An Algebraic Characterization of Vacuum States in Minkowski Space. II. Continuity Aspects

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Abstract

An algebraic characterization of vacuum states in Minkowski space is given which relies on recently proposed conditions of geometric modular action and modular stability for algebras of observables associated with wedge–shaped regions. In contrast to previous work, continuity properties of these algebras are not assumed but derived from their inclusion structure. Moreover, a unique continuous unitary representation of spacetime translations is constructed from these data. Thus the dynamics of relativistic quantum systems in Minkowski space is encoded in the observables and state and requires no prior assumption about any action of the spacetime symmetry group upon these quantities.

1 Introduction

Vacuum states in Minkowski space are ordinarily characterized by their invariance and stability properties with respect to the group of spacetime translations. This characterization has proven to be a powerful tool, both in the structural analysis of relativistic quantum field theory [29, 25] and in the construction of field theoretic models [22].

In spite of its successes, it seems desirable to reconsider this familiar characterization for several reasons. First, there is the interesting conceptual problem of whether vacuum states can be distinguished in terms of the local observables alone, *i.e.* without relying on the automorphic action of the spacetime symmetry group. An affirmative answer would corroborate the view that the full physical information of a theory is encoded in the particular "net structure" of the corresponding observables, *i.e.* the specific nesting of the algebras of observables corresponding to different spacetime regions [25]. Second, the question is of practical relevance in theories on spacetime manifolds which do not possess an isometry group as rich as that of Minkowski space. One is then forced to establish other, background–independent characterizations of vacuum states, and a fresh look at the case of Minkowski space theories could provide some clues to that effect. Finally, in the algebraic approach to the construction of quantum field theories, based on universal (free) nets of C*–algebras indexed by spacetime regions, the specification of a vacuum state amounts to the definition of a theory. One may hope that a more intrinsic characterization of vacuum states, relying only on the net structure, will also shed new light on these constructive problems. The conceptual problem of an algebraic characterization of the vacuum state on Minkowski space, therefore, has received considerable attention in recent years, cf. the survey articles [9, 28] and references quoted therein.

In the present letter we take up the approach to this problem initiated in [12] and followed up in [8, 14]. We shall show that the selection criterion for vacuum states proposed in [12] can be considerably weakened. Without relying upon any *a priori* assumptions about the presence of symmetries of the net of observables or on continuity assumptions, we shall show that states complying with our weakened criterion give rise to continuous unitary representations of the translation group which satisfy the relativistic spectrum condition. Moreover, these states are ground states for the respective dynamics. For that portion of the task carried out in [14] which we reconsider here, these results also represent a significant improvement.

The mathematical framework and our assumptions are specified in the subsequent section, where also a survey is given of results in [14] which are of relevance here. In Sec. 3 we construct from this input representations of the translations and establish the properties indicated above. Our letter concludes with remarks on further results and open problems.

2 The Condition of Geometric Modular Action

In the following, we consider families of algebras which are indexed by certain specific wedge–shaped regions W of the four–dimensional manifold $\mathbb{R}^4$. These regions can be described with the help of isotropic vectors $\ell \in \mathbb{R}^4$ which, in Cartesian coordinates, have the form $\ell = (\ell_0, \vec{\ell})$ with $\ell_0 = |\vec{\ell}|$. Given any two such vectors $\ell_\pm$ which are not parallel and a translation $\xi \in \mathbb{R}^4$, the corresponding wedge region W is given by

$$W = \{x \in \mathbb{R}^4 \mid \pm (x - \xi) \cdot \ell_\pm > 0\}.$$

The set of all these wedges W is denoted by $\mathcal{W}$. It is stable under translations and Lorentz transformations (which map isotropic vectors onto isotropic vectors). We also note that each $W \in \mathcal{W}$ has a complement $W' \in \mathcal{W}$ given by

$$W' = \{x \in \mathbb{R}^4 \mid \mp (x - \xi) \cdot \ell_\pm > 0\}.$$

It is apparent from these remarks that the manifold $\mathbb{R}^4$, equipped with the distinguished family $\mathcal{W}$ of wedges, acquires a natural interpretation as Minkowski space–time. But we shall not make use of the corresponding metric structure in the subsequent investigation.

Let $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ be a family of C^*–algebras indexed by $\mathcal{W}$, each of which is a subalgebra of some global unital C^*–algebra $\mathcal{A}$. We assume that this family satisfies the condition of isotony, *i.e.*

$$\mathcal{A}(W_1) \subset \mathcal{A}(W_2) \ \text{ if } \ W_1 \subset W_2,$$

so it constitutes a net with respect to the partially ordered index set $\mathcal{W}$. We emphasize that, for the characterization of vacuum states, we do not need any further structure on the algebraic side. As a matter of fact, the wedge algebras may be free algebras without any further relations.

Given the algebra $\mathcal{A}$, the set of positive, linear and normalized functionals (states) ω on $\mathcal{A}$ is fixed. But the states of physical interest form only a minute subset of it. So we have to solve here the problem of how to distinguish those states which describe the desired vacuum situation. To this end we consider for each state ω the corresponding GNS representation $(\mathcal{H}, \pi, \Omega)$ of $\mathcal{A}$. Within that representation we can proceed to the weak closures $\pi(\mathcal{A}(W))''$ of the wedge algebras which will be denoted by $\mathcal{R}(W)$.

Our first constraint on the states ω of interest is a condition of Reeh–Schlieder type: for any such state the GNS vector Ω has to be cyclic and separating for all von Neumann algebras $\mathcal{R}(W)$, $W \in \mathcal{W}$. We are then in a position to apply the results of Tomita–Takesaki theory, see *e.g.* [10], which yield for each pair $(\mathcal{R}(W), \Omega)$ an antiunitary involution J_W, called the modular involution, and a unitary group $\{\Delta_W^{it}\}_{t\in\mathbb{R}}$, called the modular group. The modular objects J_W and Δ_W^{it} leave Ω invariant and map $\mathcal{R}(W)$, by their adjoint action, onto $\mathcal{R}(W)'$ and $\mathcal{R}(W)$, respectively, where $\mathcal{R}(W)'$ denotes the commutant of $\mathcal{R}(W)$ in $\mathcal{B}(\mathcal{H})$.

After these preparations, we can formulate our primary condition on the states of interest, the Condition of Geometric Modular Action (henceforth, CGMA). It was introduced in the last section of [12] and its motivation and significance were explicated at length in [14].

Condition of Geometric Modular Action:

A state ω on $\mathcal{A}$ fulfills the CGMA if the corresponding net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ and vector Ω satisfy

(a) $W \mapsto \mathcal{R}(W)$ is an order-preserving bijection,

(b) if $W_1 \cap W_2 \neq \emptyset$, then Ω is cyclic and separating for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$; conversely, if Ω is cyclic and separating for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$, then $\overline{W_1} \cap \overline{W_2} \neq \emptyset$, where the bar denotes closure,

(c) for each $W \in \mathcal{W}$, the adjoint action $\mathrm{ad}J_W$ of J_W leaves the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ invariant, and

(d) the group of (anti)automorphisms generated by $\mathrm{ad}J_W$, $W \in \mathcal{W}$, acts transitively on $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$.

The first two of these conditions are based on the idea that the algebras $\mathcal{R}(W)$, $W \in \mathcal{W}$, are generated by observables which are localized in the respective wedge regions. They establish a connection between algebraic properties of the net and the lattice structure of the subsets of $\mathbb{R}^4$, cf. the discussion in [25, Ch. III.4.2]. The central part of the condition is requirement (c), which says that the modular conjugations $J_W, W \in \mathcal{W}$, generate part of the symmetric group on the set $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$. So, in view of the correspondence between algebras and regions, we say that they act geometrically. Note that no assumptions are made about the specific form of this action and the nature of the resulting group. This very general form of the condition is appropriate if one thinks of applications to theories on arbitrary spacetime manifolds, where $\mathcal{W}$ would then be other suitable collections of subregions (see [14] for a discussion of the general case). The requirement (d) of transitivity is added here for simplicity and could be relaxed.

We shall show in the following analysis that any state ω on $\mathcal{A}$ which satisfies the CGMA for the particular choice of regions $\mathcal{W}$ made above determines a Minkowski space theory which is local and covariant with respect to the action of a continuous unitary group of spacetime translations. The state itself turns out to be invariant with respect to this action, and if it also satisfies the modular stability condition, given below, it is a ground (vacuum) state. No continuity conditions are needed for the derivation of this result, in contrast to the arguments in [14]. As we shall see, the desired continuity properties are already encoded in the isotony properties of the wedge algebras.

The first part of our analysis coincides with the discussion in [14], which we recall here briefly for the convenience of the reader. Assumptions (a) and (c) imply that each J_W induces an inclusion–preserving bijection (an automorphism) τ_W on the ordered set $(\mathcal{W}, \subset)$ by its adjoint action on the elements of $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$,

$$J_W \mathcal{R}(W_0) J_W = \mathcal{R}(\tau_W(W_0)) \quad \text{for every} \quad W_0 \in \mathcal{W}.$$

Since the modular conjugations J_W are involutions, the same is true for the corresponding bijections τ_W, $W \in \mathcal{W}$. Moreover, these bijections generate, by composition, a group $\mathcal{T}$ of automorphisms of $\mathcal{W}$ with specific properties resulting from the CGMA and the modular structure [14, Lem. 2.1]. As a consequence of (a) and (b), one has, in particular, for any $\tau \in \mathcal{T}$ and pair of wedges $W_1, W_2 \in \mathcal{W}$,

(A) $W_1 \subset W_2$ if and only if $\tau(W_1) \subset \tau(W_2)$,

(B) $W_1 \cap W_2 = \emptyset$ if and only if $\tau(W_1) \cap \tau(W_2) = \emptyset$.

The concrete form of such automorphisms of the given family $\mathcal{W}$ of wedges has been determined in [14, Thm. 4.1.15]:

Any automorphism τ of $\mathcal{W}$ with the properties (A) and (B) is induced by an element λ of the Poincaré group $\mathcal{P}$ (possibly extended by dilations), i.e.

$$\tau(W) = \{\lambda x \mid x \in W\} \quad \textit{for} \quad W \in \mathcal{W}.$$

This result constitutes a significant generalization of classic results of Alexandrov [1, 2] and others [6, 35]. For the case at hand, it implies that the group $\mathcal{T}$, being generated by involutions, can be identified with some subgroup $\mathcal{P}_\mathcal{T}$ of the Poincaré group, *i.e.* nontrivial dilations do not occur.

In a next step, it was shown in [14] that subgroups of $\mathcal{P}$ which act transitively on $\mathcal{W}$ have to be large. In fact, one has [14, Prop. 4.2.9]:

Any subgroup of $\mathcal{P}$ which is generated by a family of conjugate involutions and acts transitively on $\mathcal{W}$ contains the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$.

Because of the transitivity assumption in the CGMA, the group $\mathcal{P}_\mathcal{T}$ complies with the premises of this result, so it is clear that $\mathcal{P}_\mathcal{T} \supset \mathcal{P}_+^\uparrow$. As a matter of fact, the specific properties of $\mathcal{T}$ inherited from the modular structure imply that $\mathcal{P}_\mathcal{T} = \mathcal{P}_+$, the proper Poincaré group, and the action of the elements of $\mathcal{T}$ on $\mathcal{W}$ is completely fixed.[1] More concretely, one has [14, Prop. 4.2.10]:

For any wedge $W \in \mathcal{W}$, the corresponding automorphism $\tau_W \in \mathcal{T}$ is induced by the unique involution $\lambda_W \in \mathcal{P}_+$ which maps the two defining isotropic vectors $\ell_\pm$ of W onto $-\ell_\pm$ and leaves the edge of W pointwise fixed. In particular, $\tau_W(W) = \lambda_W W = W'$, where W' is the complement of W.

So the CGMA fixes the group structure of $\mathcal{T}$ and the geometric action of its generating elements. This action is precisely that found by Bisognano and Wichmann [4, 5] in their study of the modular objects associated with the Minkowski vacuum and wedge algebras in finite–component quantum field theories satisfying the Wightman axioms. It is a remarkable fact that the much more general class of states complying with the CGMA exhibits the same properties.

Let us return now to the modular conjugations J_W which are associated with the pairs $(\mathcal{R}(W), \Omega)$, $W \in \mathcal{W}$. Making use of the preceding result, we get

$$J_W \mathcal{R}(W_0) J_W = \mathcal{R}(\tau_W(W_0)) = \mathcal{R}(\lambda_W W_0), \quad \text{for every} \quad W_0 \in \mathcal{W},$$

hence, in particular, $\mathcal{R}(W)' = J_W \mathcal{R}(W) J_W = \mathcal{R}(\lambda_W W) = \mathcal{R}(W')$. The latter relation amounts to the following statement [14, Prop. 4.3.1]:

The net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ satisfies Haag duality (and thus locality) for all complementary wedge regions $W, W' \in \mathcal{W}$.

So the CGMA induces commutation properties of the net in accord with the causal structure of $\mathbb{R}^4$ fixed by the wedges. It is therefore physically meaningful to interpret the self–adjoint elements of $\mathcal{R}(W)$ as observables in a Minkowski space theory which are localized in the wedge regions $W \subset \mathbb{R}^4$.

Since the modular conjugations associated to $(\mathcal{R}(W), \Omega)$ and $(\mathcal{R}(W)', \Omega)$ coincide, it follows from Haag duality that $J_W = J_{W'}$ for every $W \in \mathcal{W}$. Hence, as each involution λ_W uniquely determines the pair of wedges W, W' through the equation $\lambda_W W = W'$, one can consistently re–label the modular conjugations according to $J(\lambda_W) \equiv J_W$, $W \in \mathcal{W}$. Similarly, if one picks for any other $\lambda \in \mathcal{P}_+$ a fixed decomposition $\lambda = \lambda_{W_1} \cdots \lambda_{W_n}$, one can define

$$J(\lambda) \equiv J_{W_1} \cdots J_{W_n}, \quad \lambda \in \mathcal{P}_+.$$

[1] If the transitivity condition in the CGMA is relaxed, then $\mathcal{P}_\mathcal{T}$ can be one of at most five concrete subgroups of the Poincaré group [20].

These (anti)unitary operators generate a group $\mathcal{J}$ acting upon $\mathcal{H}$. As a matter of fact, one has [14, Prop. 4.3.1]:

The assignment $\lambda \mapsto J(\lambda)$ defines a projective (anti)unitary representation of $\mathcal{P}_+$ with coefficients in a subgroup $\mathcal{Z}$ contained in the center of $\mathcal{J}$. Moreover, the operators $J(\lambda)$, $\lambda \in \mathcal{P}_+$, leave the vector Ω invariant and act covariantly on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$, i.e.

$$J(\lambda)\mathcal{R}(W)J(\lambda)^{-1} = \mathcal{R}(\lambda W) \quad \textit{for} \quad W \in \mathcal{W}.$$

To summarize, every state ω on $\mathcal{A}$ which satisfies the CGMA determines a local net of wedge algebras in Minkowski space on which the proper Poincaré group acts covariantly through some (anti)unitary projective representation which leaves the state fixed. This result brings us close to our goal, the characterization of vacuum states in Minkowski space. What is missing is an argument that the projective representations obtained in this way can be lifted to continuous (true) representations, at least for the subgroup of translations. For that is what is needed in order to define the energy–momentum content of the states ω and to address the question under which circumstances they are ground states.

To this end a certain additional continuity condition on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ was introduced in [14], and it was shown that the desired representations exist in this case. In the present paper we drop this technical assumption and show that we still obtain strongly continuous unitary representations of the translation subgroup. This unexpected result [12] relies on the geometric inclusion structure (isotony) of the wedge algebras. It will enable us to define generators of spacetime translations and to determine their spectral properties with the help of a novel modular stability condition proposed in [14].

3 Representations of the Translation Group

We shall prove now that any state ω on $\mathcal{A}$ which satisfies the CGMA for the given set of wedge regions $\mathcal{W}$ determines a strongly continuous unitary representation of the translation group $\mathbb{R}^4 \subset \mathcal{P}_+$ which acts covariantly upon the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$. The building blocks of this representation are products of modular conjugations which are associated with shifted wedge regions.

As outlined in the preceding section, the CGMA implies that the modular conjugations J_W associated to the pairs $(\mathcal{R}(W), \Omega)$ induce geometric transformations of the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ which are given by specific involutions $\lambda_W \in \mathcal{P}_+$, $W \in \mathcal{W}$. Consequently, the products $J_{W+\xi}\, J_W$, $\xi \in \mathbb{R}^4$, induce the transformations $\lambda_{W+\xi}\lambda_W = (\xi - \lambda_W \xi) \in \mathbb{R}^4$, *i.e.* pure translations.

To control the algebraic properties of these products we shall make use of the fact that the projective representation J of $\mathcal{P}_+$, established above, satisfies $J(\lambda)\mathcal{R}(W)J(\lambda)^{-1} = \mathcal{R}(\lambda W)$ and $J(\lambda)\Omega = \Omega$, for $\lambda \in \mathcal{P}_+$ and $W \in \mathcal{W}$. Hence, in view of the uniqueness of the modular objects associated with a von Neumann algebra and a faithful state, the modular conjugations J_W transform covariantly under the adjoint action of the (anti)unitary operators $J(\lambda)$, *i.e.*

$$J(\lambda) J_W J(\lambda)^{-1} = J_{\lambda W}. \tag{$\star$}$$

The essential advance in our argument with respect to the results in [14] is the observation that the modular conjugations J_W enjoy certain continuity properties with respect to translations of the wedges W.

Lemma 3.1 *Given $W \in \mathcal{W}$ and $\xi \in \mathbb{R}^4$, the modular conjugations $J_{W+t\xi}$ associated to $(\mathcal{R}(W + t\,\xi), \Omega)$, $t \in \mathbb{R}$, are continuous in t in the strong operator topology.*

Proof. Note that, since W is arbitrary, it suffices to establish the asserted continuity for $t = 0$. Consider first the case where ξ is such that $W + \xi \subset W$ and, to simplify notation, set $J_t \equiv J_{W+t\xi}$. Let $\{t_n\}_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathbb{R}$ which converges to 0. Then $\{W + t_n\,\xi\}_{n\in\mathbb{N}}$ is an increasing family of wedges and Corollary A.2 of [12] implies that the sequence $\{J_{t_n}\}_{n\in\mathbb{N}}$ converges strongly to the modular conjugation J of $(\mathcal{R}, \Omega)$, where

$$\mathcal{R} \equiv \bigvee_n \mathcal{R}(W + t_n\,\xi) \;\subset\; \mathcal{R}(W).$$

According to the specific geometric action of products of the modular conjugations on the wedge algebras given above, one has for any fixed $s > 0$

$$J_0 J_{t_n}\, \mathcal{R}(W + s\,\xi)\, J_{t_n} J_0 = \mathcal{R}(W + (s - 2t_n)\,\xi) \subset \mathcal{R},$$

provided $n \in \mathbb{N}$ is sufficiently large. As $\mathcal{R}$ is weakly closed, one can proceed from this inclusion to $J_0 J\, \mathcal{R}(W + s\,\xi)\, J J_0 \subset \mathcal{R}$ and thence to $J_0 J\, \mathcal{R}\, J J_0 \subset \mathcal{R}$, so that

$$\mathcal{R} \subset \mathcal{R}(W) = J_0\, \mathcal{R}(W)'\, J_0 \subset J_0\, \mathcal{R}'\, J_0 = J_0 J\, \mathcal{R}\, J J_0 \subset \mathcal{R}.$$

But this implies $\mathcal{R} = \mathcal{R}(W)$ and hence $J = J_0$.

Next, let $\{t_n\}_{n\in\mathbb{N}}$ be an increasing sequence in $\mathbb{R}$ converging to 0. Note that $(W + t_n\,\xi)' = W' + t_n\,\xi$, so that, in view of the Haag duality of the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$, one has $J_{W+t_n\,\xi} = J_{W'+t_n\,\xi}$. Moreover, $\{W' + t_n\,\xi\}_{n\in\mathbb{N}}$ is an increasing family of wedges. The same argument presented in the first paragraph therefore yields the strong convergence of $\{J_{t_n}\}_{n\in\mathbb{N}}$ to J_0.

Finally, let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathbb{R}$ converging to 0. Since any such sequence can be decomposed into monotone subsequences, for which the strong convergence of the corresponding modular conjugations has already been established, and since the respective limits coincide, the continuity of the operators J_t at $t = 0$ follows for the special choice of ξ.

For arbitrary ξ, pick a $\zeta \in \mathbb{R}^4$ such that $W + \zeta \subset W$ and $W + \xi + 2\zeta \subset W$. Then $J_{W+t\zeta}$ and $J_{W+t(\xi+2\zeta)}$ are continuous in $t \in \mathbb{R}$. According to relation ($\star$) one has

$$J_{W+t\xi} = (J_{W+t\zeta} J_W)\, J_{W+t(\xi+2\zeta)}\, (J_{W+t\zeta} J_W)^{-1},$$

so the strong continuity of $J_{W+t\xi}$ follows from the continuity properties of the antiunitary involutions appearing on the right–hand side of this equality. □

With this information we can now proceed as in [14] and show:

Lemma 3.2 *Let $W \in \mathcal{W}$ and $\xi \in \mathbb{R}^4$ be given. The map $t \mapsto V(t) \equiv J_{W+t\xi} J_W$ is a strongly continuous homomorphism of $\mathbb{R}$ into the group of unitary operators on $\mathcal{H}$.*

Proof. Let $J_t \equiv J_{W+t\xi}$. As $V(t)$ is product of two such antiunitary involutions, it is unitary. Moreover, one has $V(t)J_0 = J_t J_0^2 = J_t = J_0^2 J_t = J_0 V(t)^{-1}$ and similarly $V(t)J_t = J_t V(t)^{-1}$. So one obtains, for $n \in \mathbb{N}$,

$$V(t)^{2n} J_0 = V(t)^n J_0 V(t)^{-n} = J_{2nt},$$

where in the second equality relation $(\star)$ has been used. Consequently, one has

$$V(t)^{2n} = V(t)^{2n} J_0^2 = J_{2nt} J_0 = V(2nt).$$

Similarly, one finds

$$V(t)^{2n+1} = V(t)^{2n} J_t J_0 = V(t)^n J_t V(t)^{-n} J_0 = J_{(2n+1)t} J_0 = V((2n+1)t).$$

From these relations one sees, in particular, that for $m_1, m_2 \in \mathbb{N}$ and $0 \neq n \in \mathbb{Z}$,

$$V(m_1/n)V(m_2/n) = V(1/n)^{m_1} V(1/n)^{m_2} = V(1/n)^{m_1+m_2} = V((m_1+m_2)/n).$$

Since V is thus a homomorphism on the subgroup of the rationals and is continuous on $\mathbb{R}$ according to the preceding lemma, it is a continuous homomorphism on $\mathbb{R}$. $\square$

As the unitary operators $V(t)$ induce the translations $t(\xi - \lambda_W \xi)$, $t \in \mathbb{R}$, on the underlying net, one obtains with the help of this lemma for every one–dimensional subgroup of the translations a continuous unitary representation. We shall show, by using methods developed in [14], that these special representations can be put together to a representation of the full translation group $\mathbb{R}^4$. To this end we fix with reference to the chosen Cartesian coordinate system the wedges

$$W_i \equiv \{ x \in \mathbb{R}^4 \mid x_i > |x_0| \}, \quad i = 1, 2, 3,$$

and consider, for $\xi \in \mathbb{R}^4$, the corresponding unitary operators

$$U_i(\xi) \equiv J_{W_i + \xi/2} J_{W_i}, \quad i = 1, 2, 3.$$

These operators induce the translations $(\xi - \lambda_{W_i}\xi)/2$ on the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$. Note that if $\xi = \xi_0 + \xi_1 + \xi_2 + \xi_3$ is the decomposition of $\xi \in \mathbb{R}^4$ into translations along the four axes of the above coordinate system, there holds in particular $(\xi_0 - \lambda_{W_i}\xi_0)/2 = \xi_0$ and $(\xi_i - \lambda_{W_i}\xi_i)/2 = \xi_i$, $i = 1, 2, 3$.

Let us first consider the restrictions of the $U_i(\,\cdot\,)$, $i = 1, 2, 3$, to the time translations ξ_0. According to the preceding remark and Lemma 3.2, these restrictions define three continuous unitary representations of the one–dimensional group of time translations. We show that these representations coincide.

Lemma 3.3 *For all time translations $\xi_0 \in \mathbb{R}^4$, one has*

$$U_i(\xi_0) = U_j(\xi_0), \quad i, j = 1, 2, 3.$$

Proof. The fact that the rotations in the time–zero hyperplane are induced by unitary operators in $\mathcal{J}$ will be employed. If ρ is a rotation by $\pi/2$ about the 1-axis, say, one obtains, first of all, from $(\star)$ the equalities

$$J(\rho)\, U_1(\xi_0)\, J(\rho)^{-1} = J(\rho)\, J_{W_1+\xi_0/2}\, J_{W_1}\, J(\rho)^{-1} = J_{\rho\,(W_1+\xi_0/2)}\, J_{\rho\, W_1} = U_1(\xi_0),$$

since $\rho\,(W_1 + \xi_0/2) = (W_1 + \xi_0/2)$. Next, one notes that the unitary operators $U_i(\xi_0)$, $U_j(\xi_0)$ induce the same time translation ξ_0 on the net. So the differences $U_i(\xi_0)\, U_j(\xi_0)^{-1}$ map, by their adjoint action, each wedge algebra $\mathcal{R}(W)$, $W \in \mathcal{W}$, onto itself and leave the vector Ω fixed. These differences therefore commute with all modular conjugations J_W, $W \in \mathcal{W}$, [10] and are thus contained in the center of $\mathcal{J}$. In particular, $U_1(\xi_0) = Z(\xi_0)\, U_2(\xi_0)$ for some central element $Z(\xi_0) \in \mathcal{J}$. Finally, one has

$$J(\rho)\, U_2(\xi_0)\, J(\rho)^{-1} = J(\rho)\, J_{W_2+\xi_0/2}\, J_{W_2}\, J(\rho)^{-1} = J_{\rho\,(W_2+\xi_0/2)}\, J_{\rho\, W_2} = U_3(\xi_0),$$

since $\rho\,(W_2 + \xi_0/2) = (W_3 + \xi_0/2)$. Putting these three facts together, one arrives at the following relations in $\mathcal{J}$

$$\begin{aligned} Z(\xi_0)\, U_2(\xi_0) &= U_1(\xi_0) = J(\rho)\, U_1(\xi_0)\, J(\rho)^{-1} = J(\rho)\, Z(\xi_0)\, U_2(\xi_0)\, J(\rho)^{-1} \\ &= Z(\xi_0)\, J(\rho)\, U_2(\xi_0)\, J(\rho)^{-1} = Z(\xi_0)\, U_3(\xi_0). \end{aligned}$$

Thus $U_2(\xi_0) = U_3(\xi_0)$, and in a similar way one proves that $U_1(\xi_0) = U_3(\xi_0)$. □

In view of this result, we can set, for arbitrary time translations $\xi_0 \in \mathbb{R}^4$,

$$U_0(\xi_0) \equiv U_i(\xi_0), \quad i = 1, 2, 3.$$

Next, we consider the operators $U_i(\xi_i)$ which, according to their geometric action on the net indicated above and Lemma 3.2, form a continuous unitary representation of the one–dimensional subgroups of spatial translations $\xi_i \in \mathbb{R}^4$, $i = 1, 2, 3$. The following result is the final step in our construction of a unitary representation of the full group $\mathbb{R}^4$ of translations.

Lemma 3.4 *Let ξ_m, $m = 0, 1, 2, 3$, be arbitrary translations in the four distinguished one–dimensional subgroups of $\mathbb{R}^4$, fixed by the chosen coordinate system. The corresponding unitary operators $U_m(\xi_m)$, $m = 0, 1, 2, 3$, commute with each other.*

Proof. Consider, for example, the operator $U_1(\xi_1)$. It leaves Ω invariant and satisfies

$$U_1(\xi_1)\mathcal{R}(W_2 + \zeta)U_1(\xi_1)^{-1} = \mathcal{R}(W_2 + \xi_1 + \zeta) = \mathcal{R}(W_2 + \zeta), \quad \zeta \in \mathbb{R}^4,$$

since $W_2 + \xi_1 = W_2$ for all translations ξ_1 along the 1–axis. But this implies that $U_1(\xi_1)$ commutes with the modular conjugations $J_{W_2+\zeta}$ [10] and hence with $U_2(\zeta)$, $\zeta \in \mathbb{R}^4$. Thus $U_1(\xi_1)$ commutes in particular with $U_2(\xi_2)$ and since $U_0(\xi_0) = U_2(\xi_0)$ according to Lemma 3.3, it also commutes with $U_0(\xi_0)$. By the same argument one can establish the commutativity of the remaining unitaries. $\square$

We now define for $\xi = \xi_0 + \xi_1 + \xi_2 + \xi_3 \in \mathbb{R}^4$ the unitary operators

$$U(\xi) \equiv U_0(\xi_0)\, U_1(\xi_1)\, U_2(\xi_2)\, U_3(\xi_3).$$

According to Lemma 3.2, each of the unitaries appearing on the right–hand side defines a continuous unitary representation of the corresponding one–dimensional subgroup of $\mathbb{R}^4$. Moreover, by Lemma 3.4 these unitaries commute with each other. Thus U is a continuous unitary representation of the group of translations $\mathbb{R}^4$ and acts geometrically correctly on the underlying net. We have thus established the following result.

Proposition 3.5 *Let ω be a state on $\mathcal{A}$ which satisfies the CGMA for the particular choice of regions $\mathcal{W}$ made above. There exists in the GNS representation $(\pi, \mathcal{H}, \Omega)$ induced by ω a continuous unitary representation U of the translations $\mathbb{R}^4$ which leaves Ω invariant and acts covariantly on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$, i.e.*

$$U(\xi)\mathcal{R}(W)U(\xi)^{-1} = \mathcal{R}(W+\xi) \quad \textit{for} \quad W \in \mathcal{W},\, \xi \in \mathbb{R}^4.$$

This result is the analogue of Lemma 4.3.5 in [14]. In view of it, we can turn now to the discussion of the energy–momentum spectrum $\mathrm{sp}\, U$ in the GNS representation induced by ω. Here again we can rely on the analysis in [14], whose outcome we recall for completeness.

As Ω is invariant under the action of U, it belongs to the point 0 in the discrete (atomic) part of $\mathrm{sp}\, U$. But, as was noticed in [14], the CGMA does not imply that Ω is necessarily a ground state. In fact, there exist examples fitting into the present framework for which $\mathrm{sp}\, U = \mathbb{R}^4$. So one has to amend the CGMA by additional conditions in order to select the desired class of vacuum states, for which one usually requires that $\mathrm{sp}\, U \subset \overline{V}_+ \equiv \{\, p \in \mathbb{R}^4 \mid p_0 \geq |\vec{p}|\,\}$ (relativistic spectrum condition).

Algebraic characterizations of the spectrum condition appeared first in [17] and [26]. More recently, the work of Borchers [7] (cf. also [19] for simpler proofs) has triggered renewed interest in this problem [32, 12, 33, 11, 24, 34]. The upshot of these latter investigations is the insight that the spectral properties of U are encoded in the modular groups $\{\Delta_W^{it}\}_{t\in\mathbb{R}}$ affiliated with the wedge algebras. But in all of these approaches the underlying framework was adapted to Minkowski space theories and does not seem to allow for a natural generalization to other space–times. A criterion which avoids this problem has been proposed in [14]. It also involves the modular groups but does not bear explicitly on the specific properties of the underlying spacetime manifold.

Modular Stability Condition:

The modular unitaries are contained in the group generated by the modular involutions, *i.e.* $\Delta_W^{it} \in \mathcal{J}$, for all $t \in \mathbb{R}$ and $W \in \mathcal{W}$.

This condition is expressed solely in terms of the algebraically determined modular objects. Hence, it can also be formulated sensibly for nets defined over arbitrary space–times, once a collection $\mathcal{W}$ of wedge regions has been selected. There are already indications from studies of nets on de Sitter space–time [14, 20] and also more general Robertson–Walker space–times [13] that this condition is indeed relevant to characterize maximally symmetric states of particular physical interest. This is true in spite of the fact that there is no translation subgroup in the isometry group of these spaces and thus the standard definition of vacuum state is inapplicable. Furthermore, for theories in Anti–de Sitter space the modular stability seems to be a characteristic feature of the corresponding vacuua [15]. For the case of interest here, Minkowski space, there holds the following assertion [14, Thm. 5.1.2].

Proposition 3.6 *Let ω be a state on $\mathcal{A}$ which satisfies the CGMA, with the above choice of wedges $\mathcal{W}$, and the Modular Stability Condition. Then the unitary representation U of the spacetime translations whose existence has been established in Proposition 3.5 satisfies* $\mathrm{sp}\, U \subset \overline{V}_+$ *or* $\mathrm{sp}\, U \subset -\overline{V}_+$.

It is a remarkable fact that although neither the CGMA nor the Modular Stability Condition contains any input about the arrow of time (note that the set $\mathcal{W}$ is invariant under time reflections), every state satisfying these two conditions breaks this symmetry. For $\mathrm{sp}\, U$ is a Lorentz invariant set as a consequence of relation ($\star$) and $\mathrm{sp}\, U \neq \{0\}$ by part (a) of the CGMA, so the state fixes one of the two cones $\pm\overline{V}_+$ in the dual of the space–time $\mathbb{R}^4$. It thereby determines a time direction. By choosing proper coordinates, we may therefore assume without loss of generality that $\mathrm{sp}\, U \subset \overline{V}_+$. With this convention, U is then the only continuous unitary representation of the spacetime translations which acts covariantly on the net and leaves Ω invariant [14, Prop. 5.1.3], cf. also [12, Prop. 2.4]. So the apparent ambiguities in our construction of spacetime translations have disappeared.

We have thus arrived at the desired characterization of vacuum states in Minkowski space in an algebraic setting which is general enough to cover also theories on other spacetime manifolds. Similar results can be established under slightly different conditions. For example, it suffices for the proof of the preceding two propositions to assume that only even products of the modular conjugations act geometrically on the net in the sense of the CGMA [20]. This result is of interest if one wants to include in the algebraic setting also non–observable quantities, such as Fermi fields, which do not satisfy the condition of spacelike commutativity. Moreover, as discussed in [14], there is also a version of the CGMA based on the modular groups, which may be regarded as a generalization of the Condition of Modular Covariance, discussed in the literature [18, 11, 24, 23, 16]. We will return to the latter issue elsewhere.

4 Further Remarks

Without any *a priori* assumptions of an action of the translation group upon the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$, we have derived from the CGMA and the Modular Stability Condition a continuous unitary representation U of the spacetime translations acting covariantly upon the net and satisfying the spectrum condition. In light of the uniqueness of U, we have thus determined the dynamics of the system from the given state.

In [14] it was shown that if the CGMA as well as a certain continuity condition of the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ are satisfied, then there also exists a strongly continuous unitary representation of the full proper Poincaré group $\mathcal{P}_+$ under which $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ transforms covariantly. In a forthcoming publication we shall extend the arguments given here and prove that this strongly continuous representation of $\mathcal{P}_+$ can also be obtained without assuming any kind of continuity of the initial data.

Since we have proven that our conditions are sufficient to entail that one has a vacuum state on Minkowski space, one may ask to which extent we have characterized such states. First, the work of Bisognano and Wichmann [4, 5] and Thomas and Wichmann [31] implies that all of our assumptions are necessary if the initial state and algebra arise from finite–component quantum fields satisfying the Wightman axioms and some natural regularity conditions. It is noteworthy that, from the corresponding nets, this underlying field theoretic structure can be recovered in an intrinsic manner [21, 30]. Second, it can be deduced from the work of Kuckert [27] that in a vacuum state on a net of observable algebras over Minkowski space, if the adjoint action of the modular objects maps local algebras onto local algebras, then our assumptions are again necessary. Therefore, in these physically natural situations, we have indeed obtained a characterization of vacuum states.

On the other hand, there exist examples of vacuum states in Minkowski space theories [3, 18] which do not fit into our setting because they are not Lorentz invariant. (As already mentioned, our conditions entail the Lorentz invariance of the corresponding vacua.) But these examples are of a rather *ad hoc* nature. Hence, although from a mathematical point of view our conditions do not characterize all vacuum states which can appear in the algebraic setting of quantum field theory, we believe that they distinguish the states of physical interest.

Let us finally comment on the significance of the choice of the index set $\mathcal{W}$ in the CGMA. A detailed discussion of this issue can be found in [14], so we do not need to reproduce it here. However, we wish to entice the reader's interest in this matter with the following remark: If another index set $\mathcal{W}$ of subregions of $\mathbb{R}^4$ is chosen and a state is found such that the CGMA is satisfied, then the group $\mathcal{T}$ will, in general, be different from the one examined in this paper and will, if it can be implemented by point transformations, induce a different subgroup of the diffeomorphism group of $\mathbb{R}^4$. Interpreting this group as the isometry group of a space–time, this suggests that it may be possible to derive geometric information about a space–time from a net of algebras with a suitable index set and a state on the net. Indeed, as sketched in the final chapter of [14], there emerges the possibility of actually deriving the space–time itself from suitable algebraic data satisfying the more general form of the CGMA presented in [14].

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