Generalized Stable Multivariate Distribution and Anisotropic Dilations

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Abstract

After having closely re-examined the notion of a Lévy's stable vector, it is shown that the notion of a stable multivariate distribution is more general than previously defined. Indeed, a more intrinsic vector definition is obtained with the help of non isotropic dilations and a related notion of generalized scale. In this framework, the components of a stable vector may not only have distinct Levy's stability indices α's, but may depend on its norm. Indeed, we demonstrate that the Levy's stability index of a vector rather correspond to a linear application than to a scalar, and we show that the former should satisfy a simple spectral property.

1 Introduction and notation

In order to define Levy stable (or α-stable) vectors valued on a real Hilbert space H and their corresponding multivariate distribution in a more generalized way than the classical and nearly isotropic definition [1], we will use a notion of scale $\| \underline{x} \|$ which, as discussed in the following section, is weaker than the canonical norm $| \underline{x} |$ defined by the scalar product $(\underline{x}, \underline{y})$;

$$| \underline{x} | = (\underline{x}, \underline{x})^{1/2} \tag{1}$$

Let $L(H,H)$ denote the set of (continuous) endomorphisms of H. Although most of the properties discussed below do not depend on the precise structure of H, it can be considered as being $H = R^d$. For the Laplace-Fourier transform, we need to consider its complexified space $\overline{H}$ ($\overline{H} = H + iH$) equipped with the hermitian extension of the scalar product:

$$\forall \underline{q} \in \overline{H}, \forall \underline{x} \in H : \quad (\underline{q}, \underline{x}) = (Re(\underline{q}), \underline{x}) + i(Im(\underline{q}), \underline{x}) \tag{2}$$

for a Fourier transform it suffices to consider iH, whereas for a Laplace transform a subspace $H^+ \subset H$, e.g. $H^+ = (R^+)^d$ for $H = R^d$.

For any $\underline{\underline{M}} \in L(H,H)$, $Spec(\underline{\underline{M}})$ denotes its spectrum, $Re(Spec(\underline{\underline{M}})$ its real part, i.e. the set of the real parts of its eigenvalues, $Ker(\underline{\underline{M}})$ its kernel, $sym(\underline{\underline{M}})$ its symmetric part, i.e. $2.sym(\underline{\underline{M}}) = \underline{\underline{M}} + \underline{\underline{M}}^*$, where $\underline{\underline{M}}^*$ denotes the hermitian conjugate of $\underline{\underline{M}}$.

2 A Generalized notion of scale

The need for a generalized notion of scale will be more apparent in sect.6, however its definition is rather straightforward [5], at least for the linear case which will be sufficient for the following. However, the subsequent developments can be skipped in a first reading.

Definition 1 *An application $\| \, . \, \|$ from a real Hilbert space H onto the set of positive real numbers R^+ is said to be a generalized notion of scale, associated to one-parameter (linear) dilation $\underline{\underline{T}}_\lambda$, when it satisfies the following:*

- *nondegeneracy*, i.e:

$$\| \underline{x} \| = 0 <=> \underline{x} = \underline{0} \tag{3}$$

- *linearity with the dilation parameter*, i.e:

$$\forall \underline{x} \in H, \forall \lambda \in R^+ : \quad \| \underline{\underline{T}}_\lambda . \underline{x} \| = \lambda \| \underline{x} \| \tag{4}$$

- *Balls defined by this scale are strictly increasing*, i.e.:

$$\forall \lambda, \lambda' \in R^+ ; \lambda \geq \lambda' : \quad B_\lambda \supseteq B_{\lambda'} \tag{5}$$

where the balls B_λ defined by the dilation $\underline{\underline{T}}_\lambda$ satisfy (due to eq.4):

$$B_\lambda \equiv \underline{\underline{T}}_\lambda (B_\lambda) = \{ \underline{x} \in H ; \| \underline{x} \| \leq \lambda \} \tag{6}$$

and have the following frontier:

$$\partial B_\lambda = \{ \underline{x} \in H ; \| \underline{x} \| = \lambda \} \tag{7}$$

It is straightforward to check that the canonical norm $| \, . \, |$ is the scale associated to the isotropic dilation $\underline{\underline{T}}_\lambda = \lambda \underline{\underline{1}}$. Whereas the two first properties are rather identical to those of a norm, the last one is weaker than the triangular inequality which is required for a norm.

The conditions of existence of a generalized scale should depend on the generator $\underline{\underline{G}}$ of the one parameter group of (linear) dilations $\underline{\underline{T}}_\lambda$:

$$\underline{\underline{T}}_\lambda = \lambda^{\underline{\underline{G}}} = e^{Ln\lambda \underline{\underline{G}}} \tag{8}$$

Indeed, we have the following theorem [5]:

Theorem 1 *Let us consider the unit ball defined as an ellipsoid associated to a positive symmetric matrix $\underline{\underline{A}}$:*

$$B_1 = \{\underline{x} \in H; (\underline{x}, \underline{\underline{A}}.\underline{x})^{1/2} \leq 1\} \tag{9}$$

The dilation group $\underline{\underline{T}}_\lambda$ defines a generalized scale, if and only if its generator $\underline{\underline{G}}$ satisfies:

$$Spec(sym(\underline{\underline{A}}.\underline{\underline{G}})) \geq 0 \tag{10}$$

Indeed, the balls B_λ are then defined by:

$$\underline{\underline{A_\lambda}} = (\underline{\underline{T}}_\lambda^{-1})^*.\underline{\underline{A}}.\underline{\underline{T}}_\lambda^{-1} : \quad B_\lambda = \{\underline{x} \in H; (\underline{x}, \underline{\underline{A}}_\lambda.\underline{x})^{1/2} \leq 1\} \tag{11}$$

and we have:

$$\frac{d}{d\lambda}(\underline{x}, \underline{\underline{A}}_\lambda.\underline{x}) = -(\underline{\underline{T}}_\lambda^{-1}.\underline{x}, sym(\underline{\underline{A}}.\underline{\underline{G}}).\underline{\underline{T}}_\lambda^{-1}.\underline{x}) \tag{12}$$

This theorem has the following corrolary, which will be sufficient for the following:

Corollary 1 *When the unit ball is efined as an ellipsoid associated to a positive symmetric matrix $\underline{\underline{A}}$ (eq.9), the dilation group $\underline{\underline{T}}_\lambda$ defines a generalized scale, if and only if its generator $\underline{\underline{G}}$ satisfies:*

$$Spec(sym(\underline{\underline{G}})) \geq 0 \Longleftrightarrow Re(Spe(\underline{\underline{G}}) \geq 0 \tag{13}$$

when $\underline{\underline{A}}$ belongs to a given neighbourhood of a scalar linear application, i.e. $\exists \mu \in R^+ : \underline{\underline{A}} = \mu \underline{\underline{1}}$.

Indeed, for the scalar case ($\underline{\underline{A}} = \mu \underline{\underline{1}}$, it is a straightforward consequence of theo.1. Due to continuity, it is also valid for a given neighbourhood of this case, i.e. when the eigenvalues of $\underline{\underline{A}}$ are not too much different. However, simple counter examples (e.g. in $L(R^2, R^2)$) are easy to find which show that this neighbourhood is indeed bounded, i.e. which generators $\underline{\underline{G}}$ satisfy eq. 13, but violate eq. 1 as soon as the eigenvalues of $\underline{\underline{A}}$ are different enough.

3 Levy stable vectors

Let us recall that a (real) Levy stable variable [1, 2, 3, 4] can be defined in the following manner, e.g. [2] ($=^d$ denotes the equality in distribution):

Definition 2 *A random variable X is said to be a Levy stable variable, iff it is stable under renormalized sum –i.e. with the rescaling factor $a(n)$ and recentring term $\gamma(n)$– of any n of its independent realizations $X_i, (i = 1, n)$. This corresponds to:*

$$i=1,n: X_i =^d X \forall n \in N, \exists a(n), \gamma(n) \in R \sum_{i=1}^{n} X_i =^d a(n)X + \gamma(n) \quad (14)$$

Furthermore, X is said strictly *stable* [1] *when the recentring term $\gamma(n)$ is* 0.

It is straightforward to check (by induction) that this definition is equivalent to the original definition given by Levy [1] which addresses the stability under any linear combination:

Definition 3 *Two identically distributed random variables X_1, X_2 are said to be stable under linear combination iff:*

$$X_1 =^d X_2 =^d X \forall a_1, a_2 \in R^+, \exists a \in R^+, \gamma \in R : a_1 X_1 + a_2 X_2 =^d aX + \gamma \quad (15)$$

Furthermore, they are said strictly *stable, when the recentring term γ is* 0.

Due to their linearity, these definitions can be extended in a rather straightforward manner to random vectors [1, 6, 7], with the only necessary modification that the (rather trivial) recentring term ($\gamma(n)$ in eq.14, γ in eq.15) is now a vector [2]. However, this extension seems rather restrictive, one of its consequences, recalled below, is that all the components have the same Levy index. We will therefore consider the following more intrinsic vector definition of a stable vector:

Definition 4 *A random vector $\underline{X}$ valued on an Hilbert H is said to be a Levy stable vector, iff it is stable under renormalized sum –i.e. with a rescaling linear application $\underline{\underline{a}}(n)$ and recentring vector $\underline{\gamma}(n)$– of any n of its independent realizations $\underline{X}_i, i = 1, n$. This corresponds to:*

$$i=1,n: \underline{X}_i =^d \underline{X}; \forall n \in N, \exists \underline{\underline{a}}(n) \in L(H,H), \underline{\gamma}(n) \in H : \sum_{i=1}^{N} \underline{X}_i =^d \underline{\underline{a}}(n).\underline{X} + \underline{\gamma}(n) \quad (16)$$

Furthermore, $\underline{X}$ is said strictly stable, when the recentring term $\underline{\gamma}(n)$ is $\underline{0}$.

The classical definition [1, 6, 7] corresponds to the scalar case for $\underline{\underline{a}}$(n):

$$\underline{\underline{a}}(n) = a(n)\underline{\underline{1}} \quad (17)$$

and therefore can be called 'quasi-scalar case' of Lévy stable vectors.

[1] at least, according to Feller's terminology

[2] In fact, this definition might have seemed so trivial to Lévy that it is not explicitely written in [1]

4 Attractivity

We can now extend the definition of attractivity to random vectors :

Definition 5 *The distribution R of the independent random vectors $\underline{X}_i$ belongs to the domain of attraction of a distribution R_a of the random vector $\underline{X}$, if the distribution of their renormalized sum –i.e.with a rescaling linear application $\underline{\underline{a}}(n)$ and recentring vector $\underline{b}(n)$– tends to R_a. This corresponds to:*

$$lim_{n\to\infty} \frac{\sum_{i=1,n} \underline{X}_i - \underline{b}(n)}{\underline{\underline{a}}(n)} =^d \underline{X} \tag{18}$$

By its very definition (def.4) each stable vector belongs to its own domain of attraction and we will show there is no other possible limit. This will demonstrate that the stable vectors are the only attractive vectors.

5 Scaling law of the rescaling factor

As for the scalar case, it is straightforward to demonstrate the following:

Lemma 1 *the rescaling factor $\underline{\underline{a}}(n)$ forms a multiplicative group:*

$$\forall m, n \in N : \quad \underline{\underline{a}}(m.n) = \underline{\underline{a}}(m)\underline{\underline{a}}(n) \tag{19}$$

In order to obtain this group property, it suffices to iterate Eq.4 over $m.n$ since it yields:

$$\underline{\underline{a}}(m.n)\underline{X} + \underline{\gamma}(m.n) = \underline{\underline{a}}(n).\underline{\underline{a}}(m)\underline{X} + \underline{\underline{a}}(n).\underline{\gamma}(m) + m\underline{\gamma}(n) \tag{20}$$

Lemma 1 yields the following:

Lemma 2 *The multiplicative group $\underline{\underline{a}}(n)$ has a generator $\underline{\underline{\alpha}}^{-1}$, i.e.:*

$$\exists \underline{\underline{\alpha}} \in L(H, H), \forall n \in N : \underline{\underline{a}}(n) = n^{\underline{\underline{\alpha}}^{-1}} = e^{Ln(n)\underline{\underline{\alpha}}^{-1}} \tag{21}$$

Due its symmetry in m and n, Eq.20 also shows that:

$$\exists \underline{b} \in H, \forall n : \underline{\gamma}(n) = [\underline{\underline{a}}(n) - n]\underline{b} \tag{22}$$

therefore, it demonstrates the following

Lemma 3 *If $\underline{X}_i$ are stable with generator $\underline{\underline{\alpha}}^{-1}$ and $1 \notin Spec(\underline{\underline{\alpha}})$, there exists $\underline{b}$ (eq.22), such that $\underline{X}_i - \underline{b}$ are strictly stable with the same generator.*

6 Characteristic functions for stable vectors

The first (resp. second) characteristic functions $Z_{\underline{X}}(\underline{q})$ (resp. $K_{\underline{X}}\underline{q})$) are defined in the following way:

$$Z_{\underline{X}}(\underline{q}) = e^{K_{\underline{X}}(\underline{q})} = \int_{H'} e^{(\underline{q}, \underline{X})} dP_{\underline{X}} \tag{23}$$

where the domain H' of the conjugate vector $\underline{q}$ is respectively $H' = H^+, iH, \overline{H}$ for Laplace characteristic functions, for Fourier characteristic functions, and Fourier-Laplace functions[3].

With the help of the lemma 3, we need only to consider the case of strictly stable vectors. In this case the group property of $\underline{\underline{a}}$(n) (eq.21) corresponds to:

Theorem 2 *The second characteristic function $K_X(\underline{q})$ of a strictly stable stable vector has the following scaling behavior*

$$\forall \lambda \in R^+, \forall \underline{q} \in H' : K_{\underline{X}}(\underline{\underline{T}}_\lambda . \underline{q}) = \lambda K_{\underline{X}}(\underline{q}) \tag{24}$$

where the dilation $\underline{\underline{T}}_\lambda$ has the generator $\underline{\underline{\alpha}}^{-1}$.

For any positive integer λ, this is an immediate consequence of def.4 and lem.2. It is readily extended to any inverse of integer λ, by considering the the intermediate vector $\underline{q}' = \underline{\underline{T}}_{-\lambda^{-1}} . \underline{q}$, and therefore to any rational λ. Finally, due to the continuity of $K_{\underline{X}}(\underline{q})$, this is true for any positive real λ. In the classical case, we have:

$$\underline{\underline{\alpha}} = \alpha \underline{\underline{1}}; \quad \parallel \underline{y} \parallel = \mid \underline{y} \mid^\alpha \tag{25}$$

and on the other hand the component $((\underline{u}, \underline{X}))$ of a stable vector $(\underline{X})$ along any given direction $\underline{u}$ ($\mid \underline{u} \mid = 1$) is a stable variable. Therefore, the characteristic of the former is obtained from the one of the latter, with the help of a positive measure $d\Sigma'(\underline{u})$ of the directions $\underline{u}$. With the help of the Lemma.3, this yield the classical result [1, 6, 7]:

Corollary 2 *The second characteristic function of a classical (or quasi-scalar) Levy stable vector corresponds to:*

$$K_{\underline{X}}(\underline{q}) = \int_{\underline{u} \in \partial B_1'} (\underline{q}, \underline{u})^\alpha d\Sigma'(\underline{u}) + (\underline{q}, \underline{b}) \tag{26}$$

where $d\Sigma'$ is a positive measure which support $\partial B_1'$ is a subset of the frontier of the unit ball ∂B_1 [4].

it is straightforward that it is solution of eq.24.

Eq.26 already points out a major difficulty with Lévy stable vectors: a classical Lévy multivariate distribution is in general not parametric, contrary to an attempt to reduce it to a parametric distribution [8], which turns out [6] to be only a very particular sub-case.

One may note that in case of a Fourier characteristic function, eq.26 yields the classical expression for $\alpha \in]0, 1[\cup]1, 2[$:

$$K_{\underline{X}}(\underline{q}) = i(\underline{q}, \underline{b}) - \int_{\underline{u} \in \partial B_1} (\underline{q}, \underline{u})^\alpha d\Sigma'(\underline{u}) + i\beta(\underline{q}) \tag{27}$$

[3] As for the scalar case the Fourier transform is defined for any type of stable vectors, whereas the Laplace transform is only defined for extremely asymmetric cases, but are more convenient for the latter case.

[4] The symmetry of the probability distribution is related to the one of $d\Sigma$, in the extreme asymmetric case, the support $\partial B_1'$ of this measure is a subset of $H^+ \cap \partial B_1$.

where the asymmetry function $\beta(\underline{q})$ is given by:

$$\beta(\underline{q}) = tan(\frac{\pi\alpha}{2}) \int_{\underline{u}\in\partial B_1} (\underline{q}, \underline{u})^{\alpha-1} d\Sigma'(\underline{u}) \tag{28}$$

7 Levy canonical measure and generation of Levy stable vectors

Let us recall that the Levy 'canonical' measure of a stable random variable can be best understood as corresponding to the distribution of hyperbolic jumps in a Poisson compound process [1, 9].

This corresponds to substituting a Poisson sum for the deterministic sum in the definition of stability (eq.14 for stable variables, respectively eq.16, for stable vectors), as well as for the definition of acttractivity (eq.18). The second characteristic function $K_{\underline{X}}(\underline{q})$ is therefore rather easily determined with the help of the is the Levy canonical measure which is the $\sigma-$ probability $F_{\underline{Y}}$ of the jumps $\underline{Y}$.

Indeed, a compound Poisson process of a random vector field $\underline{Y}$ defined by a measure $d\sigma$, generates the following (random) measures $\underline{S}(A)$ for any given borelian A (of the space on which the process takes place):

$$\underline{S}(A) = \int_A \underline{Y} d\sigma \tag{29}$$

and which have the following characteristic function:

$$K_{\underline{S}(A)}(\underline{q}) = c\sigma(A) \int [e^{(\underline{q},\underline{y})} - \omega((\underline{q},\underline{y})) - 1] dF_{\underline{y}} \tag{30}$$

where the last term under the integral of the right hand side, is classical and merely removes the divergence of the $\sigma-$ probability $F_{\underline{y}}$ in the limit $|\ \underline{y}\ |\rightarrow 0$. Indeed, a $\sigma-$ probability does need to be finite, but only to be the limit $\varepsilon \rightarrow 0$ of finite positive measures $F^{\varepsilon}_{\underline{y}}$. The term $\omega((\underline{q},\underline{y}))$ rather corresponds to a generalization of the 'Levy's trick' for removing other divergences $((\underline{q},\underline{y}) \longrightarrow 0)$ of higher order (i.e. $(\underline{q},\underline{y})$), and whose appropriate choice will be discussed below. In order to determine the latter, we have first to determine the scaling behavior of the Levy canonical measure $F_{\underline{Y}}$. With no surprise, we obtain by conjugation of theorem 2:

Theorem 3 *The Levy canonical measure $F_{\underline{Y}}$ of the jumps $\underline{Y}$ of a stable vector $\underline{X}$ has the following scaling behavior:*

$$\forall\lambda \in R^+, \forall\underline{y} \in H : dF_{\underline{\underline{T}}^*_\lambda(\underline{y})} = \lambda^{-1} dF_{\underline{y}} \tag{31}$$

where the dilation $\underline{\underline{T}}^*_\lambda$ is the conjugate of $\underline{\underline{T}}_\lambda$, i.e. has for generator: $\underline{\underline{\alpha}}^{*-1}$

In order to make a step further, let us consider the following generalized definition of the 'unitary vector' $\underline{u}(\underline{y})$ corresponding to a given vector $\underline{y}$:

Definition 6 *In the framework of a generalized scale (def.1), the unitary vector $\underline{u}(\underline{y})$ of any non-zero vector $\underline{y}$ is defined by:*

$$\forall \underline{y} \in H - \{\underline{0}\}, \exists \underline{u}(\underline{y}) \in \partial B_1 : \underline{u}(\underline{y}) = \underline{\underline{T}}^{-1}_{\|\underline{y}\|}(\underline{y}) \tag{32}$$

In the case of the norm $|\ .\ |$, it corresponds to the usual notion of unitary vector, i.e.:

$$\underline{u}(\underline{y}) = \frac{\underline{y}}{|\ \underline{y}\ |} \tag{33}$$

Theorem 4 *The Levy canonical measure factors into a measure having a density in scale and a positive measure $d\Sigma$ on the frontier of unit ball ∂B_1:*

$$dF_{\underline{y}} = d\Sigma(\underline{u}) \frac{d \parallel \underline{y} \parallel}{\parallel \underline{y} \parallel^2} \tag{34}$$

Indeed, due to its scaling property (theorem 3) the Levy canonical measure $F_{\underline{Y}}$ should factor into a scaling measure of $\parallel \underline{Y} \parallel$ (therefore a power law) and a measure invariant for any dilation $\underline{\underline{T}}^*_\lambda$. Since every ball B_λ is obtained by dilation of the unit ball B_1, it suffices to consider a measure on the frontier of the latter and to take care of the fact that the Levy canonical measure should be positive. The classical case (eq.25) does correspond to:

$$dF_{\underline{y}} = \alpha \frac{d\ |\ y\ |}{|\ y\ |^{-\alpha-1}} d\Sigma(\underline{u}) \tag{35}$$

and with the (classical) choice of either $\omega(x) = 0, (0 < \alpha \leq 1)$ or $\omega(x) = x, (1 < \alpha \leq 2)$, this yields eq.26 and the proportionality between the measure $d\Sigma'$ (eq.26) and $d\Sigma$ (eq.35) is determined with the help of the Euler Γ function:

$$d\Sigma' = \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} d\Sigma \tag{36}$$

As mentioned by Levy, another possibility is to choose [5]:

$$\omega(x) = \frac{x}{1+x^2} \tag{37}$$

which is valid for any $\alpha \in]0,2]$, since $\omega \sim x(x \longrightarrow 0)$ and $\omega \sim 0(x \longrightarrow \infty)$.

For the general case, the characteristic function is rather more involved:

Corollary 3 *The second characteristic function of a Levy stable vector has the following expression*

$$K_{\underline{X}}(\underline{q}) = \int_{\underline{u} \in \partial B_1} d\Sigma(\underline{u}) \int_{\lambda=0}^{\infty} \frac{d\lambda}{\lambda^2} [e^{(\underline{q}, \underline{\underline{T}}^*_\lambda(\underline{u}))} - \omega(\underline{q}, \underline{\underline{T}}^*_\lambda(\underline{u})) - 1] \tag{38}$$

where ω is defined by eq.37, $d\Sigma$ is a positive measure on the frontier of the unit ball ∂B_1 and the spectrum of $\underline{\underline{\alpha}}$, which is the inverse of the generator of the dilation $\underline{\underline{T}}_\lambda$, should satisfy:

[5] However, one may note that the simplified expression that in fact used is not relevant for the following

$$Re(Spec(\underline{\underline{\alpha}})) \subseteq]0,2] \tag{39}$$

Eq.39 is a rather straightforward extension of the scalar case constraint ($0 < \alpha 2$). On the one hand, the lower spectral bound 0 of $\underline{\underline{\alpha}}$ allows its inverse to be defined, whereas the upper spectral bound 2, which not surprisingly corresponds to the gaussian limit case, allows the definition of a generalized notion of scale associated to the generator $\underline{\underline{\alpha}}^{-1}$ (sect.2 and in particular theo.1) and it is required in order to ensure the convergence of the second integration involved in eq.38 for the lower bound ($\lambda \to 0$). More precisely the integrand is of order:

$$\frac{d\lambda}{\lambda^2}(\underline{q}, \underline{\underline{T}}^*_\lambda(\underline{u}))^2 \tag{40}$$

and it should be bounded by below by $\frac{d\lambda}{\lambda}$ to avoid divergences. We have the following two cases:

- *diagonalizable* $\underline{\underline{\alpha}}$:
  eq.39 merely corresponds to imposing the adequate constraint to each of eigenspace, therefore to the full space.
- *otherwise*:
  one needs to consider the Jordan form of $\underline{\underline{\alpha}}^{-1}$. And on each generalized eigenspace corresponding to the eigenvalue $\alpha^{-1} = a+ib, a, b \in R$ the component (e.g.[10]) of $\underline{y}_\lambda = \underline{\underline{T}}^*_\lambda(\underline{u})$ is a linear combination of functions of the form $\mid Log(\lambda) \mid^k \lambda^a cos(b\lambda)$ and $\mid Log(\lambda) \mid^k \lambda^a sin(b\lambda)$, where k is bounded above by the codimension of the deficiency of the eigenvalue: $k \leq c = inf_{n \in N}[codim(Ker((\underline{\underline{T}}_\lambda - \lambda \underline{\underline{1}})^n)]$ Therefore the previous result holds.

8 Conclusion

We demonstrated that the notion of Levy stable vectors can be broadly generalized, the stability index becoming a linear application $\underline{\underline{\alpha}}$ which needs to respect only one spectral constraint (eq.39). We have two main cases:

- *diagonalizable* $\underline{\underline{\alpha}}$:
  on each of eigenspace, the stable vector have a common stability index which is the corresponding eigenvalue. However, there is no need for the eigenvalues to be equal, as in fact hypothesized in the classical definition of Levy stable vectors.
- the generalization is in fact *much broader*:
  indeed complex eigenvalues of $\underline{\underline{\alpha}}$ induce rotation modulations, and deficiency of its eigenvalues introduce logarithmic modulations. As a consequence the stability index of a component of a stable vector depends on its norm.

Not only these results are fundamentaly important for multivariate analysis of random vectors, but for their stochastic simulations. Indeed, this study was in fact was motivated by the latter, more precisely by simulations of multifractal fields [11] generated by strongly anisotropic stochastic differential equations and the latter naturaly introduce the generalized scale notion used in the present paper.

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