Extended Quantum Hall edge states. General domains

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Abstract

We extend recent results about extended edge states in Quantum Hall systems, which have been obtained for a half–plane, to more general geometries.

1 Results

At first, we consider an electron of charge $e = +1$ moving in the half–plane

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < 0\} \tag{1}$$

subject to a homogeneous magnetic field $B = \partial_1 A_2 - \partial_2 A_1 > 0$. The *guiding center* of its trajectory is

$$Z = x - B^{-1}\varepsilon(p - A) \,, \qquad A = (A_1, A_2) \,, \qquad \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \,, \tag{2}$$

with ε representing a rotation of angle $\pi/2$. Classically, the electron circles clockwise around the fixed guiding center as long as its trajectory does not hit the domain boundary. If it does, then the electron follows the boundary counterclockwise by repeatedly bouncing at it. The quantum mechanical counterpart thereof is some absolutely continuous spectrum which, due to the chirality of the extended edge states, is stable under addition of small perturbations, like e.g. disorder ([1]).

The unperturbed Hamiltonian is

$$H_0 = (-\mathrm{i}\nabla - A)^2$$

on $\mathrm{L}^2(\Omega)$ with Dirichlet boundary conditions, i.e., with domain $\mathcal{D}(H_0) = \mathrm{W}^2(\Omega, A) \cap \mathrm{W}^1_0(\Omega, A)$, where the magnetic Sobolev spaces are defined using covariant derivatives, e.g.,

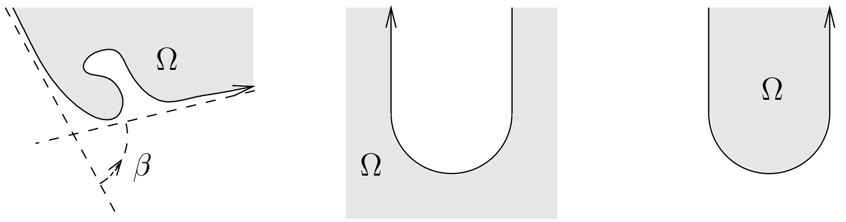
$$\mathrm{W}^1(\Omega, A) = \{\psi \in \mathrm{L}^2(\Omega) \mid (-\mathrm{i}\partial_i - A_i)\psi \in \mathrm{L}^2(\Omega),\ i = 1, 2\} \,. \tag{3}$$

The perturbation $V = V(x)$ is included through $H = H_0 + V$, where we assume V to be bounded.

Theorem 1 *Let $E/B \notin 2\mathbb{N}+1$. For sufficiently small $\|V\|_\infty/B$, the spectrum of H on the half–plane (1) is purely absolutely continuous near E.*

Here, and below, 'sufficiently small' is meant depending only on quantities mentioned before, i.e., on E/B, but not, e.g., on B. We remark that $(2\mathbb{N}+1)B$ is just the spectrum, consisting of Landau levels, of H_0 in the full plane $\mathbb{R}^2$.

A discussion of various boundary conditions for quantum Hall systems is found in [2]. Closely related to our work is [3], where a confining potential instead of Dirichlet boundary condition is considered, and the spectrum is shown to be continuous at E. (see also [4]). The proof of Theorem 1 given below is basically contained in [4, 5]; we reproduce it below in a slightly streamlined form, thereby also using ideas from [6], where the same result is established. However, we shall not rely on methods which, like Fourier decomposition along the x_2–axis, are special to the half–plane geometry. This allows for a partial extension of the result to more general unbounded domains $\Omega \subset \mathbb{R}^2$. The precise *geometric assumption (GA)* is stated in Section 3. Here we just mention that it allows for the domains (a, b) in the following figure, but not for (c).



(a) (b): $\beta = -\pi$ (c): $\beta = \pi$

Figure 1: Examples

At each collision the guiding center moves forward in the direction of the tangent vector to $\partial\Omega$ at the collision point. Yet, this may be a backward motion w.r.t. the tangent vector at the next collision point, with the result that a classical edge trajectory can get trapped. No trapping is possible however if the cyclotron radius of the electron is small compared to the radius of curvature of $\partial\Omega$ (more precisely: compared to the radius of injectivity of the tubular map associated with $\partial\Omega$). Then this bouncing motion will result in an effective counterclockwise progression along $\partial\Omega$.

Theorem 2 *Assume Ω satisfies (GA). Fix $E/B \notin 2\mathbb{N}+1$ and let $\|V\|_\infty/B$ be small enough. Then, the spectrum of H is purely absolutely continuous near E, provided B is large enough.*

Finally, no trapping is possible if Ω lies on one side of the graph of a function f,

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < f(x_2)\} , \tag{4}$$

with $f' \in \mathrm{C}^2_{\mathrm{unif}}(\mathbb{R})$. In fact, as for the half–plane, Z_2 is increasing at each reflection.

Theorem 3 *Let $E/B \notin 2\mathbb{N}+1$, and $B_0 > 0$. For sufficiently small $\|V\|_\infty/B$, the spectrum of H on the domain (4) is purely absolutely continuous near E, provided $B \geq B_0$.*

Under the scaling $x \to \lambda x$, $B \to \lambda^{-2}B$, $V \to \lambda^{-2}V$ we have $H \to \lambda^{-2}H$. Without loss of generality we may thus set

$$B = 1$$

in Theorem 1, and the same is true for Theorem 2 if its last sentence is replaced by: *Then, the spectrum of H on $\lambda\Omega$ is purely absolutely continuous near E, provided $\lambda > 0$ is large enough.* Similarly the conclusion of Theorem 3 is ... *the spectrum of H on the domain defined by $f_\lambda(x) = \lambda f(x/\lambda)$ is is purely absolutely continuous near E, provided $\lambda \geq \lambda_0 = B_0^{1/2}$.* The theorems will be proved in this form.

Theorems 1–3 do not rule out the possibility that the spectrum of H is empty near E. However, one can show that, if $E/B > 1$, then $E \in \sigma(H_0)$ under the assumptions of any of the theorems. This statement typically extends from $\sigma(H_0)$ to $\sigma(H)$, e.g., for V a disorder potential (see [7] for details).

2 The half–plane

We begin the proof of Theorem 1 with some preliminaries (1.–3.):

1. We shall work in the Landau gauge

$$A = (0, x_1)\ . \tag{5}$$

2. Let $p = -\mathrm{i}\nabla$. The component $p_1 = -\mathrm{i}\partial_1$ can not be defined self-adjointly on $\mathrm{L}^2(\Omega)$, a fact related to the integration by parts identity

$$(p_1\omega, \rho) - (\omega, p_1\rho) = \mathrm{i}\langle\omega, \rho\rangle\ , \qquad (\omega, \rho \in \mathrm{W}^1(\Omega, A))\ , \tag{6}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathrm{L}^2(\partial\Omega)$. Strictly speaking one should write $\langle L\omega, L\rho\rangle$ instead of $\langle\omega, \rho\rangle$, where L is the boundary trace operator, which is bounded as a map $L : \mathrm{W}^1(\Omega, A) \to \mathrm{L}^2(\partial\Omega)$. Below we shall need another, similar statement (see e.g. [8], Theorem 8.3 or [9], Section 7.50)

Theorem A *The map*

$$L : \mathrm{W}^2(\Omega, A) \to \mathrm{W}^{3/2}(\partial\Omega) \times \mathrm{W}^{1/2}(\partial\Omega)\ , \qquad \psi \mapsto (\psi \restriction \partial\Omega, \partial_1\psi \restriction \partial\Omega)$$

is continuous, has $\ker L = \mathrm{W}_0^2(\Omega, A)$, *and admits a continuous right inverse R: $LR = 1$.*

The statement in [8, 9] refers to $A = 0$; to extend it to the present situation, pick a function $j \in \mathrm{C}_0^\infty(-\infty, 0]$ with $j(x_1) = 1$ for x_1 near 0. Then L and R can be replaced by Lj, resp. jR. The claim now follows, since the map $\psi \mapsto j\psi$ is bounded as a map $\mathrm{W}^2(\Omega, A) \to \mathrm{W}^2(\Omega, 0)$, resp. $\mathrm{W}^2(\Omega, 0) \to \mathrm{W}^2(\Omega, A)$, for the special gauge (5).

3. The proof of Theorem 1 is based on Mourre theory ([10, 11, 12]), which we reproduce from [13] using the following pieces of notation: $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators on the Hilbert space $\mathcal{H}$; $\rho(H) \ni z$ is the resolvent set of H, where $R(z) := (H - z)^{-1}$; $E_\Delta(H)$ is the spectral projection for H onto $\Delta \subset \mathbb{R}$.

Theorem B *Let H, C be self–adjoint operators on $\mathcal{H}$, and $E \in \mathbb{R}$. Assume:*

(i) There is $z \in \rho(H)$ such that the map

$$g : \mathbb{R} \to \mathcal{B}(\mathcal{H}) \, , \qquad s \mapsto \mathrm{e}^{\mathrm{i}sC} R(z) \mathrm{e}^{-\mathrm{i}sC} \tag{7}$$

is of class $\mathrm{C}^{1+\varepsilon}$ with $0+ \leq \varepsilon \leq 1$ in the norm topology (with $\varepsilon = 0+$ meaning that $g'(s)$ is Dini continuous);

(ii) There is an open interval $\Delta \ni E$ and $\alpha > 0$ such that

$$E_\Delta(H) \mathrm{i}[H, C] E_\Delta(H) \geq \alpha E_\Delta(H) \, . \tag{8}$$

Then the spectrum of H is purely absolutely continuous near E.

According to Theorem 6.2.10 in [13], $g \in \mathrm{C}^1$ implies that $[H, C]$, defined as a quadratic form on $\mathcal{D}(H) \cap \mathcal{D}(C)$, extends to a bounded operator $[H, C] : \mathcal{D}(H) \to \mathcal{D}(H)^*$. In particular, the l.h.s. of (8) is a bounded operator on $\mathcal{H}$.

An observable classically increasing in time is

$$K = Z_2 = x_2 - p_1 \, , \tag{9}$$

which like p_1 can not be defined self–adjointly on $\mathrm{L}^2(\Omega)$. Nevertheless we consider its commutator with H_0 in the sense of quadratic forms:

Lemma 4 *For $\varphi, \psi \in \mathcal{D}(H_0) \cap \mathcal{D}(x_2)$ we have*

$$\mathrm{i}[(H_0\varphi, K\psi) - (K\varphi, H_0\psi)] = \langle p_1\varphi, p_1\psi \rangle \, . \tag{10}$$

Proof. By density we may assume that φ, ψ are of compact support. The separate contributions from the two terms in $H_0 = p_1^2 + (p_2 - x_1)^2$ are

$$\begin{aligned} \mathrm{i}[(p_1^2\varphi, K\psi) - (K\varphi, p_1^2\psi)] &= \langle p_1\varphi, p_1\psi \rangle \, , & (11) \\ \mathrm{i}[((p_2 - x_1)^2\varphi, K\psi) - (K\varphi, (p_2 - x_1)^2\psi)] &= 0 \, . & (12) \end{aligned}$$

In fact, as (11) is concerned, the contribution of x_2 to K is, using (6),

$$\mathrm{i}[(p_1\varphi, p_1 x_2\psi) - (p_1 x_2\varphi, p_1\psi)] = 0 \, ,$$

whereas that of $-p_1$ is

$$-\mathrm{i}[(p_1^2\varphi, p_1\psi) - (p_1\varphi, p_1\psi)] = \langle p_1\varphi, p_1\psi \rangle \, .$$

To verify (12), we note that $[p_2 - x_1, x_2 - p_1]\psi = 0$. Hence

$$\begin{aligned} ((p_2 - x_1)^2\varphi, (x_2 - p_1)\psi) &= ((p_2 - x_1)\varphi, (p_2 - x_1)(x_2 - p_1)\psi) \\ &= ((p_2 - x_1)\varphi, (x_2 - p_1)(p_2 - x_1)\psi) = ((x_2 - p_1)(p_2 - x_1)\varphi, (p_2 - x_1)\psi) \\ &= ((p_2 - x_1)(x_2 - p_1)\varphi, (p_2 - x_1)\psi) = ((x_2 - p_1)\varphi, (p_2 - x_1)^2\psi) \, , \end{aligned}$$

where, in the third equality, we used (6) with $(p_2 - x_1)\varphi \restriction \partial\Omega = 0$. ■

At energies away from the Landau levels, the quadratic form (10) is strictly positive.

Lemma 5 *Fix $E \notin 2\mathbb{N}+1$. Then there is an open interval $\Delta \ni E$ and $\alpha > 0$ such that*

$$|||p_1 E_\Delta(H_0)\psi|||^2 \geq \alpha \|E_\Delta(H_0)\psi\|^2 , \tag{13}$$

where we set $||| \cdot |||^2 = \langle \cdot, \cdot \rangle$.

Proof. We note that $RL\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$ and write for $\psi \in \mathcal{D}(H_0)$

$$(H_0 - E)\psi = (H_0 - E)(\psi - RL\psi) + (H_0 - E)RL\psi .$$

Since $L(\psi - RL\psi) = 0$, the function $\psi - RL\psi \in \mathrm{W}_0^2(\Omega, A)$ can be extended to $\mathbb{R}^2$ by 0, and the Dirichlet boundary condition dropped. Hence $\|(H_0 - E)(\psi - RL\psi)\| \geq \operatorname{dist}(E, 2\mathbb{N}+1)\|\psi - RL\psi\|$. Using $\|(H_0 - E)RL\psi\| \leq \text{const}\, \|RL\psi\|_2 \leq \text{const}\, (|||\psi|||_{3/2} + |||\partial_1\psi|||_{1/2})$ (with indices denoting Sobolev norms) and $\psi \restriction \partial\Omega = 0$ we obtain

$$\|(H_0 - E)\psi\| \geq \operatorname{dist}(E, 2\mathbb{N}+1)\|\psi\| - \text{const}\, |||\partial_1\psi|||_{1/2} . \tag{14}$$

For any $\varepsilon > 0$ we have

$$|||\partial_1\psi|||_{1/2} \leq \text{const}\, (\varepsilon^{-1}|||\partial_1\psi|||_0 + \varepsilon|||\partial_1\psi|||_1)$$

with $|||\partial_1\psi|||_1 \leq \text{const}\, \|\psi\|_{5/2} \leq \text{const}\, \|H_0^{5/4}\psi\|$. Here we used the boundedness of the map $\mathrm{W}^{5/2}(\Omega, A) \to \mathrm{W}^1(\partial\Omega)$, $\psi \mapsto \partial_1\psi \restriction \partial\Omega$ (see [8], Theorem 9.4 or [9], Theorem 7.58). We insert these estimates in eq. (14) and use $\|H_0^{5/4}E_\Delta(H_0)\psi\| \leq (|E| + |\Delta|)^{5/4}\|\psi\|$ and $\|(H_0 - E)E_\Delta(H_0)\psi\| \leq |\Delta|\|\psi\|$. For sufficiently small ε, $|\Delta|$ we obtain the claim. ■

Following [6] we choose $C = x_2$ as the conjugate operator to H, which in contrast to K is self–adjoint on its natural domain. The r.h.s. in

$$H(\tau) = \mathrm{e}^{\mathrm{i}\tau x_2} H \mathrm{e}^{-\mathrm{i}\tau x_2} = p_1^2 + (p_2 - \tau - x_1)^2 + V = H - \tau(p_2 - x_1) + \tau^2$$

extends from $\tau \in \mathbb{R}$ to an entire analytic family of type A. Hence assumptions (i) of Theorem B is fulfilled. Assumption (ii) holds because of the following lemma.

Lemma 6 *Let $E \notin 2\mathbb{N}+1$. Then there is an open interval $\Delta \ni E$ and $\alpha > 0$ such that*

$$E_\Delta(H)\mathrm{i}[H, x_2]E_\Delta(H) \geq \alpha E_\Delta(H) ,$$

provided $\|V\|_\infty$ is small enough.

Proof. We shall prove the statement in the case $V = 0$ first. Because of (10, 13) it suffices to show that the difference $Bx_2 - K = p_1$ contributes arbitrarily little to the commutator. In a state $\psi = E_\Delta(H_0)\psi$ this contribution is estimated as

$$\begin{aligned} |(H_0\psi, p_1\psi) - (p_1\psi, H_0\psi)| &= |((H_0 - E)\psi, p_1\psi) - (p_1\psi, (H_0 - E)\psi)| \\ &\leq 2\|(H_0 - E)\psi\|\|p_1\psi\| \leq 2|\Delta|(|E| + |\Delta|)^{1/2}\|\psi\|^2 , \end{aligned}$$

where the equality is justified by (6) and $\psi \in \mathcal{D}(H_0)$. The r.h.s. can be made arbitrarily small by letting $|\Delta| \to 0$.

We next extend [6] the result to H and note that $H'(0) = \mathrm{i}[H, x_2] = \mathrm{i}[H_0, x_2]$, which is relatively bounded w.r.t. H_0, is unchanged. Let $\alpha > 0$ and $\Delta \ni E$ be as given by Lemma 5 (we may assume Δ to be centered on E). For $\widetilde{\Delta} \ni E$, we split

$$E_{\widetilde{\Delta}}(H) = E_\Delta(H_0)E_{\widetilde{\Delta}}(H) + (1 - E_\Delta(H_0))E_{\widetilde{\Delta}}(H) ,$$

and obtain

$$\begin{aligned} E_{\widetilde{\Delta}}(H)(H'(0) - \alpha)E_{\widetilde{\Delta}}(H) \ &\geq \ E_{\widetilde{\Delta}}(H)E_\Delta(H_0)(H'(0) - \alpha)E_\Delta(H_0)E_{\widetilde{\Delta}}(H) - \\ &\quad\ 2\|(H'(0) - \alpha)(1 - E_\Delta(H_0))E_{\widetilde{\Delta}}(H)\| . \end{aligned}$$

The first term on the r.h.s. is non–negative, and the second line can be estimated by a constant times

$$\|H_0(1 - E_\Delta(H_0))(H_0 - E)^{-1}\|\|(H_0 - E)E_{\widetilde{\Delta}}(H)\| \leq (1 + 2|E||\Delta|^{-1})(|\widetilde{\Delta}| + \|V\|_\infty) .$$

Hence,

$$E_{\widetilde{\Delta}}(H)(H'(0) - \alpha)E_{\widetilde{\Delta}}(H) \geq -\alpha/2$$

for $|\widetilde{\Delta}| + \|V\|_\infty$ small enough, and therefore $E_{\widetilde{\Delta}}(H)H'(0)E_{\widetilde{\Delta}}(H) \geq (\alpha/2)E_{\widetilde{\Delta}}(H)$. ■

Proof of Theorem 1. Application of Theorem B. ■

3 General domains

The precise geometrical assumption made in Theorem 2 is as follows.

Assumption (GA) *Let $\Omega \subset \mathbb{R}^2$ be an open set.*

(i) Ω has the uniform C^3–property in the sense of [9];

(ii) Let $\partial\Omega$ consist of finitely many connected components γ, each parametrized by its arclength s (with the induced orientation). Let there be a function $s \in \mathrm{C}^2(\overline{\Omega})$ extending arclength from $\partial\Omega$ to Ω, i.e., $s(\gamma(s')) = s'$ for $s' \in \mathbb{R}$, satisfying

$$\|\partial_i s\|_\infty < \infty , \qquad \|\partial_i\partial_j s\|_\infty < \infty . \tag{15}$$

We note that, by (i), a bounded component of $\partial\Omega$ would be a closed curve, but by (ii) no such curve is allowed. Hence Ω is simply connected and unbounded. Also, (GA) is not affected by the ambiguity $s \mapsto s - s_0$ implicit in the definition of arclength.

We illustrate (GA) with an example. Let $\Omega \subset \mathbb{R}^2$ be simply connected open set with oriented boundary $\partial\Omega$ consisting of a single unbounded smooth curve $\gamma \in \mathrm{C}^3_{\mathrm{unif}}(\mathbb{R})$ parametrized by arclength $s \in \mathbb{R}$. For simplicity, we shall assume that γ is *asymptotically straight*, i.e.,

$$\ddot{\gamma} \in \mathrm{L}^1(\mathrm{d}s) , \tag{16}$$

with $\dot{} = \mathrm{d}/\mathrm{d}s$. The *overall bending* of γ is

$$\beta = \int_{-\infty}^{\infty} \ddot{\gamma}(s)\mathrm{d}s ,$$

and takes values in $[-\pi, \pi]$.

Example 7 *If*

$$\beta \neq \pi \ , \tag{17}$$

then Ω *as described satisfies (GA). In particular, the domains (a, b) in Figure 1 are allowed, but not (c).*

Proof. Only part (ii) of (GA) requires proof. Because of $\gamma \in \mathrm{C}^3_{\mathrm{unif}}(\mathbb{R})$ and of (16) we also have $\lim_{s\to\pm\infty}\ddot{\gamma}(s) = 0$. Elementary geometric considerations show that, for large enough $r > 0$, the equation $|\gamma(s)| = r$ has exactly two solutions $s = s_\pm(r)$, with $s_\pm(r) \to \pm\infty$, $(r \to \infty)$. We define functions $\varphi_\pm(r)$ through $\gamma(s_\pm(r)) = r(\cos\varphi_\pm(r), \sin\varphi_\pm(r))$ in such a way that $\varphi_\pm(r)$ are continuous and that $\varphi_-(r) - \varphi_+(r) \in (0, 2\pi)$. Their limiting values $\varphi^\infty_\pm = \lim_{r\to\infty}\varphi_\pm(r)$ exist and satisfy $\beta + (\varphi^\infty_- - \varphi^\infty_+) = \pi$. Condition (17) implies

$$\varphi^\infty_- - \varphi^\infty_+ > 0 \ . \tag{18}$$

Under our assumptions,

$$\frac{\mathrm{d}s_\pm}{\mathrm{d}r} \to \pm 1 \ , \quad \frac{\mathrm{d}^2 s_\pm}{\mathrm{d}r^2} \to 0 \ , \quad r\frac{\mathrm{d}\varphi_\pm}{\mathrm{d}r} \to 0 \ , \quad r\frac{\mathrm{d}^2\varphi_\pm}{\mathrm{d}r^2} \to 0 \ , \tag{19}$$

as $r \to \infty$. For $R > 0$ large enough we define $s(x)$ on

$$\overline{\Omega} \cap \{x \mid |x| > R\} = \{x = r(\cos\varphi, \sin\varphi) \mid r > R,\ \varphi_+(r) \leq \varphi \leq \varphi_-(r)\}$$

by linear interpolation along arcs $|x| = r$, i.e., by

$$s(x) = s_-(r)\frac{\varphi - \varphi_+(r)}{\varphi_-(r) - \varphi_+(r)} + s_+(r)\frac{\varphi_-(r) - \varphi}{\varphi_-(r) - \varphi_+(r)} \ ;$$

we then smoothly extend s further to the compact complement $\overline{\Omega} \cap \{x \mid |x| \leq R\}$. Now (15) follows from (18, 19). ■

The proof will just address the changes required for the generalization from Theorem 1 to Theorem 2. For notational simplicity we first consider the Hamiltonian on Ω. The required scaling $\Omega \to \lambda\Omega$ (see the end of Section 1) will be done later.

We define the components of the velocity as $\pi_i = -\mathrm{i}\partial_i - A_i$ with domain $\mathrm{W}_1(\Omega, A)$; see eq. (3). For $\omega, \rho \in \mathrm{W}_1(\Omega, A)$ we have the integration by parts identity

$$(\pi_i\omega, \rho) - (\omega, \pi_i\rho) = \mathrm{i}\langle\omega, n_i\rho\rangle \ , \tag{20}$$

where n_i are the components of the *outer unit normal* $n = n(s) = -\varepsilon\dot{\gamma}(s)$. Also, $H_0\psi = (\pi_1^2 + \pi_2^2)\psi$ for $\psi \in \mathcal{D}(H_0)$.

We next extend the trace theorem to the present setting.

Lemma 8 *The map*

$$L:\ \mathrm{W}^2(\Omega, A) \to \mathrm{W}^{3/2}(\partial\Omega, A\cdot\dot{\gamma}) \times \mathrm{W}^{1/2}(\partial\Omega, A\cdot\dot{\gamma}) \ , \qquad \psi \mapsto (\psi \restriction \partial\Omega, \pi\cdot n\psi \restriction \partial\Omega) \tag{21}$$

is continuous, has $\ker L = \mathrm{W}^2_0(\Omega, A)$*, and admits a continuous right inverse* R*:* $LR = 1$*. The norms of* L *and* R *depend only on the* C^3*–regularity data of* Ω*.*

Proof. The uniform C^3–property is associated with a cover $\{U_j\}$ of $\partial\Omega$ and corresponding diffeomorphisms (of class C^3) $\Psi_j : B = \{x \in \mathbb{R}^2 \mid |x| < 1\} \to U_j$, such that $\Psi_j(B \cap \{x_1 < 0\}) = \Omega \cap U_j$; see [9], Section 4.6 for details. The statement for $A = 0$ is [9], Theorem 7.53 (see also Section 4.29 allowing for unbounded $\partial\Omega$; or [14], Theorem 5.9). The proof makes use of a partition of unity for $\partial\Omega$ subordinate to $\{U_j\}$, with the effect of reducing the statement to the analogous restriction property from $B_- \equiv B \cap \{x_1 < 0\}$ to $B_0 \equiv B \cap \{x_1 = 0\}$. Similarly, in our case ($A \neq 0$) matters are reduced to (21) with the replacements $\Omega \to B_-$, $\partial\Omega \to B_0$, $A \to \tilde{A} = \Psi_j^* A$ and $A \cdot \dot{\gamma} \to \tilde{A}_2$, where $\tilde{A}$ is the pull–back of A under Ψ_j, in the sense of 1–forms. The claim so left to prove is gauge covariant, and we may choose the special gauge

$$\tilde{A}(x_1, x_2) = (0, \int_0^{x_1} \tilde{\omega}(\xi, x_2)\mathrm{d}\xi) \ ,$$

where $\tilde{\omega}(x_1, x_2)\mathrm{d}x_1 \wedge \mathrm{d}x_2 = \Psi_j^*(\mathrm{d}x_1 \wedge \mathrm{d}x_2)$ is the pull–back of the area 2–form on U_j. We note that $\tilde{\omega}, \tilde{A} \in \mathrm{C}^2(B_-)$, with bounded derivatives. Moreover, $\tilde{A} = 0$ on B_0. Now the claim again follows from the case $\tilde{A} = 0$, since the norms of $\mathrm{W}^2(B_-)$ and of $\mathrm{W}^2(B_-, \tilde{A})$ are equivalent. ■

Remark. As in proof of Lemma 5, we shall also need the boundedness of the map $\mathrm{W}^{5/2}(\Omega, A) \to \mathrm{W}^1(\partial\Omega, A \cdot \dot{\gamma})$, $\psi \mapsto \pi \cdot n\psi \restriction \partial\Omega$. The same argument applies and, in fact, it is only here that a uniform C^2–property does not suffice.

In the context of Theorem 2 a formal candidate for an observable increasing in time is $K = s(Z)$, where s is arclength as given by (GA), and Z is the guiding center (2). This definition is unsuitable, because $p = -\mathrm{i}\nabla$ and hence Z are not self–adjoint (vector) operators, and because the two components of Z do not commute. Instead, we formally linearize $s(Z)$ in $Z - x$ and set

$$K = s(x) - \nabla s \cdot \varepsilon\pi = s(x) - \varepsilon\pi \cdot \nabla s \ , \tag{22}$$

which is a well–defined (non self–adjoint) operator. The second expression follows from $\varepsilon_{ij}[\pi_j, \partial_i s] = -\mathrm{i}\varepsilon_{ij}\partial_j\partial_i s = 0$. Here, and in the following, the summation convention is understood.

Lemma 9 *Let s satisfy (15). For $\varphi, \psi \in \mathcal{D}(H_0) \cap \mathcal{D}(s)$ we have*

$$\mathrm{i}[(H_0\varphi, K\psi) - (K\varphi, H_0\psi)] = (\varphi, \pi_i(m_{ij} + m_{ji})\pi_j\psi) + \langle \pi_i\varphi, (\dot{\gamma}_j\partial_j s)\pi_i\psi \rangle \ , \tag{23}$$

where $m_{ij}(x) = \varepsilon_{jk}\partial_k\partial_i s$.

The last term in (23) can also be written as $\langle (\pi \cdot n)\varphi, (\dot{\gamma} \cdot \nabla s)(\pi \cdot n)\psi \rangle$, since $\pi_i\psi = n_i(\pi \cdot n)\psi$ on γ for $\psi \in \mathcal{D}(H_0)$. If, as in (GA), s is arclength on γ we have $\dot{\gamma} \cdot \nabla s = 1$ by definition.

Proof. We first claim that for $\rho \in \mathrm{W}_1(\Omega, A) \cap \mathcal{D}(s)$ we have

$$\mathrm{i}[(\pi_i\varphi, K\rho) - (K\varphi, \pi_i\rho)] = (\pi_j\varphi, m_{ij}\rho) \ . \tag{24}$$

Indeed, the contribution from s in (22) is

$$\mathrm{i}[(\pi_i\varphi, s\rho) - (s\varphi, \pi_i\rho)] = \mathrm{i}(\varphi, [\pi_i, s]\rho) = (\varphi, \partial_i s\rho) \ , \tag{25}$$

by using (20) and $\varphi \in \mathcal{D}(H_0)$, which makes the boundary term vanish. To compute the other contribution we note that

$$\mathrm{i}[(\pi_i\varphi, \pi_j\rho) - (\pi_j\varphi, \pi_i\rho)] = \mathrm{i}([\pi_i, \pi_j]\varphi, \rho) + \langle \pi_i\varphi, n_j\rho\rangle - \langle \pi_j\varphi, n_i\rho\rangle = -\varepsilon_{ij}(\varphi, \rho) ,$$

since $\pi_i\varphi = n_i(\pi \cdot n)\varphi$ on $\partial\Omega$, and hence, for $f_j = \varepsilon_{kj}\partial_k s$,

$$\begin{aligned} \mathrm{i}[(\pi_i\varphi, f_j\pi_j\rho) - (f_j\pi_j\varphi, \pi_i\rho)] &= \mathrm{i}[(\pi_i\varphi, \pi_j f_j\rho) - (\pi_j\varphi, \pi_i f_j\rho)] - (\pi_i\varphi, \partial_j f_j\rho) + (\pi_j\varphi, \partial_i f_j\rho) \\ &= (\varphi, \partial_i s\rho) - (\pi_j\varphi, m_{ij}\rho) , \end{aligned} \qquad (26)$$

because of $\varepsilon_{ij}f_j = -\partial_i s$, $\partial_j f_j = 0$ and of $\partial_i f_j = -m_{ij}$. Subtracting (26) from (25) yields (24), which can also be written as

$$\mathrm{i}[(\pi_i\omega, K\psi) - (K\omega, \pi_i\psi)] = (\omega, m_{ij}\pi_j\psi) , \qquad (27)$$

for $\omega \in \mathrm{W}_1(\Omega, A) \cap \mathcal{D}(s)$. Moreover, (20) yields

$$\mathrm{i}[(K\omega, \rho) - (\omega, K\rho)] = \langle \omega, \varepsilon_{ji}\partial_j s n_i\rho\rangle = \langle \omega, \dot\gamma_i\partial_i s\rho\rangle . \qquad (28)$$

Writing the l.h.s. of (23) as

$$\mathrm{i}[(\pi_i^2\varphi, K\psi) - (K\pi_i\varphi, \pi_i\psi) \ + \ (K\pi_i\varphi, \pi_i\psi) - (\pi_i\varphi, K\pi_i\psi) \ + \ (\pi_i\varphi, K\pi_i\psi) - (K\varphi, \pi_i^2\psi)]$$

and using (27, 28, 24) with $\omega = \pi_i\varphi$ and $\rho = \pi_i\psi$ concludes the proof if $\omega, \rho \in \mathcal{D}(s)$. By density this suffices. ■

Proof of Theorem 2. Statement and proof of Lemma 5 hold true with the replacement $p_1 \to \pi\cdot n$, with Δ, α depending only on the regularity data for Ω. These data are inherited by $\lambda\Omega$ with $\lambda \geq 1$. On the other hand $\lambda\Omega$ satisfies part (ii) of (GA) with $s_\lambda(x) = \lambda s(x/\lambda)$, and correspondingly $m_\lambda(x) = \lambda^{-1}m(x/\lambda)$ in eq. (23). There, the first term on the r.h.s. is then estimated in absolute value for $\varphi = \psi = E_\Delta(H_\lambda)\psi$ by a constant times

$$\lambda^{-1}\|\pi E_\Delta(H_\lambda)\psi\|^2 \leq \lambda^{-1}(|E| + |\Delta|)^{1/2}\|\psi\|^2 .$$

As a result, the positivity of the commutator (23) on $E_\Delta(H_\lambda)$ is preserved for large $\lambda > 0$. The conjugate operator $C = s(x)$ gives raise to the analytic family

$$H(\tau) = \mathrm{e}^{\mathrm{i}\tau s}H\mathrm{e}^{-\mathrm{i}\tau s} = H - \tau(\nabla s \cdot \pi + \pi \cdot \nabla s) + \tau^2(\nabla s)^2 . \qquad (29)$$

Statement and proof of Lemma 6 are changed accordingly, with $s(x) - K = \varepsilon\pi \cdot \nabla s$ replacing p_1. In (29) we suppressed the subscript λ. One should however notice that the relative bound of $H'(0) = \nabla s \cdot \pi + \pi \cdot \nabla s$ w.r.t. H is independent of λ. ■

Proof of Theorem 3. We set $s(x) = x_2$, i.e., $K = x_2 - \pi_1$ (as in (9), except for the gauge). Then, in (23), $m_{ij} = 0$ and $\dot\gamma \cdot \nabla s = (1 + f'(x_2)^2)^{-1/2} \geq \delta$ for some $\delta > 0$ and all $x_2 \in \mathbb{R}$. Upon scaling, $f'_\lambda(x_2) = f'(x_2/\lambda)$ has $\mathrm{C}^2_{\mathrm{unif}}(\mathbb{R})$–norm which can be bounded independently of $\lambda \geq \lambda_0$. In particular, bounds on the norms associated with (21), as well as δ, are independent of $\lambda \geq \lambda_0$. ■

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