{1.105}$$

The constraints (1.3) are automatically satisfied, since the field tensor is derived from potentials according to (1.102). Next we vary the world line z^μ. From S_0 we obtain the mechanical acceleration $m_{\mathrm{b}}\dot{v}^\mu$. The variation of $\mathsf{S}_{\mathrm{int}}$ is somewhat lengthy and deferred to Section 1.7. The final evolution equation reads

$$\begin{aligned} m_{\mathrm{b}}\frac{d}{ds}v^\mu(s) \;&=\; e\int d^4\xi\, f(\xi^2)\delta(v_\lambda(s)\xi^\lambda)[F^{\mu\nu}(z(s)+\xi)+F^{\mu\nu}_{\mathrm{ex}}(z(s)+\xi)] \\ &\qquad (v_\nu(s)-\eta_{\nu\sigma}(s)\xi^\sigma)\,. \end{aligned} \tag{1.106}$$

At first glance the equations of motion (1.106) look rather similar to its semi–relativistic sister. It seems natural to specify then $z(0), \dot{z}(0) = v(0)$ and the field tensor on the hyperplane determined by $(x - z(0))^\lambda v_\lambda(0) = 0$. If we assume that $R_\rho|\dot{v}| < 1$, then the future is decoupled from the past and one would hope to have a unique solution. For short times one can presumably copy the proof given for the Abraham model. We are not aware of mathematical results which ensure the existence global in time. If the acceleration becomes large and $R_\rho|\dot{v}| > 1$ for some time span, it is not even clear how to properly specify the initial data. Since such questions remain largely unexplored, the only option is to proceed as if a solution is well defined.

As seen from the laboratory frame $\mathcal{S}$ the rigid charge distribution Thomas precesses. To be physically consistent we have to allow then also for a rotation of the body fixed frame relative to $\mathcal{K}(s)$. For sure, such a rotation will be induced through the back reaction of the electromagnetic field onto the relativistically rigid charge. Let us denote by

$$\omega^\mu{}_\nu(s) \tag{1.107}$$

the angular velocities of the motion of the body fixed frame relative to $\mathcal{K}(s)$. From the point of view of the laboratory system they simply have to be added to $\eta^\mu{}_\nu$, i.e. in the current (1.94) $\eta^\mu{}_\nu$ has to be substituted by $\eta^\mu{}_\nu + \omega^\mu{}_\nu$. The variation of the action is now with respect to A^μ, z^μ, and the Euler angles of the body fixed frame relative to $\mathcal{K}(s)$. This results in the inhomogeneous Maxwell equations, as before, and in a coupled set of translational and rotational equations of motion which are nonlocal in time.

1.7 Appendix: Variation of the action

We carry out the variation $\delta \mathsf{S}_{\text{int}}$ of (1.103). Let us first list the necessary identities. Since $(ds)^2 = dz^\mu\, dz_\mu$ we have

$$\delta(ds) = \frac{dz^\sigma}{ds}\delta dz_\sigma = v^\sigma\, ds\, \frac{d}{ds}\, \delta z_\sigma \tag{1.108}$$

and

$$\delta v^\mu = \delta\frac{dz^\mu}{ds} = \frac{dz^\mu + \delta dz^\mu}{ds + \delta ds} - \frac{dz^\mu}{ds} = \frac{d}{ds}\,\delta z^\mu - v^\mu\, v_\sigma \frac{d}{ds}\,\delta z^\sigma\,. \tag{1.109}$$

Similarly

$$\delta \dot v^\mu = \frac{d}{ds}\,\delta v^\mu - \dot v^\mu v_\sigma\, \frac{d}{ds}\,\delta z^\sigma\,. \tag{1.110}$$

In the variation of $\eta^\mu{}_\nu$ we have to maintain the frames at $z^\mu(s)$ and $z^\mu(s) + \delta z^\mu(s)$ without relative spatial rotation. This leads to

$$\delta \eta^\mu{}_\nu = v^\mu \delta \dot v_\nu - v^\nu\, \delta \dot v_\mu = v^\mu\, \frac{d}{ds}\,\delta v_\nu - v^\nu\, \frac{d}{ds}\,\delta v_\mu - \eta^\mu{}_\nu\, v_\sigma\, \frac{d}{ds}\,\delta z^\sigma\,. \tag{1.111}$$

If in (1.111) we would also vary v^μ, then the constraint of zero spatial rotation is no longer satisfied.

In the following we will assume that (1.87) has a unique solution within the tube traced out by the form factor. For this we need $\dot v^\nu (x-z)_\nu < 1$ inside the tube, equivalently $R_\rho |\dot v| < 1$, i.e. the acceleration has to be sufficiently small on the scale of $1/R_\rho$. Using this condition, $\mathsf{S}_{\rm int}$ can be written more explicitly as

$$\mathsf{S}_{\rm int} = e\int\limits_{s_1}^{s_2} ds \int d^4\xi\, f(\xi^2)\delta(v_\lambda(s)\xi^\lambda)A^\mu(z(s)+\xi)(v_\mu(s)-\eta_{\mu\nu}(s)\xi^\nu)\,. \quad (1.112)$$

Then

$$\begin{aligned}
\delta\mathsf{S}_{\rm int} \quad = \quad & e\int\limits_{s_1}^{s_2} ds\,(\frac{d}{ds}\,\delta z^\sigma)v_\sigma \int d^4\xi\, f(\xi^2)\delta(v_\lambda\xi^\lambda)\,A^\mu(z+\xi)(v_\mu-\eta_{\mu\nu}\,\xi^\nu) \\
& +e\int\limits_{s_1}^{s_2} ds \int d^4\xi\, f(\xi^2)\delta(v_\lambda\xi^\lambda)\,A^\mu(z+\xi) \\
& \qquad [\delta v_\mu - v_\mu\frac{d}{ds}\,\delta v_\nu\xi^\nu + v_\nu\frac{d}{ds}\,\delta v_\mu\xi^\nu + \eta_{\mu\nu}\,\xi^\nu v_\sigma\frac{d}{ds}\,\delta z^\sigma] \\
& +e\int\limits_{s_1}^{s_2} ds \int d^4\xi\, f(\xi^2)\,[\delta(v_\lambda\xi^\lambda)\,\delta z_\sigma\,\partial^\sigma\,A^\mu(z+\xi) \\
& \qquad +\delta'(v_\lambda\xi^\lambda)\delta v_\sigma\,\xi^\sigma\,A^\mu(z+\xi)](v_\mu-\eta_{\mu\nu}\,\xi^\nu)\,. \qquad (1.113)
\end{aligned}$$

We define

$$Q^\mu = e\int d^4\xi\, f(\xi^2)\,\delta(v_\lambda\xi^\lambda)\,A^\nu(z+\xi)v_\nu\,\xi^\mu\,. \qquad (1.114)$$

Then

$$
\begin{aligned}
\delta\mathsf{S}_{\rm int} \;=\;& -\int_{s_1}^{s_2} ds\, \frac{d}{ds}\,(Q^\mu \delta v_\mu) + \int_{s_1}^{s_2} ds\, [\dot{Q}^\mu - v^\mu v_\nu \dot{Q}^\nu]\frac{d}{ds}\,\delta z_\mu \\
&+e\int_{s_1}^{s_2} ds \int d^4\xi\, f(\xi^2)\delta(v_\lambda \xi^\lambda)\, A^\mu(z+\xi)\,\frac{d}{ds}\,\delta z_\mu \\
&+e\int_{s_1}^{s_2} ds \int d^4\xi\, f(\xi^2)\delta'(v_\lambda \xi^\lambda)\, A^\nu(z+\xi)(v_\nu - \eta_{\nu\sigma}\xi^\sigma)(\xi^\mu - \xi^\sigma v_\sigma v^\mu)\,\frac{d}{ds}\,\delta z_\mu \\
&+e\int_{s_1}^{s_2} ds\, \delta z_\mu \int d^4\xi\, f(\xi^2)\delta(v_\lambda \xi^\lambda) F^{\mu\nu}(z+\xi)(v_\nu - \eta_{\nu\sigma}\xi^\sigma) \\
&+e\int_{s_1}^{s_2} ds\, \delta z_\mu \int d^4\xi\, f(\xi^2)\delta(v_\lambda \xi^\lambda)\, \partial^\nu A^\mu(z+\xi)(v_\nu - \eta_{\nu\sigma}\xi^\sigma)\,.
\end{aligned}
\tag{1.115}
$$

The last term can be rewritten as, using $x\delta(x) = 0$,

$$
\begin{aligned}
& \int d^4\xi\, f(\xi^2)\,\delta(v_\lambda \xi^\lambda)\,\partial^\nu\, A^\mu(z+\xi)\,(v_\nu - \eta_{\nu\sigma}\xi^\sigma) \\
=\;& -\int d^4\xi\, A^\mu(z+\xi)\,\partial^\nu\,[f(\xi^2)\,\delta(v_\lambda \xi^\lambda)(v_\nu - v_\nu \dot{v}_\sigma \xi^\sigma)] \\
=\;& \int d^4\xi\, A^\mu(z+\xi) f(\xi^2)\,\delta(v_\lambda \xi^\lambda) v_\nu \dot{v}^\nu \\
& -\int d^4\xi\, f(\xi^2)\,\delta'(v_\lambda \xi^\lambda)\, A^\mu(z+\xi) v^\nu v_\nu (1 - \dot{v}_\sigma \xi^\sigma) \\
=\;& -\int d^4x\, f((x-z)^2)\,\delta'(v_\lambda (x-z)^\lambda)\, A^\mu(x)(1 - \dot{v}_\sigma (x-z)^\sigma) \\
=\;& \frac{d}{ds}\int d^4x\, f((x-z)^2)\,\delta(v_\lambda (x-z)^\lambda) A^\mu(x) \\
=\;& \frac{d}{ds}\int d^4\xi\, f(\xi^2)\,\delta(v_\lambda \xi^\lambda)\, A^\mu(z+\xi)
\end{aligned}
\tag{1.116}
$$

which cancels against the second term of (1.115). Therefore

$$
\begin{aligned}
\delta\mathsf{S}_{\rm int} &= -\int\limits_{s_1}^{s_2} ds\,\frac{d}{ds}\,(Q^\mu\,\delta v_\mu) + \int\limits_{s_1}^{s_2} ds\,\delta z_\mu \Big(-\frac{d}{ds}\,(\dot Q^\mu - v^\mu v_\nu \dot Q^\nu) \\
&\quad -e\frac{d}{ds}\int d^4\xi\, f(\xi^2)\,\delta'(v_\lambda\xi^\lambda)\,A^\nu(z+\xi)(v_\nu - \eta_{\nu\sigma}\xi^\sigma)(\xi^\mu - v^\mu v_\sigma\,\xi^\sigma) \\
&\quad +e\int d^4\xi\, f(\xi^2)\,\delta(v_\lambda\xi^\lambda)\,F^{\mu\nu}(\xi+z)(v_\nu - \eta_{\nu\sigma}\,\xi^\sigma)\Big)\,. \qquad (1.117)
\end{aligned}
$$

Under variation of $\delta z_\mu(s)$ the term in the big brackets has to vanish.

The term containing $F^{\mu\nu}$ is already in its final form. The remaining summands are the time derivative of I, where

$$
I = \dot Q^\mu - v^\mu v_\nu \dot Q^\nu + e\int d^4\xi\, f(\xi^2)\,\delta'(v_\lambda\xi^\lambda)\,A^\nu(z+\xi)(v_\nu - \eta_{\nu\sigma}\,\xi^\sigma)(\xi^\mu - v^\mu v_\sigma\,\xi^\sigma)\,. \qquad (1.118)
$$

We have, using partial integration with respect to ξ, $x\delta(x) = 0$, and $x\delta'(x) = -\delta(x)$,

$$
\begin{aligned}
\dot Q^\mu &= e\int d^4\xi\, f(\xi^2)\delta(v_\lambda\xi^\lambda)(A^\nu(z+\xi)\dot v_\nu\xi^\mu + \partial^\sigma A^\nu(z+\xi)v_\sigma v_\nu\xi^\mu) \\
&\quad +e\int d^4\xi\, f(\xi^2)\delta'(v_\lambda\xi^\lambda)A^\nu(z+\xi)\dot v_\sigma\xi^\sigma v_\nu\xi^\mu \\
&= e\int d^4\xi\, f(\xi^2)\delta(v_\lambda\xi^\lambda)A^\nu(z+\xi)(\dot v_\nu\xi^\mu - v^\mu v_\nu) \\
&\quad -e\int d^4\xi\, f(\xi^2)\delta'(v_\lambda\xi^\lambda)A^\nu(z+\xi)v_\nu\xi^\mu(1-\dot v_\sigma\xi^\sigma)\,, \qquad (1.119) \\
v_\nu\dot Q^\nu &= -e\int d^4\xi\, f(\xi^2)\delta(v_\lambda\xi^\lambda)\,A^\nu(z+\xi)v_\nu\dot v_\sigma\xi^\sigma\,, \qquad (1.120) \\
\dot Q^\mu - v^\mu v_\nu\dot Q^\nu &= e\int d^4\xi\, f(\xi^2)\delta(v_\lambda\xi^\lambda)\,A^\nu(z+\xi)\dot v_\nu\xi^\mu \\
&\quad -e\int d^4\xi\, f(\xi^2)\delta(v_\lambda\xi^\lambda)\,A^\nu(z+\xi)v_\nu v^\mu(1-\dot v_\sigma\xi^\sigma) \\
&\quad -e\int d^4\xi\, f(\xi^2)\delta'(v_\lambda\xi^\lambda)\,A^\nu(z+\xi)v_\nu\,\xi^\mu(1-\dot v_\sigma\xi^\sigma)\,. \qquad (1.121)
\end{aligned}
$$

We turn to the second term in (1.118) and use $x\delta'(x) = -\delta(x)$, $x^2\delta'(x) = 0$,

$$\begin{aligned} & e\int d^4\xi\, f(\xi^2)\,\delta'(v_\lambda\xi^\lambda)\,A^\nu(z+\xi)(v_\nu - v_\nu\dot{v}_\sigma\xi^\sigma + \dot{v}_\nu v_\sigma\xi^\sigma)(\xi^\mu - v^\mu v_\sigma\xi^\sigma) \\ = \;& e\int d^4\xi\, f(\xi^2)\,\delta'(v_\lambda\xi^\lambda)\,A^\nu(z+\xi)(v_\nu\xi^\mu - v_\nu\xi^\mu\dot{v}_\sigma\xi^\sigma) \\ & +e\int d^4\xi\, f(\xi^2)\,\delta(v_\lambda\xi^\lambda)\,A^\nu(z+\xi)(v_\nu v^\mu - v_\nu v^\mu\dot{v}_\sigma\xi^\sigma - \dot{v}_\nu\xi^\mu)\,. \end{aligned} \tag{1.122}$$

Thus $I = 0$.

We conclude that the variation

$$\delta\big(\mathsf{S}_0 + \mathsf{S}_{\mathrm{int}}\big) = 0 \tag{1.123}$$

at fixed endpoints leads to the Lorentz force equation

$$m_{\mathrm{b}}\,\frac{d}{ds}\,v^\mu = e\int d^4\,\xi\, f(\xi^2)\delta(v_\lambda\xi^\lambda)\,F^{\mu\nu}(z+\xi)(v_\nu - \eta_{\nu\sigma}\xi^\sigma)\,. \tag{1.124}$$

Notes and References

ad 1.1, ad 1.2: The material discussed can be found in any textbook. Particularly useful I find Jackson (1999) and Scharf (1994).

ad 1.3: In our history chapter, Section 2, we discuss other approaches which cannot be subsumed under short distance regularization. In the literature the size of a classical electron, r_c, is usually determined through equating the rest mass with the Coulomb energy, $m_e c^2 = e^2/r_c$, which gives $r_c = 3\times 10^{-13}$cm. This is really a lower bound in the sense that an even smaller radius would be in contradiction to the experimentally observed mass of the electron (assuming a positive bare mass, cf. the discussion in Section 4.5). Milonni (1994) argues that due to quantum fluctuations the electron appears to have a classical extent, which is given by its Compton wave length $\lambda_c = r_c/\alpha$ with α the fine structure constant. Renormalization in Euclidean quantum field theory is explained by Glimm, Jaffe (1987). Effective potentials for classical fluids are discussed, e.g., in Huang (1987).

ad 1.4: The Abraham model was very popular at the beginning of this century and was studied by Lorentz (1892,1915), Abraham (1903,1905), Sommerfeld (1904,1905), and Schott (1912), amongst others. Apparently a rotating rigid charge had been considered only much later Frenkel (1926), Bhabha, Corben (1941). The proof on the existence and uniqueness of the dynamics is taken from Komech, Spohn (1999), where a much wider class of external potentials are allowed. A somewhat different technique is used by Bauer, Dürr (1999). They cover also the case of a negative bare mass and discuss the smoothness of solutions in terms of smoothness of initial data.

ad 1.5: The long time asymptotics is treated in Komech, Spohn (1999), where the details of the proof can be found. Pitt's version of the Wiener theorem is proved in Rudin (1977), Theorem 9.7(b). We remark that Theorem 1.3 gives no rate of convergence. Thus to investigate the asymptotics of the velocity and position always requires some extra considerations. Komech, Spohn (1998) study the long time asymptotics for zero external potentials in the case of a scalar wave field. In particular, by these methods one can handle the asymptotics of scattering trajectories. Presumably these results extend to the Maxwell field. They require the Wiener condition. In Section 4.3 we give a different proof without Wiener condition (W) but for a sufficiently small charge. This indicates that (W) is an artifact of the proof. One might wonder, how bounded energy dissipation works in the case of several particles. The result is somewhat disappointing. One concludes only that the center of mass acceleration vanishes in the long time limit. To show that also the relative motions come to rest requires novel techniques.

In the literature, Bohm, Weinstein (1948), in particular the review by Pearle (1982), periodic solutions of the Abraham model have been reported repeatedly for the case of a charged sphere, i.e. $\rho(\boldsymbol{x}) = e(4\pi a^2)^{-1}\delta(|\boldsymbol{x}| - a)$, which is not covered by Theorem 1.3 since (W) is violated. These computations invoke certain approximations and it is not clear whether the full model, as defined by (1.37) to (1.39), has periodic solutions. Kunze (1998) excludes periodic solutions in some small part of phase space for the scalar field without the Wiener condition (W).

ad 1.6: This section is based on the monumental work of Nodvik (1964). Nodvik includes the rotation of the body fixed frame. A complementary discussion is given by Rohrlich (1990), Chapter 7–4. The relativistic Thomas precession is discussed in Møller (1952).

Of course relativistic theories have been studied much before, e.g. Born (1909). One difficulty is to write down the proper variant of the Lorentz force equation, which may be circumvented by considering the Liénard–Wiechert fields generated by a point charge and to infer through the balance of momentum the actual motion of the charge, e.g. Rohrlich (1990), Teitelbom et.al. (1980). One then looses the notion of a true trajectory and its approximation through a comparison dynamics. In particular, the selection of physical solutions can come only through an additional postulate.

The current generated by a point charge can be written as

$$j^\mu(x) = \int_{-\infty}^{\infty} ds v^\mu(s)\delta(x - z(s))\,. \tag{1.125}$$

McManus (1948) proposed to smear out the δ-function as

$$j^\mu(x) = \int_{-\infty}^{\infty} ds v^\mu(s) f((x - z(s))^2)\,, \tag{1.126}$$

which is to be inserted in the action (1.103), (1.104). The resulting equations of motion are nonlocal in space–time. McManus did not identify the conserved four-momentum. The case of rectilinear motion only is discussed by Schwinger (1983). It seems to us that Nodvik (1964) is the only worked out example of an extended relativistic charge model. As to be explained, for slowly varying external potentials it agrees with experience over the full range of allowed velocities.

ad 1.7: The variation of Nodvik (1964), which includes the Euler angles of the body fixed frame, is adapted here to the restricted variation over the world line of the particle only.

2 Historical Notes

2.1 Extended charge models (1897-1912)

When in 1897 J.J. Thomson identified the cathode rays as consisting of particles with charge $-e$, he had not only discovered the first elementary particle, but he also challenged the theoretical physicists to compute the energy–momentum relation of the electron. To put it slightly differently, we write the equations of motion in crossed $\boldsymbol{E}$ and $\boldsymbol{B}$ fields as

$$m(\boldsymbol{v})\dot{\boldsymbol{v}} = e(\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{B}) \tag{2.1}$$

with $m(\boldsymbol{v})$ the velocity dependent mass as a 3×3 matrix. The challenge was then to determine the ratio $m(\boldsymbol{v})/e$. In a long series is of experiments starting in 1897 the velocity dependent mass was measured by W. Kaufmann (1901). With improving technology the experiments were repeated and extended to a larger range of velocities. Latest by 1914 the relativistic dependence $m(\boldsymbol{v}) = m_0\gamma(\mathbb{1} + \gamma^2 c^{-2}|\boldsymbol{v}\rangle\langle\boldsymbol{v}|), \gamma = (1 - \boldsymbol{v}^2/c^2)^{-1/2}$, and m_0 the rest mass of the electron, was regarded as well confirmed.

So which theory could be used to determine $m(\boldsymbol{v})$? In fact, there was little choice. Since the phenomenon under consideration is clearly electromagnetic, one had to use Maxwell equations, and since the trajectory of a single charge was measured, one had to couple to the Lorentz force equation. Thus the electron was pictured as a tiny sphere charged with electricity. In the inhomogeneous Maxwell equations one had to insert the current generated by that moving sphere. On the other hand the electromagnetic fields back react on the charge distribution through the Lorentz force equation. Thereby one has introduced what is called an extended charge model. Abraham (1905) adopted first a in the absolute laboratory frame rigid charge distribution. The corresponding energy–momentum relation is discussed at length in the second volume of his book on electromagnetism, compare with Section 3.1. For the Abraham model, Sommerfeld (1904,1905) obtained an exact equation of motion for the electron, which as a complicating and unfamiliar feature contained memory terms as a result of integrating over the retarded fields. Lorentz (1904a,b) advertised a charge distribution which is rigid in its momentary rest frame and therefore, as seen from the laboratory frame, is contracted parallel to its momentary velocity. Of course, it was left completely open by which forces this charge distribution is kept in place. Poincaré (1906) developed non–electromagnetic models where additional stresses counteracted the Coulomb repulsion.

In all extended charge models the velocity dependent mass has the additive structure $m(\boldsymbol{v}) = m_\mathrm{b}\mathbb{1} + m_\mathrm{f}(\boldsymbol{v})$, where m_b is the bare mechanical mass of the particle, in accordance with Newtonian mechanics taken to be velocity independent, and where $m_\mathrm{f}(\boldsymbol{v})$ is the mass due to the coupling to the field, which had to be computed from the model charge distribution. Clearly, in the experiment only the sum $m_\mathrm{b} + m_\mathrm{f}$ can be measured. Lorentz apparently favored to set $m_\mathrm{b} = 0$. His model yields then the usual relativistic velocity dependence, cf. Notes to Section 3.1. However, in these times a fully relativistic model was out of reach. Rather one relied on the semi–relativistic Abraham model of a rigid sphere and substituted at the appropriate places a relativistically contracting charge distribution.

By 1904 the theoretical predictions were worked out with the experiments not yet precise enough to clearly distinguish between them. Nevertheless the whole enterprise came to a rather sudden end, since Einstein (1905a,b) forcefully argued that just like electromagnetism in vacuum also the mechanical laws had to be Lorentz invariant. But if Einstein was right, then the energy–momentum relation of the electron had to be the relativistic one, as emphasized independently by Poincaré (1906). Thus the only free parameter was the rest mass of the electron which anyhow could not be deduced from theory, since the actual charge distribution was not known. There was simply nothing left to compute. By latest in 1913 with the atomic model of Bohr, it was obvious that a theory based on classical electromagnetism could not account for the observed stability of atoms nor for the sharp spectral lines. As a tool to explain properties of atoms, electrons, and nuclei the classical electron theory was abandoned.

The effective equation of motion for the electron as given by Equation (2.1) could not possibly have been the full story. Through the work of Larmor it was already understood that a charge looses energy through radiation roughly proportional to $\dot{\boldsymbol{v}}^2$. Lorentz observed that in the approximation of small velocities this loss could be written as the friction or radiation reaction force

$$\boldsymbol{F}_{rr} = \frac{e^2}{6\pi c^3}\,\ddot{\boldsymbol{v}}\,, \tag{2.2}$$

which had to be added to the effective Lorentz force equation (2.1). In 1904 Abraham obtained this friction force for arbitrary velocities as

$$\boldsymbol{F}_{rr} = \frac{e^2}{6\pi c^3}\,[\gamma^4 c^{-2}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} + 3\gamma^6 c^{-4}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{v} + 3\gamma^4 c^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \gamma^2\ddot{\boldsymbol{v}}]\,, \tag{2.3}$$

$\gamma = (1-\boldsymbol{v}^2/c^2)^{-1/2}$. He argued that energy and momentum is transported to infinity through the far field. On that scale the charge distribution is like a point charge and the electromagnetic fields can be computed from the Liénard–Wiechert potentials. Using conservation of energy and momentum for the total system he showed that the loss at infinity could be accounted for by the friction like force (2.3). Von Laue (1909) recognized that the radiation reaction is relativistically invariant and can be written as, $\dot{}$ denoting now differentiation with respect to the eigentime,

$$F^\mu_{rr} = \frac{e^2}{6\pi c^3}\,[\ddot{v}^\mu - c^{-2}\dot{v}^\lambda\dot{v}_\lambda v^\mu]\,. \tag{2.4}$$

This is how the radiation reaction appears in the famous 1921 review article of Pauli on relativity. But apparently, there was no incentive to study properties of the effective Lorentz force equation (2.1) including the full radiation reaction correction (2.3). According to Schott (1912) after studying the motion in a uniform electric field: *Hence the effect of the reaction due to radiation is quite inappreciable in this and probably in all practical cases.* For applications, simpler phenomenological approaches sufficed.

The first chapter on the dynamics of classical electrons closes around 1912 as compiled and worked out in great detail by Schott (1912). In essence there were two results: (i) a relativistically invariant expression for the radiation reaction and (ii) energy–momentum relations for the charged particle which were depending on the particular model charge distribution. However all models were inconsistent with Einstein's theory of special relativity. In particular, the rest mass came out to be different from the electrostatic energy of the charge distribution.

2.2 The point charge

Our second chapter consists of a single paper: "Classical theory of radiating electrons" submitted by P.A.M. Dirac on March 15, 1938. But before we have to follow up the intermission during which some research on the classical electron theory continued. We mention only the studies by Fermi (1922) and Frenkel (1925). Fermi argues that Abraham and Lorentz had not used the relativistically proper definition of energy and momentum which explained their disagreement with Einstein's theory. Frenkel, apparently influenced by Ehrenfest, proposes to consider electrons as undivisable, therefore without any extent, and studies the general structure of equations of motion. Of course, the most important event during the intermission was the development of quantum mechanics, which almost immediately after its discovery was applied to quantizing the electromagnetic field. Thereby the line shape and life time for excited states of atoms could be determined. Quantum mechanics gave a strong push to the classical theory. One had to quantize in Hamiltonian form. Thus the Lagrangian and Hamiltonian structure of the coupled Maxwell–Lorentz equations had to be explored, the role of the constraints and of the gauge freedom had to be understood.

It became apparent fairly soon that the newly born quantum electrodynamics yields infinities when one tries to remove the ultraviolet cutoff, i.e. in the limit of a point charge distribution. Thus a problem which had been dropped over 15 years before reappeared in a different guise. In the '30 and early '40 it was a fairly widespread believe that one way to overcome the difficulties of quantum electrodynamics is a better understanding of the classical theory of point charges coupled to their radiation field. Of course, this was only a vehicle to the final goal, namely a consistent quantized theory. We do not describe the various attempts, since the proper formulation of quantum electrodynamics eventually went a very different route. Dirac's paper was equally motivated by quantum electrodynamics. However, as such it is concerned only with the classical electron theory.

We have to report the findings of Dirac in fair detail, since most further activities start from there. The formal argument in the original paper can be well followed and alternative versions can be found in Rohrlich (1990), Teitelbom et. al. (1980), Thirring (1997). So there is no need for repetition and we concentrate on the conclusions. At first reading it is mandatory to disregard all philosophical claims and to concentrate on the equations. But before, let us see how Dirac himself viewed the 1897-1912 period: *The Lorentz model of the electron as a small sphere charged with electricity, possessing mass account of the energy of the electric field around it, has proved very valuable in accounting for the motion and radiation of electrons in a certain domain of problems, in which the electromagnetic field does not vary too rapidly and the accelerations of the electrons are not too great.* Dirac wanted to construct quantum electrodynamics. There the electron is regarded as an elementary particle with, almost by definition, no internal structure. Thus Dirac had to dispense with model charges and to develop a theory of *point* electrons.

What did Dirac really accomplish? Of course, he assumes the validity of the inhomogeneous Maxwell equations. The current is generated by a point charge whose motion is yet to be determined. Mechanically this point charge is relativistic with bare mass m_{b}. There is no explicit back reaction of the field onto the charge, since at no stage Dirac would invoke the Lorentz force equation. Rather conservation of energy and momentum should suffice to fix the true trajectory of the point charge. Note that this is very different from the extended charge models where the starting point is a closed systems of equations for the particle and the Maxwell field. Dirac studies the flow of energy and momentum through a thin tube of radius R around the world line of the particle. The computation simplifies by writing the retarded fields generated by the motion of the point charge as

$$F_{\rm ret} = \frac{1}{2}\left(F_{\rm ret} + F_{\rm adv}\right) + \frac{1}{2}\left(F_{\rm ret} - F_{\rm adv}\right) \tag{2.5}$$

in all of space–time. The difference term turns out to be finite on the world line of the charge and yields in the limit $R \to 0$, through a balancing of energy and momentum, the relativistic radiation reaction (2.4). Thus in retrospect one can understand why in the semi–relativistic Abraham model the radiation reaction is nevertheless of relativistic form.

The more delicate term in (2.5) is the sum, which is divergent on the world line of the particle. At the expense of ignoring other divergent terms, cf. Thirring (1997), Equation (8.4.16), Dirac obtains the expected result, namely

$$-\frac{e^2}{4\pi R c^2}\,\dot v^\mu = -m_{\rm f}\dot v^\mu\,. \tag{2.6}$$

Adding the radiation reaction (2.4) and equating with the mechanical four–momentum, the final result is an equation of motion which determines the trajectory of the particle,

$$(m_{\rm b}+m_{\rm f})\dot v^\mu = m_{\rm exp}\dot v^\mu = \frac{e}{c}F^{\mu\nu}_{\rm ex}\,v_\nu + \frac{e^2}{6\pi c^3}\left[\ddot v^\mu - c^{-2}\,\dot v^\lambda \dot v_\lambda v^\mu\right] + \mathcal{O}(R) \tag{2.7}$$

with an error of the size of the tube, where we have added in the prescribed tensor $F^{\mu\nu}_{\rm ex}$ of the external fields.

To complete the argument Dirac has to take the limit $R \to 0$. Since $m_{\rm f} \to \infty$, this amounts to

$$m_{\rm b} \to -\infty\ ,\ m_{\rm f} \to \infty\ ,\ m_{\rm exp} = m_{\rm b} + m_{\rm f} \quad \text{fixed}\ , \tag{2.8}$$

where $m_{\rm exp}$ is adjusted such that it agrees with the experimentally determined mass of the charged particle. (2.8) is the classical charge renormalization.

Dirac admits that *such a model is hardly a plausible one according to current physical ideas but this is not an objection to the theory provided we have a reasonable mathematical scheme.*

Equation (2.7), dropping the terms $\mathcal{O}(R)$, is the Lorentz–Dirac equation. Within the framework of Dirac it makes no sense to ask whether the Lorentz–Dirac equation is "exact", since there is nothing to compare to. The Lorentz–Dirac equation comes as one package, so to speak. One only could compare with real experiments, which is difficult since the radiation reaction is so small, or one could compare with higher level theories as quantum electrodynamics. But this has never been seriously attempted, since it would require to have a well defined relativistic quantum field theory which is a difficult task to begin with.

The Lorentz–Dirac equation is identical to the effective equations of motion obtained from extended charge models, if we ignore for a moment that the kinetic energy might come out differently depending on which model charge is used. In this sense Dirac has recovered the classical results through a novel approach. However there is an important distinction. For extended charge models one has a true solution for the position of the charged particle, say $\tilde{\boldsymbol{q}}(t)$. One can compare then $\tilde{\boldsymbol{q}}(t)$ with a solution of the Lorentz–Dirac equation and hope for agreement in asymptotic regimes, like slowly varying potentials. In addition for an extended charge model a negative bare mass might have drastic consequences which cannot be ignored.

Dirac continues with an observation which shattered the naive trust in the classical electron theory. He observes that even for zero external fields (2.7) has solutions where $|\boldsymbol{v}(t)/c| \to 1$ as $t \to \infty$ and $|\dot{\boldsymbol{v}}(t)|$ increases without bound. Such unphysical solutions he called runaway. If one inserts numbers, then runaways grow very fast. E.g. for an electron one has $\dot{\boldsymbol{v}}(t) = \dot{\boldsymbol{v}}(0)e^{t/\tau}$ with $\tau = 10^{-23}$ sec. If the Lorentz–Dirac equation (2.7) is a valid approximation in an extended charge model, which after all was the main consensus of the 1897-1912 period, then also there one encounters runaway solutions. It is somewhat surprising that apparently runaways went completely unnoticed before, wich only proves that no attempt was made to apply the Lorentz–Dirac equation to a concrete physical problem.

Dirac proposed to eliminate the runaway solutions by requiring the asymptotic condition

$$\lim_{t\to\infty} \dot{v}^\mu(t) = 0\,. \tag{2.9}$$

As additional bonus the problem of the missing initial condition is resolved: Since in (2.7) the third derivative appears, one has to know $z^\mu(0), \dot{z}^\mu(0)$, as in any mechanical problem, and in addition $\ddot{z}^\mu(0)$. If one accepts (2.9), the initial condition $\dot{v}^\mu(0)$ is replaced by the asymptotic condition (2.9). Dirac checked that for zero external forces and for a spatially constant but time–dependent force the asymptotic condition singles out physically meaningful solutions.

By the end of 1938 the classical electron theory was in an awkward shape, in fact in a much worse shape than by the end of the 1912. Formal, but even by strict standards careful, derivations yielded an equation with unphysical solutions. How did they come into existence? While Dirac's asymptotic condition seemed to be physically sensible, it was very much ad hoc and imposed post festum to get rid of unwanted guests. Even those physicists willing to accept the asymptotic condition as a new principle, like Haag (1955), could not be too happy. Solutions satisfying the asymptotic condition are acausal in the sense that the charge starts moving even before any force is acting. To be sure the causality violation is on the time scale of $\tau = 10^{-23}$ sec for an electron, and even shorter for a proton, and thus has no observable consequences. But acausality remains as a dark spot in a relativistic theory.

The clear recognition of runaway solutions generated a sort of consensus that the coupled Maxwell–Lorentz equations have internal difficulties.

To quote from the preface of the book by Rohrlich: *Most applications treat electrons as point particles. At the same time, there was the widespread belief that the theory of point particles is beset with various difficulties such as infinite electrostatic self–energy, a rather doubtful equation of motion which admits physically meaningless solutions, violation of causality, and others. It* *is not surprising, therefore, that the very existence of a consistent classical theory of charge particles is often questioned.*

To quote from Chapter 28 of the Feynman Lectures: *Classical mechanics is a mathematically consistent theory; it just doesn't agree with experience. It is interesting, though, that the classical theory of electromagnetism is an unsatisfactory theory all by itself. The electromagnetic theory predicts the existence of an electromagnetic mass, but it also falls on its face in doing so, because it does not produce a consistent theory.*

To quote from the textbook on mathematical physics by Thirring: *Not all solutions to (2.7) are crazy. Attempts, have been made to separate sense from nonsense by imposing special initial conditions. It is to be hoped that some day the real solution of the problem of the charge–field interaction will look differently, and the equations describing nature will not be so highly unstable that the balancing act can only succeed by having the system correctly prepared ahead of time by a convenient coincidence.*

To be sure, these issues were of concern only to theoretical physicists in search for a secure foundation. Synchroton radiation sources were built anyhow. The loss in energy of an electron during one revolution can be accounted for by Larmor's formula. This is then the amount of energy which has to be supplied in order to maintain a stationary electron current. The radiation emitted from the synchroton source is computed from the inhomogeneous Maxwell equations with a point charge source, i.e. from the Liénard–Wiechert potentials. No problem.

2.3 Wheeler–Feynman electrodynamics

To avoid the infinities of self–interaction Wheeler and Feynman (1945,1949) designed a radical solution, at least on the classical level since the quantized version of their theory was never accomplished. The Wheeler–Feynman theory departs in two essential aspects from standard electrodynamics.
(i) The only dynamical variables are the trajectories of the charges. As such there are no electromagnetic fields, even though one uses them as a familiar

and convenient notational device.
(ii) To achieve agreement with observation, the theory requires to have many particles. For example, in the two–body scattering problem there is no radiation damping. Such friction forces are understood as the result of the interaction with the charged particles in the surrounding matter.

The starting point of the Wheeler–Feynman electrodynamics is an action first written down by Fokker (1929). Let us consider N particles with mass m_i, charge e_i, and motion given by the world line $z_{(i)}(\tau_i)$, $i = 1, \dots, N$. The world line is parametrized by its eigentime τ_i and $\dot{}$ denotes differentiation with respect to that eigentime. The action functional has the form

$$\begin{aligned} \mathsf{S} \;&=\; -\sum_{i=1}^{N} m_i\, c^2 \int \sqrt{(\dot{z}_{(i)})^2}\, d\tau_i \\ &\quad + \sum_{\substack{i,j=1 \\ i\neq j}}^{N} e_i e_j \int\!\!\int \delta(z_{(i)} - z_{(j)})\dot{z}_{(i)} \cdot \dot{z}_{(j)} d\tau_i d\tau_j \,. \end{aligned} \tag{2.10}$$

A formal variation of S leads to the equations of motion

$$m_i \ddot{z}^{\mu}_{(i)} = \frac{e_i}{c} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{2}\big(F^{\mu\nu}_{\mathrm{ret}(j)}(z_{(i)}) + F^{\mu\nu}_{\mathrm{adv}(j)}(z_{(i)})\big)\dot{z}_{(i)\nu}\,. \tag{2.11}$$

Here $F^{\mu\nu}_{\mathrm{ret}(j)}$, $F^{\mu\nu}_{\mathrm{adv}(j)}$ are the retarded and advanced Liénard–Wiechert fields generated by the charge at $z_{(j)}$ and evaluated at $z_{(i)}$. They are derived from the retarded and advanced potentials

$$A^{\mu}_{\mathrm{ret}(j)}(x) \;=\; e_j \dot{z}^{\mu}_{(j)}(\tau_{j\mathrm{ret}})/(x_\sigma - z_{(j)\sigma}(\tau_{j\mathrm{ret}}))\dot{z}^{\sigma}_{(j)}(\tau_{j\mathrm{ret}})\,, \tag{2.12}$$

$$A^{\mu}_{\mathrm{adv}(j)}(x) \;=\; e_j \dot{z}^{\mu}_{(j)}(\tau_{j\mathrm{adv}})/(x_\sigma - z_{(j)\sigma}(\tau_{j\mathrm{adv}}))\dot{z}^{\sigma}_{(j)}(\tau_{j\mathrm{adv}}) \tag{2.13}$$

with $\tau_{j\mathrm{ret}}$, resp. $\tau_{j\mathrm{adv}}$, the eigentime when the trajectory $z_{(j)}$ crosses the backward, resp. the forward, light cone with apex at x.

To transform (2.11) into a familiar form, we use the decomposition (2.5) and Dirac's observation that $(F_{\mathrm{ret}} - F_{\mathrm{adv}})/2$ at the trajectory of the particle yields the radiation reaction. Then

$$
\begin{aligned}
m_i \ddot{z}^{\mu}_{(i)} &= \frac{e_i}{c} \sum_{\substack{j=1 \\ j \neq i}}^{N} F^{\mu\nu}_{\mathrm{ret}(j)}(z_{(i)}) \dot{z}_{(i)\nu} + \frac{e_i^2}{6\pi c^3} (\dddot{z}^{\mu}_{(i)} - c^{-2} \, \dddot{z}^{\nu}_{(i)} \, \ddot{z}_{(i)\nu} \, \dot{z}^{\mu}_{(i)}) \\
&\quad + \frac{e_i}{c} \sum_{j=1}^{N} \frac{1}{2} \big(F^{\mu\nu}_{\mathrm{adv}(j)}(z_{(i)}) - F^{\mu\nu}_{\mathrm{ret}(j)}(z_{(i)}) \big) \dot{z}_{(i)\nu} \, . \qquad (2.14)
\end{aligned}
$$

Of course, being symmetric in time, we could have equally transformed to the advanced fields for the force and a radiation reaction with reversed sign.

We note that in (2.14) the mass of the particle is not renormalized. The retarded force is of the usual form. The radiation reaction has runaways. So one must either impose the asymptotic condition (2.9) or have the good faith that (2.11) does not posess such unphysical solutions. The last term in (2.14) is unwanted and Wheeler and Feynman spend a considerable effort to argue that for a sum over a large number of charges in disordered motion this last term vanishes. If it is exactly zero, the condition of a perfect absorber is satisfied and the standard equations of motion for charged particles result.

As with Dirac, one can accept only the whole Wheeler–Feynman package. Consequently, there has been little further work on the theory. In particular, it has never been checked how well the assumption of a perfect absorber is satisfied.

Notes and References

ad 2: A more detailed account on the history of the classical electron theory can be found in Pais (1972,1982), Rohrlich (1973), and in the introductory chapters of Rohrlich (1990). The interconnection with quantum electrodynamics before the 1947 Shelter Island conference is well described in Schweber (1994).

ad 2.2: Kramers (1948) investigations on the mass renormalization in the classical theory were instrumental for a correct computation of the Lamb shift. We refer to Dresden (1987) and Schweber (1994).

ad 2.3: The two–body problem in Wheeler-Feynman electrodynamics is discussed by Schild (1963). The existence and classification of solutions is studied by Bauer (1997). A few explicit solutions are listed in Stephas (1992).

3 Energy–Momentum Relation

For the Abraham model we established already that its energy $\mathcal{E}$ is conserved, $\mathcal{E}$ given by Equation (1.42). If the external fields vanish, then the dynamics is invariant under spatial translations. Thus the total momentum, denoted by $\mathcal{P}$, must also be conserved. The minimum of $\mathcal{E}$ at fixed $\mathcal{P}$ defines the energy–momentum relation.

If the external forces vanish, the simplest solution to the equations of motion has the particle travelling at constant velocity $\boldsymbol{v}$ in company with its electromagnetic fields. There seems to be no accepted terminology for this object. Since it will be used as a basic building block later on, we need a short descriptive name and we call this particular solution a *charge soliton*, or simply soliton, at velocity $\boldsymbol{v}$, in analogy to solitons of nonlinear wave equations. The soliton has an energy and a momentum which are linked through the energy–momentum relation.

In the following two sections we compute the conserved energy and momentum, the charge solitons, and the energy–momentum relation for both the Abraham and the Lorentz model. We will assume $\phi_{\mathrm{ex}} = 0$, $\boldsymbol{A}_{\mathrm{ex}} = 0$ throughout.

3.1 Abraham model

The mechanical momentum of the particle is given by

$$m_{\mathrm{b}}\gamma\boldsymbol{v} \tag{3.1}$$

and the momentum of the field by

$$\mathcal{P}_f = \int d^3x \big(\boldsymbol{E}(\boldsymbol{x}) \times \boldsymbol{B}(\boldsymbol{x})\big)\,. \tag{3.2}$$

Thus we set the total momentum

$$\mathcal{P} = m_{\mathrm{b}}\gamma\boldsymbol{v} + \mathcal{P}_f \tag{3.3}$$

as a functional on $\mathcal{M}$. One easily checks that $\mathcal{P}$ is conserved by the Maxwell–Lorentz equations (1.37) - (1.39). The corresponding Lagrangian, compare with (1.41), is invariant under spatial translations and $\mathcal{P}$ is the conserved quantity which, by Nöther's theorem, corresponds to this symmetry.

We want to minimize the energy at fixed total momentum. We eliminate $\boldsymbol{v}$ between (1.42) and (3.3) and thus have to minimize

$$\big(m_{\mathrm{b}}^2 + \big(\mathcal{P} - \int d^3x(\boldsymbol{E}\times\boldsymbol{B})\big)^2\big)^{1/2} + \frac{1}{2}\int d^3x(\boldsymbol{E}^2+\boldsymbol{B}^2) \tag{3.4}$$

at fixed $\mathcal{P}$ and subject to the constraints $\nabla\cdot\boldsymbol{E}=\rho\,,\nabla\cdot\boldsymbol{B}=0$. By translation invariance we can center ρ at an arbitrary $\boldsymbol{q}\in\mathbb{R}^3$. For $\boldsymbol{q}=0$, say, the minimizer is unique and given by

$$\begin{aligned}\boldsymbol{E}_{\boldsymbol{v}}(\boldsymbol{x}) &= -\nabla\phi_{\boldsymbol{v}}(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{v}\cdot\nabla\phi_{\boldsymbol{v}}(\boldsymbol{x}))\,,\\ \boldsymbol{B}_{\boldsymbol{v}}(\boldsymbol{x}) &= -\boldsymbol{v}\times\nabla\phi_{\boldsymbol{v}}(\boldsymbol{x})\,,\end{aligned} \tag{3.5}$$

where

$$\widehat{\phi}_{\boldsymbol{v}}(\boldsymbol{k}) = [\boldsymbol{k}^2 - (\boldsymbol{v}\cdot\boldsymbol{k})^2]^{-1}\hat{\rho}(\boldsymbol{k}) \tag{3.6}$$

and in physical space

$$\phi_{\boldsymbol{v}}(\boldsymbol{x}) = \int d^3y\big(4\pi\sqrt{\gamma^{-2}(\boldsymbol{x}-\boldsymbol{y})^2 + (\boldsymbol{v}\cdot(\boldsymbol{x}-\boldsymbol{y}))^2}\,\big)^{-1}\rho(\boldsymbol{y})\,. \tag{3.7}$$

Here $|\boldsymbol{v}|<1$, i.e. $\boldsymbol{v}\in\mathbb{V}$, and $\boldsymbol{v}$ has to be adjusted such that $\mathcal{P}=\boldsymbol{P}_s(\boldsymbol{v})$ with

$$\begin{aligned}\boldsymbol{P}_s(\boldsymbol{v}) &= m_{\mathrm{b}}\gamma\boldsymbol{v} + \int d^3k|\hat{\rho}(\boldsymbol{k})|^2\big([\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-1}\boldsymbol{v}\\ &\qquad -\gamma^{-2}[\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-2}(\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{k}\big)\\ &= \boldsymbol{v}\big\{m_{\mathrm{b}}\gamma + m_{\mathrm{e}}\,|\boldsymbol{v}|^{-2}\,\big[\frac{1+\boldsymbol{v}^2}{2|\boldsymbol{v}|}\log\frac{1+|\boldsymbol{v}|}{1-|\boldsymbol{v}|} - 1\big]\big\}\,,\end{aligned} \tag{3.8}$$

where m_{e} is the electrostatic energy of the charge distribution ρ,

$$m_{\mathrm{e}} = \frac{1}{2}\int d^3x\,d^3x'\rho(\boldsymbol{x})\,\rho(\boldsymbol{x}')(4\pi|\boldsymbol{x}-\boldsymbol{x}'|)^{-1}\,. \tag{3.9}$$

The map $\mathbb{V} \ni \boldsymbol{v} \mapsto \boldsymbol{P}_s(\boldsymbol{v}) \in \mathbb{R}^3$ is one to one and therefore $\mathcal{P} = \boldsymbol{P}_s(\boldsymbol{v})$ has a unique solution. The minimizing energy is given by

$$\begin{aligned} E_s(\boldsymbol{v}) &= m_{\mathrm{b}}\gamma + \frac{1}{2}\int d^3k|\hat{\rho}(\boldsymbol{k})|^2\,[\boldsymbol{k}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-2}\big((1+\boldsymbol{v}^2)\boldsymbol{k}^2 - (3-\boldsymbol{v}^2)(\boldsymbol{v}\cdot\boldsymbol{k})^2\big) \\ &= m_{\mathrm{b}}\gamma + m_{\mathrm{e}}\,\Big[\frac{1}{|\boldsymbol{v}|}\log\frac{1+|\boldsymbol{v}|}{1-|\boldsymbol{v}|} - 1\Big]\,. \end{aligned} \tag{3.10}$$

Eliminating now $\boldsymbol{v}$ between E_s and $\boldsymbol{P}_s$ yields the *energy–momentum relation*

$$E_{\mathrm{eff}}(\boldsymbol{p}) = E_s(\boldsymbol{v}(\boldsymbol{p}))\,, \tag{3.11}$$

where $\boldsymbol{v}(\boldsymbol{P}_s)$ is the function inverse to $\boldsymbol{P}_s(\boldsymbol{v})$. As to be underlined, E_{eff} depends on the charge distribution only through its electrostatic energy.

We note that

$$\boldsymbol{P}_s(\boldsymbol{v}) = \nabla_{\boldsymbol{v}} T(\boldsymbol{v}) \tag{3.12}$$

with

$$T(\boldsymbol{v}) = -m_{\mathrm{b}}\gamma^{-1} + \frac{1}{2}\gamma^{-2}\int d^3k|\hat{\rho}(\boldsymbol{k})|^2\,[\boldsymbol{k}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-1} \tag{3.13}$$

and that

$$E_s(\boldsymbol{v}) = \boldsymbol{P}_s(\boldsymbol{v})\cdot\boldsymbol{v} - T(\boldsymbol{v})\,. \tag{3.14}$$

This suggests that T will play the role of the inertial term in an effective Lagrangian and E_s the role of an effective Hamiltonian as our notation in (3.11) indicates already. In particular,

$$\boldsymbol{v} = \nabla_{\boldsymbol{p}}\, E_{\mathrm{eff}}(\boldsymbol{p}) \tag{3.15}$$

and, equivalently,

$$\boldsymbol{v}\,\frac{\mathrm{d}\boldsymbol{P}_s(\boldsymbol{v})}{\mathrm{d}\boldsymbol{v}} = \nabla_{\boldsymbol{v}}\, E_s(\boldsymbol{v}) \tag{3.16}$$

which implies that $\boldsymbol{v}$ is to be interpreted as velocity.

For a relativistic theory one expects that

$$E_s(\boldsymbol{v}) = (m_\mathrm{b} + m_\mathrm{e})\gamma, \quad \boldsymbol{P}_s(\boldsymbol{v}) = (m_\mathrm{b} + m_\mathrm{e})\gamma\boldsymbol{v}\,. \tag{3.17}$$

Since the Abraham model is semi–relativistic, there is no reason for such a property to be satisfied. Still we found that, as in the relativistic case, the energy–momentum relation depends on the charge distribution ρ only through m_e.

For small $\boldsymbol{v}$ we have

$$E_s(\boldsymbol{v}) - E_s(0) \cong \frac{1}{2}\,(m_\mathrm{b} + \frac{4}{3}\,m_\mathrm{e})\boldsymbol{v}^2, \quad \boldsymbol{P}_s(\boldsymbol{v}) = (m_\mathrm{b} + \frac{4}{3}\,m_\mathrm{e})\boldsymbol{v}\,. \tag{3.18}$$

Thus the effective mass in the nonrelativistic approximation is

$$m_\mathrm{eff} = m_\mathrm{b} + \frac{4}{3}\,m_\mathrm{e}\,. \tag{3.19}$$

In Figure 2 we plot $E_s(\boldsymbol{v}), \boldsymbol{P}_s(\boldsymbol{v})$ for the extreme case $m_\mathrm{b} = 0$ and compare with the relativistic dispersion of mass $\frac{4}{3}\,m_\mathrm{e}$. Clearly at speeds $|\boldsymbol{v}| > 0.3$, the Abraham model looses its empirical validity. One could partially save the Abraham model by declaring the Compton wave length as the characteristic size of the charge distribution. Then $m_\mathrm{e}/m_\mathrm{b} \cong 0.01$ and the relativistic dispersion is violated only for speeds close to one.

The energy minimizer has a simple dynamical interpretation. We look for a solution, $S_{\boldsymbol{q},\boldsymbol{v}}$, of (1.37) , (1.39) travelling at constant velocity $\boldsymbol{v}$ and find

$$S_{\boldsymbol{q},\boldsymbol{v}}(t) = (\boldsymbol{E}_{\boldsymbol{v}}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{v}t),\, \boldsymbol{B}_{\boldsymbol{v}}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{v}t), \boldsymbol{q} + \boldsymbol{v}t, \boldsymbol{v}) \tag{3.20}$$

with $\boldsymbol{v} \in \mathbb{V}$, $\boldsymbol{q} \in \mathbb{R}^3$, and $\boldsymbol{E}_{\boldsymbol{v}}, \boldsymbol{B}_{\boldsymbol{v}}$ from (3.5). $S_{\boldsymbol{q},\boldsymbol{v}}$ is the *charge soliton* labeled by its center $\boldsymbol{q}$ and velocity $\boldsymbol{v}$. It has the energy $\mathcal{E}(S_{\boldsymbol{q},\boldsymbol{v}}) = E_s(\boldsymbol{v})$ and momentum $\mathcal{P}(S_{\boldsymbol{q},\boldsymbol{v}}) = \boldsymbol{P}_s(\boldsymbol{v})$.

There is an instructive alternate way to represent the charged soliton. We consider the inhomogeneous Maxwell equations (1.37) and prescribe the initial data at time τ. We require that the particle travels along the straight line $\boldsymbol{q} = \boldsymbol{v}t$. If we let $\tau \to -\infty$ and consider the solution at time $t = 0$, then in (1.16), (1.17) the initial fields will have escaped to infinity and only the retarded fields survive. Using (1.16), (1.17) this leads to

$$\begin{aligned}
\boldsymbol{E}_{\boldsymbol{v}}(\boldsymbol{x}) &= -\int_{-\infty}^{0} dt \int d^3y \, \big(\nabla G_{-t}\,(\boldsymbol{x}-\boldsymbol{y})\,\rho(\boldsymbol{y}-\boldsymbol{v}t) \\
&\qquad + \partial_t\, G_{-t}\,(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{v}\rho(\boldsymbol{y}-\boldsymbol{v}t)\big)\,, \qquad (3.21) \\
\boldsymbol{B}_{\boldsymbol{v}}(\boldsymbol{x}) &= \int_{-\infty}^{0} dt \int d^3y\, \nabla \times G_{-t}\,(\boldsymbol{x}-\boldsymbol{y})\,\boldsymbol{v}\rho(\boldsymbol{y}-\boldsymbol{v}t)\,, \qquad (3.22)
\end{aligned}$$

which can be checked either directly in Fourier space or as being a solution of the Maxwell equations travelling at constant velocity $\boldsymbol{v}$.

3.2 Lorentz model

We look for a solution travelling at constant velocity $\boldsymbol{v}$. Since the model is relativistic, we first determine the four–potential of the charge soliton in its rest frame which yields

$$A^\mu(x') = (g(x'^2),\ 0) \qquad (3.23)$$

with $g(|\boldsymbol{x}|^2) = -(\Delta^{-1}\rho)(\boldsymbol{x})$. Then in the laboratory frame $\mathcal{S}$

$$A^\mu(x) = v^\mu g(x^2 - (v_\lambda x^\lambda)^2) \qquad (3.24)$$

and the electromagnetic field tensor has the form

$$\begin{aligned}
F^{\mu\nu}(x) &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \qquad (3.25) \\
&= 2\left[(x^\mu - (v_\lambda x^\lambda)v^\mu)v^\nu - (x^\nu - (v_\lambda x^\lambda)v^\nu)v^\mu\right] g'(x^2 - (v_\lambda x^\lambda)^2)\,.
\end{aligned}$$

$F^{\mu\nu}$ indeed satisfies the Maxwell equations (1.105) with the current

$$j^\mu(x) = v^\mu\, ef(x^2 - (v_\lambda x^\lambda)^2)\,, \qquad (3.26)$$

where $ef(|\boldsymbol{x}|^2) = \rho(\boldsymbol{x})$, in accordance with (1.36), (1.85).

Expressed in terms of electric and magnetic fields we have

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x},t) &= -\nabla\phi_{\boldsymbol{v}}(\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{v}t)+\boldsymbol{v}(\boldsymbol{v}\cdot\nabla\phi_{\boldsymbol{v}}(\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{v}t))\,,\\ \boldsymbol{B}(\boldsymbol{x},t) &= -\boldsymbol{v}\times\phi_{\boldsymbol{v}}(\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{v}t)\,,\end{aligned}\tag{3.27}$$

where

$$\widehat{\phi}_{\boldsymbol{v}}(\boldsymbol{k}) = [\boldsymbol{k}^2-(\boldsymbol{v}\cdot\boldsymbol{k})^2]^{-1}\hat{\rho}(\gamma^{-1}\boldsymbol{k}_{\|}+\boldsymbol{k}_{\perp})\tag{3.28}$$

with $\boldsymbol{k}_{\|}$ parallel and $\boldsymbol{k}_{\perp}$ orthogonal to $\boldsymbol{v}$. In contrast to the nonrelativistic coupling, the charge distribution is now Lorentz contracted as seen from the laboratory frame, compare with (3.6), where we note that $\boldsymbol{k}^2-(\boldsymbol{v}\cdot\boldsymbol{k})^2 = \gamma^{-2}\boldsymbol{k}_{\|}^2+\boldsymbol{k}_{\perp}^2$ and $\hat{\rho}(\gamma^{-1}\boldsymbol{k}_{\|}+\boldsymbol{k}_{\perp}) = \hat{\rho}_r([\boldsymbol{k}^2-(\boldsymbol{v}\cdot\boldsymbol{k})^2]^{1/2})$.

To determine energy and momentum of the relativistic soliton we first have to find out how these quantities are even defined. We start from the energy–momentum tensor of the electromagnetic field

$$T^{\mu\nu} = F^{\nu\lambda}\,F_{\lambda}{}^{\mu} - g^{\mu\nu}\,F^{\alpha\beta}\,F_{\alpha\beta}\,.\tag{3.29}$$

¿From the Maxwell equations it satisfies the local balance

$$\partial_\nu\,T^{\mu\nu} = F^{\mu\nu}j_\nu\,.\tag{3.30}$$

We now claim that

$$\frac{d}{ds}\big(m_{\mathrm{b}}v^\mu + \int d^4x\,T^{\mu\nu}(x)v_\nu\,\delta(v_\lambda(x-z)^\lambda)\big) = 0\,.\tag{3.31}$$

Thus it is natural to regard

$$P^\mu = m_{\mathrm{b}}v^\mu + \int d^4x\,T^{\mu\nu}(x)v_\nu\,\delta(v_\lambda(x-z)^\lambda)\tag{3.32}$$

as the conserved four–momentum.

To derive (3.31) we multiply (3.30) by $[1-\dot{v}_\sigma(x-z)^\sigma]\,\delta(v_\lambda(x-z)^\lambda)$ and integrate over all space–time. Then

$$\begin{aligned}&\int d^4x\,F^{\mu\nu}(x)j_\nu(x)\,[1-\dot{v}_\sigma(x-z)^\sigma]\,\delta(v_\lambda(x-z)^\lambda)\\ =&\int d^4x\,\partial_\nu\,T^{\mu\nu}(x)\,[1-\dot{v}_\sigma(x-z)^\sigma]\,\delta(v_\lambda(x-z)^\lambda)\\ =&-\int d^4x\,T^{\mu\nu}(x)\big[-\dot{v}_\nu\,\delta(v_\lambda(x-z)^\lambda)+(1-\dot{v}_\sigma(x-z)^\sigma)v_\nu\,\delta'(v_\lambda(x-z)^\lambda)\big]\,.\end{aligned}\tag{3.33}$$

We have

$$
\begin{aligned}
&\frac{d}{ds}\int d^4x\, T^{\mu\nu}(x)v_\nu\, \delta(v_\lambda(x-z)^\lambda) \\
&= \int d^4x\; T^{\mu\nu}(x)\,\big(\dot v_\nu\, \delta(v_\lambda(x-z)^\lambda) - v_\nu(1-\dot v_\sigma(x-z)^\sigma)\,\delta'(v_\lambda(x-z)^\lambda)\big) \\
&= \int d^4x\, F^{\mu\nu}(x) j_\nu(x)(1-\dot v_\sigma(x-z)^\sigma)\,\delta(v_\lambda(x-z)^\lambda) \\
&= \int d^4x\, F^{\mu\nu}(x)(1-\dot v_\sigma(x-z)^\sigma)\,\delta(v_\lambda(x-z)^\lambda) \\
&\quad \times \int ds'\, [(v_\nu - v_\nu \dot v_\sigma(x-z)^\sigma) f((x-z)^2)\,\delta(v_\lambda(x-z)^\lambda)](s')\,. \qquad (3.34)
\end{aligned}
$$

Under our assumption of not too large an acceleration, compare with (1.90) below, the hyperplane $\{x:\ v_\lambda(s)(x-z(s))^\lambda = 0\}$ intersects the hyperball $\{x:\ v_\lambda(s')\ (x-z(s'))^\lambda = 0,\ f((x-z(s'))^2) > 0\}$ only if $s = s'$ and

$$
\begin{aligned}
&(1-\dot v_\sigma(s)(x-z(s))^\sigma)\,\delta(v_\lambda(s)(x-z(s))^\lambda)\,\delta(v_\lambda(s')(x-z(s'))^\lambda) \\
&= \delta(v_\lambda(s)(x-z(s))^\lambda)\delta(s-s')\,. \qquad (3.35)
\end{aligned}
$$

Thus

$$
\begin{aligned}
(3.34) \quad &= -\int d^4x\, F^{\mu\nu}(x)\,\delta(v_\lambda(x-z)^\lambda)\,(v_\nu - \eta_{\nu\sigma}(x-z)^\sigma)\, f((x-z)^2) \\
&= -m_{\mathrm{b}}\,\frac{d}{ds}\,v^\mu(s)\,, \qquad (3.36)
\end{aligned}
$$

where we used the equations of motion (1.106). This proves (3.31).

The expression (3.32) for P^μ is covariant. Thus we are allowed to work out the integral in the frame $\mathcal{S}'$ traveling with velocity $\boldsymbol{v}$ relative to $\mathcal{S}$. In this frame

$$
P'^{\,0} = \gamma\,\big(m_{\mathrm{b}} + \frac{1}{2}\int d^3x'\,\boldsymbol{E}(\boldsymbol{x}')^2\big),\ \ P'^{\,\mu} = 0 \quad \text{for} \quad \mu = 1,2,3 \qquad (3.37)
$$

and thus

$$
P^\mu = (m_{\mathrm{b}} + m_{\mathrm{e}})\,\gamma\, v^\mu\,, \qquad (3.38)
$$

which shows that the Lorentz model has the physically correct relativistic four–momentum. Of course, experimentally only the sum, $m_{\mathrm{b}} + m_{\mathrm{e}}$, can be observed.

Notes and References

ad 3.1, 3.2: Abraham (1905) computed the energy–momentum relation in essence along the same route as outlined here (except for the variational characterization). Sommerfeld (1904,1905) used the expansion of the exact self–force, as will be explained in Chapter 5. Lorentz (1904a) proposed a model charge which relativistically contracts along its momentary velocity. Thus provisionally we replace the charge distribution $\rho(\boldsymbol{x})$ by its Lorentz contracted version

$$\begin{aligned} \rho_L(\boldsymbol{x}) &= \gamma\rho_r([\boldsymbol{x}^2+\gamma^2(\boldsymbol{x}\cdot\boldsymbol{v})^2]^{1/2}), && (3.39)\\ \hat{\rho}_L(\boldsymbol{k}) &= \hat{\rho}_r([\boldsymbol{k}^2-(\boldsymbol{v}\cdot\boldsymbol{k})^2]^{1/2}). && (3.40) \end{aligned}$$

This expression is substituted in (3.6) and gives the electromagnetic fields comoving with the charge at velocity $\boldsymbol{v}$. Their energy and momentum is computed as before with the result

$$\begin{aligned} \boldsymbol{P}_L(\boldsymbol{v}) &= \boldsymbol{v}\big(m_{\mathrm{b}}\gamma(\boldsymbol{v})+\frac{4}{3}\,m_{\mathrm{e}}\gamma(\boldsymbol{v})\big), && (3.41)\\ E_L(\boldsymbol{v}) &= m_{\mathrm{b}}\gamma(\boldsymbol{v})+m_{\mathrm{e}}\gamma(\boldsymbol{v})\big(1+\frac{1}{3}\boldsymbol{v}^2\big). && (3.42) \end{aligned}$$

The momentum has the anticipated form, except for the factor 4/3 which should be 1. The energy has an unwanted $\boldsymbol{v}^2/3$. In particular the relation (3.16) does not hold, which implies that the power equation $\frac{d}{dt}E_L(\boldsymbol{v})$ differs from the force equation $\boldsymbol{v}\cdot\frac{d}{dt}\boldsymbol{P}_L(\boldsymbol{v})$. We refer to Yaghjian (1992) for a thorough discussion.

Schott (1912) employed as a model charge a deformable elastic medium. To compute the velocity dependent mass he used in principle the same method as Sommerfeld, an exact self–force and an expansion in the charge diameter. Schott considered also electron models different from those of Abraham and Lorentz.

There have been various attempts to improve on the oversimplistic version (3.39) of the Lorentz model. Fermi (1922) argues that in a relativistic theory energy and momentum have to be redefined. His argument has been rediscovered several times and is explained in Rohrlich (1990). Poincaré (1906) takes the elastic stresses into account. His theory is excellently presented in Yaghjian (1992). The material of Section 3.2 is adapted from of Nodvik (1964).

4 Adiabatic Limit

If we assume that the mass of an electron is purely electromagnetic, then by equating its rest energy and electrostatic Coulomb energy the charge distribution ρ must be concentrated in a ball of radius

$$R_\rho = \frac{e^2}{mc^2} = 3 \times 10^{-13} \text{ cm} \tag{4.1}$$

which is the so called classical electron radius. Quantum mechanically one argues that through fluctuations the electron appears to have an effective size of the order of the Compton wave length $\lambda_c = \hbar m/c = (e^2/\hbar c)^{-1} R_\rho = 137\, R_\rho$. Electromagnetic fields which can be manipulated in the laboratory vary little over that length scale. R_ρ defines a time scale through the time it takes light to cross the diameter of the charge distribution,

$$t_\rho = R_\rho/c = 10^{-23} \text{ sec}\ , \quad \text{equivalently as a frequency,}\ \omega_\rho = 10^{23}\ \text{Hz}\,. \tag{4.2}$$

Again, manufactured frequencies are much smaller than ω_ρ. Space–time variations as fast as (4.1) and (4.2) lead us deeply into the quantum regime. Thus it is a natural and physically a mandatory problem to study the dynamics of a charged particle under external potentials which vary *slowly* on the scale of R_ρ. This means we have to introduce a scale of potentials an enquire about an approximately autonomous particle dynamics with an error depending on the scale under consideration. We will introduce such a scheme formally in the following section. The resulting problem has many similarities with the derivation of hydrodynamics from Newtonian particle dynamics – with the most welcome addition that it is simpler mathematically by many order of magnitudes. Still, the comparison is instructive.

4.1 Scaling limit

We assumed that in the Lorentz force equation there are in addition to dynamical fields $\boldsymbol{E}(\boldsymbol{x}, t)$, $\boldsymbol{B}(\boldsymbol{x}, t)$ also prescribed external fields acting on the particle, which are the gradients of the external potentials $\phi_{\text{ex}}(\boldsymbol{x}), \boldsymbol{A}_{\text{ex}}(\boldsymbol{x})$, compare with Equation (1.39). We want to impose that ϕ_{ex} and $\boldsymbol{A}_{\mathrm{ex}}$ are slowly varying on the scale of R_ρ. Formally we introduce a small *dimensionless* parameter ε and consider the potentials

$$\phi_{\mathrm{ex}}(\varepsilon \boldsymbol{x}) \ , \quad \boldsymbol{A}_{\mathrm{ex}}(\varepsilon \boldsymbol{x}) \, , \tag{4.3}$$

which are slowly varying in the limit $\varepsilon \to 0$. Most of our results extend to potentials which vary also slowly in time. But for simplicity we restrict ourselves to time–independent potentials here. Clearly, ε appears as a parameter of the potential, just like ω_0 is a parameter of the harmonic potential $\frac{1}{2}\, m\omega_0^2 x^2$. But one really should think of ε as a book keeping device which orders the magnitude of the various terms and the space–time scales in powers of ε. Such a scheme is familiar in very diverse contexts and appears whenever one has to deal with a problem involving scale separation.

So how small is ε ? From the discussion above one might infer that if $\phi_{\mathrm{ex}}, \boldsymbol{A}_{\mathrm{ex}}$ vary over a scale of 1 mm, then $\varepsilon = 10^{-12}$. This is a strictly meaningless statement, because $e\phi_{\mathrm{ex}}$, $\frac{e}{c}\boldsymbol{A}_{\mathrm{ex}}$ have the dimension of an energy and thus the variation depends on the adopted energy scale. In (4.3) we fix the energy scale and merely stretch the spatial axes by a factor ε^{-1}. Since from experience this point is likely to be confusing, let us consider the specific example of a charge circling in the uniform magnetic field $(0, 0, B_0)$. Since the corresponding vector potential is linear in $\boldsymbol{x}$, to introduce ε as in (4.3) just means that the magnetic field strength equals εB_0 and the limit $\varepsilon \to 0$ is a limit of small magnetic field *relative* to some reference field B_0. Thus to obtain ε we first have to determine the reference field and compare it with the magnetic field of interest. This shows that in order to fix ε we have to specify the physical situation concretely, in particular the external potentials, the mass and charge of the particle, $\gamma(\boldsymbol{v})$, and the time span of interest.

The scaling scheme (4.3) has the enormous advantage that the analysis can be carried out in generality. In a second step one has to figure out ε for a *concrete* situation, which leads to a quantitative estimate on the error terms. E.g. if in the case above we consider an electron with velocities such that $\gamma \leq 10$, then, by comparing the Hamiltonian term and the friction term, the reference field turns out to be $B_0 = 10^{17}$ Gauss. Laboratory magnetic fields are less than 10^5 Gauss and thus $\varepsilon < 10^{-12}$. In practice, ε is always very small, less than 10^{-10}. This means that, firstly, all corrections *beyond* radiation reaction are negligible. Secondly, we do not have to go each time through the scheme indicated above and may as well set $\varepsilon = 1$ thereby returning to the conventional units. Still on an theoretical level the use of the scale parameter ε is very convenient. In Section 4.3 we will work out the example of a constant magnetic field more explicitly. If the reader feels uneasy about the scaling limit, (s)he should consult this example first.

Adopting (4.3), the Lorentz force equation reads now

$$\begin{aligned} \frac{d}{dt}\big(m_{\mathrm{b}}\gamma\boldsymbol{v}(t)\big) &= \int d^3x\, \rho(\boldsymbol{x}-\boldsymbol{q}(t))\,\big[\varepsilon\boldsymbol{E}_{\mathrm{ex}}(\varepsilon\boldsymbol{x}) + \boldsymbol{E}(\boldsymbol{x},t) \\ &\qquad + \boldsymbol{v}(t)\times\big(\varepsilon\boldsymbol{B}_{\mathrm{ex}}(\varepsilon\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x},t)\big)\big]\,, \end{aligned} \tag{4.4}$$

where

$$\boldsymbol{E}_{\mathrm{ex}} = -\nabla\phi_{\mathrm{ex}}\ ,\ \boldsymbol{B}_{\mathrm{ex}} = \nabla\times\boldsymbol{A}_{\mathrm{ex}}\,. \tag{4.5}$$

It has to be supplemented with the Maxwell equations (1.37), (1.38). Our goal is to understand the structure of the solution for small ε and as a first qualitative step one should discuss the rough order of magnitudes in powers of ε. But before we have to specify the initial data. We give ourselves $\boldsymbol{q}^0, \boldsymbol{v}^0$ as initial position and velocity of the charge. The initial fields are assumed to be Coulombic, centered at $\boldsymbol{q}^0$ with velocity $\boldsymbol{v}^0$, i.e.

Condition (I):

$$Y(0) = S_{\boldsymbol{q}^0,\boldsymbol{v}^0}(0)\,, \tag{4.6}$$

compare with (3.20). Equivalently, according to (3.21), (3.22), we may say that the particle has travelled freely with velocity $\boldsymbol{v}^0$ for the infinite time span $(-\infty,0]$. At time $t=0$ the external potentials are turned on. More geometrically we define the six–dimensional charge soliton manifold $\mathsf{S} = \{S_{\boldsymbol{q},\boldsymbol{v}},\ \boldsymbol{q}\in\mathbb{R}^3,\ \boldsymbol{v}\in\mathbb{V}\}$ as a submanifold of the phase space $\mathcal{M}$. Then our initial data are exactly on S. If there are no external forces, the solution remains on S and moves along a straight line. For slowly varying external potentials as in (4.3) we will show that the solution remains ε–close to S in the local energy distance.

On general grounds one may wonder whether such specific initial data are really required. In analogy to hydrodynamics, we call this the initial slip problem. In times of order t_ρ, the fields close to the charge acquire their Coulombic form. However, during that period the particle might gain or loose in momentum and energy and the data at time t_ρ close to the particle are approximately of the form $S_{\tilde{\boldsymbol{q}},\tilde{\boldsymbol{v}}}$, where $\tilde{\boldsymbol{q}}$ and $\tilde{\boldsymbol{v}}$ are to be computed from the full solution. Of course, at a distance ct away from the charge, the field still remembers its $t = 0$ data. Thus we see that the initial slip problem translates into the long time asymptotics of a charge at *zero* external potentials but with general initial field data. We study this point in more detail in Section 4.3. At the moment we just circumvent the initial slip by fiat.

Let us discuss the three relevant time scales, where we recall that $t_\rho = R_\rho/c$.

(i) *Microscopic scale*, $t = \mathcal{O}(t_\rho)$, $\boldsymbol{q} = \mathcal{O}(R_\rho)$. On that scale the particle moves along an essentially straight line. The electromagnetic fields adjust themselves to their comoving Coulombic form. As we will see, they do this with a precision $\mathcal{O}(\varepsilon)$ in the energy norm.

(ii) *Macroscopic scale*, $t = \mathcal{O}(\varepsilon^{-1}t_\rho)$, $\boldsymbol{q} = \mathcal{O}(\varepsilon^{-1}R_\rho)$. This scale is defined by the variation of the potentials, i.e. on that scale the potentials are $\phi_{\mathrm{ex}}(\boldsymbol{x})$, $\boldsymbol{A}_{\mathrm{ex}}(\boldsymbol{x})$. The particle follows the external forces. Since it is in company with the almost Coulombic fields, the particle responds to the forces according to the effective energy–momentum relation, which we determined in the previous section. On the macroscopic scale the motion is Hamiltonian up to errors of order ε. There is no dissipation of energy and momentum.

(iii) *Friction scale*. Accelerated charges loose energy through radiation, which means that there must be friction corrections to the effective Hamiltonian motion. According to Larmor's formula the radiation losses are proportional to $\dot{\boldsymbol{v}}(t)^2$. Since the external forces are of the order ε, these losses are proportional to ε^2 when measured in microscopic units. Integrated over a time span $\varepsilon^{-1}t_\rho$ the friction results in an effect of order ε. Thus we expect order ε dissipative corrections to the conservative motion on the macroscopic scale. Followed over the even longer time scale $\varepsilon^{-2}t_\rho$, the radiation reaction results in $\mathcal{O}(1)$ deviations from the Hamiltonian trajectory.

On the friction time scale the motion either comes to a stand still or stays uniform. In addition, as to be shown, the dissipative effective equation has the same long time behavior as the true solution. Thus we expect no further qualitatively distinct time scale beyond the friction scale.

¿From our description, in a certain sense, the most natural scale is the macroscopic scale and we transform the Maxwell–Lorentz equations to this new scale by setting

$$t' = \varepsilon t \ , \ \boldsymbol{x}' = \varepsilon \boldsymbol{x} \, . \tag{4.7}$$

We have the freedom of how to scale the amplitudes of the dynamic part of the electromagnetic fields. We require that their energy is independent of ε. Then

$$\boldsymbol{E}'(\boldsymbol{x}', t') = \varepsilon^{-3/2}\, \boldsymbol{E}(\boldsymbol{x}, t) \ , \ \boldsymbol{B}'(\boldsymbol{x}', t') = \varepsilon^{-3/2}\, \boldsymbol{B}(\boldsymbol{x}, t) \, . \tag{4.8}$$

Finally the new position and velocity are

$$\boldsymbol{q}'(t') = \varepsilon \boldsymbol{q}(t) \ , \ \boldsymbol{v}'(t') = \boldsymbol{v}(t) \, , \tag{4.9}$$

so that $\frac{d}{dt'}\boldsymbol{q}' = \boldsymbol{v}'$. There is little risk in omitting the prime. We denote then

$$\boldsymbol{q}^\varepsilon(t) = \varepsilon \boldsymbol{q}(\varepsilon^{-1}t) \ , \ \boldsymbol{v}^\varepsilon(t) = \boldsymbol{v}(\varepsilon^{-1}t) \ , \ \rho_\varepsilon(\boldsymbol{x}) = \varepsilon^{-3}\, \rho(\varepsilon^{-1}\boldsymbol{x}) \, , \tag{4.10}$$

which means that $\int d^3x\, \rho_\varepsilon(\boldsymbol{x}) = e$ independent of ε and ρ_ε is supported in a ball of radius εR_ρ. In the macroscopic coordinates the Maxwell-Lorentz equations read

$$
\begin{aligned}
\partial_t \boldsymbol{B}(\boldsymbol{x},t) &= -\nabla \times \boldsymbol{E}(\boldsymbol{x},t)\,, \\
\partial_t \boldsymbol{E}(\boldsymbol{x},t) &= \nabla \times \boldsymbol{B}(\boldsymbol{x},t) - \sqrt{\varepsilon}\rho_\varepsilon(\boldsymbol{x}-\boldsymbol{q}^\varepsilon(t))\boldsymbol{v}^\varepsilon(t)\,, \qquad (4.11)\\
\frac{d}{dt}\big(m_{\mathrm{b}}\gamma \boldsymbol{v}^\varepsilon(t)\big) &= \boldsymbol{E}_{\mathrm{ex}} * \rho_\varepsilon(\boldsymbol{q}^\varepsilon(t)) + \boldsymbol{v}^\varepsilon(t) \times \boldsymbol{B}_{\mathrm{ex}} * \rho_\varepsilon(\boldsymbol{q}^\varepsilon(t)) \\
&\quad + \int d^3x\, \sqrt{\varepsilon}\,\rho_\varepsilon(\boldsymbol{x}-\boldsymbol{q}^\varepsilon(t))\,[\boldsymbol{E}(\boldsymbol{x},t) + \boldsymbol{v}^\varepsilon(t) \times \boldsymbol{B}(\boldsymbol{x},t)]
\end{aligned}
$$

together with the constraints

$$
\nabla \cdot \boldsymbol{E} = \sqrt{\varepsilon}\,\rho(\cdot - \boldsymbol{q}^\varepsilon(t))\ , \quad \nabla \cdot \boldsymbol{B} = 0\,. \qquad (4.12)
$$

On the macroscopic scale the conserved energy is

$$
\mathcal{E}_{\mathrm{mac}} = m_{\mathrm{b}}\gamma(\boldsymbol{v}) + \phi_{\mathrm{ex}} * \rho_\varepsilon(\boldsymbol{q}) + \frac{1}{2}\int d^3x\,\big(\boldsymbol{E}(\boldsymbol{x})^2 + \boldsymbol{B}(\boldsymbol{x})^2\big)\,. \qquad (4.13)
$$

Also the initial data have to be transformed and become now

Condition (I_ε)*:*

$$
Y^\varepsilon(0) = S^\varepsilon_{\boldsymbol{q}^0,\boldsymbol{v}^0} = (\boldsymbol{E}_{\boldsymbol{v}^0}(\boldsymbol{x}-\boldsymbol{q}^0), \boldsymbol{B}_{\boldsymbol{v}^0}(\boldsymbol{x}-\boldsymbol{q}^0), \boldsymbol{q}^0, \boldsymbol{v}^0) \qquad (4.14)
$$

with

$$
\boldsymbol{E}_{\boldsymbol{v}} = -\nabla\phi^\varepsilon_{\boldsymbol{v}} + \boldsymbol{v}(\boldsymbol{v}\cdot\nabla\phi^\varepsilon_{\boldsymbol{v}})\ ,\ \boldsymbol{B}_{\boldsymbol{v}} = -\boldsymbol{v}\times\nabla\phi^\varepsilon_{\boldsymbol{v}}\,, \qquad (4.15)
$$

where now

$$
\widehat{\phi}^\varepsilon_{\boldsymbol{v}}(\boldsymbol{k}) = \sqrt{\varepsilon}\,\hat{\rho}(\varepsilon\boldsymbol{k})/[\boldsymbol{k}^2 - (\boldsymbol{v}\cdot\boldsymbol{k})^2]\,. \qquad (4.16)
$$

On the macroscopic scale, the scaling parameter ε can be absorbed into the "effective" charge distribution $\sqrt{\varepsilon}\rho_\varepsilon$. Its electrostatic energy,

$$
m_{\mathrm{e}} = \frac{1}{2}\int d^3k\,\varepsilon|\hat{\rho}_\varepsilon(\boldsymbol{k})|^2\frac{1}{\boldsymbol{k}^2} = \frac{1}{2}\int d^3k|\hat{\rho}(\boldsymbol{k})|^2\,\frac{1}{\boldsymbol{k}^2}\ , \qquad (4.17)
$$

is independent of ε, whereas its charge

$$\int d^3x \sqrt{\varepsilon}\, \rho_\varepsilon(\boldsymbol{x}) = \sqrt{\varepsilon}\, e \tag{4.18}$$

vanishes as $\sqrt{\varepsilon}$. Recall that ε is a book keeping device.

We argued that on the macroscopic scale the response to the external potentials in the motion of the charges is of order one. We thus expect that $\boldsymbol{q}^\varepsilon(t)$ tends to a nondegenerate limit as $\varepsilon \to 0$, i.e.

$$\lim_{\varepsilon\to 0} \boldsymbol{q}^\varepsilon(t) = \boldsymbol{r}(t) \ , \ \lim_{\varepsilon\to 0} \boldsymbol{v}^\varepsilon(t) = \boldsymbol{u}(t) \, . \tag{4.19}$$

The position $\boldsymbol{r}(t)$ and velocity $\boldsymbol{u}(t)$ on the macroscopic scale should be governed by an effective Lagrangian. In Section 3.1 we determined already the effective inertial term. If the potentials add in as usual, we have

$$L_{\text{eff}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T(\dot{\boldsymbol{q}}) - e\big(\phi_{\text{ex}}(\boldsymbol{q}) - \dot{\boldsymbol{q}} \cdot \boldsymbol{A}_{\text{ex}}(\boldsymbol{q})\big) \, , \tag{4.20}$$

which results in the equations of motion

$$\dot{\boldsymbol{r}} = \boldsymbol{u} \ , \quad m(\boldsymbol{u})\dot{\boldsymbol{u}} = e(\boldsymbol{E}_{\text{ex}}(\boldsymbol{r}) + \boldsymbol{u} \times \boldsymbol{B}_{\text{ex}}(\boldsymbol{r})) \, . \tag{4.21}$$

The velocity dependent mass $m(\boldsymbol{u})$ has a bare and a field contribution. ¿From (3.12) we conclude that

$$m(\boldsymbol{v}) = \frac{\mathrm{d}P_s(\boldsymbol{v})}{\mathrm{d}\boldsymbol{v}} \tag{4.22}$$

as a 3×3 matrix. If instead of the velocity we introduce the canonical momentum, $\boldsymbol{p}$, then the effective Hamiltonian reads

$$H_{\text{eff}}(\boldsymbol{r}, \boldsymbol{p}) = E_{\text{eff}}(\boldsymbol{p} - e\boldsymbol{A}_{\text{ex}}(\boldsymbol{r})) + e\phi_{\text{ex}}(\boldsymbol{r}) \tag{4.23}$$

with Hamilton's equations of motion

$$\dot{\boldsymbol{r}} = \nabla_{\boldsymbol{p}} H_{\text{eff}} \ , \quad \dot{\boldsymbol{p}} = -\nabla_{\boldsymbol{r}} H_{\text{eff}} \, . \tag{4.24}$$

Our plan is to establish the limit (4.19) and to investigate the corrections due to radiation losses.

4.2 Comparison with the hydrodynamic limit

In hydrodynamics one assumes that a small droplet of fluid with center $\boldsymbol{r}$ has its intrinsic velocity, $\boldsymbol{u}(\boldsymbol{r})$, and that relative to the moving frame the particles are distributed according to thermal equilibrium with density $\rho(\boldsymbol{r})$ and temperature $T(\boldsymbol{r})$. For such notions to be reasonably well defined, the hydrodynamic fields $\rho, \boldsymbol{u}, T$ must be slowly varying on the scale of the typical interparticle distance. This is how the analogy to the Maxwell–Lorentz equations arises. As for them we have three characteristic space–time scales.

(i) *Microscopic scale.* The microscopic scale is measured in units of a collision time, resp. interatomic distance. On that scale the hydrodynamics fields are frozen. Possible deviations from local equilibrium relax through collisions. To prove such a behavior one has to establish a sufficiently fast relaxation to equilibrium. For Newtonian particles no method is available. For the Maxwell field the situation is much simpler. Local deviations from the Coulomb field are transported off to infinity and are no longer seen.

(ii) *Macroscopic scale.* The macroscopic space–time scale is defined by the variation of the hydrodynamic fields. If, as before, we introduce the dimensionless scaling parameter ε, then space–time is $\mathcal{O}(\varepsilon^{-1})$ in microscopic units. On the macroscopic scale the time between collisions is $\mathcal{O}(\varepsilon)$, the interparticle distance $\mathcal{O}(\varepsilon)$, and the pair potential between the particles at positions $\boldsymbol{q}_i, \boldsymbol{q}_j$ is $V(\varepsilon^{-1}(\boldsymbol{q}_i - \boldsymbol{q}_j))$. On the macroscopic scale the hydrodynamic fields evolve according to the Euler equations. These are first order equations, which must be so, since space and time are scaled in the same way. The Euler equations are formally of Hamiltonian form. There is no dissipation, no entropy is produced. In fact, there is a slight complication here. Even for smooth initial data the Euler equations develop shock discontinuities. There the assumption of slow variation fails and shocks are a source of entropy.

(iii) *Friction scale.* In a real fluid there are frictional forces which are responsible for the relaxation to global equilibrium. One adds to the Euler equations diffusive like, second order in spatial derivatives, terms and obtains the compressible Navier–Stokes equations incorporating the shear and volume viscosity for friction in momentum transport and thermal conductivity for friction in energy transport. On the macroscopic scale these corrections are of order ε. In the same spirit, based on the full Maxwell–Lorentz equations, there will be dissipative terms of order ε emerging which have to be added to (4.21). Of course, in this context one only has to deal with ordinary differential equations as effective dynamics.

4.3 Initial slip

We adjusted the initial data for the electromagnetic field to be exactly on the charge soliton manifold, which physically means that without external forces the particle would travel forever at constant velocity accompanied by its comoving Coulombic fields. One may wonder whether such a rigid assumption is really necessary. Let us consider then times on the microscopic scale of the order $\varepsilon^{-1+\delta}$ with some small $\delta > 0$. The external forces, which are $\mathcal{O}(\varepsilon)$, are still negligible, but the microscopic time span diverges. Thus we are led to investigate the long time limit of the Abraham model for *zero* external fields. Roughly one has the following picture: initially there is some exchange of momentum and energy between particle and field, but, since the total energy is bounded, eventually the particle relaxes to some definite velocity and the field builds up its comoving Coulombic shape. Thus after a very short macroscopic time the state is already close to the charge soliton manifold from whereon the adiabatic dynamics applies.

To prove such a behavior we need little bit of preparation. Firstly we must have some decay and smoothness of the initial fields at infinity. We introduced already such a set of "good" initial data, $\mathcal{M}^\sigma$, in Section 1.5 and therefore require $Y(0) \in \mathcal{M}^\sigma, 0 < \sigma \leq 1$. Secondly, we need a notion of closeness of the fields. At a given time and far away from the particle the fields are determined by their initial data. Only close to the particle they are Coulombic. Therefore it is natural to measure closeness in the *local energy* *norm* defined by

$$\|(\boldsymbol{E},\boldsymbol{B})\|_R^2 = \frac{1}{2}\int\limits_{\{|\boldsymbol{x}|\le R\}} d^3x\big(\boldsymbol{E}(\boldsymbol{x})^2+\boldsymbol{B}(\boldsymbol{x})^2\big) \tag{4.25}$$

for given radius R.

The true solution is $Y(t) = (\boldsymbol{E}(\boldsymbol{x},t),\boldsymbol{B}(\boldsymbol{x},t),\boldsymbol{q}(t),\boldsymbol{v}(t))$ which is to be compared with the charge soliton approximation $\big(\boldsymbol{E}_{\boldsymbol{v}(t)}(\boldsymbol{x}-\boldsymbol{q}(t)),\boldsymbol{B}_{\boldsymbol{v}(t)}(\boldsymbol{x}-\boldsymbol{q}(t)),\boldsymbol{q}(t),\boldsymbol{v}(t)\big)$, cf. (3.5). We set $\boldsymbol{Z}_1(\boldsymbol{x},t)=\boldsymbol{E}(\boldsymbol{x},t)-\boldsymbol{E}_{\boldsymbol{v}(t)}(\boldsymbol{x}-\boldsymbol{q}(t))$, $\boldsymbol{Z}_2(\boldsymbol{x},t)=\boldsymbol{B}(\boldsymbol{x},t)-\boldsymbol{B}_{\boldsymbol{v}(t)}(\boldsymbol{x}-\boldsymbol{q}(t))$, $Z=(\boldsymbol{Z}_1,\boldsymbol{Z}_2)$ and want to establish that, for fixed R, $\|Z(\cdot+\boldsymbol{q}(t),t)\|_R\to 0$ for large times $t\to\infty$.

Proposition 4.1 *For the Abraham model with zero external potentials and satisfying (C) let $|e|\le\overline{e}$ according to Theorem 5.1 and let the initial data $Y(0)\in\mathcal{M}^\sigma$ for some $\sigma\in(0,1]$. Then for every $R>0$ we have*

$$\|Z(\cdot+\boldsymbol{q}(t),t)\|_R\le C_R(1+|t|)^{-1-\sigma}\,. \tag{4.26}$$

In particular, the acceleration is bounded as

$$|\dot{\boldsymbol{v}}(t)|\le C(1+|t|)^{-1-\sigma} \tag{4.27}$$

and there exists a $\boldsymbol{v}_\infty\in\mathbb{V}$ such that

$$\lim_{t\to\infty}\boldsymbol{v}(t)=\boldsymbol{v}_\infty\,. \tag{4.28}$$

Proof: From the Lorentz force equation and since $|\boldsymbol{v}(t)|\le\overline{v}<1$ we have

$$|\dot{\boldsymbol{v}}(t)|\le C\,e\,\|Z(\cdot+\boldsymbol{q}(t),t)\|_{R_\rho}\,. \tag{4.29}$$

Therefore (4.27) follows from (4.26). Then $\boldsymbol{v}(t)=\boldsymbol{v}(0)+\int\limits_0^t ds\,\dot{\boldsymbol{v}}(s)$ and $|\boldsymbol{v}(t)-\boldsymbol{v}_\infty|\le C\,(1+|t|)^{-\sigma}$.

It remains to establish (4.26), which uses the method described in Appendix 5.3. Since $Z(0)\neq 0$, (5.35) reads now

$$Z(t)=e^{\mathsf{A}t}Z(0)-\int\limits_0^t ds\,e^{\mathsf{A}(t-s)}g(s)\,. \tag{4.30}$$

For the integrand in the second term we have the bound, compare with (5.36),

$$\|e^{\mathsf{A}(t-s)}g(s)\|_{R_\rho} \le C(\overline{v})e^2(1+(t-s)^2)^{-1}\|Z(\cdot+\boldsymbol{q}(s),s)\|_{R_\rho}\,. \tag{4.31}$$

For the first term of (4.30) we note that $\boldsymbol{Z}_1(\boldsymbol{x},0)=\boldsymbol{E}^0(\boldsymbol{x})-\boldsymbol{E}_{\boldsymbol{v}^0}(\boldsymbol{x})\in\mathcal{M}^\sigma$ by assumption. Using the solution of the inhomogeneous Maxwell equations in position space and the bound (1.47) we have

$$\begin{aligned}
|\boldsymbol{Z}_1(\boldsymbol{x},t)|+|\boldsymbol{Z}_2(\boldsymbol{x},t)| \;\le\;& C\,t^{-2}\int d^3y\,\delta(|\boldsymbol{x}-\boldsymbol{y}|-t)\big(|\boldsymbol{Z}_1(\boldsymbol{y},0)|+|\boldsymbol{Z}_2(\boldsymbol{y},0)|\big)\\
&+C\,t^{-1}\int d^3y\,\delta(|\boldsymbol{x}-\boldsymbol{y}|-t)\big(|\nabla\boldsymbol{Z}_1(\boldsymbol{y},0)|+|\nabla\boldsymbol{Z}_2(\boldsymbol{y},0)|\big)\\
\le\;& C\,t^{-2}\int d^3y\,\delta(|\boldsymbol{x}-\boldsymbol{y}|-t)(1+|\boldsymbol{y}|)^{-1-\sigma}\\
&+C\,t^{-1}\int d^3y\,\delta(|\boldsymbol{x}-\boldsymbol{y}|-t)(1+|\boldsymbol{y}|)^{-2-\sigma}\,.\\
\le\;& C\,(1+t)^{-1-\sigma}\,.
\end{aligned} \tag{4.32}$$

We choose $R\ge R_\rho$. Then from (4.31) and (4.32)

$$\|Z(\cdot+\boldsymbol{q}(t),t)\|_R \le C(1+t)^{-1-\sigma}+C(\overline{v})e^2\int_0^t ds\,(1+(t-s)^2)^{-1}\|Z(\cdot+\boldsymbol{q}(s),s)\|_R\,. \tag{4.33}$$

Let $\kappa=\sup\limits_{t\ge0}(1+t)^{1+\sigma}\|Z(\cdot+\boldsymbol{q}(t),t)\|_R$. Then

$$\kappa\le C+C(\overline{v})e^2\Big(\int_0^t ds\,(1+(t-s)^2)^{-1}(1+s)^{-1-\sigma}\Big)\kappa\,, \tag{4.34}$$

which implies $\kappa<\infty$ provided $C(\overline{v})\,e^2$ is sufficiently small. □

4.4 Appendix: How small is ε ?

We consider an electron moving in an external magnetic field oriented along the z-axis, $\boldsymbol{B}_{\mathrm{ex}}=(0,0,B_0)$. The corresponding vector potential is $\boldsymbol{A}_{\mathrm{ex}}(\boldsymbol{x})=$

$\frac{1}{2}B_0(-x_2, x_1, 0)$. According to our convention the slowly varying vector potential is given by $\boldsymbol{A}_{\mathrm{ex}}(\varepsilon \boldsymbol{x}) = \frac{1}{2}\varepsilon B_0(-x_2, x_1, 0)$. Thus B_0 is a reference field strength, which we will determine, and $B = \varepsilon B_0$ is the physical field strength in the laboratory. The motion of the electron is assumed to be in the 1-2 plane and we set $\boldsymbol{v} = (\boldsymbol{u}, 0)$. According to Section 7.2, Example (ii), within a good approximation the motion of the electron is governed by

$$\gamma \dot{\boldsymbol{u}} = \omega_c(\boldsymbol{u}^{\perp} - \beta\omega_c \boldsymbol{u})\,. \tag{4.35}$$

Here $\boldsymbol{u}^{\perp} = (-u_2, u_1)$, $\omega_c = eB/m_0c$ is the cyclotron frequency, and $\beta = e^2/6\pi c^3 m_0$. The first term is the Lorentz force and the second term accounts for the radiation reaction.

We choose now the reference field B_0 such that both terms balance, i.e.

$$B_0 = (\beta e/m_0 c)^{-1}\,. \tag{4.36}$$

For electrons

$$B_0 = 1.1 \times 10^{17}\text{Gauss} \tag{4.37}$$

and even larger by a factor $(1836)^2$ for protons. For a laboratory field of 10^5 Gauss this yields

$$\varepsilon = 10^{-12}\,. \tag{4.38}$$

Written in units of B_0 (4.35) becomes

$$\gamma\,\dot{\boldsymbol{u}} = \varepsilon\omega_c^0(\boldsymbol{u}^{\perp} - \varepsilon\beta\omega_c^0\boldsymbol{u}) \tag{4.39}$$

with $\beta\omega_c^0 = 1, \quad \omega_c^0 = e\,B_0/m_0c = 1.6 \times 10^{28}/\text{sec}$. Thus friction is of relative order ε and higher order corrections are then of relative order ε^2. As to be demonstrated, the dimensionless scaling parameter ε merely serves as a book keeping device to keep track of the relative order of the various contributions.

4.5 Appendix: Point charge limit, negative bare mass

The convential point charge limit is to let the diameter of the total charge distribution $R_\rho \to 0$ such that the charge remains fixed. Physically, this means that the charge diameter is small in units of the variation of the external potential, since this is the only other length scale available. At first sight, one just seems to say that the potentials vary slowly on the scale set by the charge diameter and that hence point charge limit and adiabatic limit coincide.

To see the difference let us regard R_ρ as a small parameter, relative to some reference scale. As before we require that the total charge

$$\int d^3x\, \rho(\boldsymbol{x}) = e \tag{4.40}$$

is independent of R_ρ. The electrostatic energy diverges then as

$$\frac{1}{2}\int d^3k\, |\hat{\rho}(\boldsymbol{k})|^2\, \frac{1}{\boldsymbol{k}^2} \cong R_\rho^{-1}\overline{m}_\mathrm{e} \tag{4.41}$$

for small R_ρ, where $\overline{m}_\mathrm{e}$ is the electrostatic energy of the charge distribution at the reference scale which is independent of R_ρ. In particular, the ratio field mass to bare mass grows as R_ρ^{-1} in the point charge limit and remains constant in the adiabatic limit.

To display the order of magnitude of the various dynamical contributions we resort again to our standard example of an electron in a uniform magnetic field $\boldsymbol{B} = B\widehat{\boldsymbol{n}}$, $\widehat{\boldsymbol{n}} = (0,0,1)$ with B of the order of 1 Tesla $= 10^4$ Gauss, say. It suffices to consider small velocities. In the adiabatic limit we set $B = \varepsilon B_0$ where the reference field $B_0 = 1.1 \times 10^{17}$ Gauss, compare with Section 7.2. Up to higher order corrections, the motion of the electron is then governed by

$$\big(m_\mathrm{b} + \frac{4}{3}\, m_\mathrm{e}\big)\dot{\boldsymbol{v}} = \frac{e}{c}\,\varepsilon B_0(\boldsymbol{v}\times\widehat{\boldsymbol{n}}) + \frac{e^2}{6\pi c^3}\,\ddot{\boldsymbol{v}} + \mathcal{O}(\varepsilon^3) \tag{4.42}$$

on the microscopic scale. Going over to the macroscopic time scale, $t' = \varepsilon^{-1}t$, (4.42) becomes

$$\big(m_{\mathrm{b}} + \frac{4}{3}\,m_{\mathrm{e}}\big)\,\dot{\boldsymbol{v}} = \frac{e}{c}\,B_0(\boldsymbol{v}\times\widehat{\boldsymbol{n}}) + \frac{e^2}{6\pi c^3}\,\varepsilon\ddot{\boldsymbol{v}} + \mathcal{O}(\varepsilon^2)\,. \tag{4.43}$$

Setting $m_0 = m_{\mathrm{b}} + \frac{4}{3}\,m_{\mathrm{e}}$, $\omega_c^0 = e\,B_0/m_0 c$, $\beta = e^2/6\pi c^3 m_0$, and restricting to the critical manifold, as will be explained in Chapter 7, Equation (4.43) becomes

$$\dot{\boldsymbol{v}} = \omega_c^0\big(\boldsymbol{v}\times\widehat{\boldsymbol{n}} + \varepsilon\beta\omega_c^0\,(\boldsymbol{v}\times\widehat{\boldsymbol{n}})\times\widehat{\boldsymbol{n}}\big) + \mathcal{O}(\varepsilon^2)\,, \tag{4.44}$$

equivalently, on the microscopic time scale

$$\dot{\boldsymbol{v}} = \omega_c\big(\boldsymbol{v}\times\widehat{\boldsymbol{n}} + \beta\omega_c(\boldsymbol{v}\times\widehat{\boldsymbol{n}})\times\widehat{\boldsymbol{n}}\big) + \mathcal{O}(\varepsilon^3) \tag{4.45}$$

with the cyclotron frequency $\omega_c = e\,\varepsilon B_0/m_0 c = eB/m_0 c$.

For the point charge limit we rely on the Taylor expansion of Section 5.2. Then, for small velocities,

$$\big(m_{\mathrm{b}} + R_\rho^{-1}\,\frac{4}{3}\,\overline{m}_{\mathrm{e}}\big)\,\dot{\boldsymbol{v}} = \frac{e}{c}B(\boldsymbol{v}\times\widehat{\boldsymbol{n}}) + \frac{e^2}{6\pi c^3}\,\ddot{\boldsymbol{v}} + \mathcal{O}(R_\rho)\,. \tag{4.46}$$

Since based on the same expansion, as long as no limit is taken, of course, we can switch back and forth between (4.46) and (4.42), resp. (4.43), provided the appropriate units are used. This can be seen more easily if we accept momentarily the differential–difference equation

$$m_{\mathrm{b}}\dot{\boldsymbol{v}}(t) = e\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}(t)) + c^{-1}\boldsymbol{v}(t)\times\boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q}(t))\big) + \frac{e^2}{12\pi c R_\rho^2}\,\big(\boldsymbol{v}(t-2R_\rho/c) - \boldsymbol{v}(t)\big)\,, \tag{4.47}$$

cf. (5.14), as an approximate equation for the motion of the charge. If we expand in the charge diameter R_ρ, then

$$\big(m_{\mathrm{b}} + \frac{e^2}{6\pi R_\rho c^2}\big)\dot{\boldsymbol{v}} = e(\boldsymbol{E}_{\mathrm{ex}} + c^{-1}\boldsymbol{v}\times\boldsymbol{B}_{\mathrm{ex}}) + \frac{e^2}{6\pi c^3}\,\ddot{\boldsymbol{v}} + \mathcal{O}(R_\rho)\,, \tag{4.48}$$

which is the analogue of (4.46). On the other hand, if we assume that the external fields are slowly varying, as discussed in Section (4.1), then on the macroscopic scale

$$\begin{aligned}\varepsilon m_{\mathrm{b}}\dot{\boldsymbol{v}}(t) &= \varepsilon\, e\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}(t)) + c^{-1}\boldsymbol{v}(t)\times\boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q}(t))\big) \\ &\quad + \frac{e^2}{12\pi c R_\rho^2}\big(\boldsymbol{v}(t-2\varepsilon R_\rho/c) - \boldsymbol{v}(t)\big)\,, \end{aligned} \tag{4.49}$$

where R_ρ is now regarded as fixed. Taylor expansion in ε yields

$$\big(m_{\mathrm{b}} + \frac{e^2}{6\pi R_\rho c^2}\big)\dot{\boldsymbol{v}} = e(\boldsymbol{E}_{\mathrm{ex}} + c^{-1}\boldsymbol{v}\times\boldsymbol{B}_{\mathrm{ex}}) + \varepsilon\,\frac{e^2}{6\pi c^3}\,\ddot{\boldsymbol{v}} + \mathcal{O}(\varepsilon^2) \tag{4.50}$$

which is the analogue of (4.43).

As can be seen from (4.46), in the point charge limit the total mass becomes so large that the particle hardly responds to the magnetic field. The only way out seems to formally compensate the diverging $R_\rho^{-1}(4/3)\overline{m}_{\mathrm{e}}$ by setting $m_{\mathrm{b}} = -R_\rho^{-1}(4/3)\overline{m}_{\mathrm{e}} + m_{\mathrm{exp}}$. But this is asking for trouble, since the energy (1.42) is no longer bounded from below and potential energy can be transferred to kinetic mechanical energy without limit. To see this mechanism in detail we consider the Abraham model with $\boldsymbol{B}_{\mathrm{ex}} = 0$ and ϕ_{ex} varying only along the 1-axis. The bare mass of the particle is now $-m_{\mathrm{b}}$ with $m_{\mathrm{b}} > 0$, as before. We set $\boldsymbol{q}(t) = (q_t, 0, 0)$, $\boldsymbol{v}(t) = (v_t, 0, 0)$, $\boldsymbol{E}_{\mathrm{ex}} = (-\phi'(q), 0, 0)$. ϕ is assumed to be strictly convex with minimum at $q = 0$. Initially the particle is at rest at the minimum of the potential. Thus $\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(\boldsymbol{x})$ from (3.5) and $\boldsymbol{B}(\boldsymbol{x}, 0) = 0$. We now give the particle a slight push to the right, which means $q_0 = 0, v_0 > 0$. By conservation of energy

$$\begin{aligned}&-m_{\mathrm{b}}c^2\,\gamma(v_t) + e\,\phi(q_t) + \frac{1}{2}\int d^3x\,\big(\boldsymbol{E}(\boldsymbol{x},t)^2 + \boldsymbol{B}(\boldsymbol{x},t)^2\big) \\ &= \; -m_{\mathrm{b}}c^2\gamma(v_0) + e\phi(q_0) + \frac{1}{2}\int d^3x\,\boldsymbol{E}(\boldsymbol{x},0)^2\,. \end{aligned} \tag{4.51}$$

We split $\boldsymbol{E}$ into longitudinal and transverse components, $\boldsymbol{E} = \boldsymbol{E}_{\|} + \boldsymbol{E}_{\perp}$, $\widehat{\boldsymbol{E}}_{\|} =$

$|\boldsymbol{k}|^{-2}\boldsymbol{k}(\boldsymbol{k}\cdot\widehat{\boldsymbol{E}})$. Clearly $\int d^3x\, \boldsymbol{E}_{\|}\boldsymbol{E}_{\perp} = 0$ and therefore

$$\begin{aligned}\int d^3x\, \boldsymbol{E}(\boldsymbol{x},t)^2 \;&\geq\; \int d^3x\, \boldsymbol{E}_{\|}(\boldsymbol{x},t)^2 = \int d^3k\, |\boldsymbol{k}|^{-2}(\boldsymbol{k}\cdot\widehat{\boldsymbol{E}}(\boldsymbol{k},t))^2 \\ &= \int d^3k\, |\boldsymbol{k}|^{-2}\, |\hat{\rho}(\boldsymbol{k})|^2 = \int d^3x\, \boldsymbol{E}(\boldsymbol{x},0)^2\,, \end{aligned} \tag{4.52}$$

since the initial field has zero transverse component. Inserting in (4.51) yields

$$\dot{q}_t^2 \geq 1 - \big[\gamma(v_0) + (e/m_{\mathrm{b}}c^2)(\phi(q_t) - \phi(q_0))\big]^{-2}\,. \tag{4.53}$$

Since $\gamma(v_0) > 1$, $\dot{q}_t > 0$ for short times. As the particle moves to the right $(\phi(q_t)-\phi(q_0))$ is increasing and therefore $\dot{q}_t \to 1$ and $q_t \to \infty$ as $t \to \infty$. Note that v_0 and m_{b} can be arbitrarily small. Not so surprisingly, the Abraham model with a negative bare mass behaves rather unphysically. A tiny initial kick suffices to generate a runaway solution.

The point charge limit is honored through a long tradition, which however seems to have constantly overlooked that physically it is more appropriate to have the external potentials slowly varying on the scale of a fixed size charge distribution. Then there is no need to introduce a negative bare mass and there are no runaway solutions.

Notes and References

ad 4.1: The importance of slowly varying external potentials has been emphasized repeatedly. It is somewhat surprising then, that this notion was apparently never transcribed to the equations of motion. In the context of charges and the Maxwell field the adiabatic limit was first introduced in Komech, Kunze, Spohn (1999) and in Kunze, Spohn (1999).

ad 4.2: A more detailed discussion of the hydrodynamic limit can be found in Spohn (1991).

ad 4.3: The initial slip as discussed here is a side–remark in Komech, Kunze, Spohn (1999), where the adiabatic limit for a scalar wave field is studied.

Komech, Spohn (1998) prove the long–time asymptotics without the restriction $|e| < \overline{e}$ but imposing the Wiener condition instead. Orbital stability is established by Bambusi, Galgani (1993).

ad 4.4: In the early work on the classical electron theory, one simply expanded in R_ρ. R_ρ was considered to be small, but finite, say, of the order of the classical electron radius. Schott (1912) pushed the expansion to include the radiation reaction which he concluded to be "quite inappreciable in this and probably in all practical cases". According to Frenkel (1925) the electrodynamics of point, rather than extended, charges is an idea of P. Ehrenfest. The point charge limit is at the core of the famous Dirac (1938) paper, cf. Section 2.2. Since then the limit $m_\mathrm{b} \to -\infty$ is a standard piece of the theory, reproduced in textbooks and survey articles. The negative bare mass was soon recognized as a source of instability. We refer to the review by Erber (1961). On a linearized level stability is studied by Wildermuth (1955) and by Moniz, Sharp (1977). Bambusi, Noja (1993) discuss the point charge limit in the dipole approximation and show that in the limit the true solution is well–approximated by the linear Lorentz–Dirac equation with the full solution manifold, physical and unphysical, explored. The bound (4.53) is taken from Bauer, Dürr (1999), which is the only quantitative handling of the instability for the full nonlinear problem.

5 Self – Force

The inhomogeneous Maxwell equations have been solved in (1.16), (1.17). Thus it is natural to insert them in the Lorentz force equation in order to obtain a closed, albeit memory equation for the position of the particle.

According to (1.16), (1.17) the Maxwell fields are a sum of initial and retarded terms. We discuss first the contribution from the initial fields. By our specific choice of initial conditions they have the representation, for $t \geq 0$,

$$\begin{aligned} \boldsymbol{E}_{(0)}(\boldsymbol{x},t) &= -\int\limits_{-\infty}^{0} ds \int d^3y \left(\nabla G_{t-s}\left(\boldsymbol{x}-\boldsymbol{y}\right)\rho(\boldsymbol{y}-\boldsymbol{q}^0-\boldsymbol{v}^0 s)\right. \qquad (5.1)\\ &\qquad \left. +\partial_t G_{t-s}\left(\boldsymbol{x}-\boldsymbol{y}\right)\boldsymbol{v}^0\,\rho(\boldsymbol{y}-\boldsymbol{q}^0-\boldsymbol{v}^0 s)\right), \\ \boldsymbol{B}_{(0)}(\boldsymbol{x},t) &= \int\limits_{-\infty}^{0} ds \int d^3y\, \nabla\times G_{t-s}\left(\boldsymbol{x}-\boldsymbol{y}\right)\boldsymbol{v}^0\rho(\boldsymbol{y}-\boldsymbol{q}^0-\boldsymbol{v}^0 s)\,, \quad (5.2) \end{aligned}$$

compare with (3.21), (3.22). (3.12) and (1.16), can be checked by going to Fourier space and using (1.17) as initial condition in (5.1), (5.2). Since G_t is concentrated on the light cone, we conclude from (5.1), (5.2) that $\boldsymbol{E}_{(0)}(\boldsymbol{x},t)=0$, $\boldsymbol{B}_{(0)}(\boldsymbol{x},t)=0$ for $|\boldsymbol{q}^0-\boldsymbol{x}|\leq t-R_\rho$. If we would have allowed for more general initial data, such a property would hold only asymptotically for large t.

Next we note that by energy conservation the particle cannot travel too far. Using the bound on the potential, we can find a $\overline{v}<1$ such that

$$\sup_{t\in\mathbb{R}} |\boldsymbol{v}^\varepsilon(t)| < \overline{v} < 1\,, \tag{5.3}$$

cf. Equation (5.26). The charge distribution vanishes for $|\boldsymbol{x}-\boldsymbol{q}(t)|\geq R_\rho$. Since $|\dot{\boldsymbol{q}}(t)|\leq\overline{v}$, the initial fields and the charge distribution have zero overlap once

$$t \geq \overline{t}_\rho = 2R_\rho/(1-\overline{v})\,. \tag{5.4}$$

Thus for $t>\overline{t}_\rho$ the initial fields make no contribution to the self–force and it remains to discuss the effect of the retarded fields.

We insert (1.12), (1.13) into the Lorentz force equation setting the external potentials equal to zero for a while. Then on the macroscopic scale, for $t \geq \varepsilon \bar{t}_\rho$,

$$\frac{d}{dt}\big(m_{\mathrm{b}}\gamma\,\boldsymbol{v}^\varepsilon(t)\big) = \boldsymbol{F}^\varepsilon_{\mathrm{self}}\,(t) \tag{5.5}$$

with the self–force

$$\begin{aligned}\boldsymbol{F}^\varepsilon_{\mathrm{self}}\,(t) \;=\; & \int_0^t ds\,\varepsilon \int d^3k\,|\hat{\rho}(\varepsilon\boldsymbol{k})|^2\, e^{-i\boldsymbol{k}\cdot(\boldsymbol{q}^\varepsilon(t)-\boldsymbol{q}^\varepsilon(s))} \\ & \big((|\boldsymbol{k}|^{-1}\,\sin|\boldsymbol{k}|(t-s))i\boldsymbol{k} - (\cos|\boldsymbol{k}|(t-s))\boldsymbol{v}^\varepsilon(s) \\ & -(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|(t-s))\,\boldsymbol{v}^\varepsilon(t)\times(i\boldsymbol{k}\times\boldsymbol{v}^\varepsilon(s))\big)\,. \end{aligned} \tag{5.6}$$

(5.5) is exact under the stated conditions on the initial fields. No information has been discarded. The interaction with the field has been merely transcribed into a memory term. To make further progress we have to use a suitable approximation which exploits that the external forces are slowly varying. Since this corresponds to small ε, we just have to Taylor expand $\boldsymbol{F}^\varepsilon_{\mathrm{self}}\,(t)$, which is carried out in Section 5.2 with the proper justification left for Appendix 5.3. But before, also to make contact with previous work, we have a closer look at the memory term.

5.1 Memory equation

Equation (5.4) can be further simplified where we set $\varepsilon = 1$ in this subsection. By partial integration

$$
\begin{aligned}
&\int_0^t ds \int d^3k \, |\hat{\rho}(\boldsymbol{k})|^2 \, e^{-i\boldsymbol{k}\cdot(\boldsymbol{q}(t)-\boldsymbol{q}(s))} \boldsymbol{v}(s) \, \frac{d}{ds} \, |\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}|(t-s) \\
&= \; - \int d^3k \, |\hat{\rho}(\boldsymbol{k})|^2 \, e^{-i\boldsymbol{k}\cdot(\boldsymbol{q}(t)-\boldsymbol{q}(0))} \boldsymbol{v}(0) |\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}| t \\
&\quad - \int_0^t ds \int d^3k \, |\hat{\rho}(\boldsymbol{k})|^2 \, e^{-i\boldsymbol{k}\cdot(\boldsymbol{q}(t)-\boldsymbol{q}(s))} (|\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}|(t-s)) \big(\dot{\boldsymbol{v}}(s) + i(\boldsymbol{k}\cdot\boldsymbol{v}(s))\boldsymbol{v}(s)\big) \, .
\end{aligned}
\tag{5.7}
$$

Since $t \geq \bar{t}_\rho$, the boundary term vanishes. Inserting (5.7) into (5.6), returning to physical space, and setting $t - s = \tau$, we have for $t \geq \bar{t}_\rho$

$$
\begin{aligned}
\boldsymbol{F}_{\text{self}}\,(t) \;=\; -\int_0^\infty d\tau \, \big[& \dot{\boldsymbol{v}}(t-\tau) + (1 - \boldsymbol{v}(t)\cdot\boldsymbol{v}(t-\tau)) \nabla_{\boldsymbol{x}} \qquad (5.8) \\
& + \boldsymbol{v}(t-\tau)(\boldsymbol{v}(t) - \boldsymbol{v}(t-\tau)) \cdot \nabla_{\boldsymbol{x}} \big] W_t(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{q}(t)-\boldsymbol{q}(t-\tau)} \, ,
\end{aligned}
$$

where, as in (1.52),

$$
W_t(\boldsymbol{x}) = \int d^3k \, |\hat{\rho}(\boldsymbol{k})|^2 \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} |\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}| t \, . \tag{5.9}
$$

In (5.8) we have extended the integration to ∞, since the integrand vanishes for $\tau \geq \bar{t}_\rho$. Carrying out the angle integrations in (5.9) we obtain

$$
W_t(\boldsymbol{x}) = |\boldsymbol{x}|^{-1} \big(h(|\boldsymbol{x}| + t) - h(|\boldsymbol{x}| - t) \big) \, , \tag{5.10}
$$

$$
h(w) = 2\pi \int_0^\infty dk \, g(k) \cos kw \tag{5.11}
$$

with $g(|\boldsymbol{k}|) = |\hat{\rho}(\boldsymbol{k})|^2$. Since ρ vanishes for $|\boldsymbol{x}| \geq R_\rho$, $h(w) = 0$ for $|w| \geq 2R_\rho$. Note that $|\boldsymbol{q}(t) - \boldsymbol{q}(t-\tau)| \leq \bar{v}\,\tau$. Thus for $t \geq \bar{t}_\rho$ we indeed have

$W_t(\boldsymbol{q}(t) - \boldsymbol{q}(t-\tau)) = 0$, as claimed before. $\boldsymbol{F}_{\text{self}}(t)$ has a finite memory extending backwards in time up to $t - \bar{t}_\rho$.

To go beyond (5.10) one has use a specific ρ. Two, at the time popular, choices are $\rho_s(\boldsymbol{x}) = e(4\pi R_\rho^2)^{-1}\,\delta(|\boldsymbol{x}| - R_\rho)$ and $\rho_b(\boldsymbol{x}) = e\,(4\pi R_\rho^3/3)^{-1}$ for $|\boldsymbol{x}| \le R_\rho$, $\rho_b(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}| \ge R_\rho$. For the uniformly charged sphere one finds

$$h(R_\rho w) = \begin{cases} e^2(8\pi R_\rho)^{-1}(1 - |w|/2) & \text{for} \quad |w| \le 2\,, \\ 0 & \text{for} \quad |w| \ge 2\,, \end{cases} \tag{5.12}$$

and for the uniformly charged ball

$$h(R_\rho w) = \begin{cases} e^2(8\pi R_\rho)^{-1}\,\tfrac{9}{8}\,\widetilde{h} * \widetilde{h}(w) & \text{for} \quad |w| \le 2\,, \\ 0 & \text{for} \quad |w| \ge 2\,, \end{cases} \tag{5.13}$$

with $\widetilde{h}(w) = (1 - w^2)\mathbb{1}_{\{|w|\le 1\}}$.

For the charged sphere $W_t(\boldsymbol{x})$ is piecewise linear and, by first taking the gradient of W, the time integrations simplify. In the approximation of small velocities the motion of the charged particle is then governed by the differential–difference equation

$$m_{\text{b}}\dot{\boldsymbol{v}}(t) = e\big(\boldsymbol{E}_{\text{ex}}(\boldsymbol{q}(t)) + \boldsymbol{v}(t) \times \boldsymbol{B}_{\text{ex}}(\boldsymbol{q}(t))\big) + \frac{e^2}{12\pi R_\rho^2}\big(\boldsymbol{v}(t - 2R_\rho) - \boldsymbol{v}(t)\big)\,, \tag{5.14}$$

where we have reintroduced the external fields.

The memory equation (5.14) is of suggestive simplicity. However, to have a well defined dynamics one has to prescribe $\boldsymbol{q}(0)$ and $\boldsymbol{v}(t)$ for $-2R_\rho \le t \le 0$. No instruction for that choice is provided by the supporters of differential–difference equations. More importantly, R_ρ is a small parameter and we might allow on the top of small velocities a further error of $\mathcal{O}(R_\rho)$ by Taylor expanding in (5.14) to obtain

$$m_{\text{b}}\dot{\boldsymbol{v}} = e\big(\boldsymbol{E}_{\text{ex}} + \boldsymbol{v} \times \boldsymbol{B}_{\text{ex}}\big) - \frac{e^2}{6\pi R_\rho}\,\dot{\boldsymbol{v}} + \frac{e^2}{6\pi}\,\ddot{\boldsymbol{v}} + \mathcal{O}(R_\rho)\,. \tag{5.15}$$

(5.15) is a differential equation and only $\boldsymbol{q}, \boldsymbol{v}, \dot{\boldsymbol{v}}$ are needed as initial data. As to be discussed in Section 6, in fact $\dot{\boldsymbol{v}}(0)$ is determined by $\boldsymbol{q}(0), \boldsymbol{v}(0)$, since the physical solution has to lie on the critical manifold of (5.15).

5.2 Taylor expansion

We return to Equation (5.5). As will be explained in Section 5.3 we know that

$$\sup_{t\in\mathbb{R}} |\ddot{\boldsymbol{q}}^\varepsilon(t)| \le C,\ \sup_{t\in\mathbb{R}} |\dddot{\boldsymbol{q}}^\varepsilon(t)| \le C,\ \sup_{t\in\mathbb{R}} |\ddddot{\boldsymbol{q}}^\varepsilon(t)| \le C \tag{5.16}$$

uniformly in ε, provided the total charge e is sufficiently small. This smallness condition only reflects that at present we do not know how to do better mathematically. Physically we expect (5.16) to hold no matter how large e.

Because of (5.16) we are allowed to Taylor expand in (5.6). To simplify notation we set $\boldsymbol{v}^\varepsilon(t) = \boldsymbol{v}$ and $t - s = \tau$. Then

$$\boldsymbol{v}^\varepsilon(s) = \boldsymbol{v}^\varepsilon(t-\tau) = \boldsymbol{v} - \dot{\boldsymbol{v}}\tau + \frac{1}{2}\ddot{\boldsymbol{v}}\tau^2 + \mathcal{O}(\tau^3)\,, \tag{5.17}$$

$$\begin{aligned} e^{-i\boldsymbol{k}\cdot(\boldsymbol{q}^\varepsilon(t)-\boldsymbol{q}^\varepsilon(s))} &= e^{-i\boldsymbol{k}\cdot(\boldsymbol{q}^\varepsilon(t)-\boldsymbol{q}^\varepsilon(t-\tau))} = e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})\tau}\Big(1 + \frac{1}{2}\tau^2 i(\boldsymbol{k}\cdot\dot{\boldsymbol{v}}) - \frac{1}{6}\tau^3\, i(\boldsymbol{k}\cdot\ddot{\boldsymbol{v}}) \\ &\quad - \frac{1}{2}\big(\frac{1}{2}\tau^2(\boldsymbol{k}\cdot\dot{\boldsymbol{v}}) - \frac{1}{6}\tau^3(\boldsymbol{k}\cdot\ddot{\boldsymbol{v}})\big)^2 + \mathcal{O}((|\boldsymbol{k}|\tau^2)^3)\Big)\,. \end{aligned} \tag{5.18}$$

Inserting in (5.6) and substituting $s' = \varepsilon^{-1}s,\ \boldsymbol{k}' = \varepsilon\boldsymbol{k}$ yields

$$\begin{aligned}
\boldsymbol{F}^{\varepsilon}_{\mathrm{self}}(t) \;=\; & \int_0^{\varepsilon^{-1}t} d\tau\,\varepsilon^{-1}\int d^3k|\hat{\rho}(\boldsymbol{k})|^2\,e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})\tau} \\
& \Big\{(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)i\boldsymbol{k} - (\cos|\boldsymbol{k}|\tau)(\boldsymbol{v} - \varepsilon\tau\dot{\boldsymbol{v}} + \frac{1}{2}\,\varepsilon^2\tau^2\ddot{\boldsymbol{v}}) \\
& -(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\big(\boldsymbol{v}\times(i\boldsymbol{k}\times\boldsymbol{v}) - \boldsymbol{v}\times(i\boldsymbol{k}\times\varepsilon\tau\dot{\boldsymbol{v}}) + \frac{1}{2}\,\boldsymbol{v}\times(i\boldsymbol{k}\times\varepsilon^2\tau^2\ddot{\boldsymbol{v}})\big) \\
& +\frac{1}{2}\,\varepsilon\tau^2 i(\boldsymbol{k}\cdot\dot{\boldsymbol{v}})\big((|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)i\boldsymbol{k} - (\cos|\boldsymbol{k}|\tau)(\boldsymbol{v} - \varepsilon\tau\dot{\boldsymbol{v}}) \\
& -(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)(\boldsymbol{v}\times(i\boldsymbol{k}\times\boldsymbol{v}) - \boldsymbol{v}\times(i\boldsymbol{k}\times\varepsilon\tau\dot{\boldsymbol{v}}))\big) \\
& +\big(-\frac{1}{6}\,\varepsilon^2\tau^3 i(\boldsymbol{k}\cdot\ddot{\boldsymbol{v}}) - \frac{1}{8}\,\varepsilon^2\tau^4(\boldsymbol{k}\cdot\ddot{\boldsymbol{v}})^2\big)\big((|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)i\boldsymbol{k} \\
& -(\cos|\boldsymbol{k}|\tau)\boldsymbol{v} - (|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)(\boldsymbol{v}\times(i\boldsymbol{k}\times\boldsymbol{v}))\big)\Big\} + \mathcal{O}(\varepsilon^2)\,. \qquad (5.19)
\end{aligned}$$

The terms proportional to ε^{-1} cancel by symmetry. We sort all other terms,

$$
\begin{aligned}
\boldsymbol{F}^{\varepsilon}_{\mathrm{self}}(t) &= \int d^3k\,|\hat{\rho}(\boldsymbol{k})|^2 \\
&\Big\{\big(-(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\nabla_{\boldsymbol{v}}+\dot{\boldsymbol{v}}(\boldsymbol{v}\cdot\nabla_{\boldsymbol{v}})\big)\int\limits_0^{\varepsilon^{-1}t} d\tau e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})\tau}(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau) \\
&+\big(\dot{\boldsymbol{v}}+\frac{1}{2}\,\boldsymbol{v}(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})\big)\int\limits_0^{\varepsilon^{-1}t} d\tau\,\tau e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})\tau}(\cos|\boldsymbol{k}|\tau) \\
&+\varepsilon\Big(\frac{1}{2}\big[-(\boldsymbol{v}^2-1)(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})\nabla_{\boldsymbol{v}}+\boldsymbol{v}(\boldsymbol{v}\cdot\nabla_{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})+(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\nabla_{\boldsymbol{v}} \\
&-\ddot{\boldsymbol{v}}(\boldsymbol{v}\cdot\nabla_v)\big]+\frac{1}{6}\big[-(1-\boldsymbol{v}^2)(\ddot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})\nabla_{\boldsymbol{v}}-\boldsymbol{v}(\boldsymbol{v}\cdot\nabla_{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}}) \\
&+3(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})\nabla_{\boldsymbol{v}}-3\dot{\boldsymbol{v}}(\boldsymbol{v}\cdot\nabla_{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})\big]+\frac{1}{8}\big[(\boldsymbol{v}^2-1)(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})^2\nabla_{\boldsymbol{v}} \\
&-\boldsymbol{v}(\boldsymbol{v}\cdot\nabla_{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})^2\big]\Big)\int\limits_0^{\varepsilon^{-1}t} d\tau\,\tau e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})\tau}(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau) \\
&+\varepsilon\Big(-\ddot{\boldsymbol{v}}-\frac{1}{6}\big[\boldsymbol{v}(\ddot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})+3\dot{\boldsymbol{v}}(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})\big]\Big) \\
&\int\limits_0^{\varepsilon^{-1}t} d\tau\,\tau^2 e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})\tau}\cos|\boldsymbol{k}|\tau\Big\}+\mathcal{O}(\varepsilon^2)\,.
\end{aligned}
\tag{5.20}
$$

To take the limit $\varepsilon\to 0$ we go back to position space and use the fundamental solution of the wave equation. Then, for $p=0,1$,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int\limits_0^{\varepsilon^{-1}t} d\tau\int d^3k|\hat{\rho}(\boldsymbol{k})|^2\,e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})\tau}(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\,\tau^p \\
&\quad\int\limits_0^{\infty} dt\int d^3x\int d^3y\,\rho(\boldsymbol{x})\rho(\boldsymbol{y})\,\frac{1}{4\pi t}\,\delta(|\boldsymbol{x}+\boldsymbol{v}t-\boldsymbol{y}|-t)\,t^p \\
&=\begin{cases}\int d^3k\,|\hat{\rho}(\boldsymbol{k})|^2[\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-1} & \text{for}\quad p=0\,,\\ \int d^3x\,\rho(\boldsymbol{x})\int d^3y\rho(\boldsymbol{y})\,(\gamma^2/4\pi) & \text{for}\quad p=1\,.\end{cases}
\end{aligned}
\tag{5.21}
$$

By the same method

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int\limits_0^{\varepsilon^{-1}t} d\tau \int d^3k\,|\hat\rho(\boldsymbol{k})|^2\, e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})\tau}\,\tau^{1+p}\,\frac{d}{d\tau}\,(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau) && (5.22)\\
&= \; -\big(1+p+(\boldsymbol{v}\cdot\nabla_{\boldsymbol{v}})\big)\int\limits_0^{\infty} dt \int d^3k\,|\hat\rho(\boldsymbol{k})|^2\, e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})t}(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|t)t^p\\
&= \begin{cases} -\int d^3k\,|\hat\rho(\boldsymbol{k})|^2(\boldsymbol{k}^2+(\boldsymbol{k}\cdot\boldsymbol{v})^2)[\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-2} & \text{for}\quad p=0\,,\\ -\int d^3x\,\rho(\boldsymbol{x})\int d^3y\rho(\boldsymbol{y})\,(2\gamma^4/4\pi) & \text{for}\quad p=1\,. \end{cases}
\end{aligned}
$$

Collecting all terms the final result reads

$$
\begin{aligned}
\boldsymbol{F}^{\varepsilon}_{\mathrm{self}}(t) \;&=\; -m_{\mathrm{f}}(\boldsymbol{v})\dot{\boldsymbol{v}} + \varepsilon(e^2/6\pi)\,\big[\gamma^4(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} + 3\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{v}\\
&\quad +3\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \gamma^2\ddot{\boldsymbol{v}}\big] + \mathcal{O}(\varepsilon^2) \qquad (5.23)
\end{aligned}
$$

with

$$
\begin{aligned}
m_{\mathrm{f}}(\boldsymbol{v}) \;&=\; m_{\mathrm{e}}\Big[\big(|\boldsymbol{v}|^{-4}\gamma^2(3-\boldsymbol{v}^2) - (2|\boldsymbol{v}|^5)^{-1}(3+\boldsymbol{v}^2)\,\log\frac{1+|\boldsymbol{v}|}{1-|\boldsymbol{v}|}\big)|\boldsymbol{v}\rangle\langle\boldsymbol{v}|\\
&\quad +\big(-|\boldsymbol{v}|^{-2} + (2|\boldsymbol{v}|^3)^{-1}\,(1+\boldsymbol{v}^2)\,\log\frac{1+|\boldsymbol{v}|}{1-|\boldsymbol{v}|}\big)\mathbb{1}\Big]\,. \qquad (5.24)
\end{aligned}
$$

Note that $m_{\mathrm{f}}(\boldsymbol{v}) = \mathrm{d}(\boldsymbol{P}_{\mathrm{s}} - m_{\mathrm{b}}\gamma\boldsymbol{v})/\mathrm{d}\boldsymbol{v}$ as a 3×3 matrix.

Up to order ε, $\boldsymbol{F}^{\varepsilon}_{\mathrm{self}}(t)$ consists of two parts with a rather different character. The term $-m_{\mathrm{f}}(\boldsymbol{v})\dot{\boldsymbol{v}}$ is the contribution from the electromagnetic field to the change in total momentum. We computed this term already in Section 3.1 via a completely different route. As emphasized there, since the Abraham model is semi–relativistic, the velocity dependence of m_{f} has no reason to be of relativistic form and indeed it is not. The term proportional to ε in (5.23) is the *radiation reaction*. Again there is no a priori reason to expect it to be relativistic, but in fact it is. Using the four–vector notation of Section 1.6, the radiation reaction can be rewritten as

$$
\varepsilon(e^2/6\pi)[\ddot{v}^{\mu} - \dot{v}^{\lambda}\dot{v}_{\lambda}v^{\mu}]\,. \qquad (5.25)
$$

5.3 Appendix: How to bound the acceleration?

We return to the microscopic time scale. From the conservation of energy together with condition (P), we have

$$\begin{aligned} E_s(\boldsymbol{v}^0) + (\phi * \rho)(\varepsilon \boldsymbol{q}^0) &= \mathcal{E}(\boldsymbol{E}^0, \boldsymbol{B}^0, \boldsymbol{q}^0, \boldsymbol{v}^0) = \mathcal{E}(\boldsymbol{E}(t), \boldsymbol{B}(t), \boldsymbol{q}(t), \boldsymbol{v}(t)) \\ &\geq m_{\mathrm{b}}\gamma(\boldsymbol{v}(t)) + e\,\phi_{\min} \end{aligned} \tag{5.26}$$

and therefore

$$\sup_t |\boldsymbol{v}(t)| \leq \overline{v} < 1\,. \tag{5.27}$$

The external forces are of order ε. Superficially the self-force is of order one. However for a Coulombic field the self–force vanishes. Thus if we could show that the deviations from the appropriate local soliton field are of order ε, then altogether

$$\sup_t |\dot{\boldsymbol{v}}(t)| \leq C\,\varepsilon \tag{5.28}$$

with C a suitable constant. This is what we want to prove. We will not keep track of the constants and the value of C changes from equation to equation. We make sure however that C depends only on $\overline{v}$ and is thus determined by the initial conditions. Of course, to justify the Taylor expansion of Section 5.2, we also need analoguous estimates on higher derivatives, which can be obtained with more effort through the same scheme. Here we want to explain how to get (5.28) and why we need e to be sufficiently small, at least at present.

¿From the equations of motion we have

$$\begin{aligned} \dot{\boldsymbol{v}} &= m_0(\boldsymbol{v})^{-1}\Big[\varepsilon \int d^3x\,\rho(\boldsymbol{x}-\boldsymbol{q})\big(\boldsymbol{E}_{\mathrm{ex}}(\varepsilon\boldsymbol{x}) + \boldsymbol{v}\times\boldsymbol{B}_{\mathrm{ex}}(\varepsilon\boldsymbol{x})\big) \\ &\qquad + \int d^3x\,\rho(\boldsymbol{x}-\boldsymbol{q})\big(\boldsymbol{E}(\boldsymbol{x}) + \boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{x})\big)\Big]\,, \end{aligned} \tag{5.29}$$

where $m_0^{-1}(\boldsymbol{v}) = (m_{\mathrm{b}}\gamma)^{-1}(\mathbb{1} - |\boldsymbol{v}|^{-2}|\boldsymbol{v}\rangle\langle\boldsymbol{v}|)$ is the matrix inverse of $m_0(\boldsymbol{v})$. Clearly by (5.27) we have $\|m_0(\boldsymbol{v})^{-1}\| \leq C$ and, by condition (P), the first term is bounded as

$$\varepsilon\Big|\int d^3x\,\rho(\boldsymbol{x}-\boldsymbol{q})\big(\boldsymbol{E}_{\mathrm{ex}}(\varepsilon\boldsymbol{x})+\boldsymbol{v}\times\boldsymbol{B}_{\mathrm{ex}}(\varepsilon\boldsymbol{x})\big)\Big|\le C\,\varepsilon\,. \tag{5.30}$$

On the other hand the self–force looks like order one. To reduce it we have to exploit that $\boldsymbol{E},\boldsymbol{B}$ deviate only little from $\boldsymbol{E}_{\boldsymbol{v}},\boldsymbol{B}_{\boldsymbol{v}}$ close to the charge distribution, i.e. we rewrite the self–force as

$$\int d^3x\,\rho(\boldsymbol{x}-\boldsymbol{q})\big[\boldsymbol{E}(\boldsymbol{x})-\boldsymbol{E}_{\boldsymbol{v}}(\boldsymbol{x})+\boldsymbol{v}\times(\boldsymbol{B}(\boldsymbol{x})-\boldsymbol{B}_{\boldsymbol{v}}(\boldsymbol{x}))\big] \tag{5.31}$$

and have to show that the term in the square bracket is of order ε.

Let us define then

$$Z(\boldsymbol{x},t)=\begin{pmatrix}\boldsymbol{E}(\boldsymbol{x},t)-\boldsymbol{E}_{\boldsymbol{v}(t)}(\boldsymbol{x}-\boldsymbol{q}(t))\\ \boldsymbol{B}(\boldsymbol{x},t)-\boldsymbol{B}_{\boldsymbol{v}(t)}(\boldsymbol{x}-\boldsymbol{q}(t))\end{pmatrix}. \tag{5.32}$$

Using Maxwell equations and the relations $(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{E}_{\boldsymbol{v}}=-\nabla\times\boldsymbol{B}_{\boldsymbol{v}}+\rho\boldsymbol{v}$, $(\boldsymbol{v}\cdot\nabla)\boldsymbol{B}_{\boldsymbol{v}}=\nabla\times\boldsymbol{E}_{\boldsymbol{v}}$ we obtain

$$\dot{Z}(t)=\mathsf{A}Z(t)-g(t)\,, \tag{5.33}$$

where

$$\mathsf{A}=\begin{pmatrix}0 & \nabla\times\\ -\nabla\times & 0\end{pmatrix},\quad g(\boldsymbol{x},t)=\begin{pmatrix}(\dot{\boldsymbol{v}}(t)\cdot\nabla_{\boldsymbol{v}})\boldsymbol{E}_{\boldsymbol{v}}(\boldsymbol{x}-\boldsymbol{q}(t))\\ (\dot{\boldsymbol{v}}(t)\cdot\nabla_{\boldsymbol{v}})\boldsymbol{B}_{\boldsymbol{v}}(\boldsymbol{x}-\boldsymbol{q}(t))\end{pmatrix}. \tag{5.34}$$

Therefore (5.33) has again the structure of the inhomogeneous Maxwell equations. Since by our assumption on the initial data $Z(0)=0$, we have

$$Z(t)=-\int_0^t ds\,e^{\mathsf{A}(t-s)}g(s)\,. \tag{5.35}$$

We set $W(t,s)=e^{\mathsf{A}(t-s)}g(s)$. Below we prove that

$$|\boldsymbol{W}_1(t,s,\boldsymbol{q}(t)+\boldsymbol{x})|+|\boldsymbol{W}_2(t,s,\boldsymbol{q}(t)+\boldsymbol{x})|\le eC|\dot{\boldsymbol{v}}(s)|(1+(t-s)^2)^{-1} \tag{5.36}$$

for $|\boldsymbol{x}| \leq R_\rho$. Therefore inserting in (5.29) we obtain

$$|\dot{\boldsymbol{v}}(t)| \leq eC\big(\varepsilon + e\int_0^t ds\,(1+(t-s)^2)^{-1}\,|\dot{\boldsymbol{v}}(s)|\big)\,. \tag{5.37}$$

Let $\kappa = \sup_{t\geq 0} |\dot{\boldsymbol{v}}(t)|$. Then (5.37) reads

$$\begin{aligned} \kappa &\leq e\,C\big(\varepsilon + e\kappa\int_0^\infty ds\,(1+s^2)^{-1}\big)\,, \\ \kappa &\leq \frac{e\,C}{1-e^2\,C}\,\varepsilon\,. \end{aligned} \tag{5.38}$$

¿From the computation below we will see that C depends on $\overline{v}$ (and on model parameters like the form factor f), but not on e. Thus taking e sufficiently small we can ensure $e^2\,C < 1$ and $\kappa \leq C\varepsilon$ as claimed.

We still have to establish (5.36). $e^{\mathsf{A}t}$ is given in Equation (1.12), (1.13). Since $\nabla\cdot\boldsymbol{g}_1(s) = 0 = \nabla\cdot\boldsymbol{g}_2(s)$, the term proportional to $|\boldsymbol{k}\rangle\langle\boldsymbol{k}|$ drops out. In real space $|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|t$ becomes G_t from (1.15) and $\cos|\boldsymbol{k}|t$ becomes $\partial_t G_t$. Therefore

$$\begin{aligned} \boldsymbol{W}_1(t,s,\boldsymbol{x}) &= \frac{1}{4\pi(t-s)^2}\int d^3y\,\delta(|\boldsymbol{x}-\boldsymbol{y}|-(t-s)) \\ &\qquad [(t-s)\nabla\times\boldsymbol{g}_2(\boldsymbol{y},s)+\boldsymbol{g}_1(\boldsymbol{y},s)-(\boldsymbol{x}-\boldsymbol{y})\cdot\nabla\boldsymbol{g}_1(\boldsymbol{y},s)]\,, \\ \boldsymbol{W}_2(t,s,\boldsymbol{x}) &= \frac{1}{4\pi(t-s)^2}\int d^3y\,\delta(|\boldsymbol{x}-\boldsymbol{y}|-(t-s)) \\ &\qquad [-(t-s)\nabla\times\boldsymbol{g}_1(\boldsymbol{y},s)+\boldsymbol{g}_2(\boldsymbol{y},s)-(\boldsymbol{x}-\boldsymbol{y})\cdot\nabla\boldsymbol{g}_2(\boldsymbol{y},s)]\,. \end{aligned} \tag{5.39}$$

We insert g from (5.34). $\boldsymbol{E_v}$ and $\boldsymbol{B_v}$ are first order derivatives of the function $\phi_{\boldsymbol{v}}$ which according to (3.7) is given by

$$\phi_{\boldsymbol{v}}(\boldsymbol{x}) = \int d^3y\rho(\boldsymbol{x}-\boldsymbol{y})(4\pi)^{-1}\big[\big((1-\boldsymbol{v}^2)\boldsymbol{y}^2+(\boldsymbol{v}\cdot\boldsymbol{y})^2\big)\big]^{-1/2}\,. \tag{5.40}$$

Using (3.5) we have component–wise

$$\begin{aligned} |\nabla_{\boldsymbol{v}}\boldsymbol{E_v}(\boldsymbol{x})|+|\nabla_{\boldsymbol{v}}\boldsymbol{B_v}(\boldsymbol{x})| &\leq C\,(\,|\nabla\phi_{\boldsymbol{v}}(\boldsymbol{x})|+|\nabla\nabla_{\boldsymbol{v}}\phi_{\boldsymbol{v}}(\boldsymbol{x})|)\,, \qquad (5.41) \\ |\nabla\nabla_{\boldsymbol{v}}\boldsymbol{E_v}(\boldsymbol{x})|+|\nabla\nabla_{\boldsymbol{v}}\boldsymbol{B_v}(\boldsymbol{x})| &\leq C\,(\,|\nabla\nabla_{\boldsymbol{v}}\phi_{\boldsymbol{v}}(\boldsymbol{x})|+|\nabla\nabla\nabla_{\boldsymbol{v}}\phi_{\boldsymbol{v}}(\boldsymbol{x})|)\,. \end{aligned}$$

Taking now successive derivatives in (5.40) we obtain the bounds

$$\begin{aligned} |\nabla\phi_{\boldsymbol{v}}(\boldsymbol{x})| + |\nabla\nabla_{\boldsymbol{v}}\phi_{\boldsymbol{v}}(\boldsymbol{x})| &\leq e\,C\,(1+|\boldsymbol{x}|)^{-2}\,, \\ |\nabla\nabla\phi_{\boldsymbol{v}}(\boldsymbol{x})| + |\nabla\nabla\nabla_{\boldsymbol{v}}\phi_{\boldsymbol{v}}(\boldsymbol{x})| &\leq e\,C\,(1+|\boldsymbol{x}|)^{-3}\,, \end{aligned} \tag{5.42}$$

which implies

$$\begin{aligned} |\boldsymbol{g}_1(\boldsymbol{x},s)| + |\boldsymbol{g}_2(\boldsymbol{x},s)| &\leq e\,C|\dot{\boldsymbol{v}}(s)|(1+|\boldsymbol{x}-\boldsymbol{q}(s)|^2)^{-1}\,, \\ |\nabla\boldsymbol{g}_1(\boldsymbol{x},s)| + |\nabla\boldsymbol{g}_2(\boldsymbol{x},s)| &\leq e\,C|\dot{\boldsymbol{v}}(s)|(1+|\boldsymbol{x}-\boldsymbol{q}(s)|^3)^{-1}\,. \end{aligned} \tag{5.43}$$

We insert the bound (5.43) in (5.39) which results in a bound on $W(t,s,\boldsymbol{q}(t)+\boldsymbol{x})$. We use that $|\boldsymbol{x}| \leq R_\rho$ and $|\boldsymbol{q}(t)-\boldsymbol{q}(s)| \leq \overline{v}|t-s|$, which finally yields (5.36).

We summarize our findings as

Theorem 5.1 *For the Abraham model satisfying the conditions* $(C), (P)$, *and* (I) *there exist constants* $\overline{e}$ *and* C, *depending only on the initial conditions through* $\overline{v}$, *such that on the microscopic time scale we have*

$$\sup_t |\boldsymbol{v}(t)| \leq \overline{v} < 1, \;\; \sup_t \Big|\Big(\frac{d}{dt}\Big)^n \boldsymbol{v}(t)\Big| \leq C\,\varepsilon^n, \;\; n = 1,2,3\,, \tag{5.44}$$

provided the charge is sufficiently small, i.e. $e < \overline{e}$.

If we would keep track of the constant C, we would get a bound of the admissable charge in Theorem 5.1. Since we believe this restriction to be an artifact of the method anyhow, there is no point in the effort.

Notes and References

ad 5.1: Sommerfeld (1904,1905) first used systematically memory equations. In fact he considered the Abraham model with the kinetic energy $m_\mathrm{b}\boldsymbol{v}^2/2$ for the particle and wanted to understand what happens when $\boldsymbol{v}(0) > c$. He argued that the particle rapidly looses its energy to become slower than c by emitting what we call now Cherenkov radiation. The differential–difference equation was stated first by Page (1918) with its relativistic generalization Caldirola (1956). For reviews we refer to Erber (1961) and Pearle (1982). Moniz, Sharp (1974,1977) supplied a linear stability analysis and showed that the solutions to (5.14) are stable provided R_ρ is not too small. For that reason Rohrlich (1997) regards (5.14) and its relativistic sister as the fundamental starting point for the classical dynamics of extended charges. We take the Abraham model as the basic dynamical theory. Memory equations are a useful tool in analyzing its properties.

ad 5.2: Taylor expansion is taken from Kunze, Spohn (1999). Such an expansion was first used by Sommerfeld (1904,1905) and then repeated in various disguises. The traditional expansion parameter is the size of the charge distribution, which in our context is replaced by the scaling parameter ε

ad 5.3: The bound on $\dot{\boldsymbol{v}}^\varepsilon(t)$ comes from Kunze, Spohn (1999) where also higher derivatives are discussed. The contraction argument first appeared in Komech, Kunze, Spohn (1999).

6 Comparison Dynamics

If in (5.23) we simply ignore the error of order ε^2, then we obtain the following approximate equation for the motion of the charge,

$$\begin{aligned} \dot{\boldsymbol{q}} = \boldsymbol{v}, \;\; m(\boldsymbol{v})\dot{\boldsymbol{v}} = e\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}) + \boldsymbol{v}\times\boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q})\big) \\ +\varepsilon(e^2/6\pi)\left[\gamma^4(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} + 3\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{v} + 3\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \gamma^2\ddot{\boldsymbol{v}}\right]. \end{aligned} \tag{6.1}$$

Here $m(\boldsymbol{v})$ is the effective velocity dependent mass. It is the sum of the bare mass and the mass (5.24) induced by the field,

$$m(\boldsymbol{v}) = m_{\mathrm{b}}(\gamma\mathbb{1} + \gamma^3|\boldsymbol{v}\rangle\langle\boldsymbol{v}|) + m_{\mathrm{f}}(\boldsymbol{v}). \tag{6.2}$$

As anticipated in Section 3.1, via a distinct route, the leading contribution to (6.1) is derived from the effective Lagrangian

$$L_{\mathrm{eff}}(\boldsymbol{q},\dot{\boldsymbol{q}}) = T(\dot{\boldsymbol{q}}) - e\big(\phi_{\mathrm{ex}}(\boldsymbol{q}) - \dot{\boldsymbol{q}}\cdot A_{\mathrm{ex}}(\boldsymbol{q})\big), \tag{6.3}$$

equivalently from the Hamiltonian

$$E_{\mathrm{eff}}\big(\boldsymbol{p} - eA_{\mathrm{ex}}(\boldsymbol{q})\big) + e\phi_{\mathrm{ex}}(\boldsymbol{q}). \tag{6.4}$$

For later purposes it is more convenient to work with the energy function

$$H(\boldsymbol{q},\boldsymbol{v}) = E_s(\boldsymbol{v}) + e\phi_{\mathrm{ex}}(\boldsymbol{q}), \tag{6.5}$$

which is conserved by the solutions to (6.1) with $\varepsilon = 0$, compare with (3.14).

The term of order ε in (6.1) describes the radiation reaction. Globally its effect can be deduced from the energy balance. We add to (6.5) the *Schott energy*,

$$G_\varepsilon(\boldsymbol{q},\boldsymbol{v},\dot{\boldsymbol{v}}) = H(\boldsymbol{q},\boldsymbol{v}) - \varepsilon\,(e^2/6\pi)\,\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}}). \tag{6.6}$$

Then, along the solution trajectories of (6.1),

$$\frac{d}{dt}G_\varepsilon\,(\boldsymbol{q},\boldsymbol{v},\dot{\boldsymbol{v}}) = -\varepsilon\,(e^2/6\pi)\left[\gamma^4\dot{\boldsymbol{v}}^2 + \gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\right]. \tag{6.7}$$

Thus G_ε is decreasing in time. Integrating both sides of (6.7) we have

$$\begin{aligned} &-G_\varepsilon\left(\boldsymbol{q}(t), \boldsymbol{v}(t), \dot{\boldsymbol{v}}(t)\right) + G_\varepsilon\left(\boldsymbol{q}(0), \boldsymbol{v}(0), \dot{\boldsymbol{v}}(0)\right) \\ &= \ \varepsilon\,(e^2/6\pi) \int_0^t ds \left[\gamma^4\,\dot{\boldsymbol{v}}(s)^2 + \gamma^6\left(\boldsymbol{v}(s)\cdot\dot{\boldsymbol{v}}(s)\right)^2\right]. \end{aligned} \tag{6.8}$$

The mechanical energy is bounded from below, but the Schott energy does not have a definite sign. *If* (!) the Schott energy remains bounded in the course of time, then

$$\int_0^\infty dt \left[\gamma^4\,\dot{\boldsymbol{v}}(t)^2 + \gamma^6\left(\boldsymbol{v}(t)\cdot\dot{\boldsymbol{v}}(t)\right)^2\right] < \infty \tag{6.9}$$

which implies

$$\lim_{t\to\infty} \dot{\boldsymbol{v}}(t) = 0\,. \tag{6.10}$$

Equation (6.9) corresponds to the finite energy dissipation (1.67) in Section 1.5 and we can repeat verbatim the discussion there. In essence the limit (6.10) allows only two scenarios.

(i) $\lim\limits_{t\to\infty} \boldsymbol{v}(t) = \boldsymbol{v}_\infty \neq 0\,$. This corresponds to a scattering situation where the particle escapes into a region with $\boldsymbol{E}_{\text{ex}} = 0 = \boldsymbol{B}_{\text{ex}}$ and then travels with velocity $\boldsymbol{v}_\infty$ along a straight line.

(ii) $\lim\limits_{t\to\infty} \boldsymbol{v}(t) = 0$. This corresponds to a bounded motion where the particle eventually comes to rest. At such a rest point, $(\boldsymbol{q}^*, 0)$, we have

$$\nabla\phi_{\text{ex}}\left(\boldsymbol{q}^*\right) = 0 \tag{6.11}$$

by (6.1).

As noted already in Section 1.5, in general, (6.10) and (6.11) carry too little information for determining the true long–time behavior, as can be seen from the case of the motion in a uniform magnetic field.

Unfortunately the energy balance does not tell the full story. As noticed apparently first by Dirac (1938), Equation (6.1) has solutions which run away exponentially fast. There is no contradiction to (6.7). Since the Schott energy does not have a definite sign, in (6.8) both $G_\varepsilon(t)$ and the time–integral diverge as $t \to \infty$. The occurence of runaway solutions can be seen most easily in the approximation of small velocities, setting $\boldsymbol{B}_{\mathrm{ex}} = 0$, and linearizing ϕ_{ex} around a stable minimum, say at $\boldsymbol{q} = 0$. Then (6.1) becomes

$$m\dot{\boldsymbol{v}} = -m\,\omega_0^2\,\boldsymbol{q} + \varepsilon\,km\,\ddot{\boldsymbol{v}} \tag{6.12}$$

with $km = e^2/6\pi$. The three components of the linear equation (6.12) decouple and for each component there are three modes of the form e^{zt}. The characteristic equation is $z^2 = -\omega_0^2 + \varepsilon\,kz^3$ and to leading order the eigenvalues are $z_\pm = \pm\,i\omega_0 - \varepsilon\,(k\omega_0^2/2)$, $z^3 = (1/\varepsilon k) + \mathcal{O}(1)$. Thus in the 9–dimensional phase space for (6.12) there is a stable 6–dimensional hyperplane, $\mathcal{C}_\varepsilon$. On $\mathcal{C}_\varepsilon$ the motion is weakly damped, friction coefficient $\varepsilon\,(k\omega_0^2/2)$, and relaxes as $t \to \infty$ to rest at $\boldsymbol{q} = 0$. Transverse to $\mathcal{C}_\varepsilon$ the solution runs away as $e^{(t/\varepsilon k)}$.

Clearly such runaway solutions violate the stability estimates (5.16). Thus the full Maxwell–Lorentz equations do not have runaways. They somehow appear as an artifact of the Taylor expansion in (5.6). Dirac simply postulated that physical solutions must satisfy the *asymptotic condition*

$$\lim_{t\to\infty} \dot{\boldsymbol{v}}(t) = 0\,. \tag{6.13}$$

In the linearized version (6.12) this means that the initial conditions have to lie on $\mathcal{C}_\varepsilon$. In Theorem 1.3 we proved the asymptotic condition to hold for the Abraham model. Thus only those solutions to (6.1) satisfying the asymptotic condition can serve as a comparison dynamics to the true solution. We then have to understand how the asymptotic conditions arises, even better the global structure of the solution flow to Equation (6.1).

We note that in (6.1) the highest derivative is multiplied by a small prefactor. Such equations have been studied in great detail under the heading of (geometric) singular perturbation theory. The main conclusion is that the structure found for the linear equation (6.12) persists for the nonlinear equation (6.1). Of course the hyperplane $\mathcal{C}_\varepsilon$ is now deformed to some manifold, the critical (or center) manifold. We plan to explain the standard example in the following section and then to apply the theory to (6.1).

6.1 An example for singular perturbation theory

As purely mathematical example we consider the coupled system

$$\dot{x} = f(x, y) \ , \quad \varepsilon\, \dot{y} = y - h(x) \, . \tag{6.14}$$

h and f are bounded, smooth functions. The phase space is $\mathbb{R}^2$. One wants to understand how the solutions to (6.14) behave for small ε. If we just set $\varepsilon = 0$, then $y = h(x)$ and we obtain the autonomous equation

$$\dot{x} = f(x, h(x)) \, . \tag{6.15}$$

Geometrically this means that the two–dimensional phase space has been squeezed to the line $y = h(x)$ and the base point, $x(t)$, is governed by (6.15). $\{y = h(x), x \in \mathbb{R}\} = \mathcal{C}_0$ is the critical manifold to zero–th order in ε.

To see some motion in the phase space ambient to $\mathcal{C}_0$ we change from t to the slow time scale $\tau = \varepsilon^{-1} t$. Denoting differentiation with respect to τ by $'$, (6.14) goes over to

$$x' = \varepsilon\, f(x, y) \ , \quad y' = y - h(x) \, . \tag{6.16}$$

In the limit $\varepsilon \to 0$ we now have $x' = 0$, i.e. $x(\tau) = x_0$ and $y' = y - h(x_0)$ with solution $y(t) = (y_0 - h(x_0))e^t + h(x_0)$. Thus on that time scale, $\mathcal{C}_0$ consist exclusively of repelling fixed points. This is why $\mathcal{C}_0$ is called critical. The linearization at $\mathcal{C}_0$ has the eigenvalue one transverse and the eigenvalue zero tangential to $\mathcal{C}_0$. In the theory of dynamical systems zero eigenvalues in the linearization are linked to center manifolds and $\mathcal{C}_0$ is also called the center manifold (at $\varepsilon = 0$).

The basic result of singular perturbation theory is that for small ε the critical manifold deforms smoothly into $\mathcal{C}_\varepsilon$. Thus $\mathcal{C}_\varepsilon$ is invariant under the solution flow to (6.14). Its linearization at $(x, y) \in \mathcal{C}_\varepsilon$ has an eigenvalue of

$\mathcal{O}(1)$ with eigenvector tangential to $\mathcal{C}_\varepsilon$ and an eigenvalue $1/\varepsilon$ with eigenvector transverse to $\mathcal{C}_\varepsilon$. Thus for an initial condition slightly off $\mathcal{C}_\varepsilon$ the solution very rapidly diverges to infinity. Since $\mathcal{C}_0$ is deformed by order ε, also $\mathcal{C}_\varepsilon$ is of the form $\{y = h_\varepsilon(x),\ x \in \mathbb{R}\}$. According to (6.14) the base point evolves as

$$\dot{x} = f(x, h_\varepsilon(x)) \,. \tag{6.17}$$

Since h_ε is smooth in ε it can be Taylor expanded as

$$h_\varepsilon(x) = \sum_{j=0}^{m} \varepsilon^j \, h_j(x) + \mathcal{O}(\varepsilon^{m+1}) \,. \tag{6.18}$$

By (6.14) and (6.17) we have the identity

$$\varepsilon \, \partial_x h_\varepsilon(x) \, f(x, h_\varepsilon(x)) = h_\varepsilon(x) - h(x) \,. \tag{6.19}$$

Substituting (6.18) and comparing powers of ε we can thus determine recursively $h_j(x)$. To lowest order we obtain

$$h_0(x) = h(x) \ , \ \ h_1(x) = h'(x) \, f(x, h(x)) \tag{6.20}$$

and to order ε the base point is governed by

$$\dot{x} = f(x, h(x)) + \varepsilon \, \partial_y f(x, h(x)) \, h'(x) \, f(x, h(x)) \,. \tag{6.21}$$

Given the geometric picture of the center manifold, the stable (not run-away) solutions to (6.14) can be determined up to a set precision.

6.2 The critical manifold

Our task is to cast (6.1) into the canonical form used in singular perturbation theory. We set $(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{v}) \in \mathbb{R}^3 \times \mathbb{V}$, $\ \boldsymbol{y} = \dot{\boldsymbol{v}} \in \mathbb{R}^3$,

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}_2, \boldsymbol{y}) \in \mathbb{V} \times \mathbb{R}^3 \tag{6.22}$$

and

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \varepsilon) \quad = \quad & \gamma^{-2} \kappa(\boldsymbol{x}_2)^{-1} \big((6\pi/e^2) \, [m(\boldsymbol{x}_2) \boldsymbol{y} - \boldsymbol{F}_{\mathrm{ex}}(\boldsymbol{x})] \\ & - \varepsilon \, [3\gamma^6 (\boldsymbol{x}_2 \cdot \boldsymbol{y})^2 \, \boldsymbol{x}_2 + 3\gamma^4 \, (\boldsymbol{x}_2 \cdot \boldsymbol{y}) \boldsymbol{y}] \big) \,, \end{aligned} \tag{6.23}$$

where $\gamma = (1 - \boldsymbol{x}_2^2)^{-1/2}$ as before, $\boldsymbol{F}_{\text{ex}}(\boldsymbol{x}) = e(\boldsymbol{E}_{\text{ex}}(\boldsymbol{x}_1) + \boldsymbol{x}_2 \times \boldsymbol{B}_{\text{ex}}(\boldsymbol{x}_1))$, and $\kappa(\boldsymbol{v})$ is the 3×3 matrix $\kappa(\boldsymbol{v}) = \mathbb{1} + \gamma^2 \, |\boldsymbol{v}\rangle\langle\boldsymbol{v}|$ with inverse matrix $\kappa(\boldsymbol{v})^{-1} = \mathbb{1} - |\boldsymbol{v}\rangle\langle\boldsymbol{v}|$. With this notation Equation (6.1) reads

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}), \ \varepsilon \, \dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \varepsilon) \, . \tag{6.24}$$

We set $\boldsymbol{h}(\boldsymbol{x}) = m(\boldsymbol{x}_2)^{-1} \, \boldsymbol{F}_{\text{ex}}(\boldsymbol{x})$. Then for $\varepsilon = 0$ the critical manifold, $\mathcal{C}_0$, is given by

$$\mathcal{C}_0 = \{(\boldsymbol{x}, \boldsymbol{h}(\boldsymbol{x})), \ \boldsymbol{x} \in \mathbb{R}^3 \times \mathbb{V}\} = \{(\boldsymbol{q}, \boldsymbol{v}, \dot{\boldsymbol{v}}) : m(\boldsymbol{v})\dot{\boldsymbol{v}} = \boldsymbol{F}_{\text{ex}}(\boldsymbol{q}, \boldsymbol{v})\} \, , \tag{6.25}$$

which means that the critical manifold for $\varepsilon = 0$ is spanned by the solutions of the leading Hamiltonian part of Equation (6.1). Linearizing at $\mathcal{C}_0$ the repelling eigenvalue is dominated by $\gamma^{-2}\kappa(\boldsymbol{x}_2)^{-1} \, m(\boldsymbol{x}_2)$ which tends to zero as $|\boldsymbol{x}_2| \to 1$. Therefore $\mathcal{C}_0$ is not uniformly hyperbolic, which is needed to use the results from Sakamoto (1990).

To overcome this difficulty we modify $\boldsymbol{g}$ to $\boldsymbol{g}_\delta$, δ small, which agrees with $\boldsymbol{g}$ on $\mathbb{R}^3 \times \{\boldsymbol{v}, |\boldsymbol{v}| \leq 1 - \delta\} \times \mathbb{R}^3$ and which is constantly extended to values $|\boldsymbol{v}| \geq 1 - \delta$. Thus for $|\boldsymbol{x}_2(t)| \leq 1 - \delta$ the solution to $\dot{\boldsymbol{x}} = \boldsymbol{f}$, $\varepsilon\dot{\boldsymbol{y}} = \boldsymbol{g}_\delta$ agrees with the solution to $\dot{\boldsymbol{x}} = \boldsymbol{f}$, $\varepsilon\dot{\boldsymbol{y}} = \boldsymbol{g}$. For sufficiently small ε the modified equation has then a critical manifold $\mathcal{C}_\varepsilon$ with the properties as discussed in the example of Section 6.1. We only have to make sure that the modification is never seen. Thus, for the initial condition $|\boldsymbol{v}(0)| \leq \overline{v}$, we have to find a $\delta = \delta(\overline{v})$ such that $|\boldsymbol{v}(t)| \leq 1 - \delta$ for all times. To do so we need the energy balance (6.7).

We consider the modified evolution with vector field $(\boldsymbol{f}, \boldsymbol{g}_\delta)$ and we choose the initial velocity such that $|\boldsymbol{v}(0)| \leq \overline{v} < 1$. For ε small enough this dynamics has a critical manifold of the form $\dot{\boldsymbol{v}} = \boldsymbol{h}_\varepsilon(\boldsymbol{q}, \boldsymbol{v})$ and $|\boldsymbol{h}_\varepsilon(\boldsymbol{q}, \boldsymbol{v})| \leq c_1 = c_1(\delta)$. We start the dynamics on $\mathcal{C}_\varepsilon$. According to (6.7), for all $t \geq 0$,

$$\begin{aligned} G_\varepsilon\left(\boldsymbol{q}(t), \boldsymbol{v}(t), \boldsymbol{h}_\varepsilon(t)\right) &\leq G_\varepsilon(0) = H(\boldsymbol{q}(0), \boldsymbol{v}(0)) - \varepsilon(e^2/6\pi)(\boldsymbol{v}(0) \cdot \boldsymbol{h}_\varepsilon(0)) \\ &\leq E_s(\overline{v}) + e\phi_{\text{ex}}\left(\boldsymbol{q}(0)\right) + \varepsilon c_1 \, . \end{aligned} \tag{6.26}$$

We choose now δ such that $\overline{v} \leq 1 - 2\delta$. Since the initial conditions are on $\mathcal{C}_\varepsilon$, the solution will stay for a while on $\mathcal{C}_\varepsilon$ until the first time, τ, when

$|\boldsymbol{v}(\tau)| = 1 - \delta$. After that time the modification becomes visible. At time τ we have, using the lower bound on the energy and (6.26),

$$\begin{aligned} E_s(\boldsymbol{v}(\tau)) + e\overline{\phi} &\leq H(\boldsymbol{q}(\tau), \boldsymbol{v}(\tau)) = G_\varepsilon(\tau) + \varepsilon(e^2/6\pi)\,\gamma^4\,(\boldsymbol{v}(\tau)\cdot\boldsymbol{h}_\varepsilon(\tau)) \\ &\leq E_s(\overline{v}) + e\phi_{\mathrm{ex}}\,(\boldsymbol{q}(0)) + 2\varepsilon c_1 \end{aligned} \tag{6.27}$$

and therefore

$$E_s(1-\delta) \leq E_s(1-2\delta) + e\,(\phi_{\mathrm{ex}}\,(\boldsymbol{q}(0)) - \overline{\phi}) + 2\varepsilon c_1\,. \tag{6.28}$$

$E_s(1-\delta) \cong 1/\sqrt{\delta}$ for small δ, which implies

$$\frac{1}{\sqrt{\delta}} \leq c_2 + 4\,\varepsilon c_1 \tag{6.29}$$

with $c_2 = 2e\,(\phi(\boldsymbol{q}(0)) - \overline{\phi})$. We choose now δ so small that $1/\sqrt{\delta} \geq c_2 + 1$ and then ε so small that $4\varepsilon c_1 < 1$. Then (6.29) is a contradiction to the assumption that $|\boldsymbol{v}(\tau)| = 1 - \delta$. We conclude that $\tau = \infty$ and the solution trajectory stays on $\mathcal{C}_\varepsilon$ for all times.

Equipped with this information we have for small ε the critical manifold

$$\dot{\boldsymbol{v}} = \boldsymbol{h}_\varepsilon\,(\boldsymbol{q}, \boldsymbol{v})\,. \tag{6.30}$$

On the critical manifold the Schott energy is bounded and from the argument leading to (6.10) we conclude that Dirac's asymptotic condition holds on $\mathcal{C}_\varepsilon$. On the other hand, slightly off $\mathcal{C}_\varepsilon$ the solution diverges with a rate of order $1/\varepsilon$. Therefore the asymptotic condition singles out, for given $\boldsymbol{q}(0), \boldsymbol{v}(0)$, the *unique* $\dot{\boldsymbol{v}}(0)$ on $\mathcal{C}_\varepsilon$.

The motion on the critical manifold is governed by an effective equation which can be determined in approximation as in Section 6.1. We define

$$\boldsymbol{h}(\boldsymbol{q}, \boldsymbol{v}) = m(\boldsymbol{v})^{-1}\,e\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}) + \boldsymbol{v}\times\boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q})\big)\,. \tag{6.31}$$

Then, up to errors of order ε^2,

$$\begin{aligned} &m(\boldsymbol{v})\dot{\boldsymbol{v}} = e\,\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}) + \boldsymbol{v}\times\boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q})\big) \qquad (6.32) \\ &+\varepsilon\,(e^2/6\pi)\,\big[\gamma^2\kappa(\boldsymbol{v})\big(\boldsymbol{v}\cdot\nabla_{\boldsymbol{q}} + \boldsymbol{h}\cdot\nabla_{\boldsymbol{v}}\boldsymbol{h} + (3\gamma^6(\boldsymbol{v}\cdot\boldsymbol{h})^2\boldsymbol{v} + 3\gamma^4\,(\boldsymbol{v}\cdot\boldsymbol{h})\boldsymbol{h})\big)\big]\,. \end{aligned}$$

The physical solutions of (6.1), in the sense of the asymptotic condition, are governed by Equation (6.32), which thus should be regarded as the true comparison dynamics to the microscopic equation (4.52). Note that the error made in going from (6.1) to (6.32) is of the same order as the error made in the derivation of Equation (6.1).

On a formal level (6.32) is easily deduced from (6.1). We regard $m(\boldsymbol{v})\dot{\boldsymbol{v}} = e\left(\boldsymbol{E}_{\text{ex}}(\boldsymbol{q}) + \boldsymbol{v} \times \boldsymbol{B}_{\text{ex}}(\boldsymbol{q})\right)$ as the "unperturbed" equation and substitute for the terms inside the square bracket, which means to replace $\dot{\boldsymbol{v}}$ by $\boldsymbol{h}$ and $\ddot{\boldsymbol{v}}$ by $\dot{\boldsymbol{h}} = (\boldsymbol{v} \cdot \nabla_{\boldsymbol{q}})\boldsymbol{h} + (\boldsymbol{h} \cdot \nabla_{\boldsymbol{v}})\boldsymbol{h}$. While yielding the correct answer, one misses the geometrical picture of the motion in phase space and of the critical manifold.

For a numerical integration of the comparison dynamics it is advantageous to use directly (6.32). The only other practical option would be to solve (6.1) backwards in time. Then the trajectory is pushed rapidly towards the critical manifold. On $\mathcal{C}_\varepsilon$ one solves however the time–reversed dynamics which means a final rather than an initial value problem. Instead of weakly damped the motion is now slowly accelerating.

6.3 Tracking of the true solution

¿From (4.11) we have the true solution $\boldsymbol{q}^\varepsilon(t), \boldsymbol{v}^\varepsilon(t)$ with initial conditions $\boldsymbol{q}^0, \boldsymbol{v}^0$ and correspondingly adapted field data. We face the problem of how well this solution is tracked by the comparison dynamics (6.1). Let us first disregard the radiation reaction. From our a priori estimates we know that

$$\dot{\boldsymbol{q}}^\varepsilon = \boldsymbol{v}^\varepsilon,\ m(\boldsymbol{v}^\varepsilon)\dot{\boldsymbol{v}}^\varepsilon = e\left(\boldsymbol{E}_{\text{ex}}(\boldsymbol{q}^\varepsilon) + \boldsymbol{v}^\varepsilon \times \boldsymbol{B}_{\text{ex}}(\boldsymbol{q}^\varepsilon)\right) + \mathcal{O}(\varepsilon) \tag{6.33}$$

which should be compared to

$$\dot{\boldsymbol{r}} = \boldsymbol{u}\ ,\ m(\boldsymbol{u})\dot{\boldsymbol{u}} = e\left(\boldsymbol{E}_{\text{ex}}(\boldsymbol{r}) + \boldsymbol{u} \times \boldsymbol{B}_{\text{ex}}(\boldsymbol{u})\right). \tag{6.34}$$

We switched to the variables $\boldsymbol{r}, \boldsymbol{u}$ instead of $\boldsymbol{q}, \boldsymbol{v}$ so to more clearly distinguish between the true and comparison dynamics.

Theorem 6.1 *For the Abraham model satisfying the conditions* (C), (P), *and* (I) *let* $e \leq \overline{e}$ *and* $\varepsilon \leq \varepsilon_0$ *be sufficiently small. Let* $\boldsymbol{r}(t), \boldsymbol{u}(t)$ *be the solution to the comparison dynamics (6.34) with initial conditions* $\boldsymbol{r}(0) = \boldsymbol{q}^0, \boldsymbol{u}(0) = \boldsymbol{v}^0$. *Then for every* $\tau > 0$ *there exist constants* $c(\tau)$ *such that*

$$|\boldsymbol{q}^\varepsilon(t) - \boldsymbol{r}(t)| \leq c(\tau)\varepsilon \ , \ |\boldsymbol{v}^\varepsilon(t) - \boldsymbol{u}(t)| \leq c(\tau)\varepsilon \tag{6.35}$$

for $0 \leq t \leq \tau$.

Proof: One converts (6.33), (6.34) into a first order equation in its integral form. The difference is then estimated by using Gronwall's lemma, which yields an error as εe^{Ct}. □

Theorem 6.1 states that, up to an error of order ε, the true solution is well approximated by the Hamiltonian dynamics (6.34). In the next order the comparison dynamics is

$$\begin{aligned} &\dot{\boldsymbol{r}}^\varepsilon = \boldsymbol{u}^\varepsilon, \ m(\boldsymbol{u}^\varepsilon)\dot{\boldsymbol{u}}^\varepsilon = e\left(\boldsymbol{E}_{\text{ex}}(\boldsymbol{r}^\varepsilon) + \boldsymbol{u}^\varepsilon \times \boldsymbol{B}_{\text{ex}}(\boldsymbol{r}^\varepsilon)\right) \\ &+\varepsilon(e^2/6\pi)\left[\gamma^4\,(\boldsymbol{u}^\varepsilon \cdot \ddot{\boldsymbol{u}}^\varepsilon)\boldsymbol{u}^\varepsilon + 3\gamma^6(\boldsymbol{u}^\varepsilon \cdot \dot{\boldsymbol{u}}^\varepsilon)^2\,\boldsymbol{u}^\varepsilon + 3\gamma^4(\boldsymbol{u}^\varepsilon \cdot \dot{\boldsymbol{u}}^\varepsilon)\dot{\boldsymbol{u}}^\varepsilon + \gamma^2\ddot{\boldsymbol{u}}^\varepsilon\right] \end{aligned} \tag{6.36}$$

restricted to its critical manifold $\mathcal{C}_\varepsilon$ and one might expect that

$$|\boldsymbol{q}^\varepsilon(t) - \boldsymbol{r}^\varepsilon(t)| + |\boldsymbol{v}^\varepsilon(t) - \boldsymbol{u}^\varepsilon(t)| = \mathcal{O}(\varepsilon^2)\,. \tag{6.37}$$

Because of the improved precision one has the possibility to resolve the radiation reaction correction to (6.36).

An alternative option to keep track of the correction would be to consider longer times, of the order $\varepsilon^{-1}t$ on the macroscopic time scale. Then the radiative effects add up to deviations of order one from the Hamiltonian trajectory. Thus

$$|\boldsymbol{q}^\varepsilon(t) - \boldsymbol{r}^\varepsilon(t)| \cong \mathcal{O}(\varepsilon) \quad \text{for} \quad 0 \leq t \leq \varepsilon^{-1}\tau\,. \tag{6.38}$$

One should be somewhat careful here. In a scattering situation the charged particle reaches after a finite macroscopic time the force free region. According to (6.37) the error in the velocity is then $\mathcal{O}(\varepsilon^2)$ which builds up to an error in the position of order ε over a time span $\varepsilon^{-1}\tau$. Thus we cannot hope to do better than (6.38). On the other hand when the motion remains bounded, as e.g. in a uniform external magnetic field, the charge comes to rest at some point $\boldsymbol{q}^*$ in the long time limit and the rest point $\boldsymbol{q}^*$ is the same for the true and the comparison dynamics. At least for an external electrostatic potential with a discrete set of critical points we have already established such a behavior and presumably it holds in general. Thus for large τ we have $\boldsymbol{q}^\varepsilon(\varepsilon^{-1}\tau) \cong \boldsymbol{q}^*$ and also $\boldsymbol{r}^\varepsilon(\varepsilon^{-1}\tau) \cong \boldsymbol{q}^*$. Therefore we conjecture that (6.38) holds for *all* times.

Conjecture 6.2 *For the Abraham model satisfying* $(C),(P)$, *and* (I) *let* $\boldsymbol{q}^\varepsilon(t)$ *be bounded, i.e.* $|\boldsymbol{q}^\varepsilon(t)| \le C$ *for all* $t \ge 0$, $\varepsilon \le \varepsilon_0$. *Then there exists* $(\boldsymbol{r}^\varepsilon(0), \boldsymbol{u}^\varepsilon(0),$
$\dot{\boldsymbol{u}}^\varepsilon(0)) \in \mathcal{C}_\varepsilon$ *such that*

$$\sup_{t\ge 0} |\boldsymbol{q}^\varepsilon(t) - \boldsymbol{r}^\varepsilon(t)| = \mathcal{O}(\varepsilon)\,, \tag{6.39}$$

where $\boldsymbol{r}^\varepsilon(t)$ *is the solution to (6.36) with said initial conditions.*

At present we are far from such strong results. The problem is that an error of order ε^2 in (6.36) is generically amplified as $\varepsilon^2 e^{t/\varepsilon}$. Although such an increase violates the a priori bounds, it renders a proof of (6.39) difficult. We seem to be back to (6.35) which carries no information on the radiation reaction. Fortunately the radiation correction in (6.36) can be seen in the energy balance.

Theorem 6.3 *Under the assumptions of Theorem 6.1 we have*

$$\big|[E_s(\boldsymbol{v}^\varepsilon(t)) + e\,\phi_{\mathrm{ex}}(\boldsymbol{q}^\varepsilon(t))] - [E_s(\boldsymbol{u}^\varepsilon(t)) + e\phi_{\mathrm{ex}}(\boldsymbol{r}^\varepsilon(t))]\big| \le Cc(\tau)\varepsilon^2 \tag{6.40}$$

for $0 \le t \le \tau$, *where* $\boldsymbol{r}^\varepsilon(t), \boldsymbol{u}^\varepsilon(t)$ *is the solution to (6.36) with initial data* $\boldsymbol{r}^\varepsilon(0) = \boldsymbol{q}^0$, $\boldsymbol{u}^\varepsilon(0) = \boldsymbol{v}^0$, $\dot{\boldsymbol{u}}^\varepsilon(0) = \boldsymbol{h}_\varepsilon(\boldsymbol{q}^0, \boldsymbol{v}^0)$.

Proof: We use the estimate (5.23) on the self–force, where $|\boldsymbol{f}^\varepsilon(t)| \le C\varepsilon^2$ for $\varepsilon\bar{t}_\rho \le t$ and $|\boldsymbol{f}^\varepsilon(t)| \le C\varepsilon$ for $0 \le t \le \varepsilon\bar{t}_\rho$. Then, as in (6.7),

$$\frac{d}{dt}\,G_\varepsilon\,(\boldsymbol{q}^\varepsilon, \boldsymbol{v}^\varepsilon, \dot{\boldsymbol{v}}^\varepsilon) = \boldsymbol{f}^\varepsilon(t)\cdot\boldsymbol{v}^\varepsilon - \varepsilon\,(e^2/6\pi)[\gamma^4(\dot{\boldsymbol{v}}^\varepsilon)^2 + \gamma^6(\boldsymbol{v}^\varepsilon\cdot\dot{\boldsymbol{v}}^\varepsilon)^2] \tag{6.41}$$

and therefore

$$
\begin{aligned}
&|H(\boldsymbol{q}^\varepsilon, \boldsymbol{v}^\varepsilon) - H(\boldsymbol{r}^\varepsilon, \boldsymbol{u}^\varepsilon)| \\
&\leq \varepsilon\,(e^2/6\pi)|\gamma(\boldsymbol{v}^\varepsilon)^4(\boldsymbol{v}^\varepsilon \cdot \dot{\boldsymbol{v}}^\varepsilon) - \gamma(\boldsymbol{u}^\varepsilon)^4(\boldsymbol{u}^\varepsilon \cdot \dot{\boldsymbol{u}}^\varepsilon)| \\
&+ \int_0^t ds\, \big(|\boldsymbol{f}^\varepsilon \cdot \boldsymbol{v}^\varepsilon| + \varepsilon\,(e^2/6\pi)|\gamma(\boldsymbol{v}^\varepsilon)^4\,(\dot{\boldsymbol{v}}^\varepsilon)^2 + \gamma(\boldsymbol{v}^\varepsilon)^6\,(\boldsymbol{v}^\varepsilon \cdot \dot{\boldsymbol{v}}^\varepsilon)^2 \\
&\qquad\qquad -\gamma(\boldsymbol{u}^\varepsilon)^4\,(\dot{\boldsymbol{u}}^\varepsilon)^2 - \gamma(\boldsymbol{u}^\varepsilon)^6\,(\boldsymbol{u}^\varepsilon \cdot \dot{\boldsymbol{u}}^\varepsilon)^2|\big)\,. \qquad (6.42)
\end{aligned}
$$

Since $|\boldsymbol{v}^\varepsilon|, |\boldsymbol{u}^\varepsilon|$ remain bounded away from 1, we can use the bound $|\boldsymbol{v}^\varepsilon(t) - \boldsymbol{u}(t)| \leq c(\tau)\varepsilon$ from Theorem 6.1. Reinserting (6.35) into (6.33) and (6.34) we obtain $|\dot{\boldsymbol{v}}^\varepsilon(t) - \dot{\boldsymbol{u}}^\varepsilon(t)| \leq c(\tau)\varepsilon$. Furthermore $\int_0^t ds\,|\boldsymbol{f}^\varepsilon(s)| \leq Ct\varepsilon^2$. We conclude that

$$
|H(\boldsymbol{q}^\varepsilon(t), \boldsymbol{v}^\varepsilon(t)) - H(\boldsymbol{r}^\varepsilon(t), \boldsymbol{u}^\varepsilon(t))| \leq C(t + c(t))\varepsilon^2\,. \qquad (6.43)
$$

□

6.4 Electromagnetic fields in the adiabatic limit

So far we have concentrated on the Lorentz equation with retarded fields and have obtained approximate evolution equations for the charged particle. Such an approximate solution can be reinserted into the inhomogeneous Maxwell equations in order to obtain the electromagnetic fields in the adiabatic limit.

As before, let $\boldsymbol{q}^\varepsilon(t), \boldsymbol{v}^\varepsilon(t)), t \geq 0$, be the true solution. We extend it to $\boldsymbol{q}^\varepsilon(t) = \boldsymbol{q}^0 + \boldsymbol{v}^0 t,\ \boldsymbol{v}^\varepsilon(t) = \boldsymbol{v}^0$ for $t \leq 0$. According to (3.21), (3.22) and using the scaled fields as in (4.8), we have

$$
\frac{1}{\sqrt{\varepsilon}}\,\boldsymbol{E}(t) = -\int_{-\infty}^t ds \int d^3y\,\big(\nabla G_{t-s} * \rho_\varepsilon(s) + \partial_t G_{t-s} * \boldsymbol{j}_\varepsilon(s)\big) \qquad (6.44)
$$

with $\rho_\varepsilon(\boldsymbol{x},t)=\rho_\varepsilon(\boldsymbol{x}-\boldsymbol{q}^\varepsilon(t)), \boldsymbol{j}_\varepsilon(\boldsymbol{x},t)=\rho_\varepsilon(\boldsymbol{x}-\boldsymbol{q}^\varepsilon(t))\boldsymbol{v}^\varepsilon(t)$. Inserting from (1.15) and by partial integration

$$\frac{1}{\sqrt{\varepsilon}}\boldsymbol{E}(\boldsymbol{x},t) = -\int\limits_{-\infty}^{t} ds \int d^3y\, \frac{1}{4\pi(t-s)}\,\delta(|\boldsymbol{x}-\boldsymbol{y}|-(t-s))\,\nabla\rho_\varepsilon(\boldsymbol{y},s)$$

$$-\int\limits_{-\infty}^{t} ds \int d^3y\, \frac{1}{4\pi(t-s)^2}\,\delta(|\boldsymbol{x}-\boldsymbol{y}|-(t-s))\,[(\boldsymbol{y}-\boldsymbol{x})\cdot\nabla\boldsymbol{j}_\varepsilon(\boldsymbol{y},s)+\boldsymbol{j}_\varepsilon(\boldsymbol{y},s)]$$

$$\begin{aligned} = \; & -\int d^3y\,\Big(\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,\nabla\rho_\varepsilon(\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|))\boldsymbol{v}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|) \\ & +\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^2}\,\boldsymbol{v}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|)(1+(\boldsymbol{y}-\boldsymbol{x})\cdot\nabla)\,\rho_\varepsilon(\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|))\Big)\,. \end{aligned} \tag{6.45}$$

In the same fashion

$$\frac{1}{\sqrt{\varepsilon}}\boldsymbol{B}(\boldsymbol{x},t) = -\int d^3y\, \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,\boldsymbol{v}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|)\times\nabla\rho_\varepsilon(\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|))\,. \tag{6.46}$$

In the limit $\varepsilon\to 0$ we have $\rho_\varepsilon(\boldsymbol{x})\to\delta(\boldsymbol{x})$ and, by Theorem 6.1, $\boldsymbol{q}^\varepsilon(t)\to\boldsymbol{r}(t)$, $\boldsymbol{v}^\varepsilon(t)\to\boldsymbol{u}(t)$, where $\boldsymbol{r}(t)=\boldsymbol{q}^0+\boldsymbol{v}^0t$, $\boldsymbol{u}(t)=\boldsymbol{v}^0$ for $t\le 0$. We substitute $\boldsymbol{y}'=\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|)$ with volume element $\det(\mathrm{d}\boldsymbol{y}/\mathrm{d}\boldsymbol{y}')=[1-\boldsymbol{v}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|)\cdot(\boldsymbol{x}-\boldsymbol{y})/|\boldsymbol{x}-\boldsymbol{y}|]^{-1}$. Then $\delta(\boldsymbol{y}')$ leads to the constraint $0=\boldsymbol{y}-\boldsymbol{r}(t-|\boldsymbol{x}-\boldsymbol{y}|)$ which has the unique solution $\boldsymbol{y}=\boldsymbol{r}(t_{\mathrm{ret}})$, compare with (1.19). In particular the volume element $\det(\mathrm{d}\boldsymbol{y}/\mathrm{d}\boldsymbol{y}')$ becomes in the limit $[1-\widehat{\boldsymbol{n}}\cdot\boldsymbol{u}(t_{\mathrm{ret}})]^{-1}$ with $\widehat{\boldsymbol{n}}=\widehat{\boldsymbol{n}}(\boldsymbol{x},t)=(\boldsymbol{x}-\boldsymbol{r}(t_{\mathrm{ret}}))/|\boldsymbol{x}-\boldsymbol{r}(t_{\mathrm{ret}})|$.

We conclude that

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon}}\boldsymbol{E}(\boldsymbol{x},t) = \overline{\boldsymbol{E}}(\boldsymbol{x},t)\,, \tag{6.47}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon}}\boldsymbol{B}(\boldsymbol{x},t) = \overline{\boldsymbol{B}}(\boldsymbol{x},t)\,, \tag{6.48}$$

where $\overline{\boldsymbol{E}},\overline{\boldsymbol{B}}$ are the Liénard–Wiechert fields (1.21), (1.22) generated by a point charge moving along the trajectory $t\mapsto\boldsymbol{r}(t)$. The convergence in (6.47), (6.48) is pointwise if one excludes the Coulomb singularity at $\boldsymbol{x}=\boldsymbol{r}(t)$.

6.5 Larmor's formula

We want to determine the energy per unit time radiated to infinity and consider, for this purpose, a ball of radius R centered at $\boldsymbol{q}^\varepsilon(t)$. At time $t+R$ the energy in this ball is

$$\mathcal{E}_{R,\boldsymbol{q}^\varepsilon(t)}(t+R) = \mathcal{E}(0) - \frac{1}{2} \int\limits_{\{|\boldsymbol{x}-\boldsymbol{q}^\varepsilon(t)|\geq R\}} d^3x \left(\boldsymbol{E}(\boldsymbol{x},t+R)^2 + \boldsymbol{B}(\boldsymbol{x},t+R)^2\right) \tag{6.49}$$

using conservation of total energy. The radiation emitted from the charge at time t reaches the surface of the ball at time $t+R$ and the energy loss per unit time is given by

$$\begin{aligned} sI_{R,\varepsilon}(t) &= \frac{d}{dt}\mathcal{E}_{R,\boldsymbol{q}^\varepsilon(t)} \\ &= \int d^3x\, \delta(|\boldsymbol{x}-\boldsymbol{q}^\varepsilon(t)|-R) \Big(\frac{1}{2}\left(\boldsymbol{E}(\boldsymbol{x},t+R)^2 + \boldsymbol{B}(\boldsymbol{x},t+R)^2\right) \\ &\quad (\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{v}^\varepsilon(t)) + \boldsymbol{E}(\boldsymbol{x},t+R)\cdot[\boldsymbol{n}(\boldsymbol{x})\times\boldsymbol{B}(\boldsymbol{x},t+R)]\Big) \\ &= \frac{1}{2}R^2\int d^2\omega \Big(\left(\boldsymbol{E}(\boldsymbol{q}^\varepsilon(t)+R\boldsymbol{\omega},t+R)^2 + \boldsymbol{B}(\boldsymbol{q}^\varepsilon(t)+R\boldsymbol{\omega},t+R)^2\right) \\ &\quad (\boldsymbol{\omega}\cdot\boldsymbol{v}^\varepsilon(t)) + 2\boldsymbol{E}(\boldsymbol{q}^\varepsilon(t)+R\boldsymbol{\omega},t+R)\cdot[\boldsymbol{\omega}\times\boldsymbol{B}(\boldsymbol{q}^\varepsilon(t)+R\boldsymbol{\omega},t+R)]\Big), \end{aligned} \tag{6.50}$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the outer normal of the ball and $|\boldsymbol{\omega}|=1$ with $d^2\omega$ the integration over the unit sphere. (6.50) holds for sufficiently large R, since we used that $\{\boldsymbol{x}|\ |\boldsymbol{x}-\boldsymbol{q}^\varepsilon(t)|\geq R\}\cap\{\boldsymbol{x}|\ |\boldsymbol{x}-\boldsymbol{q}^\varepsilon(t+R)|\leq\varepsilon R_\rho\}=\emptyset$, which is the case for $(1-\overline{v})R\geq\varepsilon R_\rho$.

(6.50) still contains the reversible energy transport between the considered ball and its complement. To isolate that part of the energy which is irreversibly lost we have to take the limit $R\to\infty$. For this purpose we first partially integrate in (6.45), (6.46) by using the identity

$$\nabla\rho = \nabla_{\boldsymbol{y}}\,\rho - \frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|}\left(1+\frac{(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{v}^\varepsilon}{|\boldsymbol{y}-\boldsymbol{x}|}\right)^{-1}(\boldsymbol{v}^\varepsilon\cdot\nabla_{\boldsymbol{y}})\rho \tag{6.51}$$

at the argument $\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t-|\boldsymbol{y}-\boldsymbol{x}|)$. For large R the fields in (6.50) become then

$$
\begin{aligned}
&R\boldsymbol{E}(\boldsymbol{q}^\varepsilon(t)+R\boldsymbol{\omega},t+R) \cong \sqrt{\varepsilon}\int d^3y\,\frac{1}{4\pi}\,\rho_\varepsilon(\boldsymbol{y}-\boldsymbol{q}^\varepsilon)\\
&\big[-(1-\boldsymbol{\omega}\cdot\boldsymbol{v}^\varepsilon)^{-1}\,\dot{\boldsymbol{v}}^\varepsilon-(1-\boldsymbol{\omega}\cdot\boldsymbol{v}^\varepsilon)^{-2}\,(\boldsymbol{\omega}\cdot\dot{\boldsymbol{v}}^\varepsilon)(\boldsymbol{v}^\varepsilon-\boldsymbol{\omega})\big]|_{t+\boldsymbol{\omega}\cdot(\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t))}\,, \qquad (6.52)\\
&R\boldsymbol{B}(\boldsymbol{q}^\varepsilon(t)+R\boldsymbol{\omega},t+R) \cong \sqrt{\varepsilon}\int d^3y\,\frac{1}{4\pi}\,\rho_\varepsilon(\boldsymbol{y}-\boldsymbol{q}^\varepsilon)\\
&\big[-(1-\boldsymbol{\omega}\cdot\boldsymbol{v}^\varepsilon)^{-1}\,(\boldsymbol{\omega}\times\dot{\boldsymbol{v}}^\varepsilon)-(1-\boldsymbol{\omega}\cdot\boldsymbol{v}^\varepsilon)^{-2}\,(\boldsymbol{\omega}\cdot\dot{\boldsymbol{v}}^\varepsilon)(\boldsymbol{\omega}\times\boldsymbol{v}^\varepsilon)\big]|_{t+\boldsymbol{\omega}\cdot(\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t))}\\
&=\boldsymbol{\omega}\times R\boldsymbol{E}(\boldsymbol{q}^\varepsilon(t)+R\boldsymbol{\omega},t+R)\,, \qquad (6.53)
\end{aligned}
$$

where we used that $t+R-|\boldsymbol{q}^\varepsilon(t)+R\boldsymbol{\omega}-\boldsymbol{y}|=t+\boldsymbol{\omega}\cdot(\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t))+\mathcal{O}(1/R)$ for large R. Inserting in (6.50) yields

$$
\begin{aligned}
&\lim_{R\to\infty} I_{R,\varepsilon}(t)=I_\varepsilon(t)\\
&=-\varepsilon\int d^2\boldsymbol{\omega}\,(1-\boldsymbol{\omega}\cdot\boldsymbol{v}^\varepsilon(t))\big(R\boldsymbol{E}(\boldsymbol{q}^\varepsilon(t)+R\boldsymbol{\omega},t+R)\big)^2 \qquad (6.54)\\
&=-\varepsilon\int d^2\boldsymbol{\omega}\,(1-\boldsymbol{\omega}\cdot\boldsymbol{v}^\varepsilon(t))\Big(\big[\frac{1}{4\pi}\int d^3y\,\rho_\varepsilon(\boldsymbol{y}-\boldsymbol{q}^\varepsilon)(1-\boldsymbol{\omega}\cdot\boldsymbol{v}^\varepsilon)^{-2}(\boldsymbol{\omega}\cdot\dot{\boldsymbol{v}}^\varepsilon)\big]^2\\
&-\big[\frac{1}{4\pi}\int d^3y\rho_\varepsilon(\boldsymbol{y}-\boldsymbol{q}^\varepsilon)(1-\boldsymbol{\omega}\cdot\boldsymbol{v}^\varepsilon)^{-1}\dot{\boldsymbol{v}}_\varepsilon\\
&\qquad+(1-\boldsymbol{\omega}\cdot\boldsymbol{v}^\varepsilon)^{-2}(\boldsymbol{\omega}\cdot\dot{\boldsymbol{v}}^\varepsilon)\boldsymbol{v}^\varepsilon\big]^2\Big)|_{t+\boldsymbol{\omega}\cdot(\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t))}. \qquad (6.55)
\end{aligned}
$$

$I_\varepsilon(t)$ is the energy radiated per unit time at ε fixed. As argued before it is of order ε. The expression (6.54) shows that $I_\varepsilon(t)\le 0$.

(6.55) is not yet Larmor's formula. For this we have to go to the adiabatic limit $\varepsilon\to 0$. Then $\boldsymbol{q}^\varepsilon(t)\to\boldsymbol{r}(t)$. Since $\rho_\varepsilon(\boldsymbol{x})\to e\delta(\boldsymbol{x})$ we have $\boldsymbol{y}\cong\boldsymbol{q}^\varepsilon(t)\cong\boldsymbol{r}(t)$ in (6.55). From the d^3y volume element we get an additional factor of

$(1-\boldsymbol{\omega}\cdot\boldsymbol{v}^{\varepsilon})^{-1}$. Thus

$$
\begin{aligned}
\lim_{\varepsilon\to 0} I_\varepsilon(t) = I(t) &= -e^2\int d^2\boldsymbol{\omega}\,(1-\boldsymbol{\omega}\cdot\boldsymbol{u}(t))\big(4\pi(1-\boldsymbol{\omega}\cdot\boldsymbol{u}(t))^{-3}\big)^2 \\
&\quad \big((\boldsymbol{\omega}\cdot\dot{\boldsymbol{u}}(t))^2-[(1-\boldsymbol{\omega}\cdot\boldsymbol{u}(t))\dot{\boldsymbol{u}}(t)+(\boldsymbol{\omega}\cdot\dot{\boldsymbol{u}}(t))\boldsymbol{u}(t)]^2\big) \\
&= -(e^2/6\pi)\big[\gamma^4\dot{\boldsymbol{u}}(t)^2+\gamma^6(\boldsymbol{u}(t)\cdot\dot{\boldsymbol{u}}(t))^2\big] \\
&= -(e^2/6\pi)\gamma^4\big[\dot{\boldsymbol{u}}(t)^2-(\boldsymbol{u}(t)\times\dot{\boldsymbol{u}}(t))^2\big]\,,
\end{aligned}
\tag{6.56}
$$

which is the standard textbook formula of Larmor. Note that the same energy loss per unit time was obtained already in (6.7) using only the energy balance for the comparison dynamics.

Starting from (6.50) we could alternatively first take the limit $\varepsilon^{-1}I_{R,\varepsilon}(t)\to I_{R,0}(t)$, which is the change of energy in a ball of radius R centered at the particle's position $\boldsymbol{r}(t)$ in the adiabatic limit. As before we have to isolate the irreversible energy loss through

$$
\lim_{R\to\infty} I_{R,0}(t) = I(t)\,. \tag{6.57}
$$

The energy loss does not depend on the order of limits, as it should be.

Notes and References

ad 6: The radiation damped harmonic oscillator is discussed in Jackson (1999) with a variety of physical applications. The asymptotic condition is first stated in Dirac (1938). It has been reemphasized by Haag (1955) in analogy to a similar condition in quantum field theory.

ad 6.1: Singular, or geometric, perturbation theory is a standard tool in the theory of dynamical systems. We refer to Jones (1995) for a review with many applications. In the context of synergetics, Haken (1983), one talks of slow and fast variables and the slaving principle, which means that fast variables are slaved by the slow ones. Within our context this would correspond to an attractive critical manifold. The renormalization group flows in critical phenomena have a structure similar to the one discovered here. The critical surface corresponds to critical couplings which flow then to some fixed point governing the universal critical behavior. The critical surface is repelling and slightly off that surface the trajectory moves towards either the high temperature or low temperature fixed points.

ad 6.2: Particular cases have been studied before, most extensively the one–dimensional potential of finite width and with linear interpolation, Haag (1955), Carati, Galgani (1993), Carati et al (1995), Blanco (1995), in addition head on collision in the two–body problem, Huschilt and Baylis (1976), and motion in a uniform magnetic field, Endres (1993). These authors emphasize that there can be several solutions to the asymptotic condition. From the point of view of singular perturbation theory such a behavior is generic. If ε is increased, then the critical manifold is strongly deformed and no longer given as a graph of a function. For specified $\boldsymbol{q}(0), \boldsymbol{v}(0)$ there are then several $\dot{\boldsymbol{v}}(0)$ on $\mathcal{C}_\varepsilon$ which means that the solution to the asymptotic condition is not unique. However these authors miss to underline that the nonuniquess in the examples occurs only at such high field strengths where a classical theory has long lost its empirical validity. At moderate field strengths the worked out examples confirm our findings. The applicability of singular perturbation theory for a general class of potentials is first recognized in Spohn (1998).

ad 6.3, 6.4, 6.5: These results are adapted from Kunze, Spohn (1999).

7 The Lorentz–Dirac Equation

In relativistic notation the Lorentz–Dirac equation reads

$$m_0\, \dot{v}^\mu = (e/c)F^{\mu\nu}(z)v_\nu + (e^2/6\pi c^3)\,[\ddot{v}^\mu - c^{-2}\dot{v}^\lambda \dot{v}_\lambda\, v^\mu]\,, \tag{7.1}$$

where we reintroduced the speed of light, c. m_0 is the experimental rest mass of the charged particle. $F^{\mu\nu}$ is the electromagnetic field tensor of the *external* fields. In this section we omit the index "ex" for better readability. Formally, Equation (7.1) can be derived from the Lorentz model in the adiabatic limit. To conform with the usual notation we have set the adiabatic scale parameter $\varepsilon = 1$. But it should be kept in mind that the radiation reaction in (7.1) is a small correction to the Hamiltonian part.

If we fix a frame of reference and go over to three–vectors, then the Lorentz–Dirac equation becomes

$$\begin{aligned} m_0\,\gamma\,\kappa(\boldsymbol{v})\dot{\boldsymbol{v}} \;&=\; e(\boldsymbol{E}(\boldsymbol{q}) + c^{-1}\boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{q})) \\ &\quad +(e^2/6\pi c^3)\,\gamma^2\kappa(\boldsymbol{v})\,[\ddot{\boldsymbol{v}} + 3\gamma^2\,c^{-2}\,(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\,\dot{\boldsymbol{v}}]\,, \end{aligned} \tag{7.2}$$

with the 3×3 matrix $\kappa(\boldsymbol{v}) = \mathbb{1} + c^{-2}\,\gamma^2\,|\boldsymbol{v}\rangle\langle\boldsymbol{v}|$ and its inverse $\kappa(\boldsymbol{v})^{-1} = \mathbb{1} - c^{-2}|\boldsymbol{v}\rangle\langle\boldsymbol{v}|$. The Lorentz–Dirac equation (7.1) differs from (6.1) only through a proper relativistic kinetic energy. Clearly, qualitative properties of the solution flow should not depend on such a detail. In the analysis of (6.1) we only used the critical manifold to be a uniform repeller, except for $|\boldsymbol{v}|/c$ close to one, and the energy balance

$$\begin{aligned} &\frac{d}{dt}\big(m_0 c^2\,\gamma(\boldsymbol{v}) + e\phi(\boldsymbol{q}) - (e^2/6\pi c^3)\,\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\big) \\ &\qquad = -(e^2/6\pi c^3)\,\gamma^4(\dot{\boldsymbol{v}}, \kappa(\boldsymbol{v})\dot{\boldsymbol{v}})\,, \end{aligned} \tag{7.3}$$

which ensured that the solution stays for all times on the critical manifold, provided the radiation reaction term is sufficiently small. Thus we can follow the blueprint of Section 6.1 to obtain an effective second order equation for the motion on the critical manifold.

In Section 7.2 we work out some examples of experimental interest. While at present an actual test is rather indirect at best, the examples should convince the reader that the effective second order equation can be handled with ease. As an extra bonus we will make some predictions on the motion of the charge which could not have been guessed on the basis of Larmor's formula.

7.1 Critical manifold, the Landau–Lifshitz equation

We write (7.2) in the standard form of singular perturbation theory, compare with Section 6.2. Then

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}), \quad \varepsilon\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y},\varepsilon) \tag{7.4}$$

with

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{x}_2,\boldsymbol{y}), \tag{7.5}$$

$$\begin{aligned}\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y},\varepsilon) = {} & (6\pi c^3/e^2)\big(m_0\,\gamma^{-1}\boldsymbol{y} - e\,\gamma^{-2}\kappa(\boldsymbol{x}_2)^{-1}(\boldsymbol{E}(\boldsymbol{x}_1) + c^{-1}\boldsymbol{x}_2\times\boldsymbol{B}(\boldsymbol{x}_1))\big)\\ & -3\varepsilon\gamma^2c^{-2}(\boldsymbol{x}_2\cdot\boldsymbol{y})\boldsymbol{y}\,.\end{aligned} \tag{7.6}$$

To conform with (6.1) we reintroduced the small parameter ε. At zeroth order the critical manifold is $\{\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x})\}$ with $\boldsymbol{h}(\boldsymbol{q},\boldsymbol{v}) = (e/m_0)\gamma^{-1}\,\kappa(\boldsymbol{v})^{-1}\,\big(\boldsymbol{E}(\boldsymbol{q}) + c^{-1}\boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{q})\big)$. Linearizing (7.5), (7.6) at $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x})$ the repelling eigenvalue is $(6\pi c^3/e^2)\,m_0\gamma^{-1} + \mathcal{O}(\varepsilon)$, which vanishes as $|\boldsymbol{v}|/c \to 1$. Thus we have to rely on the same construction as in Section 6.2.

To order ε the effective second order equation is given by (6.32), except that now $m(\boldsymbol{v}) = m_0\,\gamma\kappa(\boldsymbol{v})$. We work out the various terms and set $\varepsilon = 1$. Then the motion on the critical manifold of the Lorentz–Dirac equation is governed by

$$
\begin{aligned}
\dot{\boldsymbol{q}} &= \dot{\boldsymbol{v}}\,, \\
m_0\,\gamma\,\kappa(\boldsymbol{v})\dot{\boldsymbol{v}} &= e(\boldsymbol{E}+c^{-1}\boldsymbol{v}\times\boldsymbol{B}) \\
&\quad +\frac{e^2}{6\pi c^3}\Big[\frac{e}{m_0}\,\gamma\,(\boldsymbol{v}\cdot\nabla_{\boldsymbol{q}})(\boldsymbol{E}+c^{-1}\boldsymbol{v}\times\boldsymbol{B})+\big(\frac{e}{m_0}\big)^2 c^{-1}\Big((\boldsymbol{E}\times\boldsymbol{B}) \\
&\quad +c^{-1}(\boldsymbol{v}\cdot\boldsymbol{E})\boldsymbol{E}+c^{-1}(\boldsymbol{v}\cdot\boldsymbol{B})\boldsymbol{B}+\big(-\boldsymbol{E}^2-\boldsymbol{B}^2 \\
&\quad +c^{-2}(\boldsymbol{v}\cdot\boldsymbol{E})^2+c^{-2}(\boldsymbol{v}\cdot\boldsymbol{B})^2+2c^{-1}\boldsymbol{v}\cdot(\boldsymbol{E}\times\boldsymbol{B})\big)\gamma^2c^{-1}\boldsymbol{v}\Big)\Big] \qquad (7.7)
\end{aligned}
$$

While singular perturbation theory provides a systematic method, Equation (7.7) can also be derived formally. In (7.5) we regard $m_0\gamma\,\kappa(\boldsymbol{v})\dot{\boldsymbol{v}} = e\,(\boldsymbol{E}+c^{-1}\boldsymbol{v}\times\boldsymbol{B})$ as unperturbed equation, differentiate it once, and substitute $\ddot{\boldsymbol{v}}$ inside the square brackets of (7.2). Resubstituting $\dot{\boldsymbol{v}}$ from the unperturbed equation results in Equation (7.7). This argument is carried out more easily in the covariant form of the Lorentz–Dirac equation. The unperturbed part is

$$
m_0\dot{v}^\mu = (e/c)F^{\mu\nu}(z)v_\nu \qquad (7.8)
$$

and differentiating with respect to the eigentime,

$$
(m_0c/e)\ddot{v}^\mu = v_\lambda\partial^\lambda F^{\mu\nu}(z)v_\nu + F^{\mu\nu}(z)\dot{v}_\nu\,. \qquad (7.9)
$$

Substituting (7.8) and (7.9) in (7.1) yields

$$
\begin{aligned}
m_0\dot{v}^\mu &= \frac{e}{c}F^{\mu\nu}v_\nu+\frac{e^2}{6\pi c^3}\Big[\frac{e}{m_0c}v_\lambda\partial^\lambda F^{\mu\,\nu}v_\nu \\
&\quad +\big(\frac{e}{m_0c}\big)^2\big(F^{\mu\nu}F_\nu{}^\lambda v_\lambda+c^{-2}F^{\alpha\lambda}F_\lambda{}^\beta v_\alpha v_\beta v^\mu\big)\Big]\,. \qquad (7.10)
\end{aligned}
$$

Written in three–vectors Equation (7.10) coincides with (7.7) together with the equation for the energy balance.

Of course, the justification of Equation (7.7) comes only from the structure of the solution flow to (7.2). Higher order corrections, although rather unimportant in our context, would have to be computed by the method explained in Section 6.1.

Equation (7.10) appears for the first time in the second volume of the Landau–Lifshitz Course in Theoretical Physics. It seems to be appropriate to call then Equation (7.10) the Landau–Lifshitz equation. The error in going from (7.1) to (7.10) is of the same order as the one in the derivation of the Lorentz–Dirac equation itself. *Thus we regard the Landau–Lifshitz equation as the effective equation governing the motion of a charged particle in the adiabatic limit.*

7.2 Some applications

(i) *Zero magnetic field.* For zero magnetic field the Landau–Lifshitz equation simplifies to

$$\begin{aligned} m_0\,\gamma\,\kappa(\boldsymbol{v})\dot{\boldsymbol{v}} &= e\,\boldsymbol{E} + \frac{e^2}{6\pi c^3}\Big[\frac{e}{m_0}\gamma\,(\boldsymbol{v}\cdot\nabla_{\boldsymbol{q}})\boldsymbol{E} \\ &\quad + \big(\frac{e}{m_0 c}\big)^2\big((\boldsymbol{v}\cdot\boldsymbol{E})\boldsymbol{E} - \gamma^2\,\boldsymbol{E}^2\boldsymbol{v} + \gamma^2 c^{-2}(\boldsymbol{v}\cdot\boldsymbol{E})^2\boldsymbol{v}\big)\Big]\,. \end{aligned} \tag{7.11}$$

Of interest is a central potential. We set $\boldsymbol{q}=\boldsymbol{r},\ |\boldsymbol{r}|=r,\ \hat{\boldsymbol{r}}=\boldsymbol{r}/|\boldsymbol{r}|, \phi_{\mathrm{ex}}(\boldsymbol{q}) = \phi(r)$ which implies $\boldsymbol{E}=-\phi'\hat{\boldsymbol{r}}$. Then (7.11) becomes

$$\begin{aligned} m_0\,\gamma\,\kappa(\boldsymbol{v})\dot{\boldsymbol{v}} &= -e\,\phi'\hat{\boldsymbol{r}} + \frac{e^2}{6\pi c^3}\Big[\frac{e}{m_0}\gamma\,(-(\boldsymbol{v}\cdot\hat{\boldsymbol{r}})\phi''\hat{\boldsymbol{r}} \\ &\quad -\frac{1}{r}(\boldsymbol{v}-(\boldsymbol{v}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}})\phi' + \big(\frac{e}{m_0 c}\big)^2\,\phi'^{\,2}\big((\boldsymbol{v}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \gamma^2\,\boldsymbol{v} \\ &\quad +\gamma^2\,c^{-2}(\boldsymbol{v}\cdot\hat{\boldsymbol{r}})^2\,\boldsymbol{v}\big)\Big]\,. \end{aligned} \tag{7.12}$$

The angular momentum $\boldsymbol{L}=\boldsymbol{r}\times m_0\gamma\boldsymbol{v}$ satisfies

$$\dot{\boldsymbol{L}} = \frac{e^2}{6\pi c^3}\Big[-\frac{e}{m_0}\,\frac{1}{r}\,\phi' - \big(\frac{e}{m_0 c}\big)^2\,\gamma^2\,(1-c^{-2}(\boldsymbol{v}\cdot\hat{\boldsymbol{r}})^2)\phi'^{\,2}\Big]\boldsymbol{L}\,. \tag{7.13}$$

Thus the orientation of $\boldsymbol{L}$ is conserved and the motion lies in the plane orthogonal to $\boldsymbol{L}$. No further reduction seems to be possible and one would have to rely on a numerical integration. Only for the harmonic oscillator, $\phi(r) = \frac{1}{2}\, m_0\omega_0^2 r^2$, a closed form solution can be achieved.

A somewhat more tractable case is to assume that ϕ_{ex} varies only along the 1–axis. Setting $\boldsymbol{v} = (v, 0, 0)$, $\boldsymbol{q} = (x, 0, 0)$, and $\boldsymbol{E} = (-\phi', 0, 0)$, Equation (7.11) becomes

$$m_0\,\gamma^3\,\dot{v} = -e\phi'(x) - \frac{e^2}{6\pi c^3}\,\frac{e}{m_0}\,\gamma\,\phi''(x)v\,. \tag{7.14}$$

The radiation reaction is proportional to $-\phi''(x)v$, which we can be regarded as a spatially varying friction coefficient proportional to $\phi''(x)$. For a convex potential, $\phi'' > 0$, like an oscillator potential, this friction coefficient is strictly positive and the resulting motion is damped until the minimum of ϕ is reached. In general however, ϕ'' will not have a definite sign, like the double well potential $\phi(x) \simeq (x^2-1)^2$, or the washboard potential $\phi(x) \simeq -\cos x$. At locations where $\phi''(x) < 0$ one has antifriction and the mechanical energy increases. This gain is always dominated by losses as can be seen from the energy balance

$$\begin{aligned}
&\frac{d}{dt}\big[m_0\,\gamma + e\phi + \frac{e^2}{6\pi c^3}\,\frac{e}{m_0}\,\gamma\,\phi' v\big] \\
&= -\frac{e^2}{6\pi c^3}\,\big(\frac{e}{m_0}\big)^2\,\phi'^2 - \frac{1}{m_0}\,\big(\frac{e^2}{6\pi c^3}\,\frac{e}{m_0}\big)^2\,\gamma\,\phi'\phi'' v\,.
\end{aligned} \tag{7.15}$$

The last term in (7.15) does not have a definite sign. But its prefactor is down by one order in ε and therefore it is outweighed by $-\phi'^2$.

Equation (7.14) has one peculiar feature. If $\phi(x) = -a_0 x$, $a_0 > 0$, over some some interval $[a_-, a_+]$, then $\phi'' = 0$ over that interval and the friction term vanishes. The particle entering at a_- is uniformly accelerated to the right until it reaches a_+. From Larmor's formula we know that the energy radiated per unit time equals $(e^2/6\pi c^3)(e/m_0)^2 a_0^2$. This energy must come entirely from the near field without a mechanical contribution. The same behavior is found for the Lorentz–Dirac equation. If, locally, $\boldsymbol{E} = \mathit{const}$ and $\boldsymbol{B} = 0$, then the Hamiltonian part is solved by the hyperbolic motion, i.e. a constantly accelerated relativistic particle. For this solution the radiation reaction vanishes which means that locally the critical manifold happens to be independent of ε. The radiated energy originates from the near field only.

(ii) *Zero electrostatic field and constant magnetic field.* We set $\boldsymbol{B} = (0, 0, B)$ with constant B. Then (7.7) simplifies to

$$\begin{aligned} m_0\,\gamma\,\kappa(\boldsymbol{v})\dot{\boldsymbol{v}} \;&=\; \frac{e}{c}(\boldsymbol{v}\times\boldsymbol{B}) + \frac{e^2}{6\pi c^3}\,\big(\frac{e}{m_0 c}\big)^2\,\big[(\boldsymbol{v}\cdot\boldsymbol{B})\boldsymbol{B} - \gamma^2\boldsymbol{B}^2\boldsymbol{v} \\ &\quad +\gamma^2\,c^{-2}(\boldsymbol{v}\cdot\boldsymbol{B})^2\boldsymbol{v}\big]\,. \end{aligned} \tag{7.16}$$

We multiply by $\kappa(\boldsymbol{v})^{-1}$ and obtain

$$m_0\,\gamma\,\dot{\boldsymbol{v}} = \frac{e}{c}(\boldsymbol{v}\times\boldsymbol{B}) + \frac{e^2}{6\pi c^3}\,\big(\frac{e}{m_0 c}\big)^2\,[(\boldsymbol{v}\cdot\boldsymbol{B})\boldsymbol{B} - \boldsymbol{B}^2\boldsymbol{v}]\,. \tag{7.17}$$

The motion parallel to $\boldsymbol{B}$ decouples with $\dot{v}_3 = 0$. We set $v_3 = 0$ and $\boldsymbol{v} = (\boldsymbol{u}, 0)$, $\boldsymbol{u}^\perp = (-u_2, u_1)$. Then the motion in the plane orthogonal to $\boldsymbol{B}$ is governed by

$$\gamma\,\dot{\boldsymbol{u}} = \omega_c(\boldsymbol{u}^\perp - \beta\omega_c\boldsymbol{u})\,, \tag{7.18}$$

with cyclotron frequency $\omega_c = eB/m_0c$ and $\beta = e^2/6\pi c^3 m_0$. (7.18) holds over the entire velocity range. For an electron $\beta\omega_c = 8.8\times 10^{-18}B$ [Gauss]. Thus even for very strong fields the friction is small compared to the inertial terms.

(7.18) can be integrated as

$$\frac{d}{dt}\,\gamma = -\beta\omega_c^2\,(\gamma^2 - 1) \tag{7.19}$$

with solution

$$\gamma_t = [\gamma_0 + 1 + (\gamma_0 - 1)e^{-2\beta\omega_c^2 t}][\gamma_0 + 1 - (\gamma_0 - 1)e^{-2\beta\omega_c^2 t}]^{-1}\,, \tag{7.20}$$

which tells us how $\boldsymbol{u}(t)^2$ shrinks to zero. To determine the angular dependence we introduce polar coordinates as $\boldsymbol{u} = u(\cos\varphi, \sin\varphi)$. Then

$$\frac{du}{d\varphi} = -\beta\omega_c u\ ,\ \frac{d\varphi}{dt} = \gamma^{-1}\omega\,. \tag{7.21}$$

Thus $u(\varphi)$ shrinks exponentially,

$$u(\varphi) = u(0)\, e^{-\beta\omega_c\varphi}\,. \tag{7.22}$$

Since $\beta\omega_c = 8.8 \times 10^{-18} B$ [Gauss] for an electron, even for strong fields the change of u in one revolution is tiny.

To obtain the evolution of the position $\boldsymbol{q} = (\boldsymbol{r}, 0)$, $|\boldsymbol{r}| = r$, we use that for zero radiation reaction, $\beta = 0$,

$$r = \frac{u}{\omega_c}\gamma\,. \tag{7.23}$$

By (7.22) this relation remains approximately valid for non–zero β. Inserting $u(t)$ from (7.20) we obtain

$$r(t) = r_0\, e^{-\beta\omega_c^2 t}[1 + ((\gamma_0 - 1)/2)(1 - e^{-2\beta\omega_c^2 t})]^{-1} \tag{7.24}$$

with r_0 the initial radius and $u(0)/c = (\gamma_0 - 1)^{1/2}/\gamma_0$ the initial speed which are related through (7.23). In the ultra–relativistic regime, $\gamma_0 \gg 1$, and for times such that $\beta\omega_c^2 t \ll 1$, (7.24) simplifies to

$$r(t) = r_0\,\frac{1}{1 + \gamma_0\beta\omega_c^2 t} \tag{7.25}$$

and the initial decay is according to the power law t^{-1} rather than exponential.

For an electron $\beta\omega_c^2 = 1.6 \times 10^{-6}(B\ [\text{Gauss}])^2/\text{sec}$. Therefore if we choose a field strength $B = 10^3$ Gauss and an initial radius of $r_0 = 10$ cm, which corresponds to the ultra–relativistic case of $\gamma = 6 \times 10^4$, then the radius shrinks within 0.9 sec to $r(t) = 1\ \mu$m by which time the electron has made 2×10^{14} revolutions.

(iii) *The Penning trap.* An electron can be trapped for a very long time in the combination of a homogeneous magnetic field and an electrostatic quadrupole potential, which has come to be known as a Penning trap. Its design has been optimized towards high precision measurements of the gyromagnetic g-factor of the electron. Our interest here is that the motion in the plane orthogonal to the magnetic field consists of two coupled modes, which means that the damping cannot be guessed by pure energy considerations using Larmor's formula. One really needs the full power of the Landau–Lifshitz equation.

An ideal Penning trap has the electrostatic quadrupole potential

$$e\phi(\boldsymbol{x}) = \frac{1}{2}\, m\omega_z^2 (-\frac{1}{2} x_1^2 - \frac{1}{2}\, x_2^2 + x_3^2)\,, \tag{7.26}$$

which satisfies $\triangle\phi = 0$, superimposed with the uniform magnetic field

$$\boldsymbol{B} = (0, 0, B)\,. \tag{7.27}$$

The quadrupole field provides an axial restoring force whereas the magnetic field is responsible for the radial restoring force, which however could be outweighed by the inverted part of the harmonic electrostatic potential.

We insert $\boldsymbol{E} = -\nabla\phi$ and $\boldsymbol{B}$ in the Landau–Lifshitz equation. The terms proportional to $(\boldsymbol{v}\cdot\nabla_{\boldsymbol{q}})\boldsymbol{E}$, $\boldsymbol{E}\times\boldsymbol{B}$, $(\boldsymbol{v}\cdot\boldsymbol{B})\boldsymbol{B}$, and $\boldsymbol{B}^2\boldsymbol{v}$ are linear in $\boldsymbol{v}$, resp. $\boldsymbol{q}$. The remaining terms are either cubic or quintic and will be neglected. This is justified provided

$$\frac{|\boldsymbol{v}|}{c} \ll 1 \tag{7.28}$$

and

$$(m_0\omega_z^2/e)\, r_{\max} \ll B, \text{ i.e. } \quad r_{\max} \ll c(\omega_c/\omega_z^2)\,, \tag{7.29}$$

if $r_{\max}$ denotes the maximal distance from the trap center. With these assumptions the Landau–Lifshitz equation decouples into an in–plane motion and an axial motion governed by

$$\dot{\boldsymbol{u}} = \frac{1}{2}\,\omega_z^2\boldsymbol{r} + \omega_c\boldsymbol{u}^\perp - \beta\big[(\omega_c^2 - \frac{1}{2}\,\omega_z^2)\boldsymbol{u} + \frac{1}{2}\,\omega_c\omega_z^2\boldsymbol{r}^\perp\big]\,, \tag{7.30}$$

$$\ddot{z} = -\omega_z^2 z - \beta\omega_z^2\dot{z}\,. \tag{7.31}$$

Here $\boldsymbol{q} = (\boldsymbol{r}, z)$, $\boldsymbol{v} = (\boldsymbol{u}, \dot{z})$, $(x_1, x_2)^\perp = (-x_2, x_1)$.

The axial motion is just a damped harmonic oscillator with frequency ω_z and friction coefficient

$$\gamma_z = \beta\omega_z^2\,. \tag{7.32}$$

The in–plane motion can be written in matrix form as

$$\frac{d}{dt}\psi = (A + \beta V)\psi \tag{7.33}$$

with $\psi = (\boldsymbol{r}, \boldsymbol{u})$ and $A_{11} = 0,\ A_{12} = \mathbb{1},\ A_{21} = \omega_z^2 \mathbb{1},\ A_{22} = i\omega_z \sigma_y,\ V_{11} = 0,\ V_{12} = 0,\ V_{21} = i\omega_c \omega_z^2 \sigma_y,\ V_{22} = (\omega_z^2 - \omega_c^2)\mathbb{1}$, where σ_y is the Pauli spin matrix with eigenvectors $\chi_\pm,\ \sigma_y \chi_\pm = \pm \chi_\pm$. The unperturbed motion is governed by the 4×4 matrix A. It has the eigenvectors $\psi_{+,\pm} = (\pm i(1/\omega_+)\chi_\mp, \chi_\mp)$ with eigenvalues $\pm i\omega_+$ and $\psi_{-,\pm} = (\pm i(1/\omega_-)\chi_\mp, \chi_\mp)$ with eigenvalues $\pm i\omega_-$, where

$$\omega_\pm = \frac{1}{2}\left(\omega_c \pm \sqrt{\omega_c^2 - 2\omega_z^2}\,\right). \tag{7.34}$$

The mode with frequency ω_+ is called cyclotron mode and the one with ω_- magnetron mode. Experimentally $\omega_c \gg \omega_z$ and therefore $\omega_+ \ll \omega_-$. The orbit is then an epicycle with rapid cyclotron and slow magnetron motion.

The adjoint matrix A^* has eigenvectors orthogonal to the $\psi's$. They are given by $\varphi_{+,\pm} = (\mp i(\omega_z^2/\omega_+)\chi_\mp, \chi_\mp)$ with eigenvalues $\pm i\omega_+$ and $\varphi_{-,\pm} = (-(\omega_z^2/\omega_-)\chi_\mp,\ \chi_\mp)$ with eigenvalue $\mp i\omega_-$.

Since β is small, the eigenfrequencies of $A + \beta V$ can be computed in first order perturbation. The cyclotron modes attains a negative real part corresponding to the friction coefficient

$$\gamma_+ = \frac{e^2}{6\pi c^3 m_0}\,\frac{\omega_+^3}{\omega_+ - \omega_-} \tag{7.35}$$

and the magnetron mode attains a positive real part corresponding to the antifriction coefficient

$$\gamma_- = \frac{e^2}{6\pi c^3 m_0}\,\frac{\omega_-^3}{\omega_- - \omega_+}\,. \tag{7.36}$$

As the electron radiates it lowers its potential energy by increasing the magnetron radius.

Experimentally $B = 6 \times 10^4$ Gauss and the voltage drop across the trap is 10V. This corresponds to $\omega_z = 4 \times 10^8$ Hz, $\omega_+ = 1.1 \times 10^{12}$ Hz, $\omega_- =$

7.4×10^4 Hz. The conditions (7.28), (7.29) are easily satisfied. For the life–times one obtains $(1/\gamma_z) = 5 \times 10^8$ sec, $(1/\gamma_+) = 8 \times 10^{-2}$ sec, and $-(1/\gamma_-) = 2 \times 10^{23}$ sec. Thus the magnetron motion is stable, as observed by keeping a single electron trapped over weeks. The cyclotron motion decays within fractions of a second. The axial motion is in fact damped by coupling to the external circuit and decays also within a second.

The variation with the magnetic field is more clearly discussed in terms of the dimensionless ratio $(\omega_c/\omega_z) = \lambda$. Then

$$\begin{aligned} \omega_\pm &= \omega_z \, \frac{1}{2} \, (\lambda \pm \sqrt{\lambda^2 - 2}) \, , \\ \gamma_\pm &= \pm \beta \omega_z^2 (\lambda \pm \sqrt{\lambda^2 - 2})^3 / 8 \sqrt{\lambda^2 - 2} \, . \end{aligned} \tag{7.37}$$

For large λ, $\omega_+ \cong \lambda$, $\omega_- \cong \lambda^{-1}$, whereas $\gamma_+ \cong \lambda^2, \gamma_- \cong \lambda^{-4}$. As $\lambda \to \sqrt{2}$, we have $\omega_+ = \omega_- = \omega_z/\sqrt{2}$. However the friction coefficients diverge as $(\lambda - \sqrt{2})^{-1/2}$. Let us call B_c the critical field at which the mechanical motion becomes unstable. For $B > B_c$, one has still periodic motion with frequency $\omega_z/\sqrt{2}$, but the coming instability is disclosed through the vanishing lifetime. In the mentioned experiment $\lambda = 2.7 \times 10^3$ and for fixed ω_z the critical field strength would be $B_c = 30$ Gauss.

Notes and References

ad 7: The name Lorentz–Dirac is standard but historically inaccurate. Some authors, e.g. Rohrlich (1997), therefore propose Abraham–Lorentz–Dirac instead. The radiation reaction term was first derived by Abraham (1905), compare with Sections 5 and 6 von Laue (1909) realized its covariant form. In the Pauli Handbuch article on relativity the equation is stated as in (7.1). The contribution of Dirac is explained in Section 2.2.

ad 7.1: The literature on the critical manifold of the Lorentz–Dirac equation is listed in *ad 6.3*. The Landau–Lifshitz equation appears in all editions of their Course in Theoretical Physics. They provide no hint on the geometrical picture of the solution flow nor on the errors involved in their approximation. It is rather surprising that the contribution of Landau and Lifshitz is ignored in essentially all discussions of radiation reaction, one notable exception being Teitelbom et al (1980). For that reason the Landau–Lifshitz equation was rederived independently in Spohn (1998). There have been other attempts to replace the Lorentz–Dirac equation by a second order equation, Mo, Papas (1971), Bonnor (1974), Parrot (1987), Ford, O'Connell (1991,1993). Based on Ford, O'Connell (1991), Jackson (1999) uses the substitution in the case of a radiation damped harmonic oscillator and discusses applications. In the general case only Landau and Lifshitz obtain the correct center manifold equation.

ad 7.2: Uniform acceleration is discussed in Rohrlich (1990). Constant magnetic field is important for synchroton sources. Since the electron is kept on its circular orbit, Larmor's formula is precise enough. Landau and Lifshitz (1959) give a brief discussion. The power law for the ultra–relativistic case is noted in Spohn (1999a). Shen (1972a,1978) discusses at which field strengths quantum corrections will become important. His results are only partially reliable, since he does not start from the Landau–Lifshitz equation. The Penning trap is reviewed by Brown and Gabrielse (1986), which includes a discussion of the classical orbits and their life–times. They state the results (7.35), (7.36) as based on a quantum resonance computation. Since the result is classical, it must follow from the Landau–Lifshitz equation, Spohn (1999a). In the classical framework, more general trap potentials can be handled through numerical integration routines for ordinary differential equations.

8 Spinning Charges

If an electron is modelled as a classical lump of highly concentrated charge, then merely by the interaction with its own radiation field the charge distribution will start to rotate. A proper mechanical description must include then the angular velocity of the internal rotation and an equation for the torque. The argument seems to leave little choice and in this chapter we will progress a few steps in the direction of including the classical spin, which leads to unexplored and interesting territory.

If we take the quantum mechanical description as starting point, however, as we should do, then the situation is more ambivalent. To be a little bit more specific we consider a quantum particle subject to slowly varying external forces, which is the standard semiclassical limit. The center of the wave packet evolves then according to an effective classical evolution equation of the form (6.34). Of course, the energy–momentum relation $E_s(\boldsymbol{P})$ has to computed now from the quantum hamiltonian. Only if the model is fully relativistic, we can be sure a priori that $E_s(\boldsymbol{P}) = (\boldsymbol{P}^2 + m^2)^{1/2}$. Let us assume that in addition the particle carries a spin $\frac{1}{2}$. The corresponding spinor, $\psi_t \in \mathbb{C}^2$, is governed by

$$i\hbar \frac{d}{dt}\,\psi_t = H_S(t)\psi_t \tag{8.38}$$

with $H_S(t)$ the time–dependent spin hamiltonian. In a relativistic theory we have

$$\begin{aligned} H_S(t) \;\; = \;\; & \frac{e}{mc}\,\frac{\hbar}{2}\boldsymbol{\sigma}\cdot\big[\big(\frac{g}{2} - 1 + \frac{1}{\gamma}\big)\boldsymbol{B}_{\text{ex}}(\boldsymbol{r}(t)) - \big(\frac{g}{2} - 1\big)\,\frac{\gamma}{1+\gamma}c^{-2} \\ & (\boldsymbol{u}(t)\cdot\boldsymbol{B}_{\text{ex}}(\boldsymbol{r}(t)))\boldsymbol{u}(t) - \big(\frac{g}{2} - \frac{\gamma}{1+\gamma}\big)c^{-1}\boldsymbol{u}(t)\times E_{\text{ex}}(\boldsymbol{r}(t))\big]\,. \end{aligned} \tag{8.39}$$

Here $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli spin matrices with $\sigma_i^2 = 1$ and commutation relations $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k, i, j, k = 1, 2, 3$. g is the gyromagnetic ratio of the quantum particle, $g \cong 2$ for an electron. The spin passively adjusts itself to the fields along the semiclassical orbit $t \mapsto (\boldsymbol{r}(t), \boldsymbol{u}(t))$ traced out by the particle. To leading order there is no back reaction onto the translational degrees of freedom. As we will see this is not the case for a classical spin and both degrees of freedom are coupled, a property which is shared by relativistically covariant Lagrangians for a particle with spin. We conclude that, in contrast to the translational degrees of freedom, a model including the classical spin serves only within limits as a phenomenological description for a quantum spin.

There is another, physically more basic objection. $H_S(t)$ is linear in $\boldsymbol{\sigma}$. Thus defining the average spin $\boldsymbol{s}_t = < \psi_t, \boldsymbol{\sigma}\psi_t >$ we see that in (8.39) $\hbar$ drops out and $\boldsymbol{s}_t$ satisfies a classical spin equation. In fact, (8.39) becomes the BMT equation for $\boldsymbol{s}_t$, an equation originally obtained on purely classical grounds. However, such an approximation by a classical angular momentum is valid only in the large spin number limit. In the semiclassical the spin degree if freedom remains fully quantum. Of course, in the standard polarization experiments, as for example the high precision measurements of the gyromagnetic ratio, interference is not probed and the classical picture serves well.

8.1 Abraham model with spin

Abraham models the charge as a nonrelativistic rigid body. Clearly, a complete mechanical description must specify both its center of mass and its angular velocity, which we denote by $\boldsymbol{\omega}(t) \in \mathbb{R}^3$. The spinning charge generates the current

$$\boldsymbol{j}(\boldsymbol{x},t) = \big(\boldsymbol{v}(t) + \boldsymbol{\omega}(t) \times (\boldsymbol{x} - \boldsymbol{q}(t))\big)\rho(\boldsymbol{x} - \boldsymbol{q}(t)) \,. \tag{8.40}$$

Therefore the source term in Maxwell equations is modified as

$$\begin{aligned} \partial_t \boldsymbol{B}(\boldsymbol{x},t) &= -\nabla \times \boldsymbol{E}(\boldsymbol{x},t), \\ \partial_t \boldsymbol{E}(\boldsymbol{x},t) &= \nabla \times \boldsymbol{B}(\boldsymbol{x},t) - \big(\boldsymbol{v}(t) + \boldsymbol{\omega}(t) \times (\boldsymbol{x} - \boldsymbol{q}(t))\big)\rho(\boldsymbol{x} - \boldsymbol{q}(t)) \,, \\ \nabla \cdot \boldsymbol{E}(\boldsymbol{x},t) &= \rho(\boldsymbol{x} - \boldsymbol{q}(t), \ \nabla \cdot \boldsymbol{B}(\boldsymbol{x},t) = 0 \,, \end{aligned} \tag{8.41}$$

which satisfies charge conservation, since ρ is radial.

The mass distribution of the rigid body is assumed to have the same form factor as the charge distribution, $\rho_m(\boldsymbol{x}) = m_\mathrm{b} f(\boldsymbol{x}^2)$. The bare moment of inertia is then

$$I_\mathrm{b} = \frac{2}{3}\int d^3x \rho_m(\boldsymbol{x})\boldsymbol{x}^2 \tag{8.42}$$

with corresponding angular momentum $\boldsymbol{S} = I_\mathrm{b}\,\boldsymbol{\omega}$. The electric dipole moment of the charge distribution ρ vanishes by symmetry. The magnetic dipole moment of the current (8.40) is given by

$$\begin{aligned} \mu &= \frac{1}{2}\int d^3x \boldsymbol{x} \times \big(\boldsymbol{v} + (\boldsymbol{\omega}\times\boldsymbol{x})\big)\rho(\boldsymbol{x}) \\ &= \frac{1}{2}\frac{e}{m_\mathrm{b}} I_\mathrm{b}\,\boldsymbol{\omega} = g_\mathrm{b}\frac{e}{2m_\mathrm{b}}\,\boldsymbol{S} \end{aligned} \tag{8.43}$$

with bare gyromagnetic ratio $g_\mathrm{b} = 1$. As in the case of the bare mass, I_b and g_b will be renormalized through the self–interaction.

The Lorentz force equation comes now in two parts, one for the linear and one for the angular momentum. To be consistent we stick to the nonrelativistic form and have

$$\begin{aligned} \frac{d}{dt} m_\mathrm{b}\boldsymbol{v}(t) &= \int d^3x \rho(\boldsymbol{x}-\boldsymbol{q}(t))\big[\boldsymbol{E}(\boldsymbol{x},t) + \big(\boldsymbol{v}(t) + \boldsymbol{\omega}(t)\times(\boldsymbol{x}-\boldsymbol{q}(t))\big)\times\boldsymbol{B}(\boldsymbol{x},t)\big]\,, \\ \frac{d}{dt} I_\mathrm{b}\,\boldsymbol{\omega}(t) &= \int d^3x \rho(\boldsymbol{x}-\boldsymbol{q}(t))(\boldsymbol{x}-\boldsymbol{q}(t))\times\big[\boldsymbol{E}(\boldsymbol{x},t) \qquad (8.44) \\ &\qquad + \big(\boldsymbol{v}(t) + \boldsymbol{\omega}(t)\times(\boldsymbol{x}-\boldsymbol{q}(t))\big)\times\boldsymbol{B}(\boldsymbol{x},t)\big]\,. \qquad (8.45) \end{aligned}$$

If in addition there are external forces acting on the charge, then $\boldsymbol{E}$ and $\boldsymbol{B}$ in (8.44), (8.45) would have to be replaced by $\boldsymbol{E} + \boldsymbol{E}_\mathrm{ex}$ and $\boldsymbol{B} + \boldsymbol{B}_\mathrm{ex}$, respectively.

The Abraham model of Section 1.4 is obtained by formally setting $\boldsymbol{\omega}(t) = 0$. Note that this is not consistent with the Lorentz torque equation (8.45), since $\dot{\boldsymbol{\omega}} \neq 0$, in general, even for $\boldsymbol{\omega} = 0$.

The Abraham model with spin has the conserved energy

$$\mathcal{E} = \frac{1}{2}m_\mathrm{b}\boldsymbol{v}^2 + \frac{1}{2}I_\mathrm{b}\boldsymbol{\omega}^2 + \frac{1}{2}\int d^3x(\boldsymbol{E}^2 + \boldsymbol{B}^2) \tag{8.46}$$

and the conserved linear momentum

$$\mathcal{P} = m_\mathrm{b}\boldsymbol{v} + \int d^3x \boldsymbol{E} \times \boldsymbol{B} \,. \tag{8.47}$$

In addition the total angular momentum

$$\mathcal{J} = \boldsymbol{q} \times m_\mathrm{b}\boldsymbol{v} + I_\mathrm{b}\boldsymbol{\omega} + \int d^3x \boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{B}) \tag{8.48}$$

is conserved. Of course, also the Abraham model without spin is invariant under rotations and must therefore have a correspondingly conserved quantity. Only it does not have the standard form of the total angular momentum, which from a somewhat different perspective indicates that inner rotations must be included.

As good tradition already, we assume that the external forces are slowly varying and want to derive in this adiabatic limit an effective equation of motion for the particle including its spin. As a first step of this program we have to determine the charge solitons. We set

$$\boldsymbol{q}(t) = \boldsymbol{v}t \ , \ \boldsymbol{\omega}(t) = \boldsymbol{\omega} \ , \tag{8.49}$$

$$\boldsymbol{E}(\boldsymbol{x},t) = \boldsymbol{E}(\boldsymbol{x} - \boldsymbol{v}t) \ , \ \boldsymbol{B}(\boldsymbol{x},t) = \boldsymbol{B}(\boldsymbol{x} - \boldsymbol{v}t) \tag{8.50}$$

and have to determine the solutions of

$$\begin{aligned} &-\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{B} = -\boldsymbol{\nabla} \times \boldsymbol{E} \ , \\ &-\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{E} = \boldsymbol{\nabla} \times \boldsymbol{B} - \rho(\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}) \,, \\ &\boldsymbol{\nabla} \boldsymbol{E} = \rho \ , \ \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \,, \end{aligned} \tag{8.51}$$

$$0 = \int d^3x \rho(\boldsymbol{x})[\boldsymbol{E}(\boldsymbol{x}) + (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}) \times \boldsymbol{B}(\boldsymbol{x})] \,, \tag{8.52}$$

$$0 = \int d^3x \rho(\boldsymbol{x}) \boldsymbol{x} \times [\boldsymbol{E}(\boldsymbol{x}) + (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}) \times \boldsymbol{B}(\boldsymbol{x})] \,. \tag{8.53}$$

We will solve (8.51) to (8.53) in Fourier space.

The inhomogeneous Maxwell equations (8.51) are solved by

$$\widehat{\boldsymbol{E}} = \widehat{\boldsymbol{E}}_1 + \widehat{\boldsymbol{E}}_2 \ , \ \widehat{\boldsymbol{B}} = \widehat{\boldsymbol{B}}_1 + \widehat{\boldsymbol{B}}_2 \tag{8.54}$$

with

$$\widehat{\boldsymbol{E}}_1(\boldsymbol{k}) = -i[\boldsymbol{k}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-1}(\boldsymbol{k} - (\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{v})\hat{\rho}(\boldsymbol{k})\,, \tag{8.55}$$

$$\widehat{\boldsymbol{E}}_2(\boldsymbol{k}) = -[\boldsymbol{k}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-1}(\boldsymbol{\omega}\times\boldsymbol{k})(\boldsymbol{v}\cdot\nabla_{\boldsymbol{k}})\hat{\rho}(\boldsymbol{k})\,, \tag{8.56}$$

and

$$\widehat{\boldsymbol{B}}_1(\boldsymbol{k}) = i[\boldsymbol{k}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-1}(\boldsymbol{k}\times\boldsymbol{v})\hat{\rho}(\boldsymbol{k})\,, \tag{8.57}$$

$$\widehat{\boldsymbol{B}}_2(\boldsymbol{k}) = -[\boldsymbol{k}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-1}(\boldsymbol{k}\times(\boldsymbol{\omega}\times\nabla_{\boldsymbol{k}}))\hat{\rho}(\boldsymbol{k})\,. \tag{8.58}$$

Note that $\widehat{\boldsymbol{E}}_1,\ \widehat{\boldsymbol{B}}_1$ are odd and $\widehat{\boldsymbol{E}}_2,\ \widehat{\boldsymbol{B}}_2$ are even in $\boldsymbol{k}$.

Using that odd terms vanish, in the Lorentz force equation we have

$$\begin{aligned}
& -\int d^3k\hat{\rho}^*[\boldsymbol{k}^2 - (\boldsymbol{v}\cdot\boldsymbol{k})^2]^{-1}(\boldsymbol{\omega}\times\boldsymbol{k})(\boldsymbol{v}\cdot\nabla_{\boldsymbol{k}})\hat{\rho} \\
& -\int d^3k\hat{\rho}^*[\boldsymbol{k}^2 - (\boldsymbol{v}\cdot\boldsymbol{k})^2]^{-1}\boldsymbol{v}\times(\boldsymbol{k}\times(\boldsymbol{\omega}\times\nabla_{\boldsymbol{k}}))\hat{\rho} \\
& +\int d^3k((\boldsymbol{\omega}\times\nabla_{\boldsymbol{k}})\hat{\rho}^*)[\boldsymbol{k}^2 - (\boldsymbol{v}\cdot\boldsymbol{k})^2]^{-1}\times(\boldsymbol{k}\times\boldsymbol{v})\hat{\rho} \\
= \; & -\int d^3k\hat{\rho}^*[\boldsymbol{k}^2 - (\boldsymbol{v}\cdot\boldsymbol{k})^2]^{-1}|\boldsymbol{k}|^{-1}\hat{\rho}_r' \\
& \qquad \Big((\boldsymbol{\omega}\times\boldsymbol{k})(\boldsymbol{v}\cdot\boldsymbol{k}) + \boldsymbol{v}\times(\boldsymbol{k}\times(\boldsymbol{\omega}\times\boldsymbol{k})) - ((\boldsymbol{\omega}\times\boldsymbol{k})\cdot\boldsymbol{v})\boldsymbol{k}\Big) = 0
\end{aligned} \tag{8.59}$$

for every $\boldsymbol{v}$ and $\boldsymbol{\omega}$, where we took into account that $\hat{\rho}$ is radial.

The Lorentz torque equation requires more work. Using again that odd terms vanish, we have

$$
\begin{aligned}
& i\int d^3k\hat{\rho}^*\big(\nabla_{\boldsymbol{k}}\times\widehat{\boldsymbol{E}}_1+\nabla_{\boldsymbol{k}}\times(\boldsymbol{v}\times\widehat{\boldsymbol{B}}_1)+\nabla_{\boldsymbol{k}}\times((\boldsymbol{\omega}\times i\nabla_{\boldsymbol{k}})\times\widehat{\boldsymbol{B}}_2\big) \\
= & -\int d^3k|\boldsymbol{k}|^{-1}\hat{\rho}_r^{*\prime}[\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-1}\hat{\rho} \\
& \qquad \big(\boldsymbol{k}\times(\boldsymbol{k}-(\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{v})-\boldsymbol{k}\times(\boldsymbol{v}\times(\boldsymbol{k}\times\boldsymbol{v}))\big) \\
& +\int d^3k|\boldsymbol{k}|^{-1}\hat{\rho}_r^{*\prime}\boldsymbol{k}\times((\boldsymbol{\omega}\times\nabla_{\boldsymbol{k}})\times\widehat{\boldsymbol{B}}_2) \\
= & \int d^3k|\boldsymbol{k}|^{-1}\hat{\rho}_r^{*\prime}\boldsymbol{k}\times(\nabla_{\boldsymbol{k}}(\boldsymbol{\omega}\cdot\widehat{\boldsymbol{B}}_2)-\boldsymbol{\omega}\nabla_{\boldsymbol{k}}\cdot\widehat{\boldsymbol{B}}_2) \\
= & -\int d^3k|\boldsymbol{k}|^{-1}\hat{\rho}_r^{*\prime}(\boldsymbol{k}\times\boldsymbol{\omega})\nabla_{\boldsymbol{k}}\cdot\widehat{\boldsymbol{B}}_2\,. \qquad (8.60)
\end{aligned}
$$

For the divergence of $\widehat{\boldsymbol{B}}_2$ we find

$$
\nabla_{\boldsymbol{k}}\cdot\widehat{\boldsymbol{B}}_2=2[\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-2}\boldsymbol{k}^2(\boldsymbol{\omega}\cdot\nabla_{\boldsymbol{k}}-(\boldsymbol{v}\cdot\boldsymbol{\omega})(\boldsymbol{v}\cdot\nabla_{\boldsymbol{k}}))\hat{\rho} \qquad (8.61)
$$

and therefore zero Lorentz torque results in the equation

$$
\int d^3k|\nabla_{\boldsymbol{k}}\hat{\rho}|^2 2[\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-2}(\boldsymbol{k}\times\boldsymbol{\omega})(\boldsymbol{\omega}\cdot\boldsymbol{k}-(\boldsymbol{v}\cdot\boldsymbol{\omega})(\boldsymbol{v}\cdot\boldsymbol{k}))=0\,. \qquad (8.62)
$$

Taking into account that $\hat{\rho}$ is radial, the torque vanishes only if either $\boldsymbol{\omega}\parallel\boldsymbol{v}$ or $\boldsymbol{\omega}\perp\boldsymbol{v}$. If $\boldsymbol{v}=0$, the torque vanishes always. For $\boldsymbol{\omega}$ oblique to $\boldsymbol{v}$ Equations (8.51) to (8.53) have no solution.

Physically the charge distribution is rigid, but the electromagnetic fields are Lorentz contracted along $\boldsymbol{v}$. This mismatch yields a nonvanishing torque unless $\boldsymbol{\omega}\parallel\boldsymbol{v}$, resp. $\boldsymbol{\omega}\perp\boldsymbol{v}$. Clearly, this mismatch is an artifact of the semi–relativistic Abraham model. For a relativistic extended charge distribution there is a charged soliton for every $\boldsymbol{v}$ and $\boldsymbol{\omega}$.

In the adiabatic limit, there must be two disjoint effective equations of motion. If $\boldsymbol{\omega}\parallel\boldsymbol{v}$ initially, then it will remain so approximately and there is a closed equation for $\boldsymbol{q},\boldsymbol{v}$ and $\boldsymbol{\omega}\cdot\boldsymbol{v}$. Similarly, if $\boldsymbol{\omega}\perp\boldsymbol{v}$ initially, this property is almost preserved in time. $\boldsymbol{v}$ and $\boldsymbol{\omega}-(\boldsymbol{v}\cdot\boldsymbol{\omega})\boldsymbol{\omega}$ vary slowly on the same time scale. We expect that through the emission of radiation an $\boldsymbol{\omega}$ initially oblique to $\boldsymbol{v}$ will rapidly relax to either being parallel or orthogonal to $\boldsymbol{v}$, depending on the initial conditions. This admittedly rather sketchy picture raises some interesting dynamical questions, in particular how precisely the spinning particle succeeds in slowly turning its axis of rotation. We have not persued this issue, since it leads away from the quantum spin.

8.2 Relativistic dynamics of charged particle with spin

To be supplied.

Notes and References

ad 8: BMT is an acronym for Bargmann, Michel, Telegdi (1955). The BMT equation is explained in Jackson (1999). Bailey, Picasso (1970) is an informative article on how the BMT equation is used in the analysis of the high precision measurements of the electron and muon g-factor. The BMT equation with $g = 2$ is the semiclassical limit of the Dirac equation, Rubinow, Keller (1963), Bolte, Keppeler (1999), Spohn (1999c).

ad 8.1: Kiessling (1999) observes that the usual form of the total angular momentum is conserved only if the inner rotation of the charged particles is included. The charge soliton solutions with spin are determined in Spohn (1999b).

ad 8.2: Just as for translational degrees of freedom, one way to guess the correct effective spin dynamics is to impose Lorentz invariance. In addition, one could require that the equations of motion come from a Lagrangian action. In full generality, including an electric dipole moment, this program was carried out by Bhabha, Corben (1941). Alternative approaches are compared in Corben (1961), Nyborg (1962). A concise and useful summary is given by Barut (1964) who discusses also how the BMT equation fits into the general scheme. In the relativistic extended charge model of Nodvik (1964) spin is included and the effective equation of motion is derived formally. Recent work on this model is Appel, Kiessling (1999).

9 Many Charges

There is little effort in extending the Abraham model to several particles. We label their positions and velocities as $\boldsymbol{q}_j(t), \boldsymbol{v}_j(t), j = 1, \ldots, N$. The j-th particle has the bare mass $m_{\mathrm{b}j}$ and the charge e_j, where for simplicity all particles have the same form factor f, i.e. $\rho_j(\boldsymbol{x}) = e_j f(\boldsymbol{x}^2)$. The motion of each particle is governed by the Lorentz force equation, as before, and the current in the Maxwell equations becomes now the sum over the single particle currents. Therefore the equations of motion read

$$\begin{aligned}
\partial_t \boldsymbol{B}(\boldsymbol{x},t) &= -\nabla \times \boldsymbol{E}(\boldsymbol{x},t)\,, \\
\partial_t \boldsymbol{E}(\boldsymbol{x},t) &= \nabla \times \boldsymbol{B}(\boldsymbol{x},t) - \sum_{j=1}^{N} \rho_j(\boldsymbol{x}-\boldsymbol{q}_j(t))\boldsymbol{v}_j(t)\,, \qquad (9.1)\\
\nabla \cdot \boldsymbol{E}(\boldsymbol{x},t) &= \sum_{j=1}^{N} \rho_j(\boldsymbol{x}-\boldsymbol{q}_j(t)) \ , \ \nabla \cdot \boldsymbol{B}(\boldsymbol{x},t) = 0\,,
\end{aligned}$$

$$\frac{d}{dt}\big(m_{\mathrm{b}i}\,\gamma_j \boldsymbol{v}_j(t)\big) = \int d^3x \rho(\boldsymbol{x}-\boldsymbol{q}_i(t))\big(\boldsymbol{E}(\boldsymbol{x},t) + \boldsymbol{v}_i(t) \times \boldsymbol{B}(\boldsymbol{x},t)\big)\,, \qquad (9.2)$$

$i = 1, \ldots, N$, with $\gamma_i = (1 - \boldsymbol{v}_i^2)^{-1/2}$.

There are no external forces. Thus the force acting on a given particle is due to the other particles as mediated through the Maxwell field. In addition, there is the self–force which we have discussed already at length. Physically we trust our model only if particles are very far apart on the scale set by R_ρ. If two particles are at a distance of a few R_ρ, then there are strong forces which depend on the details of the phenomenological and unknown charge distribution. Thus we assume that initially

$$|\boldsymbol{q}_i^0 - \boldsymbol{q}_j^0| = \mathcal{O}(\varepsilon^{-1} R_\rho) \ , \ i \neq j\,. \qquad (9.3)$$

We emphasize that the scale parameter ε enters only through the initial conditions. ε^{-1} is the typical distance of particles measured in units of R_ρ. For the initial fields it is natural to again impose the condition of no slip. Then they are a linear superposition of charge soliton fields corresponding to $\boldsymbol{q}_i^0,\ \boldsymbol{v}_i^0,\ i = 1, \ldots, N$.

We expect (9.3) to remain valid at least over a certain macroscopic time span and we want to understand whether in this limiting regime there is a closed dynamics for the particles by themselves. Since the particles are far apart, it takes a time of order $\varepsilon^{-1}R_\rho/c = \varepsilon^{-1}t_\rho$ for light to travel inbetween and the force on a given particle depends on the other particles at a macroscopically retarded time. This means that the effective equations of motion are closed, but nonlocal with a structure to be explained in the following section. In many circumstances the velocities can be regarded as small, $|\boldsymbol{v}_j|/c \ll 1$, and retardation effects are neglible. To lowest order this yields then the static Coulomb interaction. Somewhat unexpected even the first order correction has still the form of an effective Lagrangian.

9.1 Retarded interaction

We insert the solution of the inhomogeneous Maxwell equation (9.1) into the Lorentz force equation (9.2). The forces are additive and the force on particle i naturally splits into self–force ($j = i$) and a mutual force ($j \neq i$). For the self–force we use the Taylor expansion of Section 5. Thereby the mass is renormalized and the next order is the radiation reaction. For the mutual force we recall that in Section 6.4 we showed already that to leading order the field generated by charge j is the Liénard–Wiechert field. Thus, ignoring radiation reaction, we obtain as retarded equations of motion

$$m_i(\boldsymbol{v}_i)\dot{\boldsymbol{v}}_i = \sum_{\substack{j=1\\ j\neq i}}^{N} e_j\big(\boldsymbol{E}_{\mathrm{ret}j}(\boldsymbol{q}_i,t) + \boldsymbol{v}_i \times \boldsymbol{B}_{\mathrm{ret}j}(\boldsymbol{q}_i,t)\big)\ ,\ t \geq 0\,. \tag{9.4}$$

Here m_i is the effective mass of particle i as defined in (6.1). $\boldsymbol{E}_{\mathrm{ret}j}(\boldsymbol{x},t)$ equals (1.21) with $\boldsymbol{q}$ replaced by $\boldsymbol{q}_j$ and t_{ret} replaced by $t_{\mathrm{ret}j}$ which is defined by

$$t_{\mathrm{ret}j} = t - |\boldsymbol{x} - \boldsymbol{q}_j(t_{\mathrm{ret}j})|\,. \tag{9.5}$$

For $\boldsymbol{x} = \boldsymbol{q}_i$ the retarded time is $\mathcal{O}(\varepsilon^{-1}t_\rho)$. Similarly $\boldsymbol{B}_{\mathrm{ret}j}(\boldsymbol{x},t)$ equals (1.22) with $\boldsymbol{q}$ replaced by $\boldsymbol{q}_j$ and t_{ret} replaced by $t_{\mathrm{ret}j}$. Note that the equations of motion in (2.14) have the same structure.

To solve (9.4) one needs the trajectories for the whole past. Our assumption of no initial slip is equivalent to

$$\boldsymbol{q}_i(t) + \boldsymbol{q}_i^0 + t\boldsymbol{v}_i^0\ ,\ i = 1, \ldots, N\ ,\ t \leq 0 \tag{9.6}$$

which must be added to (9.4).

Using (9.4) we can estimate the size of the various contributions. The far field contributions to $\boldsymbol{E}_{\mathrm{ret}j}$ and $\boldsymbol{B}_{\mathrm{ret}j}$ are of $\mathcal{O}(\varepsilon^2)$ and the near field contributions are $\mathcal{O}(\varepsilon)\dot{\boldsymbol{v}}_j$. Thus $\dot{\boldsymbol{v}}_i = \mathcal{O}(\varepsilon^2)$ and the first order correction from the near field is $\mathcal{O}(\varepsilon^3)$. In the next order we see the radiation reaction which is proportional to $\dot{\boldsymbol{v}}_i^2$ and thus $\mathcal{O}(\varepsilon^4)$. In (9.4) we would have to add the Lorentz–Dirac term of (6.1) for particle i. As a consequence (9.4) will pick up runaway solutions and we have to restrict to the critical manifold. Thereby the friction force becomes retarded. This is not surprising. The friction comes from acceleration which is due to the retarded motion of all other charges.

The issues raised here remain largely unexplored, at present. One would like to know how well (9.4) approximates the true dynamics and over what time scale.

9.2 Limit of small velocities

We impose that the initial velocities are small. The natural scale turns out to be

$$|\dot{\boldsymbol{v}}_j| = \mathcal{O}(\sqrt{\varepsilon}c) \tag{9.7}$$

and, of course, we have to show that this order is maintained, at least for a certain time span. To preserve the relation $\dot{\boldsymbol{q}} = \boldsymbol{v}$ we have to adjust the time scale as $\varepsilon^{-1/2}$ relative to the macroscopic time scale. The accumulated force is then of order $\sqrt{\varepsilon}$ which just balances the velocity in (9.7). Therefore we arrive at the following scale transformation

$$\begin{aligned} &t = \varepsilon^{-3/2} t',\ \boldsymbol{q}_j = \varepsilon^{-1} \boldsymbol{q}'_j\ ,\ \boldsymbol{v}_j = \sqrt{\varepsilon}\, \boldsymbol{v}'_j\,, \\ &\boldsymbol{x} = \varepsilon^{-1} \boldsymbol{x}',\ \boldsymbol{E} = \varepsilon^{3/2} \boldsymbol{E}'\ ,\ \boldsymbol{B} = \varepsilon^{3/2} \boldsymbol{B}'\,, \end{aligned} \tag{9.8}$$

where the primed quantities are considered to be of $\mathcal{O}(1)$. The field amplitudes are scaled by $\varepsilon^{3/2}$ so to preserve the field energy.

There is little risk of confusion in omitting the primes. We set

$$\boldsymbol{q}_j^\varepsilon(t) = \varepsilon \boldsymbol{q}_j\big(\varepsilon^{-3/2} t\big)\ , \quad \boldsymbol{v}_j^\varepsilon(t) = \varepsilon^{-1/2} \boldsymbol{v}_j(\varepsilon^{-3/2} t)\,. \tag{9.9}$$

Then the rescaled Maxwell–Lorentz equations are

$$\begin{aligned} \sqrt{\varepsilon}\, \partial_t \boldsymbol{B}(\boldsymbol{x},t) &= -\nabla \times \boldsymbol{E}(\boldsymbol{x},t)\,, \\ \sqrt{\varepsilon}\, \partial_t \boldsymbol{E}(\boldsymbol{x},t) &= \nabla \times \boldsymbol{B}(\boldsymbol{x},t) - \sum_{j=1}^N \sqrt{\varepsilon} \boldsymbol{v}_j^\varepsilon(t) \sqrt{\varepsilon} \rho_{j,\varepsilon}(\boldsymbol{x} - \boldsymbol{q}_j^\varepsilon(t))\,, \\ \nabla \cdot \boldsymbol{E}(\boldsymbol{x},t) &= \sum_{j=1}^N \sqrt{\varepsilon} \rho_{j,\varepsilon}(\boldsymbol{x} - \boldsymbol{q}_j^\varepsilon(t))\ ,\ \nabla \cdot \boldsymbol{B}(\boldsymbol{x},t) = 0\,, \end{aligned} \tag{9.10}$$

$$\begin{aligned} &\varepsilon \frac{d}{dt}\big(m_{\mathrm{b}i}(1 - \varepsilon \boldsymbol{v}_i^\varepsilon(t)^2)^{-1/2} \boldsymbol{v}_i^\varepsilon(t)\big) \\ &= \int d^3x \sqrt{\varepsilon} \rho_{\varepsilon,i}(\boldsymbol{x} - \boldsymbol{q}_i^\varepsilon(t)) \big(\boldsymbol{E}(\boldsymbol{x},t) + \sqrt{\varepsilon} \boldsymbol{v}_i^\varepsilon(t) \times \boldsymbol{B}(\boldsymbol{x},t)\big)\,. \end{aligned} \tag{9.11}$$

On the new scale the velocity of light tends to infinity as $c/\sqrt{\varepsilon}$ and the charge distribution has total charge $\sqrt{\varepsilon}$, finite electrostatic energy m_{e}, and shrinks to a δ-function as $\sqrt{\varepsilon}\rho_{\varepsilon,j}(\boldsymbol{x}) = \sqrt{\varepsilon}\,\varepsilon^{-3}\rho_j(\varepsilon \boldsymbol{x})$. Recall that the scale parameter ε is just a convenient way to order the magnitudes of the various contributions.

Before entering into more specific computations, it is useful to first sort out what should be expected. We follow our practice from before and denote positions and velocities of the comparison dynamics by $\boldsymbol{r}_j, \boldsymbol{u}_j, j = 1, \dots, N$, i.e. $\boldsymbol{q}_j^\varepsilon(t) \cong \boldsymbol{r}_j(t), \boldsymbol{v}_j^\varepsilon(t) \cong \boldsymbol{u}_j(t)$. Since the velocities are small, the kinetic energy takes its nonrelativistic limit

$$T_0(\boldsymbol{u}_j) = \frac{1}{2}\big(m_{\mathrm{b}j} + \frac{4}{3}\,m_{\mathrm{e}j}\big)\boldsymbol{u}_j^2\,, \tag{9.12}$$

up to a constant, compare with (3.19). Note that the mass of the particle is renormalized through the interaction with the field. For small velocities magnetic fields are small and retardation effects can be neglected. Thus the potential energy of the effective dynamics should be purely Coulombic and be given by

$$U_0(\boldsymbol{r}_1, \dots, \boldsymbol{r}_N) = \frac{1}{2}\sum_{i\neq j=1}^{N} \frac{e_i e_j}{4\pi|\boldsymbol{r}_i - \boldsymbol{r}_j|}\,. \tag{9.13}$$

So what is the next order? For the kinetic energy we merely expand in (3.19) with the result

$$T_1(\boldsymbol{u}_j) = \varepsilon\big(\frac{1}{8}m_{\mathrm{b}j} + \frac{2}{15}m_{\mathrm{e}j}\big)\boldsymbol{u}_j^4\,. \tag{9.14}$$

The next order correction to the Coulomb forces requires more explicit considerations, which will be explained in the following section. There are corrections due to retardation and to the magnetic field, which combine into a velocity dependent potential as

$$U_1(\boldsymbol{r}_1, \boldsymbol{u}_1, \dots, \boldsymbol{r}_N, \boldsymbol{u}_N) = -\varepsilon\frac{1}{4}\sum_{i\neq j=1}^{N} \frac{e_i e_j}{4\pi|\boldsymbol{r}_i - \boldsymbol{r}_j|}\big(\boldsymbol{v}_i\cdot\boldsymbol{v}_j + (\boldsymbol{v}_i\cdot\widehat{\boldsymbol{r}}_{ij})(\widehat{\boldsymbol{r}}_{ij}\cdot\boldsymbol{v}_j)\big) \tag{9.15}$$

with $\widehat{\boldsymbol{r}}_{ij} = (\boldsymbol{r}_i - \boldsymbol{r}_j)/|\boldsymbol{r}_i - \boldsymbol{r}_j|$.

In principle we could continue the expansion. It is of interest to see at what scale radiation effects will be important. They are proportional to $\dot{\boldsymbol{v}}^2$. On the microscopic scale $\dot{\boldsymbol{v}} \cong \varepsilon^2$, as argued before. In rescaled velocities and accumulated over the time span $\varepsilon^{-3/2}$ this results in a loss of energy of the order $\varepsilon^{3/2}$. Thus the next order correction to the comparison dynamics is dissipative and of order $\varepsilon^{3/2}$.

We recall that $|\boldsymbol{v}_j|/c = \mathcal{O}(\sqrt{\varepsilon})$. Thus we may set $\varepsilon = 1$ at the expense of reintroducing the velocity of light, c. Then up to an error of order $(|\boldsymbol{v}_j|/c)^{3/2}$ the effective dynamics of the N charges is conservative and is governed by the Lagrangian

$$L_{\text{Darwin}} = \sum_{j=1}^{N} \big((m_{\text{b}j} + \frac{4}{3}m_{\text{e}j})\frac{1}{2}\boldsymbol{u}_j^2 + (\frac{1}{8}m_{\text{b}j} + \frac{2}{15}m_{\text{e}j})c^{-2}\boldsymbol{u}_j^4\big)$$

$$-\frac{1}{2}\sum_{i\neq j=1}^{N} \frac{e_i e_j}{4\pi|\boldsymbol{r}_i - \boldsymbol{r}_j|}\big[1 - \frac{1}{2c^2}\left(\boldsymbol{u}_i \cdot \boldsymbol{u}_j + (\boldsymbol{u}_i \cdot \widehat{\boldsymbol{r}}_{ij})(\boldsymbol{u}_j \cdot \widehat{\boldsymbol{r}}_{ij})\right)\big]\,.$$

L_{Darwin} is known as the Darwin Lagrangian and widely used in plasma physics.

9.3 The Darwin Lagrangian

As can be seen from (9.1), (9.2) the forces are additive. Thus it suffices to consider two particles only. As initial conditions we choose the linear superposition of the two charge solitons corresponding to the initial data $\boldsymbol{q}_i^0, \boldsymbol{v}_i^0$, $i = 1, 2$. We solve Maxwell equations and insert in the Lorentz force. As already explained, in the self–interaction the contribution from the initial fields vanishes for $t \geq \bar{t}_\rho$. In the mutual interaction the initial fields take a time of order $\sqrt{\varepsilon}$ to reach the other particle and their contribution vanishes for $t \geq \sqrt{\varepsilon}|\boldsymbol{q}_1^0 - \boldsymbol{q}_2^0|$. Thus for larger times we are allowed to insert in (9.2) the retarded fields only, which yields

$$\varepsilon\frac{d}{dt}\big(m_{\text{b}1}\,\gamma_1\boldsymbol{v}_1^\varepsilon(t)\big) = \boldsymbol{F}_{\text{ret},11}(t) + \boldsymbol{F}_{\text{ret},12}(t)\,, \tag{9.16}$$

$$\varepsilon\frac{d}{dt}\big(m_{\text{b}2}\,\gamma_2\boldsymbol{v}_2^\varepsilon(t)\big) = \boldsymbol{F}_{\text{ret},21}(t) + \boldsymbol{F}_{\text{ret},22}(t)\,, \tag{9.17}$$

where

$$\begin{aligned}\boldsymbol{F}_{\text{ret},ij}(t) \;=\; & \int_0^t ds \int d^3k\, \hat{\rho}_i^*(\varepsilon\boldsymbol{k})\hat{\rho}_j(\varepsilon\boldsymbol{k})e^{i\boldsymbol{k}\cdot(\boldsymbol{q}_i^\varepsilon(t)-\boldsymbol{q}_j^\varepsilon(s))}\\ & \Big(-\varepsilon^{1/2}(|\boldsymbol{k}|^{-1}\sin(|\boldsymbol{k}|(t-s)/\sqrt{\varepsilon}))i\boldsymbol{k} - \varepsilon(\cos(|\boldsymbol{k}|(t-s)/\sqrt{\varepsilon}))\boldsymbol{v}_j^\varepsilon(s)\\ & +\varepsilon^{3/2}(|\boldsymbol{k}|^{-1}\sin(|\boldsymbol{k}|(t-s)/\sqrt{\varepsilon})\boldsymbol{v}_i^\varepsilon(t) \times \big(i\boldsymbol{k} \times \boldsymbol{v}_j^\varepsilon(s)\big)\Big)\,,\end{aligned} \tag{9.18}$$

$i,j=1,2$.

For the self–interaction we set $\varepsilon\boldsymbol{k}=\boldsymbol{k}'$, $\varepsilon^{-3/2}t=t'$. Then

$$\begin{aligned}\boldsymbol{F}_{\mathrm{ret},11}(t) &= \varepsilon^{3/2}\int_0^\infty d\tau\int d^3k|\hat{\rho}_1(\boldsymbol{k})|^2 e^{i\boldsymbol{k}\cdot(\boldsymbol{q}_1^\varepsilon(t)-\boldsymbol{q}_1^\varepsilon(t-\varepsilon^{3/2}\tau))/\varepsilon}\\ &\quad\Big(-\sqrt{\varepsilon}(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)i\boldsymbol{k}-\varepsilon(\cos|\boldsymbol{k}|\tau)\boldsymbol{v}_1^\varepsilon(t-\varepsilon^{3/2}\tau)\\ &\quad+\varepsilon^{3/2}(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\boldsymbol{v}_1^\varepsilon(t)\times\big(i\boldsymbol{k}\times\boldsymbol{v}_1^\varepsilon(t-\varepsilon^{3/2}\tau)\big)\Big)\,. \end{aligned}\tag{9.19}$$

We Taylor expand as

$$\begin{aligned}\varepsilon^{-1}(\boldsymbol{q}_1^\varepsilon(t)-\boldsymbol{q}_1^\varepsilon(t-\varepsilon^{3/2}\tau)) &= \varepsilon^{1/2}\tau\boldsymbol{v}-\frac{1}{2}\varepsilon^2\tau^2\dot{\boldsymbol{v}}\,,\\ \boldsymbol{v}_1^\varepsilon(t-\varepsilon^{3/2}\tau) &= \boldsymbol{v}-\varepsilon^{3/2}\tau\dot{\boldsymbol{v}}\,. \end{aligned}\tag{9.20}$$

Then, up to errors of order $\varepsilon^{5/2}$,

$$\begin{aligned}\boldsymbol{F}_{\mathrm{ret},11}(t) &= \int_0^\infty d\tau\int d^3k|\hat{\rho}_1(\boldsymbol{k})|^2\Big\{\varepsilon\big[-(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\,\frac{1}{2}\tau^2(\boldsymbol{k}\cdot\dot{\boldsymbol{v}})\boldsymbol{k}\\ &\quad+(\cos|\boldsymbol{k}|\tau)\tau\dot{\boldsymbol{v}}\big]+\varepsilon^2\big[\big(-(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\frac{1}{2}\tau^2(\boldsymbol{k}\cdot\dot{\boldsymbol{v}})\boldsymbol{k}\\ &\quad+(\cos|\boldsymbol{k}|\tau)\tau\dot{\boldsymbol{v}}\big)\big(-\frac{1}{2}\tau^2(\boldsymbol{k}\cdot\boldsymbol{v})^2\big)\\ &\quad-(\cos|\boldsymbol{k}|\tau)\frac{1}{2}\tau^3(\boldsymbol{k}\cdot\dot{\boldsymbol{v}})(\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{v}+(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\\ &\quad\big(\tau^2(\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{v}\times(\boldsymbol{k}\times\dot{\boldsymbol{v}})+\frac{1}{2}\tau^2(\boldsymbol{k}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\times(\boldsymbol{k}\times\boldsymbol{v}))\big)\big]\Big\}, \end{aligned}\tag{9.21}$$

which, upon integration, agrees with (9.12), (9.14).

For the mutual interaction we leave the k-integration and set $\varepsilon^{1/2}t=t'$. Then

$$\begin{aligned}\boldsymbol{F}_{\mathrm{ret},12}(t) &= \sqrt{\varepsilon}\int_0^\infty d\tau\int d^3k\hat{\rho}_1^*(\varepsilon\boldsymbol{k})\hat{\rho}_2(\varepsilon\boldsymbol{k})e^{i\boldsymbol{k}\cdot(\boldsymbol{q}_1^\varepsilon(t)-\boldsymbol{q}_2^\varepsilon(t-\sqrt{\varepsilon}\tau))}\\ &\quad\Big(-\varepsilon^{1/2}(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)i\boldsymbol{k}-\varepsilon(\cos|\boldsymbol{k}|\tau)\boldsymbol{v}_2^\varepsilon(t-\sqrt{\varepsilon}\tau)\\ &\quad+\varepsilon^{3/2}(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\boldsymbol{v}_1^\varepsilon(t)\times\big(i\boldsymbol{k}\times\boldsymbol{v}_2^\varepsilon(t-\sqrt{\varepsilon}\tau)\big)\Big)\,. \end{aligned}\tag{9.22}$$

We Taylor expand as

$$\begin{aligned}
&\boldsymbol{q}_1^\varepsilon(t) - \boldsymbol{q}_2^\varepsilon(t-\sqrt{\varepsilon}\tau) = \boldsymbol{r} + \sqrt{\varepsilon}\tau\boldsymbol{v}_2 - \frac{1}{2}\,\varepsilon\tau^2\dot{\boldsymbol{v}}_2\,,\\
&\boldsymbol{v}_1^\varepsilon(t) = \boldsymbol{v}_1\ ,\ \boldsymbol{v}_2^\varepsilon(t-\sqrt{\varepsilon}\tau) = \boldsymbol{v}_2 - \sqrt{\varepsilon}\tau\dot{\boldsymbol{v}}_2
\end{aligned} \tag{9.23}$$

with $\boldsymbol{r} = \boldsymbol{q}_1^\varepsilon(t) - \boldsymbol{q}_2^\varepsilon(t)$. Then, up to errors of order $\varepsilon^{5/2}$,

$$\begin{aligned}
\boldsymbol{F}_{\mathrm{ret},12} &= \int_0^\infty d\tau \int d^3k\hat{\rho}_1^*(\varepsilon\boldsymbol{k})\hat{\rho}_2(\varepsilon\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\Big\{-\varepsilon(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\,i\boldsymbol{k}\\
&\quad+\varepsilon^2\big[(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\big(-\frac{1}{2}\,\tau^2(\boldsymbol{k}\cdot\dot{\boldsymbol{v}}_2)\boldsymbol{k} + \frac{1}{2}\,\tau^2(\boldsymbol{k}\cdot\boldsymbol{v}_2)^2 i\boldsymbol{k}\\
&\quad+\boldsymbol{v}_1\times(i\boldsymbol{k}\times\boldsymbol{v}_2)\big) + (\cos|\boldsymbol{k}|\tau)(\tau\dot{\boldsymbol{v}}_2 - i\tau(\boldsymbol{k}\cdot\boldsymbol{v}_2)\boldsymbol{v}_2)\big]\Big\}\\
&= (e_1e_2/4\pi)\Big(-\varepsilon\nabla_{\boldsymbol{r}}|\boldsymbol{r}|^{-1} + \varepsilon^2\big[\big(\frac{1}{2}\,\nabla_{\boldsymbol{r}}(\dot{\boldsymbol{v}}_2\cdot\nabla_{\boldsymbol{r}}) - \frac{1}{2}\nabla_{\boldsymbol{r}}(\boldsymbol{v}_2\cdot\nabla_{\boldsymbol{r}})^2\big)|\boldsymbol{r}|\\
&\quad-(\dot{\boldsymbol{v}}_2 - \boldsymbol{v}_2(\boldsymbol{v}_2\cdot\nabla_{\boldsymbol{r}}))|\boldsymbol{r}|^{-1} + (\boldsymbol{v}_1\times(\nabla_{\boldsymbol{r}}\times\boldsymbol{v}_2))|\boldsymbol{r}|^{-1}\big]\Big)\,.
\end{aligned} \tag{9.24}$$

We define the potential part of the Darwin–Lagrangian for two particles as

$$L_P = (e_1e_2/4\pi)\Big(-\frac{1}{|\boldsymbol{r}|} + \frac{\varepsilon}{2}\,\frac{\boldsymbol{v}_1\cdot\boldsymbol{v}_2}{|\boldsymbol{r}|} + \frac{\varepsilon}{2}\frac{(\boldsymbol{v}_1\cdot\boldsymbol{r})(\boldsymbol{v}_2\cdot\boldsymbol{r})}{|\boldsymbol{r}|^3}\Big)\,. \tag{9.25}$$

Then

$$\boldsymbol{F}_{\mathrm{ret},12}(t) = -\varepsilon\Big(\frac{d}{dt}\,(\nabla_{\boldsymbol{v}_1}L_P) - \nabla_{\boldsymbol{r}}L_P\Big) + \mathcal{O}(\varepsilon^{5/2})\,. \tag{9.26}$$

Inserting (9.21) and (9.26) into the Lorentz force equation (9.16), (9.17), we conclude that, upon neglecting contributions of order $\varepsilon^{5/2}$, the dynamics of the charges is governed by the Darwin Lagrangian (9.16). To control our Taylor expansion one has to resort to the contraction argument of Section 5.3. It becomes now considerably more involved. We note that in the effective equations of motion there is no mechanism which would preclude head on collisions. Thus the dynamics governed by L_{Darwin} can hold only until the first collision. For the subsequent motion one has to go back to the full microscopic evolution.

Notes and References

ad 9.2 and 9.3: The Darwin Lagrangian is discussed in Jackson (1999). In Kunze, Spohn (1999b) the errors relative to the motion governed by the Darwin Lagrangian are estimated.

References

Abraham M. (1903): Prinzipien der Dynamik des Elektrons, Ann. Physik **10**, 105-179.

Abraham M. (1904): Die Grundhypothesen der Elektronentheorie, Physikalische Zeitschrift **5**, 576-579.

Abraham M. (1905): Theorie der Elektrizität, Vol II: Elektromagnetische Theorie der Strahlung. Teubner, Leipzig, 2nd edition (1908).

Appel W. and Kiessling M.K.-H. (1999): In preparation.

Bailey J. and Picasso E. (1970): The anomalous magnetic moment of the muon and related topics, Progress in Nuclear Physics **12**, 43–75.

Bambusi D. (1994): A Nekhoroshev–type theorem for the Pauli–Fierz model of classical electrodynamics, Ann. Inst. H. Poincaré, Phys. Théor. **60**, 339-371.

Bambusi D. (1996): A proof of the Lorentz–Dirac equation for charged point particles, preprint, Univ. Milano.

Bambusi D. and Galgani L. (1993): Some rigorous results on the Pauli–Fierz model of classical electrodynamics, Ann. Inst. H. Poincaré, Phys. Théor. **58**, 155-171.

Bambusi D. and Noja D. (1996): On classical electrodynamics of point particles and mass renormalization, some preliminary results, Lett. Math. Phys. **37**, 449-460.

Bargmann V., Michel L., and Telegdi V.L. (1959): Precession of the polarization of particles moving in a homogeneous electromagnetic field, Phys. Rev. Lett. **2**, 435-436.

Barut A.O. (1964): Electrodynamics and Classical Theory of Fields and Particles. Dover, New York.

Barut A.O. (1980): Foundations of Radiation Theory and Quantum Electrodynamics, ed.. Plenum, New York.

Bauer G. (1997): Ein Existenzsatz für die Wheeler–Feynman Elektrodynamik. Dissertation, LMU München, unpublished.

Bauer G. and Dürr D. (1999): The Maxwell-Lorentz system of a rigid charge distribution, preprint, LMU München.

Baylis W.E. and Huschilt J. (1976): Nonuniqueness of physical solutions to the Lorentz–Dirac equation, Phys. Rev. D **13**, 3237-3239.

Bhabha H.J. (1939): Classical theory of electrons, Proc. Indian Acad. Sci. A **10**, 324-332.

Bhabha H.J. and Corben H.C. (1941): General classical theory of spinning particles in a Maxwell field. Proc. Roy. Soc. (London) **A 178**, 273-314.

Blanco R. (1995): Nonuniqueness of the Lorentz–Dirac equation with the free–particle asymptotic condition, Phys. Rev. E **51**, 680-689.

Bohm D. and Weinstein M. (1948): The self–oscillations of a charged particle, Phys. Rev. **74**, 1789-1798.

Bolte J. and Keppeler S. (1999): A semiclassical approach to the Dirac equation, Annals Physics **274**, 125-162.

Bonnor W.B. (1974): A new equation of motion for a radiating charged particle, Proc. Roy. Soc. Lond. **A 337**, 591-598.

Born M. (1909): Die Theorie des starren Elektrons in der Kinematik des Relativitätsprinzips, Ann. Physik **30**, 1-56.

Brown L.S. and Gabrielse G. (1986): Geonium theory: physics of a single electron or ion in a Penning trap, Rev. Mod. Phys. **58**, 233-278.

Bucherer A.H. (1909): Die experimentelle Bestätigung des Relativitätsprinzips, Ann. Physik **28**, 513-536.

Caldirola P. (1956): A new model of classical electron, Nuov. Cim. **3**, Supplemento 2, 297-343.

Carati A., Delzanno P., Galgani L., and Sassarini J. (1995): Nonuniqueness properties of the physical solutions of the Lorentz–Dirac equation, Nonlinearity **8**, 65-79.

Carati A. and Galgani L. (1993): Asymptotic character of the series of classical electrodynamics and an application to bremsstrahlung, Nonlinearity **6**, 905-914.

Coleman S. (1982): Classical electron theory from a modern standpoint, Chapter 6 in: Electromagnetism: Paths to Research, D. Teplitz, ed.. Plenum, New York.

Corben H.C. (1961): Spin in classical and quantum theory, Phys. Rev. **121**, 1833-1839.

Cushing J.T. (1981): Electromagnetic mass, relativity, and the Kaufmann experiments, Am. J. Phys. **49**, 1133-1149.

Dirac P.A.M. (1938): Classical theory of radiating electrons, Proc. Roy. Soc. A **167**, 148-169.

Dresden M. (1987): H. A. Kramers. Between Tradition and Revolution. Springer, New York.

Einstein A. (1905a): Ist die Trägheit eines Körpers von seinem Energiegehalt abhängig?, Ann. Physik **17**, 639-641, translation in: The Principle of Relativity. Dover, New York, 1952.

Einstein A. (1905b): Zur Elektrodynamik bewegter Körper, Ann. Physik **17**, 891-921, translation in: The Principle of Relativity. Dover, New York, 1952.

Eliezer C.J. (1950): A note on electron theory, Proc. Camb. Phil. Soc. **46**, 199-201.

Endres D.J. (1993): The physical solution to the Lorentz–Dirac equation for planar motion in a constant magnetic field, Nonlinearity **6**, 953-971.

Erber T. (1961): The classical theories of radiation reaction, Fortschritte der Physik **9**, 343-392.

Fermi E. (1922): Über einen Widerspruch zwischen der elektrodynamischen und der relativistischen Theorie der elektromagnetischen Masse, Physikalische Zeitschrift **23**, 340-344.

Feynman R.P., Leighton R.B., and Sands M. (1963): The Feynman Lectures in Physics. Addison–Wesley, Reading, Mass.

Fokker A.D. (1929): Ein invarianter Variationssatz für die Bewegung mehrerer elektrischer Massenteilchen, Z. Physik **58**, 386-393.

Ford G.W. and O'Connell R.F. (1991): Radiation reaction in electrodynamics and the elimination of runaway solutions, Phys. Letters A **157**, 217-220.

Ford G.W. and O'Connell R.F. (1993): Relativistic form of radiation reaction, Phys. Letters A **174**, 182-184.

Frenkel J. (1925): Zur Elektrodynamik punktförmiger Elektronen, Z. Physik **32**, 518-534.

Frenkel J. (1926): Die Elektrodynamik des rotierenden Elektrons, Z. Physik **37**, 243-262.

Galgani L., Angaroni C., Forti L., Giorgilli A., and Guerra F. (1989): Classical electrodynamics as a nonlinear dynamical system, Phys. Letters A **139**, 221-230.

Glimm J. and Jaffe A. (1987): Quantum Physics, A Functional Integral Point of View. Springer, Berlin, 2nd edition.

Grotch H., Kazes E., Rohrlich F., and Sharp D.H. (1982): Internal retardation, Acta Phys. Austr. **54**, 31-38.

Haag R. (1955): Die Selbstwechselwirkung des Elektrons, Z. Naturforsch, **10a**, 752-761.

Haken H. (1983): Advanced Synergetics. Instability Hierarchies of Self–Organizing Systems and Devices. Springer, Berlin.

Herglotz G. (1903): Zur Elektronentheorie, Nachr. K. Ges. Wiss. Göttingen, (6), 357-382.

Huang K. (1987): Statistical Mechanics. Wiley, New York, 2nd edition.

Huschilt J. and Baylis W.E. (1976): Numerical solutions to two–body problems in classical electrodynamics: head on collisions with retarded fields and radiation reaction, Phys. Rev. D **13**, 3256-3261 and D **13**, 3262-3268.

Jackson J.D. (1999): Classical Electrodynamics. Wiley, New York, 3rd edition.

Jones C. (1995): Geometric singular perturbation theory, in Dynamical Systems, Proceedings, Montecatini Terme 1994, Johnson, ed., Lect. Notes Math. 1609, 44-118. Springer, New York.

Kaufmann W. (1901): Series of papers in Nachr. K. Ges. Wiss. Göttingen, (2), 143-155 (1901); (5), 291-296 (1902); (3), 90-103 (1903). Physikalische Zeitschrift **4**, 54-57 (1902). Sitzungsber. K. Preuss. Akad. Wiss. **2**, 449-956 (1905). Ann. Physik **19**, 487-553 (1906).

Kiessling M. (1999): Classical electron theory and conservation laws, Phys. Letters A, to appear.

Komech A., Spohn H., and Kunze M. (1997): Long–time asymptotics for a classical particle interacting with a scalar wave field, Commun. PDE **22**, 307-335.

Komech A., Kunze M., and Spohn H. (1999): Effective dynamics of a mechanical particle coupled to a wave field, Commun. Math. Phys. **203**, 1-19.

Komech A. and Spohn H. (1998): Soliton–like asymptotics for a classical particle interacting with a scalar wave field, Nonlin. Analysis **33**, 13-24.

Komech A. and Spohn H. (1999): Long–time asymptotics for coupled Maxwell–Lorentz equations, J. Diff. Eq., to appear.

Kramers H.A. (1948): Nonrelativistic quantum electrodynamics and correspondence principle. Collected Scientific Papers, 845-869, North–Holland, Amsterdam 1956.

Kunze M. (1998): Instability of the periodic motion of a particle interacting with a scalar wave field, Comm. Math. Phys. **195**, 509-523.

Kunze M. and Spohn H. (1998): Radiation reaction and center manifolds, SIAM J. Math. Anal., submitted.

Kunze M. and Spohn H. (1999a): Adiabatic limit of the Maxwell–Lorentz equations, preprint, TU München.

Kunze M. and Spohn H. (1999b): In preparation.

Landau L.D. and Lifshitz E.M. (1959): The Classical Theory of Fields. Addison–Wesley, Reading MA, and Pergamon Press, London.

Laue M. von (1909): Die Wellenstrahlung einer bewegten Punktladung nach dem Relativitätsprinzip, Ann. Physik **28**, 436-442.

Levine H., Moniz E.J., and Sharp D.H. (1977): Motion of extended charges in classical electrodynamics, Am. J. Phys. **45**, 75-78.

Lorentz H.A. (1892): La théorie électromagnetique de Maxwell et son application aux corps mouvants, Arch. Néerl. Sci. Exactes Nat. **25**, 363-552.

Lorentz H.A. (1904a): Electromagnetic phenomena in system moving with any velocity less than that of light, Proceedings of the Academy of Sciences of Amsterdam, **6**, 809-831; contained in: The Principle of Relativity. Dover, New York, (1952).

Lorentz H.A. (1904b): Weiterbildung der Maxwell'schen Theorie: Elektronentheorie, Enzyklopädie der Mathematischen Wissenschaften V2, 145-280.

Lorentz H.A. (1909): Versuch einer Theorie der elektrischen und optischen Erscheinungen in bewegten Körpern, Teubner, Leipzig, orig. Leyden (1895).

Lorentz H.A. (1915): The Theory of Electrons and its Applications to the Phenomena of Light and Radiant Heat, 2nd edition. Reprinted by Dover, New York (1952).

McManus H. (1948): Classical electrodynamics without singularities,Proceed. Royal Soc. London **195**, 323-336.

Minkowski H. (1908): Die Grundgleichungen für elektromagnetische Vorgänge in bewegten Körpern, Göttinger Nachr. 53.

Milonni P.W. (1994): The Quantum Vacuum, an Introduction to Quantum Electrodynamics. Academic Press, San Diego.

Møller C. (1952): The Theory of Relativity. Oxford University Press.

Mo T.C. and Papas C.H. (1971): New equation of motion for classical charged particles, Phys. Rev. D **4**, 3566-3571.

Moniz E.J. and Sharp D.H. (1974): Absence of runaways and divergent self–mass in nonrelativistic quantum electrodynamics, Phys. Rev. D **10**, 113-1136.

Moniz E.J. and Sharp D.H. (1977): Radiation reaction in nonrelativistic quantum electrodynamics, Phys. Rev. D **15**, 2850-2865.

Neumann G. (1914): Die träge Masse schnell bewegter Elektronen, Ann. Physik, **45**, 529-579.

Nodvik J.S. (1964): A covariant formulation of classical electrodynamics for charges of finite extension, Ann. Phys. (N.Y.) **28**, 225-319.

Noja D. and Posilicano A. (1998): The wave equation with one point interaction and the (linearized) classical electrodynamics of a point particle, Ann. Inst. H. Poincaré, Phys. Théor. **68**, 351-377.

Nyborg P. (1962): On classical theories of spinning particles, Nuov. Cim. **23**, 47-62.

Page L. (1918): Is a moving mass retarded by the reaction of its own radiation?, Phys. Rev. **11**, 377-400.

Pais A. (1972): The early history of the theory of the electron: 1897-1947. In: Aspects of Quantum Theory, A. Salam and E.P. Wigner, eds.. Cambridge University Press.

Pais A. (1982): 'Subtle is the Lord'. The Science and Life of Albert Einstein. Oxford University Press.

Panofsky W.K.H. and Phillips M. (1962): Classical Electricity and Magnetism. Reading, MA, Addison–Wesley, 2nd edition.

Parrot S. (1987): Relativistic Electrodynamics and Differential Geometry. Springer, Berlin.

Pauli W. (1921): Relativitätstheorie, Enzyklopädie der Mathematischen Wissenschaften **V**19, 543-775. Translated as: Theory of Relativity. Pergamon, New York, 1958.

Pearle P. (1977): Absence of radiationless motions of relativistically rigid classical electron, Found. Phys. **7**, 931-945.

Pearle P. (1982): Classical electron models, Chapter 7 in: Electromagnetism: Paths to Research, D. Teplitz, ed.. Plenum, New York.

Plass G.N. (1961): Classical electrodynamic equations of motion with radiative reaction, Rev. Mod. Phys. **33**, 37-62.

Poincaré H. (1906): Sur la dynamique de l'électron, Rendiconti del Circolo Matematico di Palermo **21**, 129-176. Translated by H.M. Schwartz, Am J. Phys. **39**, 1287-1294, **40**, 862-872, and **40**, 1282-1287.

Richardson O.W. (1916): The Electron Theory of Matter. Cambridge University Press, 2nd edition.

Rohrlich F. (1960): Self–energy and stability of the classical electron, Am. J. Phys. **28**, 639-643.

Rohrlich F. (1973): The electron: development of the first elementary particle theory. In: The Physicist's Conception of Nature, J. Mehra, ed.. D. Reidel, Dordrecht.

Rohrlich F. (1990): Classical Charged Particles. Addison Wesley, Redwood City, CA, 2nd edition.

Rohrlich F. (1997): The dynamics of a charged sphere and the electron, Am. J. Phys. **65**, 1051-1056.

Rubinow S.I. and Keller J.B. (1963): Asymptotic solution of the Dirac equation, Phys. Rev. **131**, 2789-2796.

Rudin W. (1977): Functional Analysis. McGraw Hill, New York.

Sakamoto K. (1990): Invariant manifolds in singular perturbation problems for ordinary differential equations, Proc. Roy. Soc. Edinburgh, Sect. A **116**, 45-78.

Scharf G. (1994): From Electrostatics to Optics. Springer, Heidelberg.

Schild A. (1963): Electromagnetic two–body problem, Phys. Rev. **131**, 2762.

Schott G.A. (1912): Electromagnetic Radiation. Cambridge University Press.

Schott G.A. (1915): On the motion of the Lorentz electron, Phil. Mag. **29**, 49-62.

Schweber S. (1994): QED and the Men Who Made It: Dyson, Feynman, Schwinger, and Tomonaga. Princeton University Press.

Schwinger J. (1949): On the classical radiation of accelerated electrons, Phys. Rev. **75**, 1912-1925.

Schwinger J. (1983): Electromagnetic mass revisited, Found. Phys. **13**, 373-383.

Shen C.S. (1972a): Magnetic bremsstrahlung in an intense magnetic field, Phys. Review D **6**, 2736-2754.

Shen C.S. (1972b): Comment on the new equation of motion for classical charged particles, Phys. Rev. D **6**, 3039-3040.

Shen C.S. (1978): Radiation and acceleration of a relativistic charged particle in an electromagnetic field, Phys. Rev. D **17**, 434-445.

Sommerfeld A. (1904): Zur Elektronentheorie, 1. Allgemeine Untersuchung des Feldes einer beliebig bewegten Ladung, 2. Grundlagen für eine allgemeine Dynamik des Elektrons, 3. Über Lichtgeschiwindigkeits- und Überlichtgeschwindigkeitselektronen, Nachr. der Kgl. Ges. der Wiss. Göttingen, Math.-phys. Klasse, S. 99-130, S. 363-469, (1905) S. 201-235.

Sommerfeld A. (1905): Simplified deduction of the field and the forces of an electron moving in any given way, Akad. van Wetensch. te Amsterdam, **7**, 346-367.

Spohn H. (1991): Large Scale Dynamics of Interacting Particles. Springer, Berlin.

Spohn H. (1998): Runaway charged particles and center manifolds, unpublished manuscript.

Spohn H. (1999a): The critical manifold of the Lorentz–Dirac equation, preprint.

Spohn H. (1999b): privates notes.

Spohn H. (1999c) Semiclassics of the Dirac equation, in preparation.

Stephas P. (1992): Analytic solutions for Wheeler–Feynman interaction, J. Math. Phys. **33**, 612.

Teitelbom C., Villarroel D., and van Weert Ch. G. (1980): Classical electrodynamics of retarded fields and point particles. Rev. Nuov. Cim. **3**, 1-64.

Thirring W. (1997): Classical Mathematical Physics, Dynamical Systems and Field Theory. Springer, New York, 3rd edition.

Thomas L.H. (1926): The motion of the spinning electron, Nature **117**, 514.

Thomas L.H. (1927): On the kinematics of an electron with an axis, Phil. Mag. **3**, 1-22.

Thomson J.J. (1897): Cathode rays, Phil. Mag. **44**, 294-316.

Valentini A. (1988): Resolution of causality violation in the classical radiation reaction, Phys. Rev. Lett. **61**, 1903-1905.

Wheeler J.A. and Feynman R.P. (1945): Interaction with the absorber as the mechanism of radiation, Rev. Mod. Phys. **17**, 157-181.

Wheeler J.A. and Feynman R.P. (1949): Classical electrodynamics in terms of direct interparticle action, Rev. Mod. Phys. **21**, 425-433.

Wildermuth K. (1955): Zur physikalischen Interpretation der Elektronenselbstbeschleunigung, Z. Naturf. **10 a**, 450-459.

Yaghjian A.D. (1992): Relativistic Dynamics of a Charged Sphere. Lect. Notes in Physics **m 11**, Springer, Berlin.