SUPERSYMMETRY, WITTEN COMPLEX AND ASYMPTOTICS FOR DIRECTIONAL LYAPUNOV EXPONENTS IN $\mathbf{Z}^d$

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Abstract. By using a supersymmetric Gaussian representation, we transform the averaged Green's function for random walks in random potentials into a 2-point correlation function of a correponding lattice field theory. We study the resulting lattice field theory using the Witten Laplacian formulation. We obtain the asymptotics for the directional Lyapunov exponents.

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I. Introduction

1. The analytical formulation and the main results.

Beginning in this paper, we start our quantitative study of random walks in random potentials in dimensions $d \geq 1$. In analytical language, which is the language we adopt, this corresponds to the study of the operator

$$H = \Delta + \gamma V, \text{ on } \ell^2(\mathbf{Z}^d), \tag{1.1}$$

where

$$\begin{aligned} \Delta_{ij} &= 1 \quad |i-j|_{\ell_1} = 1 \\ &= 0 \quad \text{otherwise}; \end{aligned} \tag{1.2}$$

γ is a positive parameter, the potential function V is a diagonal matrix: $V = \operatorname{diag}(v_j)$, $j \in \mathbf{Z}^d$, where $\{v_j\}$ is a family of independently identically distributed (iid) real random variables with distribution dg. (dg is only assumed to be a measure.) From now on, we write $|\,|$ for $|\,|_{\ell_1}$. ℓ_2 norms will be denoted by $\|\,\|$. The probability measure is taken to be $P = \prod_{j \in \mathbf{Z}^d} dg(v_j)$. We use the notation $\langle\,\rangle_g$ to denote the expectation value with respect to P.

As is well known, the spectrum $\sigma(\Delta)$ is $[-2d, 2d]$. (Note that the Laplacian defined here differs from the one usually used by the probabilists by the constant $2d$ times the identity. So the spectrum differs by $2d$.) Let supp dg be the support of dg, then we have the well established fact (See e.g., [CFKS,PF].) that

$$\sigma(H) = [-2d, 2d] + \operatorname{supp}\, dg \tag{1.3}$$

almost surely. Assume $E \notin \sigma(H)$, then we define $G(E) = (E-H)^{-1}$ and

$$G(E, \mu, \nu) = (\delta_\mu, (E-H)^{-1}\delta_\nu), \qquad \mu, \nu \in \mathbf{Z}^d, \tag{1.4}$$

to be the Green's function (or correlation function) of H at energy E. In this paper, we take $E \in \mathbf{R}\backslash\sigma(H)$ (assuming $\mathbf{R}\backslash\sigma(H) \neq \emptyset$) and study the asymptotics of $\langle G(E,\mu,\nu)\rangle_g$, as $|\mu - \nu| \to \infty$ for $\gamma << 1$.

Our result, which is to be stated precisely in Theorem 7.2 below, shows that $\langle G(E,\mu,\nu)\rangle_g$ can be expressed as the inverse of an effective *convolution matrix.* Our work in this paper can therefore be categorized as *quantitative* homogenization. But in order to achieve that, we need some technical conditions on dg.

We need to impose conditions on dg to ensure that the resolvent set: $\mathbf{R}\backslash\sigma(H) \neq \emptyset$. This is satisfied if supp dg is bounded either from below or above. We define the Laplace transform of dg, $\hat{g}(t)$ for $t \geq 0$ to be

$$\hat{g}(t) = \int e^{-tv} dg(v), \tag{1.5}$$

if supp dg is bounded from below; and

$$\hat{g}(t) = \int e^{tv} dg(v), \tag{1.6}$$

if supp dg is bounded from above. We also require that dg has bounded moments, in order that the derivatives of $\hat{g}$ have the required properties at infinity.

Remark. This is needed for the Witten Laplacian construction and subsequently for computing the asymptotics of the Lyapunov exponents. For the present paper as we do not work to all orders of γ, it is sufficient to assume that dg has finite n^{th} moment, for some fixed n. But as the construction could be extended to all orders, we assume that all moments of dg are finite.

Due to the presence of the parameter E, without loss of generality, we may then assume that

(H1) supp $dg \subseteq (-\infty, 0]$ or supp $dg \subseteq [0, +\infty)$

(H2) dg has finite moments, i.e.,

$$|\int v^n dg(v)| < \infty, \qquad \text{for all } n \geq 0.$$

Define

$$g(a) = \int_{-a}^{0^+} dg(v), \qquad (a \geq 0) \tag{1.7}$$

in the first case of (H1) and

$$g(a) = \int_{0^-}^{a} dg(v), \qquad (a \geq 0) \tag{1.8}$$

in the second case of (H1). We assume further

(H3) dg is such that g is of regular variation at 0 with exponent ρ $(0 \leq \rho < \infty)$, i.e.,

$$\frac{g(Ca)}{g(a)} \to f(C) = C^\rho \tag{1.9}$$

as $a \to 0$ for all $C > 0$.

Remark. Since dg is a probability measure, g is a positive monotone function on $[0, \infty)$. For such functions, condition (1.9) is much less restrictive than it seems at first sight. This is because once we assume the limit of $g(Ca)/g(a)$ exists and is finite as $a \to 0$ for a dense set of C's in $\mathbf{R}^+$, then the limit $f(C)$ is necessarily of the form C^ρ for some $0 \leq \rho < \infty$. For more details on this, see p. 268 of [F].

(H3) enables us to use Tauberian type theorems to deduce the desired properties of the derivatives of $\hat{g}$ as $t \to \infty$. We note in particular that the Bernoulli distribution

$$\begin{aligned} dg(v) &= \frac{\delta(v) + \delta(v+1)}{2} \\ &= \frac{dh(v) + dh(v+1)}{2} \end{aligned} \tag{1.10}$$

where δ is the Dirac distribution at 0 and h is the Heaviside function, is of regular variation with exponent $\rho = 0$.

For all $\delta > 0$, define

$$I_\delta = (2d+\delta, \infty) \qquad \text{if supp } dg \subseteq (-\infty, 0], \tag{1.11}$$
$$= (-\infty, -2d-\delta) \qquad \text{if supp } dg \subseteq [0, \infty).$$

Assuming (H1-3), our main result is

Theorem 7.2. *For all $\delta > 0$, there exists $\gamma_0 > 0$, such that for all $0 < \gamma < \gamma_0$, all $E \in I_\delta$, all μ, $\nu \in \mathbf{Z}^d$,*

(1.12)

$$\begin{aligned}
\log\langle G(E,\mu,\nu)\rangle_g =& \log\langle G(E,\mu-\nu,0)\rangle_g \\
=& \log \big(E - \Delta - \gamma\langle v\rangle_g - \gamma^2\langle (v-\langle v\rangle_g)^2\rangle_g[(E-\Delta-\gamma\langle v\rangle_g)^{-1}(0,0)] \\
& \qquad - \gamma^3\langle (v-\langle v\rangle_g)^3\rangle_g[(E-\Delta-\gamma\langle v\rangle_g)^{-1}(0,0)]^2\big)^{-1}(\mu-\nu,0) \\
& + \mathbf{O}(\gamma^4)(|\mu-\nu|+1) \\
=& \log(\widetilde{E} - \Delta)^{-1}(\mu-\nu,0) + \mathbf{O}(\gamma^4)(|\mu-\nu|+1),
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{E} \stackrel{\text{def}}{=} & E - \gamma\langle v\rangle_g - \gamma^2\langle (v-\langle v\rangle_g)^2\rangle_g[(E-\Delta-\gamma\langle v\rangle_g)^{-1}(0,0)] \\
& \qquad - \gamma^3\langle (v-\langle v\rangle_g)^3\rangle_g[(E-\Delta-\gamma\langle v\rangle_g)^{-1}(0,0)]^2 \\
\notin & [-2d, 2d].
\end{aligned} \tag{1.13}$$

We stress here that the order of the quantifiers are important, in particular γ_0 is *uniform* in $|\mu-\nu|$.

Remark. We do not require any regularity conditions on dg other than (H1-3) for Theorem 7.2 to hold, in marked contrast to the random Schrödinger case, see e.g., [SW1,2]. Instead the conditions on dg, in particular (H3) are reminisent of the conditions that one imposes to obtain Lifshitz tails, see e.g., [PF]. The main reason for this contrast as we will see in (1.30, 1.31) is that when $E \in \mathbf{R}\backslash\sigma(H)$, it is the Laplace transform of dg that is relevant; while when $E \in \mathbf{C}\backslash\sigma(H)$, $\Re E \in \sigma(H)$, it is the Fourier transform of dg.

Note the interesting $\mathbf{O}(\gamma^2)$, $\mathbf{O}(\gamma^3)$ terms, which we will give a simple explanation in (1.15)-(1.17). The $\mathbf{O}(\gamma)$ term just corresponds to a shift in E. Note also that

$$|E - \gamma\langle v\rangle_g| = |E| + \gamma|\langle v\rangle_g| > |E| \tag{1.14}$$

for $E \in I_\delta$. So the rate of decay is better than the naive one; recall that the almost sure spectrum is contained in $[-2d, \infty)$ or $(-\infty, 2d]$. Hence in some sense, the "effective" spectrum is further away than the almost sure one. This is related to the so called Lifshitz tails for the density of states in random Schrödinger operators, see e.g., [PF].

We now give a simple explanation for the terms $\mathbf{O}(\gamma)$, $\mathbf{O}(\gamma^2)$ and $\mathbf{O}(\gamma^3)$ in (1.12). Let $G_0 = (E - \Delta - \gamma\langle v\rangle_g)^{-1}$ and $\bar{V} = V - \gamma\langle v\rangle_g$. By the resolvent series, we have (1.15)

$$\begin{aligned} G(E,\mu,\nu) =& G_0(\mu,\nu) + \gamma(G_0\bar{V}G_0)(\mu,\nu) + \gamma^2(G_0\bar{V}G_0\bar{V}G_0)(\mu,\nu) \\ &+ \gamma^3(G_0\bar{V}G_0\bar{V}G_0\bar{V}G_0)(\mu,\nu) + \gamma^4(G_0\bar{V}G_0\bar{V}G_0\bar{V}G_0\bar{V}G)(\mu,\nu), \end{aligned}$$

Let $G_0 = (E - \Delta - \gamma\langle v\rangle_g)^{-1}$ and $\bar{V} = V - \gamma\langle v\rangle_g$. Note that the last factor in order $\mathbf{O}(\gamma^4)$ is G itself. So

$$\begin{aligned} \langle G(E,\mu,\nu)\rangle_g =& G_0(\mu,\nu) + \gamma^2\langle (v - \langle v\rangle_g)^2\rangle_g G_0(0,0)(G_0G_0)(\mu,\nu) \\ &+ \gamma^3\langle (v - \langle v\rangle_g)^3\rangle_g (G_0(0,0))^2(G_0G_0)(\mu,\nu) + \mathbf{O}(\gamma^4). \end{aligned} \tag{1.16}$$

We hence obtain that for all μ, ν with $|\mu - \nu|$ fixed (1.17)

$$\begin{aligned} \langle G(E,\mu,\nu)\rangle_g = \big(& E - \Delta - \gamma\langle v\rangle_g - \gamma^2\langle (v - \langle v\rangle_g)^2\rangle_g (E - \Delta - \gamma\langle v\rangle_g)^{-1}(0,0) \\ &- \gamma^3\langle (v - \langle v\rangle_g)^3\rangle_g ((E - \Delta - \gamma\langle v\rangle_g)^{-1}(0,0))^2\big)^{-1}(\mu,\nu) + \mathbf{O}(\gamma^4). \end{aligned}$$

We see that indeed, the $\mathbf{O}(\gamma)$, $\mathbf{O}(\gamma^2)$, $\mathbf{O}(\gamma^3)$ terms are identical to the ones in (1.12). The trouble comes when we let $|\mu - \nu| \to \infty$, as the estimate for the remainder $\mathbf{O}(\gamma^4)$ is not in the appropriate weighted space due to the appearance of G there. We see therefore that Theorem 7.2 extends the result obtained from perturbation series, which is valid only for *fixed* $|\mu - \nu|$.

Using the usual method in field theory, in order to obtain the correct behaviour for $\langle G(E,\mu,\nu)\rangle_g$ for $|\mu - \nu| \to \infty$, one needs to expand the infinite series and then resum it. The advantage of our method is that one establishes first the existence of an effective convolution matrix. It is only when one wants to derive an expression for this convolution matrix that one expands. Therefore to obtain an accuracy of the given order, one only needs to expand a finite number of terms. It is *not* an infinite series.

For $E \notin \sigma(H)$, define

$$\beta_E(j) = \lim_{n\to\infty} -\frac{\log\langle G(E,0,nj)\rangle_g}{n}, \tag{1.18}$$

if the limit in the RHS exists. Using the FKG inequality [FKG, Li] as stated in Remark 3.2 on p. 241-242 of [Sz1] and subadditivity ergodic theorem, it can be shown, similar to [Sz1, Z] that for all $E \notin \sigma(H)$, the limit in (1.18) exists and moreover by patching of limits:

$$\lim_{j\to\infty} \frac{|\beta_E(j) + \log\langle G(E,0,j)\rangle|}{\|j\|} = 0. \tag{1.19}$$

The $\beta_E(j)$ are called the annealed directional Lyapunov exponents. They are in some sense, natural generalizations of the traditional Lyapunov exponent to higher dimensions. For a fixed E, β_E defines a norm on $\mathbf{R}^d$ [Sz1, Z]. One can similarly define α_E, the directional Lyapunov exponents in the almost sure case, which we will not address in this paper. α_E are called the quenched Lyapunov exponents. For more detailed statements, see Theorem 3.4, p. 244 of [Sz1] and Theorem A of [Z]. (We remark here that for $E \in I_\delta$, (1.19) can also be proved by using the constructions in sects. III-VI.)

Remark. There also appears to be some connection between the Lyapunov exponents defined here and the Lyapunov exponents of d-dimensional lattices of coupled, non-linear oscillators. See [EW1,2] and references therein for a discussion and some confirmation of this connection.

Define

$$D_{|\widetilde{E}|} = \{y \in R^{d^*} \,|\, 2\sum_{i=1}^{d} \cosh y_i < |\widetilde{E}|\}, \tag{1.20}$$

where $\widetilde{E}$ is as in (1.13). $D_{|\widetilde{E}|}$ is a convex set. We then obtain as a direct consequence of Theorem 7.2

Corollary. *For all $\delta > 0$, there exists $\gamma_0 > 0$, such that for all $0 < \gamma < \gamma_0$, all $E \in I_\delta$,*

$$\beta_E(j) = \sup_{y \in \bar{D}_{|\widetilde{E}|}} y \cdot j + \mathbf{O}(\gamma^4)\|j\|, \tag{1.21}$$

where $\tilde{E}$ is as in (1.13). (Note that $\|y\| = \mathbf{O}(1)$.) Let

$$\hat{j} = \frac{j}{\|j\|} \in S^{d-1}. \tag{1.22}$$

$\beta_E(\hat{j})$ extends to an analytic function in E for E such that $\Re E \in I_\delta$.

The above result is new for $d > 1$. For $d = 1$, there are similar results in the continuum by using probabilistic methods [Fr, Po, Sz2]. The case $d = 1$ is special, as the (random walk) process is additive. We use lattice field theory (see subsect. 3) to prove Theorem 7.2; the Corollary is a straight forward consequence.

Proof of the Corollary. (1.21) follows directly from Theorem 7.2 and standard convex analysis as in [Sj1], so we do not repeat it here. (For a general reference on convex analysis, see e.g., [H].) Analyticity in E ($\Re E \in I_\delta$) for $\beta_E(\hat{j})$ is a direct consequence of Theorem 7.2, (1.21) and the Cauchy Theorem. □

Remark. We note from the Corollary that, to $\mathbf{O}(\gamma^4)$ the Lyapunov exponent is the support function of a convex set corresponding to the effective convolution matrix: $\widetilde{E} - \Delta$. Hopefully it will become clear after the proof of Theorem 7.2 that to *all orders* in γ, the Lyapunov exponent is the support function of a convex set corresponding to an effective convolution matrix by iterating the procedure used here. (cf. [Sj2] for a related situation.) Starting from $\mathbf{O}(\gamma^4)$, the off-diagonal elements of the effective matrix will be different from Δ, i.e., there will be corrections to the generator itself.

2. The probabilistic aspect.

Before explaining the proof of Theorem 7.2, we first take a brief detour to elaborate on the probabilistic content of the problem. The proof of Theorem 7.2 and its Corollary are purely analytical, so they could be understood independently. However in order to put the results here in a more general context, it is the probabilitic point of view which is most natural and most useful. This is not surprising as the problem originates there.

Definition of random walks in random potentials.

Define P on $\mathbf{Z}^d \times \mathbf{Z}^d$ by

$$P(i,j) = \frac{\Delta_{ij}}{2d}, \tag{1.23}$$

for all $(i,j) \in \mathbf{Z}^d \times \mathbf{Z}^d$, where Δ_{ij} is as defined in (1.2). We have from (1.2) that

$$P(i,j) = P(0, i-j), \quad P(0,i) \geq 0, \quad \sum_{i \in \mathbf{Z}^d} P(0,i) = 1, \tag{1.24}$$

i.e., $P(i,j)$ defines the transition probability of a (simple) random walk (see e.g., [Spi]).

Using this point of view, the Green's function for Δ (i.e., the free Green's function) has the following path representation (see e.g., p. 169 of [La]):

$$G_0(E,\mu,\nu) = (E-\Delta)^{-1}(\mu,\nu) = \sum_{w:\,\mu\to\nu} E^{-|w|}, \qquad E > 2d, \tag{1.25}$$

where the sum is over all walks w from μ to ν and $|w|$ is the length (i.e., total number of steps) of the walk w. Similarly the Green's function for H defined earlier in (1.4) has the representation (see e.g., (1) of [Z]):

$$\begin{aligned} G(E,\mu,\nu) &= (E-H)^{-1}(\mu,\nu) \\ &= \sum_{n=0}^{\infty} \sum_{w:\,\mu\to\nu;\,|w|=n} \prod_{j=0}^{n} (E-\gamma V(w_j))^{-1}, \qquad E > 2d, \end{aligned} \tag{1.26}$$

where w_j is the position of the walk w after j steps. Note that due to the discrete time formulation, (1.25) and (1.26) are slightly different from the usual Feynmann-Kac formula. Clearly (1.26) reduces to (1.25) when $V = 0$, and can be seen as defining a weighted (by V) path measure. (In fact, (1.25), (1.26) are, in analytical language, the resolvent (or Neumann) series about E or $E-\gamma V$ written in a slightly different way.) We say that H defines a random walk in the random potential γV.

Remark. The book *Brownian Motion, Obstacles and Random Media* by A. S. Sznitman [Sz1] provides an excellent in-depth reference on the subject. There is also a Bourbaki seminar [Kom] by Komorowski centered on the work of Sznitman, which can serve as a short introduction to the subject. The method that we use to prove Theorem 7.2 is substantially different–it is purely analytical. However much of our understanding of the subject comes from reading the book, which we frequently refer to in this section. Also when we make references to works of Sznitman covered in the book [Sz1], we do not in general trace back to the original reference.

For a given realization of the potential V, $G(E, \mu, \nu)$ is the probability of finding the random walk at site ν conditioned that it starts at site μ. (Recall that the Green's function is the integral over all time t of the heat kernel.) For more details, see e.g.,[La,MS,Spi]. It follows then $\langle G(E, \mu, \nu)\rangle_g$ is the *expected* (with respect to the random potential V) probability of finding the random walk at site ν conditioned that it starts at site μ. In the limit $|\mu - \nu| \to \infty$, $\langle G(E, \mu, \nu)\rangle_g$ is also essentially the normalization constant for the measure defined as the tensor product of the probability measure $P = \prod_{i \in \mathbf{Z}^d} dg(v_i)$ with the path measure for H in (1.26) (see (7), p. XI and (2.7), p. 323 of[Sz1]).

Comments on Theorem 7.2.

We believe quantitative results like (1.12, 1.13) were not know before for random walks in (time-independent) random potentials in $d > 1$. If one considers random walks in time-*dependent* random potentials (cf. [Bol,IS]) , which is a type of the so called *directed* random walks, then Sinai in [Sin] obtained a result which is similar to Theorem 7.2. (See also (2.22) on p. 327 of [Sz1].) The term directed refers to the fact that the walk is parametrized by time and that the graph of the walk in $\mathbf{Z} \times \mathbf{Z}^{d-1}$ moves at a constant rate in the time direction.

The main and also the crucial difference between the directed case and our (non-directed) case is that in the directed case, one can always, in some sense, reduce to a transfer matrix type of situation, as there is a prefered direction; while in our case, the *non*-directed case, one cannot avoid treating walks that return (loops or self-intersections), even though they are unlikely due to the condition $E \notin \sigma(H)$. (See (1.26).)

In spite of the difference stressed above, for high dimensions and potentials which have small variations ($\gamma << 1$), the result for the correlation function in the non-directed case (1.12) is similar to that obtained in the directed case in [Sin]. This partially proves a commonly held belief (see e.g., p.327-328 of [Sz1]).

In the mathematical physics literature, there are results which are similar to ours, e.g., [CC,P-L,Sch]. We mention in particular the result obtained by Chayes and Chayes in [CC] for self-avoiding walks (walks with no self-intersections), as the setting is sufficiently close to ours. (For a precise definition and general references, see [La,MS].) In [CC], the correlation function is defined to be:

$$G(\beta, \mu, \nu) = \sum_{\omega:\, \mu \to \nu} e^{-\beta|\omega|}, \qquad \beta \in [\beta_c, \infty), \tag{1.27}$$

where the sum is over all *self-avoiding* walks ω from μ to ν, $|\omega|$ is the length (i.e., total steps) of ω, β_c is the critical parameter beyond which the sum in (1.27) converges.

Clearly G defined in (1.27) is translationally invariant. So we may take $\nu = 0$. In [CC], for $\nu \to \infty$ in a conic neighborhood of a given direction, which apparently without loss of generality may be chosen to be one of the axis in $\mathbf{Z}^d$, it is shown that for all $\beta < \beta_c$, the correlation function for walks that are conditioned so that their projections onto the axis of the cone contains only points of self-intersections (for more precise definition, see [CC]) decays *strictly* faster than the correlation function for the unconditioned walk. (Questions of uniformity with respect to different directions do not seem to be addressed explicitly in the paper.) From this they deduce the equality of the upper and lower bound of the rate of decay for the correlation function, although no explicit expression is given for this rate.

The decay rate of the walk is customarily called *the gap*; the decay rate of the conditioned walk is called *the upper gap*. The key result of [CC] can therefore be rephrased as stating that the upper gap is strictly greater than the gap "in a cone".

We will come back to this question of gap and upper gap later in subsect. 3, when we describe our method, which reduces the problem to the study of the spectral theory of a self-adjoint operator, the so called Witten Laplacian. As we will see, our construction is *uniform* in all directions. Gap and upper gap appear naturally in this construction. We only preview now that the fact that we have an asymptotics for $\log\langle G(E,\mu,\nu)\rangle_g$ uniformly in $|\mu-\nu|$ as in (1.12), and not just an estimate is a direct consequence of our precise control not only over the gap, but also over the upper gap, or more precisely the spectrum beyond the gap.

Comments on the Corollary.

As mentioned earlier, the results in the Corollary are not known before for $d>1$. For $d=1$, in the continuum, there are some results due to Friedlin, Povel and Sznitman [Fr,Po,Sz2], which are summarized on p. 233,325 of [Sz1]. The case $d=1$ is special as the Brownian motion or the random walk process is additive. Theorem 7.2 and its Corollary confirm a few conjectures raised in the past in the annealed (averaged) case. For a sample of such conjectures in the quenched (almost sure) case, see p. 219, 233 of [Sz1].

In [Sz2], Sznitman proved that in $d=1$ and for Brownian motions in Poissonian potentials, i.e., for

$$H=-\frac{d^2}{dx^2}+V,$$

where $V\geq 0$ is a random potential with Poisson distributions, the quenched Lyapunov exponent $\alpha_E(1)$ is analytic in E for E such that $\Re E>0$. He used an explicit formula obtained in [Fr] to derive the result. The analyticity of $\beta_E(\hat{j})$ proved in the above Corollary extends analyticity results to the annealed case in $\mathbf{Z}^d$ for all $d\geq 1$, when $\gamma << 1$ and for E such that $\Re E\in I_\delta$.

However, the analyticity of $\beta_E(\hat{j})$ for E such that $\Re E\in I_\delta$, as it stands (contrary to the almost sure case) does not yet contain much information. This is because for E complex, $G(E,0,j)$ is in general complex. A priori, the signs of its real and imaginary part depend on V. However, since there is good reason [Spe] and also p. 326 of [Sz1] to believe that $\alpha_E=\beta_E$ almost surely for $d>2$, the Corollary would then imply that $\alpha_E(\hat{j})$ is analytic for $d>2$, almost surely, which certainly carries interesting informations.

In the case of Brownian motions in Poissonian potentials, Sznitman [Sz1] derived upper and lower bounds for α_E and β_E. In the case of random walks in random potentials (same setting as in this paper), Zerner [Z] derived upper and lower bounds for α_E. Our aim is actually to *compute* the *asymptotics* of β_E as $\gamma\searrow 0$. Our method is constructive.

Due to the lack of rotational symmetry in $\mathbf{Z}^d$, the directional dependence of the Lyapunov exponents are more complicated than in $\mathbf{R}^d$. (In $\mathbf{R}^d$, it is known to be proportional to the Euclidean norm, although the constant of proportionality is not known, see p. 219 of [Sz1].) In particular, the Corollary shows that the unit ball in the norm defined by β_E is *not* rotationally invariant. In [Z], when V is a *constant*, i.e., H is a convolution matrix, Zerner obtained a closed expression for the Lyapunov exponents. Moreover for $d = 2$, using this expression, Zerner numerically computed the Lyapunov exponents in the case where the potential V is a *constant.* Combining the Corollary with this numerical result, we see explicitly that for $d = 2$, the unit ball in the norm given by β_E approaches instead the shape of a diamond.

We mention another application of the Corollary to random walks in random potentials with a constant drift $h \in \mathbf{R}^d$. More precisely, we define the first order difference operator ∇ componentwise as

$$(\nabla_{e_\alpha} u)(n) = u(n + e_\alpha) - u(n),$$

where $e_\alpha \in \mathbf{Z}^d$, $e_\alpha(\beta) = \delta_{\alpha\beta}$, α, $\beta = 1, \cdots, d$, and we replace the generator Δ by $\Delta + \sum_{\alpha=1}^d h_\alpha \nabla_{e_\alpha}$. We define the dual norm β_E^* as

$$\beta_E^*(\ell) = \sup_{x\neq 0} \{\frac{\ell \cdot x}{\beta_E(x)}\}, \qquad \ell \in \mathbf{R}^d, \quad E \notin \sigma(H).$$

The unit ball in the norm β_E^* is the critical unit ball [Sz1]: if h is such that $\beta_E^*(h) > 1$, then the motion is ballistic; if h is such that $\beta^*(h) < 1$, then the motion is sub-ballistic. The Corollary then allows us to compute the critical drifts h_c, which are directional dependent, for $E \in I_\delta$.

3. The constructions toward the proof of Theorem 7.2.

We now describe briefly the construction of the proof of Theorem 7.2. As usual we use finite dimensional approximations and take Λ to be a finite set in $\mathbf{Z}^d$ and define H_Λ to be the restricted operator with appropriate boundary conditions. For concreteness, assume $\operatorname{supp} dg \subseteq (-\infty, 0]$ and $E > 2d$. (The other case works in exactly the same way.) Our starting point as in [SW1] is the following Gaussian integral representation of $G_\Lambda(E, \mu, \nu)$:

$$G_\Lambda(E, \mu, \nu) = \int x_\mu \cdot x_\nu [\det(E - H_\Lambda)]^{-1} e^{-\sum_{i,j}(E-H_\Lambda)_{ij} x_i \cdot x_j} \prod_{i\in\Lambda} \frac{d^2 x_i}{\pi}, \tag{1.28}$$

where $x_i \in \mathbf{R}^2$ and $\cdot$ is the usual scalar product in $\mathbf{R}^2$. Using Grassmann variables (see sect. II), the determinant can be further expressed as a Gaussian integral (of Grassmann variables). (This is first used in this context in [BCKP,KS].) This is the so called supersymmetry. Using this representation, we can explicitly take the expectation value of $G_\Lambda(E, \mu, \nu)$ with respect to dg. Let

$$\hat{g}(t) = \int e^{tv} dg(v) \tag{1.29}$$

be the Laplace transform of dg. Since $\hat{g}(t) > 0$, we define

$$k(t) = \log \hat{g}(t).$$

We obtain (see (2.8,2.9))

$$\langle G_\Lambda(E,\mu,\nu)\rangle_g = \int x_\mu \cdot x_\nu e^{-2\phi(x)} \prod_{i\in\Lambda} \frac{d^2 x_i}{\pi}, \tag{1.30}$$

where

(1.31)
$$\phi(x) = \frac{1}{2}[\sum_{i,j}(E-\Delta_\Lambda)_{ij} x_i \cdot x_j - \sum_{j\in\Lambda} k(\gamma x_j \cdot x_j) - \log\det(E-\Delta_\Lambda - \gamma \operatorname{diag} k'(\gamma x_j \cdot x_j))].$$

We see that the above supersymmetric-Gaussian transform has mapped the averaged correlation function of a random walk in random potentials to a correlation function of statistical mechanics. The measure $\prod_{i\in\Lambda} dg(v_i)$ on $\mathbf{R}^\Lambda$ has been transformed to $e^{-2\phi}$ on $(\mathbf{R}^2)^\Lambda \stackrel{\text{def}}{=} \mathbf{R}^{2\Lambda}$. From now on we denote by $\langle\,\rangle$ without subscript, the expectation with respect to the measure $e^{-2\phi}$.

Remark. In the physics literature (see e.g., [Ef]), supersymmetry has always played an important role in the study of disordered systems. In [BCKP,KS], supersymmetry was incorporated in a mathematically rigorous manner for the first time. One of the main differences between the present paper (also the earlier paper [SW1]) with [BCKP,KS], is that the Grassmann variables are further integrated over (see (1.30,1.31)), so that "conventional" analysis becomes feasible.

By a now well known integration by parts initiated in [HS,Sj1], which for completeness we rederive in sect. II, we have further (formally)

$$\langle G_\Lambda(E,\mu,\nu)\rangle_g = \sum_{i=1}^{2} \int (e^{-\phi} dx_\mu^{(i)}, (\Delta_\phi^{(1)})^{-1}(e^{-\phi} dx_\nu^{(i)})), \tag{1.32}$$

where

(1.33)
$$\begin{aligned} &\Delta_\phi^{(1)} = \Delta_\phi^{(0)} \otimes I + 2\phi'' \qquad \text{on } \mathbf{C}_0^\infty(\mathbf{R}^{2\Lambda}; \wedge \mathbf{R}^{2\Lambda}) \\ &\Delta_\phi^{(0)} = \sum z_j^* z_j, \text{where } z_j = \frac{\partial}{\partial x_j} + \frac{\partial\phi}{\partial x_j}, \quad z_j^* = -\frac{\partial}{\partial x_j} + \frac{\partial \phi}{\partial x_j} \qquad \text{on } \mathbf{C}_0^\infty(\mathbf{R}^{2\Lambda}), \end{aligned}$$

z_j^* is the formal adjoint of z_j, we have identified 1-forms with functions with values in $\mathbf{R}^{2\Lambda}$ and $(\,,\,)$ denotes the inner product on $L^2(\mathbf{R}^{2\Lambda}; \wedge\mathbf{R}^{2\Lambda})$. (We will address the questions of conditions on ϕ, domaines of $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ etc., in sect. III. At this stage it suffices to mention that under conditions (H1-3) on g, $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ have self-adjoint extensions.) $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ are respectively the Witten Laplacians on 0, 1-forms. Note that when ϕ is quadratic, $\Delta_\phi^{(0)}$ is just the usual harmonic oscillator on $L^2(\mathbf{R}^{2\Lambda})$.

More generally

$$(1.34)\qquad \begin{aligned} \Delta_\phi &= d_\phi^* d_\phi + d_\phi d_\phi^*,\\ d_\phi &= e^{-\phi} d e^{\phi} = \sum_j z_j dx_j^{\wedge},\\ d_\phi^* &= e^{\phi} d^* e^{-\phi} = \sum_j z_j^* dx_j^{\lfloor}, \end{aligned}$$

where d is the usual exterior differential, d^* its formal adjoint (and consequently d_ϕ^* is the formal adjoint of d_ϕ).

The spectra of $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ play a crucial role in our construction. For now it suffices to mention that $\Delta_\phi^{(0)} \geq 0$ and that $\Delta_\phi^{(0)}$ has 0 as an eigenvalue of multiplicity 1 with $e^{-\phi}$ the unique eigenfunction. If ϕ is strictly convex, i.e., $\phi'' > c > 0$ *uniformly* in Λ as an operator, which is the case for $E \in I_\delta$, $(\delta > 0)$, we obtain that $\Delta_\phi^{(1)} \geq c > 0$ *uniformly* in Λ. This is the so called spectral gap, which is responsable for the exponential decay in Theorem 7.2. Hence to obtain asymptotics for $\langle G_\Lambda\rangle$, we need to compute this spectral gap as an asymptotic series in γ. This requires precise control over the spectrum *beyond* the gap. The main difficulty here is that we need estimates which are uniform in Λ, so that we can pass to the limit $\Lambda \nearrow \mathbf{Z}^d$. This is achieved in sects. IV, V by using appropriate weighted spaces. We also note that for $E \notin \sigma(H)$,

$$\lim_{\Lambda\nearrow\mathbf{Z}^d} \langle G_\Lambda(E,\mu,\nu)\rangle_g = \langle G(E,\mu,\nu)\rangle_g$$

exists a priori by resolvent series and spectral theory, so we only need to ensure that we have uniform estimates in this paper.

We use a Grushin problem to reduce the study of $\Delta_\phi^{(1)}$ near the lower part of its spectrum to that of an effective operator. From the representation in (1.32), we are only interested in $(\Delta_\phi^{(1)} - z)^{-1}$ for $z = 0$. We show that to order $\mathbf{O}(\gamma^4)$ in appropriate weighted spaces, this effective operator is just $\langle 2\phi''\rangle = \int e^{-2\phi} 2\phi''$.

Sketchily, this is accomplished as follows. Let $\bar\Lambda = \Lambda \times \{1,2\}$ which we identify with $\{1, 2, \cdots, |\Lambda|, \cdots, 2|\Lambda|\}$. We first use the set of orthonormal 1-forms $A_1 = \{e^{-\phi} dx_j, j \in \bar\Lambda\}$ to pose the first Grushin problem for $\Delta_\phi^{(1)}$. (Recall that $e^{-\phi}$ is the unique eigenfunction of $\Delta_\phi^{(0)}$ with eigenvalue 0.) Consequently, we obtain that the effective operator is $\langle 2\phi''\rangle$ modulo $\mathbf{O}(\gamma^2)$, valid for a certain spectral interval I_1 at 0. (See Propositions 4.1, 4.3.) This step is similar to that of [Sj1] (see also [BJS]) in the statistical mechanics context, where the effective operator is constructed to order $\mathbf{O}(h^{3/2})$, where h is the semi-classical parameter. The order $\mathbf{O}(h^{3/2})$ there approximately corresponds to the order $\mathbf{O}(\gamma^2)$ here.

However as it is clear from (1.12,1.13) and as we explained earlier, the terms which reveal the structure of the problem start at $\mathbf{O}(\gamma^2)$. So we need to carry the Grushin constructions a bit further. To $\mathbf{O}(\gamma^4)$, this is accomplished by enlarging the set of orthonormal forms to

$$A_3 = \{e^{-\phi} dx_j, j \in \bar\Lambda\} \cup \{z_k^* e^{-\phi} dx_j,\ z_k^* z_\ell^* e^{-\phi} dx_j,\ j, k, \ell \in \bar\Lambda,\ k \leq \ell\}^\dagger,$$

where $\{\quad\}^\dagger$ denotes the set of orthonormal 1-forms obtained from $\{\quad\}$ by orthonormalization. These two sets are of the same dimension here. (The restriction $k \leq \ell$ is due to the commutativity of the z_j^* $(j \in \bar{\Lambda})$, defined in (1.33).)

More precisely we show that the well-posedness of the first Grushin problem for $\Delta_\phi^{(1)}$ using $\{e^{-\phi}dx_j, j \in \bar{\Lambda}\}$, implies the well-posedness of a Grushin problem for $\Delta_\phi^{(0)}$ using

$$\{e^{-\phi}\} \cup \{z_k^* e^{-\phi},\, k \in \bar{\Lambda}\}^\dagger.$$

So we obtain an effective operator for $\Delta_\phi^{(0)}$ valid in the same interval I_1. (See Proposition 4.4.) Here we used in a crucial way that d_ϕ is a complex and therefore the spectrum of $\Delta_\phi^{(1)}$ and $\Delta_\phi^{(0)}$ are related.

Using the first equation of (1.33) and $\phi'' > c > 0$, we obtain an effective operator for $\Delta_\phi^{(1)}$ valid in a *larger* spectral interval $I_2 \supset I_1$ by using

$$A_2 = \{e^{-\phi}dx_j, j \in \bar{\Lambda}\} \cup \{z_k^* e^{-\phi}dx_j,\, j, k \in \bar{\Lambda}\}^\dagger.$$

(See Proposition 4.7.)

Iterating this once more (Propositions 4.9-4.14), we obtain an effective operator for $\Delta_\phi^{(1)}$ modulo $\mathbf{O}(\gamma^4)$ when restricted to the subspace spanned by A_1, in an even larger interval $I_3 \supset I_2 \supset I_1$ by using A_3. (Although we do not pursue it in this paper, it is clear to us that this procedure can in fact be iterated to all orders.)

We stress that since we are interested in the asymptotics as $|\mu - \nu| \to \infty$, it is crucial that I_3 is large enough, as this translates into estimates which are in appropriate weighted spaces. Theorem 7.2 is then obtained by computing explicitly $\langle 2\phi'' \rangle^{-1}$. The Corollary follows by using Theorem 7.2 and standard results on the inverse of convolution matrices.

We note also that for γ small, ϕ is close to a quadratic form, and $e^{-2\phi}$ is close to a Gaussian. So it should not be surprising that the set of functions

$$\{e^{-\phi},\, z_k^* e^{-\phi},\, z_k^* z_\ell^* e^{-\phi}, \cdots k, \ell \in \bar{\Lambda},\, k \leq \ell, \cdots\}$$

play a special role here. This is because when $\gamma = 0$, this set of functions after proper orthonormalization is just a set of Hermite functions. What we did can therefore be seen as a perturbation theory around an "infinite" dimensional Gaussian or harmonic oscillator. In other words, what we are using is a Witten complex formulation of Euclidean lattice field theory, whose foundation was laid down by Sjöstrand in his seminal paper [Sj1].

After we finished the paper, we realized that the connections between random walks and lattice field theory are in fact well known, see e.g., [FFS], in particular [BF1,2], where they use the term oscillation modes, which roughly corresponds to the eigenfunctions of $\Delta_\phi^{(1)}$ in our vocabulary. The novelty here is that we have an *explicit* transformation of random walks in random potentials to a specific operator: $\Delta_\phi^{(1)}$, where ϕ is determined by the random walk and the random potential as in (1.31). $\Delta_\phi^{(1)}$ can in turn be interpreted as giving rise to a field theory. It is due to this equality in (1.32) that we are able to *compute* various quantities.

Finally, we like to add that we also hope to address the almost sure (quenched) behaviour of G by similar constructions in the future. As mentioned earlier, it is conjectured that $\alpha_E = \beta_E$ a.s., for $d > 2$. If it is true, then the Lyapunov exponents obtained here are also the Lyapunov exponents in the quenched case for $d > 2$.

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II. The Averaged Green's Function

We start by recapturing the problem that we study:

$$H = \Delta + \gamma V, \text{ on } \quad \ell^2(\mathbf{Z}^d), \tag{2.1}$$

where

$$\begin{aligned} \Delta_{ij} &= 1 \quad |i-j|_{\ell_1} = 1 \\ &= 0 \quad \text{otherwise;} \end{aligned} \tag{2.2}$$

γ is a positive parameter, $(Vu)_j = v_j u_j$, v_j is a family of iid random variables with common distribution dg satisfying (H1-3) of sect. I, and $\sigma(H) = [-2d, 2d] + \gamma \operatorname{supp} dg$. Let $E \in \mathbf{R} \backslash \sigma(H)$. For definitiveness assume supp $dg \subseteq (-\infty, 0]$ and $E > 0$. (supp $dg \subseteq [0, +\infty)$ and $E < 0$ works the same way.) We are interested in computing the asymptotics of $\langle G(E, \mu, \nu)\rangle_g = \langle (E-H)^{-1}_{\mu\nu}\rangle_g$ as $|\mu - \nu| \to \infty$ for γ sufficiently small.

We use finite dimensional approximations. We take Λ to be a finite set in $\mathbf{Z}^d$ or a large torus of the form $(\mathbf{Z}/N\mathbf{Z})^d$. We denote by $|\Lambda|$ the number of lattice points of $\mathbf{Z}^d$ in Λ. We identify Λ with the set $\{1, 2 \cdots |\Lambda|\}$. The corresponding discrete Laplacian Δ_Λ is then defined as in (2.2) for $i, j \in \Lambda$. Similarly we define V_Λ. Define

$$H_\Lambda = \Delta_\Lambda + \gamma V_\Lambda$$

on $\ell^2(\Lambda)$. Clearly for all Λ, $\sigma(H_\Lambda) \subseteq [-2d, 2d] + \gamma \operatorname{supp} dg$. So $G_\Lambda = (E - H_\Lambda)^{-1}$ is well defined. As mentioned earlier in sect. I, by using spectral theory $(E-H)^{-1}$

exists a priori. So to infer the behavior of $\langle (E-H)^{-1}_{\mu\nu}\rangle$, it is sufficient to obtain estimates on $\langle (E-H_\Lambda)^{-1}_{\mu\nu})\rangle$ which are uniform in Λ. For the rest of the paper, we will only work with H_Λ etc., so for simplicity of notations, we will drop the subscript Λ. We reserve Δ (or Δ_Λ) to denote the discrete Laplacians only; the Witten Laplacians will always be denoted as Δ_ϕ.

The main goal of this section is to obtain a convenient representation of $\langle (E-H)^{-1}_{\mu\nu}\rangle_g$. This is achieved in Proposition 2.1. Our principal tools are the Gaussian integrals. For any symmetric $n\times n$ matrix A with $\Re A > 0$, we recall the following well-known formula

$$A^{-1}_{\mu\nu} = \int x_\mu \cdot x_\nu \det A e^{-\sum_{i,j=1}^n A_{ij} x_i\cdot x_j} \prod_{i=1}^n \frac{d^2 x_i}{\pi}, \tag{2.3}$$

where $x_i \in \mathbf{R}^2$ and $x_i \cdot x_j$ is the usual scalar product in $\mathbf{R}^2$. Substituting $E-H$ for A in the above formula, we then have

$$G(E,\mu,\nu) = \int x_\mu \cdot x_\nu \det(E-H) e^{-\sum_{i,j\in\Lambda}(E-H)_{ij} x_i\cdot x_j} \prod_{i\in\Lambda} \frac{d^2 x_i}{\pi}. \tag{2.4}$$

Let $\hat g$ be the Laplace transform of dg:

$$\hat g(t) = \int e^{tv} dg(v), \qquad t \ge 0 \tag{2.5}$$

For example, for the Bernoulli measure:

$$dg(v) = \frac{\delta(v+1)+\delta(v)}{2} \tag{2.6}$$

introduced earlier, $\hat g(t) = e^{-t/2}\cosh(t/2)$. Using (H1), $\hat g$ is well defined for $t \ge 0$, moreover $\hat g$ is a real analytic function in t for $\Re t > 0$. Since dg is a probability measure, $\hat g(t) > 0$ for $t \ge 0$. We write

$$\hat g(t) = e^{k(t)},$$

for $t \ge 0$. Hence

$$k(t) = \log \hat g(t). \tag{2.7}$$

For the Bernoulli measure in (2.6), $k(t) = \log\cosh(t/2) - t/2$.

Proposition 2.1.

$$\langle G(E,\mu,\nu)\rangle_g = \int x_\mu \cdot x_\nu e^{-2\phi(x)} \prod_{i\in\Lambda} \frac{d^2 x_i}{\pi}, \tag{2.8}$$

where

(2.9)
$$\phi(x) = \frac{1}{2}[\sum_{i,j}(E-\Delta)_{ij} x_i\cdot x_j - \sum_{j\in\Lambda} k(\gamma x_j\cdot x_j) - \log\det(E-\Delta-\gamma \operatorname{diag} k'(\gamma x_j\cdot x_j))],$$
$x_i \in \mathbf{R}^2$.

Starting from sect. III, we will be using the representation in (2.8) to study $\langle (E-H)^{-1}_{\mu\nu}\rangle\rangle_g$. The method that we will use (aside from some estimates) is essentially independent of the derivation of (2.8). So those readers who are not particularly interested in the derivation could skip the following proof and proceed directly to sect. III. The proof that we give below is almost identical to that given in [SW1]. For completeness, we resummarize it here:

Proof of Proposition 2.1. There are presumably a few ways to derive (2.8). We continue to use the method in [SW1], which uses Grassmann algebra. (For another related way, see appx. B of [SW1].) For a general proper formulation, see e.g., [Be,V]; and in this context see appx. A of [SW1]. In the following we will only write the few lines that are absolutely necessary for the proof of (2.8).

Let $|\Lambda|$ be the number of points in Λ. We use the Grassmann algebra of $2|\Lambda|$ generators to express $\det[E-H]$. This algebra is generated by $2|\Lambda|$ anticommuting variables ξ_i, η_i, $i \in \Lambda$ satisfying the relations:

$$\begin{aligned} [\xi_i, \eta_j] &= \xi_i\eta_j + \eta_j\xi_i = 0, \\ [\xi_i, \xi_j] &= \xi_i\xi_j + \xi_j\xi_i = 0, \\ [\eta_i, \eta_j] &= \eta_i\eta_j + \eta_j\eta_i = 0, \end{aligned} \tag{2.10}$$

where we write $[a, b] = ab+ba$ for the anti-commutator. It is denoted by $\mathit{\Lambda}[\xi_1, \eta_1, .., \xi_{|\Lambda|}, \eta_{|\Lambda|}]$ (if we identify Λ with $\{1, .., |\Lambda|\}$). From (2.4), we see in particular that $\xi_i^2 = \eta_i^2 = 0$. We write these anticommutative variables collectively as ξ, η.

To make connections with better known objects, ξ_j, η_j can be thought of as the two 1-forms on the j^{th} copy of $\mathbf{R}^2$. "C^∞ functions" $F(\xi_i, \eta_j)$ of these anticommuting variables are defined by Taylor's formula at $(0, 0)$ which contains a finite number of terms because of nilpotency. In this way $F(\xi, \eta)$ becomes an element of the Grassmann algebra. For example if

$$F(\xi, \eta) = e^{A_{ij}\xi_i\eta_j}, \tag{2.11}$$

then

$$F(\xi, \eta) = 1 + A_{ij}\xi_i\eta_j. \tag{2.12}$$

This is the function that we need in writing the determinant. We also need to define the notions of differentiation and integration. Define:

$$\begin{aligned} \frac{\partial}{\partial \xi_i}(\xi_i) &= 1, \\ \frac{\partial}{\partial \eta_i}(\eta_i) &= 1. \end{aligned} \tag{2.13}$$

We require further that these differentiations be linear operators and that Leibnitz' rule hold. We can then define integrals (with respect to ∂) as follows:

$$\int 1 d\xi_i = 0, \ \int \xi_i d\xi_i = 1, \ \int 1 d\eta_i = 0, \ \int \eta_i d\eta_i = 1. \tag{2.14}$$

A multiple integral is defined to be a repeated integral. For example,

$$\int \xi_i \eta_j d\xi_i d\eta_j = -\int \eta_j \xi_i d\xi_i d\eta_j = -\int \eta_j d\eta_j = -1. \tag{2.15}$$

Using (2.11)–(2.15), we obtain

$$\det(E-H) = \int e^{-\sum_{j,k\in\Lambda}(E-H)_{jk}\eta_j\xi_k} \prod_{j\in\Lambda}(d\eta_j d\xi_j), \tag{2.16}$$

$$(E-H)^{-1}_{\mu\nu}\det(E-H) = \int \xi_\mu \eta_\nu e^{-\sum_{j,k\in\Lambda}(E-H)_{jk}\eta_j\xi_k} \prod_{j\in\Lambda}(d\eta_j d\xi_j), \tag{2.17}$$

We illustrate (2.16) in the case where $(E-H) = M$ is a 2×2 matrix. The integrand in the RHS of (2.16) is

$$e^{-\sum_{j,k=1,2} M_{jk}\eta_j\xi_k} = \prod_{j,k=1,2}(1 - M_{jk}\eta_j\xi_k), \tag{2.18}$$

where we used the commutativity of $e^{-M_{jk}\eta_j\xi_k}$ and (2.11, 2.12). Doing the integral in the RHS of (2.18) upon using (2.14) and (2.15), we obtain

$$\text{RHS} = M_{11}M_{22} - M_{12}M_{21} = \det M, \tag{2.19}$$

as expected. Similarly, we obtain for the RHS of (2.17):

$$\text{RHS} = (-1)^{|\mu|+|\nu|}\det(M^{\mu\nu}) = M^{-1}_{\mu\nu}\det M, \qquad \mu,\, \nu = 1,\, 2, \tag{2.20}$$

where $M^{\mu\nu}$ is the comatrix with the μth row and νth column removed. (In the 2×2 case it is just the one matrix element that is left.)

Combining (2.4) with (2.16), we obtain the following expression:

$$G(E,\mu,\nu) = \int x_\mu \cdot x_\nu e^{-\sum_{j,k\in\Lambda}(E-H)_{jk}X_j\cdot X_k} \prod_{j\in\Lambda} d^2X_j, \tag{2.21}$$

where

$$\begin{aligned} x_j &\in \mathbf{R}^2, \\ X_j &\stackrel{\text{def}}{=} (x_j, \xi_j, \eta_j), \\ X_j \cdot X_k &\stackrel{\text{def}}{=} x_j \cdot x_k + \frac{1}{2}(\eta_j\xi_k + \eta_k\xi_j), \\ d^2X_j &\stackrel{\text{def}}{=} \frac{d^2x_j}{\pi} d\eta_j d\xi_j, \end{aligned} \tag{2.22}$$

and we used that $(E-H)$ is a symmetric matrix. Hence,
(2.23)

$$\begin{aligned}\langle G(E,\mu,\nu)\rangle_g &= \int x_\mu \cdot x_\nu e^{-(\sum_{j\in\Lambda} EX_j\cdot X_j - \sum_{j,k\in\Lambda,|j-k|_1=1} X_j\cdot X_k - \gamma\sum_{j\in\Lambda} v_j X_j\cdot X_j)} \\ &\qquad \prod_{j\in\Lambda} dg(v_j) \prod_{j\in\Lambda} d^2X_j \\ &= \int x_\mu\cdot x_\nu e^{-(\sum_{j\in\Lambda} EX_j\cdot X_j - \sum_{j,k\in\Lambda,|j-k|_1=1} X_j\cdot X_k - \sum_{j\in\Lambda} k(\gamma X_j\cdot X_j))} \prod_{j\in\Lambda} d^2X_j \\ &\overset{\text{def}}{=} \int x_\mu\cdot x_\nu e^{-2\Phi}\prod_{j\in\Lambda} d^2X_j,\end{aligned}$$

where

$$\begin{aligned} k(\gamma X_j\cdot X_j) &= k(\gamma x_j\cdot x_j + \gamma\eta_j\xi_j) \\ &= k(\gamma x_j\cdot x_j) + \gamma k'(x_j\cdot x_j)\eta_j\xi_j, \end{aligned} \tag{2.24}$$

by using (2.12) to expand $e^{\gamma v_j\xi_j\eta_j}$ and direct computations, and

$$2\Phi \overset{\text{def}}{=} \sum_{j\in\Lambda} EX_j\cdot X_j - \sum_{j,k\in\Lambda,|j-k|_1=1} X_j\cdot X_k - \sum_{j\in\Lambda} k(\gamma X_j\cdot X_j). \tag{2.25}$$

Using (2.23) and integrating over the anticommutative variables ξ, η according to (2.14, 2.15), we then obtain the proposition. □

We notice from the proof of Proposition 2.1 that in (2.21), apart from the factor $x_\mu\cdot x_\nu$, the integrand is only a "function" of $X_j\cdot X_k$. More precisely, we consider the algebra $C^\infty(\mathbf{R}^{2|\Lambda|})\otimes\Lambda[\xi_1,\eta_1,..,\xi_{|\Lambda|},\eta_{|\Lambda|}]$. We call elements of this algebra, (super)functions. Clearly,

$$\begin{aligned} D(X) = X_j\cdot X_k &= x_j\cdot x_k + \frac{1}{2}(\eta_j\xi_k+\eta_k\xi_j) \\ &\in C^\infty(\mathbf{R}^{2|\Lambda|})\otimes\Lambda[\xi_1,\eta_1,..,\xi_{|\Lambda|},\eta_{|\Lambda|}], \end{aligned}$$

and is hence a (super)function. We call the scalars (which are functions in $C^\infty(\mathbf{R}^{2|\Lambda|})$) in front of the Grassmann generators ξ, η, coefficients. In the case of $X_j\cdot X_k$, the coefficients are $x_j\cdot x_k$ and $1/2$. Those (super)functions which are only functions of $X_j\cdot X_k$ (j, $k\in\Lambda$) are called supersymmetric functions. More precisely, supersymmetries are defined to be the set of transformations (in the alegebraic sense) that leave the dot products $X_j\cdot X_k$ (j, $k\in\Lambda$) invariant. Two obvious transformations that leave the dot product invariant are the usual rotations O in $\mathbf{R}^2$,

$$x_i = x_i'O \qquad (i=1,\cdots,n)$$

and the transformations $A\in Sp(2)$ acting on ξ_i, η_i ($i=1,\cdots,n$) such that

$$(\xi_i,\eta_i) = (\xi_i',\eta_i')A,$$

where $\{x_i',\xi_i',\eta_i'\}_{i=1}^n$ is another set of coordinates, x_i' being the even ones and ξ_i', η_i' the odd ones. We put $X_i'=(x_i',\xi_i',\eta_i')$ $\quad (i=1,\cdots,n)$. Aside from these two linear transformations, supersymmetries also include transformations generated by (super)vector fields of the type:

$$\begin{aligned} V &= \sum_i V_i \\ &= \sum_i (\xi_i a + \eta_i b)\frac{\partial}{\partial x_i} + 2(b \cdot x_i)\frac{\partial}{\partial \xi_i} - 2(a \cdot x_i)\frac{\partial}{\partial \eta_i}, \end{aligned}$$

where $a,\ b \in \mathbf{R}^2$, and

$$\begin{aligned} a\frac{\partial}{\partial x_i} &\stackrel{\text{def}}{=} a^{(1)}\frac{\partial}{\partial x_i^{(1)}} + a^{(2)}\frac{\partial}{\partial x_i^{(2)}}, \\ b\frac{\partial}{\partial x_i} &\stackrel{\text{def}}{=} b^{(1)}\frac{\partial}{\partial x_i^{(1)}} + b^{(2)}\frac{\partial}{\partial x_i^{(2)}}. \end{aligned}$$

(Note that it is the same transformation in all the X_i.) The above transformation is to be understood in the algebraic sense. We check that $VD(X_i, X_j) = 0$. Analogous to the usual change of variables, the (super) change of variables also generates a Jacobian, which is called a Berezinian. We will not go into the details, except mentioning that the Berezinian here is 1.

Let τ be a supersymmetric transformation. Let $X_i = \tau X_i'$ $(i = 1, \cdots, n)$.

Definition. *A superfunction F is supersymmetric if it is invariant under all supersymmetries:*

$$F(X_1, \cdots, X_n) = F(X_1', \cdots, X_n'),$$

for all τ supersymmetric transformations.

Clearly, supersymmetric functions belong to a rather restricted class of functions. For example, in $\mathbf{R}^2$, with two generators ξ, η, F is supersymmetric if and only if there exists f: $[0, \infty) \mapsto \mathbf{R}$ of class C^∞, such that

$$F(X) = f(X \cdot X) = f(x \cdot x) + f'(x \cdot x)\eta\xi.$$

For the general case, see *e.g.*, [KS].

Define $dX_i = (d^2x_i/\pi)d\eta_i d\xi_i$ $(i = 1, \cdots, n)$. One of the most useful properties of the supersymmetric functions is the following:

Proposition 2.2. *If F is supersymmetric with all of its coefficients in $\mathcal{S}(\mathbf{R}^{2n})$, then*

$$\int F(X_1, \cdots, X_n)dX_1 \cdots dX_n = F(0, \cdots 0). \tag{2.26}$$

For a proof, see e.g., [K,KS]. Proposition 2.2 is related to the so called "localization formula" in geometry, see e.g., [BGV]. Let $F(X) = X_\mu \cdot X_\nu e^{-2\Phi}$. Using Proposition 2.2, (2.23), $F(0) = 0$ and integrating over the anti-commuting variables, we then obtain another expression for $\langle G(E,\mu,\nu)\rangle_g$ (to be compared to (2.8)):

$$
(2.27)\qquad
\begin{aligned}
\langle G(E,\mu,\nu)\rangle_g &= \int \xi_\mu \eta_\nu e^{-2\Phi} \prod_{j\in\Lambda} d^2 X_j \\
&= \int (E-\Delta-\gamma\,\mathrm{diag}\,k')^{-1}_{\mu\nu} e^{-2\phi} \prod_{j\in\Lambda} \frac{d^2 x_j}{\pi},
\end{aligned}
$$

where we used (2.17,2.9) to reach the last equality.

Proposition 2.2, (2.17) and (2.27) will be our main tool in estimating various integrals with respect to the measure $e^{-2\phi}$. It helps us to obtain the correct order of magnitude in γ. It also allows us to skirt the use of maximum principles as in [Sj1] (or Witten Laplacians as in [BJS].) In spite of all of its virtues, supersymmetry is not central for the main constructions in this paper. So if the readers wish, they could skip the details of such estimates of the integrals and only use the end results. Finally we remark that the measure $e^{-2\phi}$ with ϕ defined in (2.9) is normalized for all Λ as $\prod_{j\in\Lambda} dg(v_j)$ is normalized, which can also be verified by using (2.25,2.9) and Proposition 2.2.

III. The Witten Laplacian Representation of the Averaged Green's Function

Recall that in the Gaussian integral representation (2.4), we doubled the dimension, we therefore introduce the set $\bar\Lambda = \Lambda \times \{1,2\}$ as in sect. I, which we identify with $\{1, 2\cdots|\Lambda|\} \times \{1,2\}$ and sometimes also with $\{1,2,\cdots,|\Lambda|,\cdots,2|\Lambda|\}$. Our main aim is to write the correlation

$$
(3.1)\qquad \langle G(E,\mu,\nu)\rangle_g = \int x_\mu \cdot x_\nu e^{-2\phi(x)} \prod_{i\in\Lambda} d^2 x_i
$$

in (2.8) using Witten Laplacians as initiated in [Sj1]. In order to use the Witten Laplacian construction, we need to have some control over k and its derivatives. To that end, we use (H3) and apply a classical Tauberian theorem, see Theorems 2, 3 on p. 418-422 of [W], which we summarize below as Theorem 3.0. Recall also the notion of regularly varying functions introduced earlier in (H3).

Let dU be a measure defined on $[0,\infty)$ and such that its Laplace transform

$$
(3.2)\qquad \omega(\lambda) = \int_0^\infty e^{-\lambda x} dU(x)
$$

exists for $\lambda > 0$.

Theorem 3.0. *Let $\rho \in [0,\infty)$. If $L \geq 0$ defined on $[0,\infty)$ is regularly varying at 0 with exponent 0, then each of the relations*

$$
(3.3)\qquad \omega(\lambda) \sim \lambda^{-\rho} L(\frac{1}{\lambda}), \qquad \lambda \to \infty,
$$

and

$$U(t) \sim \frac{1}{\Gamma(\rho+1)} t^{\rho} L(t), \qquad t \to 0 \tag{3.4}$$

implies the other.

Using the above Tauberian theorem, we arrive at

Lemma 3.1. *Assume (H1-3), then*

$$k^{(n)}(t) = \frac{\mathbf{O}_n(1)}{(1+|t|)^n}, \qquad (n \geq 0). \tag{3.5}$$

Proof. For simplicity we work out the case supp $dg \subseteq (-\infty, 0]$, the other case works in the same way. Since

$$\hat{g}(t) = \int e^{tv} dg(v) = e^{k(t)},$$

(3.5) is obviously true when $n = 0$ by using (H1). We now look at $n \geq 1$.

$$\hat{g}'(t) = \int v e^{tv} dg(v) = k'(t) e^{k(t)}.$$

So

$$k'(t) = \frac{\int v e^{tv} dg(v)}{\int e^{tv} dg(v)} = \frac{\int e^{tv} d(g(v)v) - \int e^{tv} g(v) dv}{\int e^{tv} dg(v)},$$

by integration by parts, as g is of regular variation. Since g is of regular variation with exponent ρ, vg, $\int_0^v g(s)ds$ are both of regular variations with exponent $\rho + 1$. The Tauberian theorem 3.0, then gives

$$k'(t) = \frac{\mathbf{O}_1(1)}{(1+|t|)} \qquad \text{as}\, t \to \infty.$$

Similarly

$$\begin{aligned} k''(t) &= \frac{\int v^2 e^{tv} dg(v)}{\int e^{tv} dg(v)} - (k'(t))^2 \\ &= \frac{\int e^{tv} d(g(v)v^2) - 2\int e^{tv} v g(v) dv}{\int e^{tv} dg(v)} - (k'(t))^2 \\ &= \frac{\mathbf{O}_2(1)}{(1+|t|)^2}, \\ &\vdots \\ k^{(n)}(t) &= \frac{\mathbf{O}_n(1)}{(1+|t|)^n} \\ &\vdots \end{aligned}$$

□

Using (2.9), we compute that

$$\begin{aligned}(2\phi'_{i^{(\alpha)}})(x) = {} & 2\sum_{\ell}(E-\Delta-\gamma\,\mathrm{diag}\ k'(\gamma x_j\cdot x_j))_{i\ell}x_\ell^{(\alpha)} \\ & + 2\gamma^2 k''(\gamma x_i\cdot x_i)(E-\Delta-\gamma\,\mathrm{diag}\ k'(\gamma x_j\cdot x_j))^{-1}_{ii}x_i^{(\alpha)},\end{aligned} \tag{3.6}$$

and

(3.7)

$$\begin{aligned}(2\phi'')_{i^{(\alpha)}j^{(\beta)}} = {} & 2E\delta_{ij}\delta_{\alpha\beta} - 2\Delta_{ij}\delta_{\alpha\beta} - 2\gamma k'(\gamma x_i\cdot x_i)\delta_{ij}\delta_{\alpha\beta} - 4\gamma^2 x_i^{(\alpha)}x_i^{(\beta)}k''(\gamma x_i\cdot x_i)\delta_{ij} \\ & + 2\gamma^2 k''(\gamma x_i\cdot x_i)(E-\Delta-\gamma\,\mathrm{diag}\,k')^{-1}_{ii}\delta_{ij}\delta_{\alpha\beta} \\ & + 4\gamma^3 x_i^{(\alpha)}x_i^{(\beta)}k'''(\gamma x_i\cdot x_i)(E-\Delta-\gamma\,\mathrm{diag}\,k')^{-1}_{ii}\delta_{ij} \\ & + 4\gamma^4 x_i^{(\alpha)}x_j^{(\beta)}k''(\gamma x_i\cdot x_i)k''(\gamma x_j\cdot x_j)[(E-\Delta-\gamma\,\mathrm{diag}\,k')^{-1}_{ij}]^2,\end{aligned}$$

For all $\delta>0$, $E\in I_\delta$, (I_δ as defined in (1.11),) using Lemma 3.1, we deduce that $\phi\in C^\infty(\mathbf{R}^{2|\Lambda|},\mathbf{R})$, satisfies

$$\begin{aligned}&\phi(x)=\mathbf{O}(1+x^2),\ \phi'(x)=\mathbf{O}(1+|x|),\\ &x\cdot\phi'(x)\sim|x|^2,\quad \text{for } |x|>>1,\quad \phi^{(n)}(x)=\mathbf{O}_n(1)\quad\text{for}\quad n\geq 2,\end{aligned} \tag{3.8}$$

and

$$\phi''(x)\geq r_E>0 \qquad \text{in} \quad \mathcal{L}(\ell^2(\bar\Lambda),\ell^2(\bar\Lambda)) \tag{3.9}$$

where r_E (depending on δ) is uniform in Λ. The results to be stated below (except the last subsection when ϕ is an exact quadratic form) are general and in particular do not require that $\phi''>c>0$.

For notational simplicity, we make a slight change of point of view. We make the identification:

$$(\mathbf{R}^2)^\Lambda\sim(\mathbf{R})^{2|\Lambda|}\sim(\mathbf{R})^{\bar\Lambda}.$$

(In fact, careful readers probably have already remarked that we made this identification earlier in the introduction.) Hence unless specified otherwise, the i, j, $k\cdots$ etc. will now be in $\bar\Lambda$ and x_i, x_j, $x_k\cdots$ etc. will now be in $\mathbf{R}$. We will generally keep this point of view unless we need to estimate integrals with respect to $e^{-2\phi}$, where it is more convenient to revert back to the point of view of $(\mathbf{R}^2)^\Lambda$ and use supersymmetry.

The Witten complex.

We introduce the Witten complex (originally introduced in [W]) as in [Sj1] (cf. also [BJS,HS2]):

$$d_\phi=e^{-\phi}de^{\phi}=d+d\phi^\wedge=\sum_{j\in\bar\Lambda}z_j dx_j^\wedge, \tag{3.10}$$

acting (for the moment) on $C_0^\infty(\mathbf{R}^{2\Lambda};\wedge^\ell\mathbf{R}^{2\Lambda})$ $(0\le\ell\le 2|\Lambda|)$, where $d=\sum_{j\in\bar\Lambda}\partial_{x_j}dx_j^\wedge$ is the usual exterior differentiation and

$$z_j=\frac{\partial}{\partial x_j}+\frac{\partial\phi}{\partial x_j} \tag{3.11}$$

can be viewed as an annihilation operator. Let

$$d_\phi^*=e^\phi d^* e^{-\phi}=\sum_{j\in\bar\Lambda} z_j^* dx_j^\rfloor, \tag{3.12}$$

be the formal complex adjoint, where

$$z_j^*=-\frac{\partial}{\partial x_j}+\frac{\partial\phi}{\partial x_j} \tag{3.13}$$

can be viewed as a creation operator. We have the commutation relation:

$$[z_j,z_k^*]=2\partial_j\partial_k\phi, \tag{3.14}$$

which plays an important role in our later constructions. We check easily that indeed

$$d_\phi d_\phi=d_\phi^* d_\phi^*=0. \tag{3.15}$$

Using d_ϕ, ${d_\phi}^*$, we define the Witten Laplacian, which is a deformed Hodge Laplacian:

$$\Delta_\phi=d_\phi^* d_\phi+d_\phi d_\phi^*, \tag{3.16}$$

on $C_0^\infty(\mathbf{R}^{2\Lambda};\wedge^\ell\mathbf{R}^{2\Lambda})$ $(0\le\ell\le 2|\Lambda|)$. Notice that

$$d_\phi\Delta_\phi=\Delta_\phi d_\phi,\ d_\phi^*\Delta_\phi=\Delta_\phi d_\phi^* \tag{3.17}$$

by using (3.15). If we let $\Delta_\phi^{(\ell)}$ be the restriction of Δ_ϕ to forms of degree ℓ, we obtain more precisely:

$$d_\phi\Delta_\phi^{(\ell)}=\Delta_\phi^{(\ell+1)}d_\phi,\ d_\phi^*\Delta_\phi^{(\ell+1)}=\Delta_\phi^{(\ell)}d_\phi^*. \tag{3.18}$$

This is the same as in the case of Hodge Laplacians which corresponds to taking $\phi=0$. We have explicitly

$$\Delta_\phi^{(0)}={d_\phi}^*d_\phi=\sum_{j\in\bar\Lambda}z_j^*z_j=-\sum_{j\in\bar\Lambda}\frac{\partial^2}{\partial x_j^2}+\|d\phi\|^2-\operatorname{Tr}\operatorname{Hess}\phi. \tag{3.19}$$

For example, if ϕ is a non-degenerate quadratic form, then $\Delta_\phi^{(0)}$ is a $2|\Lambda|$-dimensional harmonic oscillator. z_j and z_j^* are just the famous annihilation and creation operators for the harmonic oscillator. More generally, we have

$$\begin{aligned}\Delta_\phi&=\sum\sum z_jz_k^*dx_j^\wedge dx_k^\rfloor+\sum\sum z_k^*z_jdx_k^\rfloor dx_j^\wedge\\&=\sum\sum z_k^*z_j(dx_j^\wedge dx_k^\rfloor+dx_k^\rfloor dx_j^\wedge)+[z_j,z_k^*]dx_j^\wedge dx_k^\rfloor\\&=\sum z_j^*z_j+2\sum\sum(\partial_{x_j}\partial_{x_k}\phi)dx_j^\wedge dx_k^\rfloor\\&=\Delta_\phi^{(0)}\otimes I+2\sum\sum(\partial_{x_j}\partial_{x_k}\phi)dx_j^\wedge dx_k^\rfloor,\end{aligned} \tag{3.20}$$

where to obtain the third line from the second, we used (3.14). In particular, if we identify 1-forms with $\mathbf{R}^{2\Lambda}$ valued functions, we obtain

$$\Delta_\phi^{(1)} = \Delta_\phi^{(0)} \otimes I + 2\phi''. \tag{3.21}$$

Also if we identify 2-forms with functions valued in the space of real anti-symmetric matrices, we have

$$\Delta_\phi^{(2)} u = \Delta_\phi^{(0)} u + 2(\phi'' \circ u + u \circ \phi''). \tag{3.22}$$

Below we describe briefly some general properties of $\Delta_\phi^{(\ell)}$ for ϕ satisfying (3.8). ((3.9) is not needed for these general statements.) There is a well written paper by Johnsen [Jo] on the subject. (3.8) ensures that most of the results there are valid in the present case. So for more details, the readers are refered to [Jo].

Since $(\Delta_\phi^{(\ell)} u, u) \geq 0$ for all $u \in C_0^\infty(\mathbf{R}^{2\Lambda}; \wedge^\ell(\mathbf{R}^{2\Lambda})^*)$, we can define $\Delta_\phi^{(\ell)}$ as a self-adjoint operator by taking the Friedrichs extension. Using (3.8), we also see that $\Delta_\phi^{(\ell)}$ has compact resolvent. Moreover $\Delta_\phi^{(\ell)}$ has discrete spectrum contained in $[0, \infty)$. The lowest eigenvalue of $\Delta_\phi^{(0)}$ is zero and a corresponding eigenfuction is $e^{-\phi}$, since this function is annihilated by d_ϕ. This eigenvalue is simple, for if u is another eigenfunction associated to the same eigenvalue, then $0 = (\Delta_\phi^{(0)} u, u) = \|d_\phi u\|^2$ and hence $d_\phi u = 0$, which means precisely that u is a multiple of $e^{-\phi}$.

For ϕ satisfying (3.8), we also have the important property that $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ have closed range (Theorem 1.7 of [Jo]), i.e.,

$$\begin{aligned} \text{Ran } \Delta_\phi^{(0)} &= \overline{\text{Ran } \Delta_\phi^{(0)}} \\ \text{Ran } \Delta_\phi^{(1)} &= \overline{\text{Ran } \Delta_\phi^{(1)}}. \end{aligned} \tag{3.23}$$

We will need this property to prove the well-posedness of the various Grushin problems in sect. IV.

We next repeat the argument given in [Sj1] to show that the lowest eigenvalue of $\Delta_\phi^{(1)}$ on $L^2(\mathbf{R}^{2|\Lambda|}, \wedge \mathbf{R}^{2|\Lambda|})$ is positive. (For more detailed arguments, see Theorem 7.5 of [Jo] and its proof.) Assume that $u \in \mathcal{D}(\Delta_\phi^{(1)})$ and that $\Delta_\phi^{(1)} u = 0$. Then we have $d_\phi u = 0$, $d_\phi^* u = 0$. Since we are in $\mathbf{R}^{2\Lambda}$, the first equation implies that $e^{-\phi} u$ is an exact form: $e^\phi u = d(e^\phi v)$, i.e., $u = d_\phi v$, where v is given by

$$v(x) = \int_0^x e^{-(\phi(x) - \phi(y))} u(y), \tag{3.24}$$

a line integral. From (3.8), it follows by using weighted Lithner-Agmon estimates as in [HS1] that $u \in \mathcal{S}(\mathbf{R}^{2\Lambda})$ and then using (3.24) and (3.8) once more, that $v \in \mathcal{S}(\mathbf{R}^{2\Lambda})$. Using that $d_\phi^* u = 0$, we obtain $\Delta_\phi^{(0)} v = 0$, so $v = \text{const.}\, e^{-\phi}$. This implies that $u = d_\phi v = 0$.

Remark. The above argument does not require ϕ to be convex. It holds for all Λ which are finite subsets of $\mathbf{Z}^d$. The point is that the lower bound obtained above is *not* uniform in Λ.

The relations (3.17,3.18) yield connections between the spectra of Witten Laplacians on various forms. For instance, if $\lambda \neq 0$ is an eigenvalue for $\Delta_\phi^{(\ell)}$ with eigenform $u \in \mathcal{S}(\mathbf{R}^{2\Lambda}; \wedge^\ell \mathbf{R}^{2\Lambda})$, then $d_\phi u \in \mathcal{S}(\mathbf{R}^{2\Lambda}; \wedge^{\ell+1}\mathbf{R}^{2\Lambda})$ and $d_\phi^* u \in \mathcal{S}(\mathbf{R}^{2\Lambda}; \wedge^{\ell-1}\mathbf{R}^{2\Lambda})$ (if they are non-zero) are eigenforms for $\Delta_\phi^{(\ell+1)}$ and $\Delta_\phi^{(\ell-1)}$ respectively. (When $\lambda = 0$, both $d_\phi u$ and $d_\phi^* u$ are zero, so there is no such straightforward relation between the zero eigenvalue. These are basic properties of a complex, see e.g., chap. 11 of [CFKS].

The following observation, which will be used later in the paper, is a manifestation of the above fundamental fact. Let $\lambda > 0$ be the smallest eigenvalue of $\Delta_\phi^{(1)}$. Then the gap between the first eigenvalue of $\Delta_\phi^{(0)}$ which is zero, and the second one is $\geq \lambda$. This is because if $\mu < \lambda$ is the second eigenvalue for $\Delta_\phi^{(0)}$ with eigenfunction $u \in \mathcal{S}(\mathbf{R}^{2\Lambda}; \mathbf{R})$, then $0 \neq d_\phi u \in \mathcal{S}(\mathbf{R}^{2\Lambda}; \mathbf{R}^{2\Lambda})$ is an eigenform for $\Delta_\phi^{(1)}$ with eigenvalue $\mu < \lambda$, which is in contradiction with the assumption. The Grushin reductions in sect. IV can be seen as using the above argument for higher parts of the spectrum.

We mentioned earlier that in the case of a Gaussian (ϕ a quadratic form), $\Delta_\phi^{(0)}$ is just a $2|\Lambda|$-dimensional harmonic oscillator. This observation can be generalized to note that if $e^{-\phi}$ is the eigenfunction of the Schrödinger operator

$$-\sum \frac{\partial^2}{\partial x_j^2} + V$$

for the lowest eigenvalue λ, then

$$-\sum \frac{\partial^2}{\partial x_j^2} + V - \lambda = \Delta_\phi^{(0)}.$$

From this point of view, the Witten Laplacian can be seen as, in some sense, a natural generalization of harmonic oscillators.

The Witten Laplacian Representation.

We begin by rewriting the averaged Green's function in (3.1) in the new notation as

$$\begin{aligned} \langle G(E,\mu,\nu)\rangle_g &= 2\int x_\mu x_\nu e^{-2\phi(x)} \prod_{i\in\bar\Lambda} \frac{dx_i}{\sqrt{\pi}} \\ &= 2(x_\mu e^{-\phi}, x_\nu e^{-\phi}), \end{aligned} \tag{3.25}$$

where μ, $\nu \in \Lambda \subset \bar\Lambda$, x_μ, x_ν, $x_i \in \mathbf{R}$ and the extra factor of 2 comes from symmetry. We note that due to symmetry $(e^{-\phi}, x_\nu e^{-\phi}) = 0$ for all $\nu \in \Lambda$. We look for a 1-form ω, such that

$$x_\nu e^{-\phi} - (e^{-\phi}, x_\nu e^{-\phi})e^{-\phi} = x_\nu e^{-\phi} = d_\phi^* \omega. \tag{3.26}$$

Further we look for a particular type of 1-form $\omega = d_\phi u$. (3.25) then leads to the following equation for u

$$x_\nu e^{-\phi} = \Delta_\phi^{(0)} u, \tag{3.27}$$

where $u \in \mathcal{D}(\Delta_\phi^{(0)})$. Using (3.27) in (3.25), we obtain

$$\begin{aligned} \langle G(E,\mu,\nu)\rangle_g &= (x_\mu e^{-\phi}, \Delta_\phi^{(0)} u) \\ &= (e^{-\phi} dx_\mu, d_\phi u). \end{aligned} \tag{3.28}$$

Applying d_ϕ to (3.27), we obtain

$$e^{-\phi} dx_\nu = \Delta_\phi^{(1)}(d_\phi u). \tag{3.29}$$

Since $e^{-\phi} dx_\nu \in \mathcal{S}$, by using a partition of unity, weighted L^2 estimates and repeated differentiation of (3.27), it can be shown that $u \in \mathcal{S}$, therefore $d_\phi u \in \mathcal{S} \subset \mathcal{D}(\Delta_\phi^{(1)})$ (see also [Sj1]). Hence

$$d_\phi u = (\Delta_\phi^{(1)})^{-1}(e^{-\phi} dx_\nu). \tag{3.30}$$

Using (3.21) in (3.20), we obtain finally

$$\langle G(E,\mu,\nu)\rangle_g = 2(e^{-\phi} dx_\mu, (\Delta_\phi^{(1)})^{-1}(e^{-\phi} dx_\nu)), \tag{3.31}$$

where x_μ, $x_\nu \in \mathbf{R}$.

Remark. For all finite sets $\Lambda \subset \mathbf{Z}^d$, $\Delta_\phi^{(1)} > 0$ as discussed previously. So (3.31) is well defined. The main work starting in sect. IV is to study the lower part of the spectrum of $\Delta_\phi^{(1)}$ *uniformly* in Λ as $\Lambda \nearrow \mathbf{Z}^d$. Looking for a 1-form ω as in (3.26) is dual to looking for a vector field (for integration by parts) as in [HS2,SW1]. In fact $e^\phi \Delta_\phi^{(1)} e^{-\phi}$ is the vector heat operator obtained there after identifying 1-forms with vector fields.

$\Delta_\phi^{(0)}$ and $\Delta_\phi^{(1)}$ for ϕ an exact quadratic form–the Fock space representation.

When ϕ is a non-degenerate quadratic form, $\Delta_\phi^{(0)}$ is a 2Λ-dimensional harmonic oscillator, as mentioned earlier. Its properties are well-known. However, since most of our insights in the general case come from the quadratic case, below we mention briefly the spectral properties of $\Delta_\phi^{(0)}$ and $\Delta_\phi^{(1)}$ when $\gamma = 0$, i.e., for

$$\phi(x) = \frac{1}{2}(E - \Delta)x \cdot x - \frac{1}{2}\log\det(E - \Delta), \tag{3.32}$$

where $E \notin \sigma(\Delta)$, and for definitiveness we have assumed $E > 0$; the second term in ϕ is just the normalization constant. We will also express $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ in the basis provided by the associated Hermite functions and thus obtain the so called Fock space representations in (3.38),(3.42).

As we will realize later, most of the essential features in the general case are already present in the quadratic case. Since in the latter, ϕ decouples into the sum of two identical terms corresponding to the two copies of $\mathbf{R}$, for simplicity, in the rest of this subsection (and only there) we assume $j \in \Lambda$ instead of $\bar{\Lambda}$.

Substituting (3.23) in (3.12) and (3.14), we have

$$\begin{aligned} \Delta_\phi^{(0)} &= -\sum_{j\in\Lambda} \frac{\partial^2}{\partial x_j^2} + \sum_{j,k\in\Lambda} [(E-\Delta)^2]_{jk} x_j x_k - 2|\Lambda| E \\ \Delta_\phi^{(1)} &= \Delta_\phi^{(0)} \otimes I + 2(E-\Delta), \end{aligned} \tag{3.33}$$

where as earlier, we have identified 1-forms with $\mathbf{R}^\Lambda$ valued functions and we now further identify $\mathbf{R}^\Lambda$ valued functions with $\ell^2(\Lambda)$.

Let ω_i, $(i = 1, \cdots, |\Lambda|)$ be the eigenvalues of $E - \Delta$, in increasing order. Clearly $E - 2d \le \omega_1 < \cdots < \omega_{|\Lambda|} \le E + 2d$. As $\Lambda \nearrow \mathbf{Z}^d$, the equalities are reached and in the limit the spectrum $\sigma(E-\Delta) = [E-2d, E+2d]$. For any finite Λ, the spectrum of $\Delta_\phi^{(0)}$ is discrete. Its eigenvalues are:

$$\sum_{i=1}^{|\Lambda|} 2m_i\omega_i \qquad m_i = \{0, 1\cdots\}.$$

The first spectral gap $= 2\omega_1 > 0$ and approaches $E - 2d > 0$ as $|\Lambda| \to \infty$. The higher gaps generally diminish in size as $|\Lambda|$ increases and the ω_i's approach a continum.

Let v_i be the eigenvector of $(E - \Delta)$ corresponding to the eigenvalue ω_i $(i = 1, \cdots \Lambda)$. Then clearly $e^{-\phi}v_i$ is an eigenform of $\Delta_\phi^{(1)}$ with eigenvalue $2\omega_i$. Let ψ_1 denote the eigenfunction of $\Delta_\phi^{(0)}$ corresponding to the eigenvalue $2\omega_1$. Similarly, we have that $\psi_1 v_i$ (component-wise multiplication) are eigenforms for $\Delta_\phi^{(1)}$ corresponding to eigenvalues $2\omega_1 + 2\omega_i \ge 4\omega_1$ $(i = 0, 1\cdots|\Lambda|)$. Clearly, we can keep on iterating and obtain the whole spectrum of $\Delta_\phi^{(1)}$ this way.

Let Π be the projection onto the subspace in $L^2(\mathbf{R}^\Lambda; \mathbf{R}^\Lambda)$ spanned by $e^{-\phi}v_i$. The feature to notice is that

$$\Delta_\phi^{(1)} \ge 2\omega_1$$
$$\Delta_\phi^{(1)}(1 - \Pi) \ge 4\omega_1.$$

Customarily, $2\omega_1$ is called "the gap" and $4\omega_1$ "the upper gap". Note that the upper gap is twice the gap. As we will see these two features remain in the general case and manifest themselves when we compute the correlation function. In fact in order to compute the asymptotics of the correlation function, it is necessary to have precise control over the spectrum beyond the gap. For example, in our case, to order $\mathbf{O}(\gamma^2)$, it is the upper gap; and to $\mathbf{O}(\gamma^4)$, it is the spectrum beyond the upper gap.

Coming back to the quadratic case, where ϕ is as in (3.23), we know that the Hermite functions provide a basis for $L^2(\mathbf{R}^\Lambda)$. It is instructive to express $\Delta_\phi^{(0)}$ in that basis. Let

$$\zeta_j^* = \sum_k [(2\phi'')^{-1/2}]_{jk} z_k^*. \tag{3.34}$$

The relevant orthonormal set of Hermite functions are:

$$e^{-\phi};\ \zeta_j^* e^{-\phi},\ (j \in \Lambda);\ \zeta_j^* \zeta_k^* e^{-\phi},\ (j,k \in \Lambda, j > k),\ \frac{(\zeta_j^*)^2}{\sqrt{2}} e^{-\phi},\ (j \in \Lambda), \cdots \tag{3.35}$$

For any vector space $\mathcal{E}$, we consider the symmetric tensor product $\mathcal{E} \otimes_s \mathcal{E}$ as a subspace of $\mathcal{E} \otimes \mathcal{E}$. Therefore $\ell^2(\Lambda)^{\otimes_s n}$ is a subspace of $\ell^2(\Lambda)^{\otimes n}$. We further identify $\mathbf{R}e^{-\phi}$ with $\mathbf{R}$, the span of $\{\zeta_j^* e^{-\phi}\}$ with $\mathbf{R}^\Lambda \sim \ell^2(\Lambda)$, the span of$\{\zeta_j^* \zeta_k^* e^{-\phi}$, $(j > k)$, $\frac{(\zeta_j^*)^2}{\sqrt{2}} e^{-\phi}\}$ with $\ell^2(\Lambda) \otimes_s \ell^2(\Lambda)$, $\cdots$ etc., where $\otimes_s$ denotes the symmetric tensor product as $[\zeta_j^*, \zeta_k^*] = 0$ for all j, $k \in \Lambda$.

Let $L^2(\mathbf{R}^\Lambda)_n$ be the span of $\zeta_{j_1}^* \cdots \zeta_{j_n}^* e^{-\phi}$. We have the decomposition (Hilbert direct sum):

$$\begin{aligned} L^2(\mathbf{R}^\Lambda) &= \oplus_{n=0}^\infty L^2(\mathbf{R}^\Lambda)_n \\ &= \oplus_{n=0}^\infty \ell^2(\Lambda)^{\otimes_s n} \\ &= \mathbf{R} \oplus \oplus_{n=1}^\infty \ell^2(\Lambda)^{\otimes_s n}. \end{aligned} \tag{3.36}$$

Since $\Delta_\phi^{(0)}$ leaves each $L^2(\mathbf{R}^\Lambda)_n$ invariant, we write $\Delta_{\phi,n}^{(0)}$ for the restriction. We identify $\Delta_{\phi,n}^{(0)}$ with an element of $\left(\ell^2(\Lambda)^{\otimes_s n}\right)^* \otimes \left(\ell^2(\Lambda)^{\otimes_s n}\right)$. Therefore $\Delta_{\phi,n}^{(0)}$ can be seen as a tensor in $\otimes^{2n} \ell^2(\Lambda)$, (seperately) symmetric in the first n indices and the last n indices.

We have further that

$$\begin{aligned} \Delta_{\phi,0}^{(0)} =& 0 \\ \Delta_{\phi,1}^{(0)} =& E - \Delta \\ \Delta_{\phi,2}^{(0)} =& I \otimes (E - \Delta) + (E - \Delta) \otimes I \\ & \quad \text{restricted to } \ell^2(\lambda) \otimes_s \ell^2(\Lambda) \\ \Delta_{\phi,3}^{(0)} =& I \otimes I \otimes (E - \Delta) + I \otimes (E - \Delta) \otimes I + (E - \Delta) \otimes I \otimes I \\ & \quad \text{restricted to } \ell^2(\lambda) \otimes_s \ell^2(\Lambda) \otimes_s \ell^2(\Lambda) \\ & \quad \vdots \end{aligned} \tag{3.37}$$

We summarize by writing $\Delta_\phi^{(0)}$ as a block diagonal matrix:

$$\Delta_\phi^{(0)} = 2\begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & (E-\Delta) & 0 & \dots \\ 0 & 0 & (E-\Delta)\otimes_s I + I\otimes_s (E-\Delta) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.38}$$

where $(E-\Delta)$, I are once contra, once covariante tensors. In fact, we have the identification

$$(E-\Delta)^i_j = (E-\Delta)_{ij}; \quad I^i_j = I_{ij} = \delta_{ij}; \tag{3.39}$$

$\otimes_s$ (with a slight abuse of notation) denotes the restriction of the tensors to the symmetric subspace.

(3.37) and (3.38) can be proved by direct inspection of the corresponding matrix element, for example

$$\begin{aligned} &((E-\Delta)\otimes_s I + I\otimes_s (E-\Delta))^{ij}_{kl} \\ =&(E-\Delta)^i_k\delta^j_\ell + (E-\Delta)^i_\ell\delta^j_k + \delta^i_k(E-\Delta)^j_\ell \\ &\quad + \delta^i_\ell(E-\Delta)^j_k \qquad (i\neq j, k\neq \ell) \\ =&(E-\Delta)^i_k\delta^j_\ell + \delta^i_\ell(E-\Delta)^j_k \qquad (i\neq j, k=\ell) \\ =&(E-\Delta)^i_k\delta^i_\ell + \delta^i_k(E-\Delta)^i_\ell \qquad (i=j, k\neq\ell) \\ =&(E-\Delta)^i_i \qquad (i=j=k=\ell). \end{aligned} \tag{3.40}$$

By using the identification (3.39), we have the same equality as in (3.38) with all the indices down. (3.38) is just the usual Fock space representation of $\Delta_\phi^{(0)}$. (See e.g., [Be].)

By the same reasoning, $\Delta_\phi^{(1)}$ operates on

$$L^2(\mathbf{R}^\Lambda;\mathbf{R}^\Lambda) \sim \left(\mathbf{R}\oplus\oplus_{n=1}^\infty \ell^2(\Lambda)^{\otimes_s^n}\right)\otimes\ell^2(\Lambda), \tag{3.41}$$

where the tensor product outside of the parenthesis is unsymmetrized. It has the following Fock space representation:

$$\Delta_\phi^{(1)} = 2\begin{pmatrix} (E-\Delta) & 0 & 0 & \dots \\ 0 & \begin{matrix}(E-\Delta)\otimes I \\ +I\otimes(E-\Delta)\end{matrix} & 0 & \dots \\ 0 & 0 & \begin{matrix}[(E-\Delta)\otimes_s I + I\otimes_s(E-\Delta)]\otimes I \\ +(I\otimes_s I)\otimes(E-\Delta)\end{matrix} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.42}$$

We notice also that the Witten Laplacian representation gives in this case

$$\langle G(E,\mu,\nu)\rangle_g = 2(e^{-\phi}dx_\mu, (\Delta_\phi^{(1)})^{-1}(e^{-\phi}dx_\nu)) = (E-\Delta)^{-1}(\mu,\nu), \tag{3.43}$$

which is undoubtedly the correct answer. (cf. (1.1,1.4) with V set to 0.)

We end this section by commenting that for given μ, ν, the sign of $G(E;\mu,\nu)$ is independent of V for $E \notin [-2d, 2d] + \gamma \operatorname{supp} dg$. Hence $\langle G(E;\mu,\nu)\rangle_g$ has statistical significance. This can be seen as follows:

(i) $E > 2d$,

$$G = (E-H)^{-1} = (E - \Delta - \gamma V)^{-1}$$

Since

$$\frac{\partial}{\partial E} G(E,\mu,\nu) = -\sum_k G(\mu,k)G(k,\nu),$$

we see that it is enough to prove $G(E;\mu,\nu) > 0$ for some large enough E. For such E's, we can simply use the resolvent series about $(E-V)^{-1}$ to conclude that $G(E;\mu,\nu) > 0$.

(ii) $E < -2d$,

Let U be the unitary multiplication operator: $(Uf)(i) = (-1)^{|i|_1} f(i)$. Then

$$U^{-1}HU = -\Delta + \gamma V.$$

Hence

$$G(E;\mu,\nu) = (E-H)^{-1}(\mu,\nu) = (-1)^{|\mu|_1 - |\nu|_1}(E - \gamma V + \Delta)^{-1}(\mu,\nu).$$

Now $(E - \gamma V + \Delta)^{-1}(\mu,\nu) < 0$ by using the same argument as before. So for given μ, ν, $G(E;\mu,\nu)$ has a sign which is independent of V.

IV. Grushin Problems and the Spectrum of $\Delta_\phi^{(1)}$

We start the study of $\Delta_\phi^{(1)}$, where ϕ is as defined in (2.9). Recall from (3.8,3.9) that for all $\delta > 0$, $E \in I_\delta$ (I_δ as defined in (1.11)), we have uniformly in Λ,

$$\begin{aligned} &\phi(x) = \mathbf{O}(1+x^2),\ \phi'(x) = \mathbf{O}(1+|x|), \\ &x\cdot\phi'(x) \sim |x|^2, \quad \text{for } |x| >> 1 \quad, \phi^{(n)}(x) = \mathbf{O}_n(1) \text{ for } n \geq 2, \\ &\phi''(x) \geq r_E > 0 \qquad \text{in} \quad \mathcal{L}(\ell^2(\Lambda), \ell^2(\Lambda)), \end{aligned} \tag{4.1}$$

where r_E (depending on δ) is independent of Λ. We recall the expression for $2\phi''$, which first appeared in (3.7) of sect. III, and where we used the point of view of $(\mathbf{R}^2)^\Lambda$:

(4.2)

$$\begin{aligned} (2\phi'')_{i^{(\alpha)}j^{(\beta)}} =& 2E\delta_{ij}\delta_{\alpha\beta} - 2\Delta_{ij}\delta_{\alpha\beta} - 2\gamma k'(\gamma x_i\cdot x_i)\delta_{ij}\delta_{\alpha\beta} - 4\gamma^2 x_i^{(\alpha)} x_i^{(\beta)} k''(\gamma x_i \cdot x_i)\delta_{ij} \\ &+ 2\gamma^2 k''(\gamma x_i\cdot x_i)(E - \Delta - \gamma \operatorname{diag} k')^{-1}_{ii}\delta_{ij}\delta_{\alpha\beta} \\ &+ 4\gamma^3 x_i^{(\alpha)} x_i^{(\beta)} k'''(\gamma x_i\cdot x_i)(E - \Delta - \gamma\operatorname{diag} k')^{-1}_{ii}\delta_{ij} \\ &+ 4\gamma^4 x_i^{(\alpha)} x_j^{(\beta)} k''(\gamma x_i\cdot x_i)k''(\gamma x_j\cdot x_j)[(E-\Delta-\gamma\operatorname{diag} k')^{-1}_{ij}]^2, \end{aligned}$$

where $k = \log \hat{g}$ as in (2.7) and $\hat{g}$ is the Laplace transform of dg; i, $j \in \Lambda$, x_i, $x_j \in \mathbf{R}^2$, α, $\beta \in \{1, 2\}$. Unless specified otherwise, all our estimates below are uniform in Λ.

From (3.31), we see that we are mainly interested in the spectrum of $\Delta_\phi^{(1)}$ around 0. Hence we use the method of Grushin problems to obtain an effective operator (around 0). Similar to the quadratic case, the family of functions: $e^{-\phi}$, $\{z_j^* e^{-\phi}\, j \in \bar{\Lambda}\}$, $\{z_j^* z_k^* e^{-\phi},\ j, k \in \bar{\Lambda},\ j \geq k\} \cdots$ etc., after orthonormalization will play a special role and provide a "basis" for the Grushin problems.

Our main goal is to compute the asymptotics for the averaged Green's function in (2.8) as $|\mu - \nu| \to \infty$. In order to capture the interesting effect of randomness, we need the error term to be of at most $\mathbf{o}(\gamma^2)$. (To $\mathbf{O}(1)$, the asymptotics only reflects $E - \Delta$. The $\mathbf{O}(\gamma)$ term simply corresponds to the shift in energy: $\gamma \langle v \rangle_g$.) In the following we study $\Delta_\phi^{(1)}$ to $\mathbf{O}(\gamma^4)$. However, it will become clear (we hope) that the procedure that we describe below can be iterated to get the asymptotics to all orders.

1. The initial Grushin problem for $\Delta_\phi^{(1)}$ and approximations to $\mathbf{O}(\gamma^2)$.

We begin our study of the spectrum $\Delta_\phi^{(1)}$ around 0. From (3.21,3.19,4.1) we have, a priori, for all u,

$$(\Delta_\phi^{(1)} u, u) \geq 2r_0 ||u||^2 \tag{4.3}$$

for some $r_0 \geq r_E > 0$. Hence the second eigenvalue (spectral gap) of $\Delta_\phi^{(0)} \geq r_0 > 0$. (Recall that we are working with a complex). Therefore if $u \perp e^{-\phi} dx_j$ for all $j \in \bar{\Lambda}$, then

$$(u, \Delta_\phi^{(1)} u) \geq 2(r_0 + r_E) \|u\|. \tag{4.4}$$

As we will see by the end of the section in order that the error in the decay rate in the asymptotics of $\langle G_\Lambda(E, \mu, \nu) \rangle$ as $|\mu - \nu| \to \infty$ is of $\mathbf{o}(\gamma^2)$, we need to use not only $e^{-\phi}$, but also $\{z_j^* e^{-\phi},\ j \in \bar{\Lambda}\}$, $\{z_j^* z_k^* e^{-\phi},\ j, k \in \bar{\Lambda},\ j \geq k\}$ (properly orthonormalized) as our Grushin basis. In order to get a more optimal ("correct") lower bound than in (4.4) for u in the corresponding orthogonal space, we need to do a preliminary study of $\Delta_\phi^{(1)}$ around 0. To that end, we pose our first Grushin problem.

Let $e^{-\phi} dx_j$ (all $j \in \bar{\Lambda}$) be our initial Grushin basis. We introduce the projection operator $R_+ : L^2(\mathbf{R}^{2\Lambda}; \wedge \mathbf{R}^{2\Lambda}) \to \ell^2(\bar{\Lambda})$ by

$$R_+ u(j) = (u, e^{-\phi} dx_j) = (u_j, e^{-\phi}), \quad j \in \bar{\Lambda}, \tag{4.5}$$

where $u = \sum_{j \in \bar{\Lambda}} u_j dx_j$. Let $R_- = R_+^t : \ell^2(\bar{\Lambda}) \to L^2(\mathbf{R}^{2\Lambda}; \wedge \mathbf{R}^{2\Lambda})$:

$$R_- u_- = \sum_j e^{-\phi} u_-(j) dx_j. \tag{4.6}$$

Note that $R_+R_- = I$. We pose our initial Grushin problem :

$$
\begin{pmatrix} \Delta_\phi^{(1)} - z & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{4.7}
$$

where $u \in \mathcal{D}(\Delta_\phi^{(1)})$, $v \in L^2$, $u_-, v_+ \in \ell^2$. $\begin{pmatrix} v \\ v_+ \end{pmatrix}$ is the given and $\begin{pmatrix} u \\ u_- \end{pmatrix}$ is the unknown.

Using (4.2), we have the following proposition,

Proposition 4.1. *For every $C > 0$, there exists $\gamma_0 > 0$ such that the problem (4.4) is well posed for all $z \leq 2\big(r_0 + r_E - \frac{1}{C}\big)$, $0 \leq \gamma \leq \gamma_0$. Moreover*

$$
\|u\| + \|u_-\| \leq \mathbf{O}(1)(\|v\| + \|v_+\|). \tag{4.8}
$$

The proof of the above proposition (and indeed the study of $\Delta_\phi^{(1)}$ to $\mathbf{O}(\gamma^2)$, see Proposition 4.3) depends on the following fundamental fact.

Lemma 4.2. *Let $w \in \ell^2$, R_- be as defined in (4.4). Then*

$$
\|(\phi'' - \langle\phi''\rangle)R_- w\| \leq \mathbf{O}(\gamma^2)\|w\|. \tag{4.9}
$$

Proof.

$$
\|(\phi'' - \langle\phi''\rangle)R_- w\|^2 = \sum_{ijk} \Big((\phi''_{ik} - \langle\phi''_{ik}\rangle)e^{-\phi} w_k, (\phi''_{ij} - \langle\phi''_{ij}\rangle)e^{-\phi} w_j \Big).
$$

We now write

$$
\phi''_{ik} e^{-\phi} - \langle\phi''_{ik}\rangle e^{-\phi} = \Delta_\phi^{(0)} \tilde{u}
$$

for some $\tilde{u}$. Proceeding exactly as in (3.17), (3.22) in sect. III, we obtain

$$
\begin{aligned}
\text{RHS} &= \sum_{ijk} \Big(e^{-\phi} d\phi''_{ik}, (\Delta_\phi^{(1)})^{-1} e^{-\phi} d\phi''_{ij} \Big) w_j w_k \\
&= \mathbf{O}(1) \sum_{j,k} w_j w_k \sum_i \|e^{-\phi} d\phi''_{ik}\| \ \|e^{-\phi} d\phi''_{ij}\|,
\end{aligned} \tag{4.10}
$$

where we used (4.3) to bound $\|(\Delta_\phi^{(1)})^{-1}\|$.

To estimate the RHS of (4.10), we need to estimate $\langle(\phi''')^2\rangle$. For this purpose, it is convenient to revert back to the point of view of $(\mathbf{R}^2)^\Lambda$ and use supersymmetry. (We will have more occasions to do such computations in sect. VII, where we need more precise results.) Hence elements of $\bar{\Lambda}$ will be denoted by $i^{(\alpha)}$, $i \in \Lambda$, $\alpha \in \{1, 2\}$ and $x_i \in \mathbf{R}^2$. (Those readers who are not particularly interested in the details of this estimate could go directly to the end result (4.21) and continue with the reading of the proof.)

From (4.2), there are 7 terms in $2\phi''$, which we denote by a_n, $n = 1 \cdots 7$. So

$$
\begin{aligned}
\langle (\phi''')^2 \rangle = \langle (\sum_{n=1}^{7} a_n')^2 \rangle &= \sum_{n=3}^{7} \langle (a_n')^2 \rangle + 2 \sum_{n>m>2} \langle a_n' a_m' \rangle \\
&\le 5 \sum_{n=3}^{7} (a_n')^2.
\end{aligned} \tag{4.11}
$$

as a_1, a_2 are constants. The estimates on $\langle (\phi''')^2 \rangle$, and indeed $\langle (\phi^{(m)})^2 \rangle$ for all $m \ge 3$ rely on the following estimate:

$$
\int x_i^{2p} x_j^{2q} e^{-2\phi} = \mathbf{O}_{p,q}(1), \qquad p,\, q = 0,\, 1 \cdots \tag{4.12}
$$

where $\mathbf{O}_{p,q}(1)$ only depends on p and q and is uniform in Λ. Clearly () holds when $p = q = 0$. Assume $p,\, q \ge 1$. (The argument works the same way if either p or $q = 0$.) () can be easily seen by using Proposition 2.2 as follows. Since

$$
\int X_i^{2p} X_j^{2q} e^{-2\Phi} = 0, \tag{4.13}
$$

where $X_i^2 \overset{\text{def}}{=} X_i \cdot X_i = x_i \cdot x_i + \eta_i \xi_i$ as in (2.22) and Φ is as defined in (2.25),

$$
\begin{aligned}
\text{LHS} &= \int (x_i \cdot x_i + \eta_i \xi_i)^p (x_j \cdot x_j + \eta_j \xi_j)^q e^{-2\Phi} \\
&= 0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int x_i^{2p} x_j^{2q} e^{-2\Phi} = &- 2q \int x_i^{2p} x_j^{2(q-1)} \eta_j \xi_j e^{-2\Phi} \\
&- 2p \int x_i^{2(p-1)} x_j^{2q} \eta_i \xi_i e^{-2\Phi} \\
&- 4pq \int x_i^{2(p-1)} x_j^{2(q-1)} \eta_i \xi_i \eta_j \xi_j e^{-2\Phi}.
\end{aligned} \tag{4.14}
$$

Let $M = E - \Delta - \gamma \operatorname{diag} k'$. Integrating over the antisymmetric variables ξ, η using (2.14,2.15), we have

$$
\begin{aligned}
\int e^{-2\Phi} \prod_{\ell \in \Lambda} d\eta_\ell d\xi_\ell &= e^{-2\phi} \\
\int \eta_i \xi_i e^{-2\Phi} \prod_{\ell \in \Lambda} d\eta_\ell d\xi_\ell &= M_{ii}^{-1} e^{-2\phi} \\
\int \eta_j \xi_j e^{-2\Phi} \prod_{\ell \in \Lambda} d\eta_\ell d\xi_\ell &= M_{jj}^{-1} e^{-2\phi} \\
\int \eta_i \xi_i \eta_j \xi_j e^{-2\phi} \prod_{\ell \in \Lambda} d\eta_\ell d\xi_\ell &= M_{ii}^{-1} (M^{(ii)})_{jj}^{-1} e^{-2\phi},
\end{aligned} \tag{4.15}
$$

where $M^{(ii)}$ is the matrix obtained from M after deleting the i^{th} row and the i^{th} column. Using (4.15) in (4.14), we are left with integrals over $\mathbf{R}^{2\Lambda}$ only and (4.15) becomes

(4.16)
$$\begin{aligned}\int x_i^{2p}x_j^{2q}e^{-2\phi}\prod_{\ell\in\Lambda}d^2x_\ell =&-2q\int M_{jj}^{-1}x_i^{2p}x_j^{2(q-1)}e^{-2\phi}\prod_{\ell\in\Lambda}d^2x_\ell\\&-2p\int M_{ii}^{-1}x_i^{2(p-1)}x_j^{2q}e^{-2\phi}\prod_{\ell\in\Lambda}d^2x_\ell\\&-4pq\int M_{ii}^{-1}(M^{(ii)})_{jj}^{-1}x_i^{2(p-1)}x_j^{2(q-1)}e^{-2\phi}\prod_{\ell\in\Lambda}d^2x_\ell.\end{aligned}$$

Since M_{ii}^{-1}, $(M^{(ii)})_{jj}^{-1}=\mathbf{O}(1)$, we obtain

$$\begin{aligned}|\int x_i^{2p}x_j^{2q}e^{-2\phi}\prod_{\ell\in\Lambda}d^2x_\ell|\leq \mathbf{O}(1)[&\int x_i^{2p}x_j^{2(q-1)}e^{-2\phi}\prod_{\ell\in\Lambda}d^2x_\ell\\&+\int x_i^{2(p-1)}x_j^{2q}e^{-2\phi}\prod_{\ell\in\Lambda}d^2x_\ell\\&+\int x_i^{2(p-1)}x_j^{2(q-1)}e^{-2\phi}\prod_{\ell\in\Lambda}d^2x_\ell].\end{aligned}\tag{4.17}$$

We note that the integrand in the RHS of (4.17) is of degree $\leq 2(p+q-1)$ in x, while that of the LHS is of $2(p+q)$. Clearly the above procedure (4.14)-(4.17) can be repeated on the integrals in the RHS of (4.17) until all the integrands become of degree 0, i.e., constants. We therefore obtain (4.12). (It is also clear that we have similar estimates, when the integrand is of the form $\prod_{i=1}^n x_i^{2p_i}$, $n\in\mathbf{N}$, although we do not need it here for $n>2$.)

The other ingredient that we need to estimate $\langle(\phi''')^2\rangle$ is the following equality obtained from a straight forward computation:

(4.18)
$$\begin{aligned}&\frac{\partial}{\partial x_\ell^{(\alpha)}}(E-\Delta-\gamma\,\mathrm{diag}\,k')_{ij}^{-1}\\=&2(E-\Delta-\gamma\,\mathrm{diag}\,k')_{i\ell}^{-1}\gamma^2x_\ell^{(\alpha)}k''(\gamma x_\ell\cdot x_\ell)(E-\Delta-\gamma\,\mathrm{diag}\,k')_{\ell j}^{-1},\qquad(\alpha=1,2).\end{aligned}$$

Using (4.2), Lemma 3.1, the bound $\|(E-\Delta-\gamma\,\mathrm{diag}\,k')^{-1}\|_{\ell^2}=\mathbf{O}(1)$, we see that pointwise (before integration)

$$|a_n'|\leq\mathbf{O}(\gamma^2)$$

for $n=5$, 6, 7. So we only need to estimate $\langle(a_n')^2\rangle$ for $n=3$, 4. Since

$$\frac{\partial a_3}{\partial x_i^{(\alpha)}}=-4\gamma^2x_i^{(\alpha)}k''(\gamma x_i\cdot x_i),$$

$$\begin{aligned}(\frac{\partial a_3}{\partial x_i^{(\alpha)}})^2&=16\gamma^4(x_i^{(\alpha)})^2(k''(\gamma x_i\cdot x_i))^2\\&\leq 16\gamma^4(x_i\cdot x_i)(k''(\gamma x_i\cdot x_i))^2.\end{aligned}$$

So

$$\langle (\frac{\partial a_3}{\partial x_i^{(\alpha)}})^2 \rangle \leq \mathbf{O}(\gamma^4),$$

by using Lemma 3.1 and the estimate in (). Similarly we obtain

$$\langle (\frac{\partial a_4}{\partial x_i^{(\alpha)}})^2 \rangle \leq \mathbf{O}(\gamma^4).$$

So

$$\langle (\phi''')^2 \rangle = \mathbf{O}(\gamma^4). \tag{4.19}$$

Let π: $\bar{\Lambda} = \Lambda \times \{1,2\} \to \Lambda$. Since $(E - \Delta - \gamma \operatorname{diag} k')^{-1}$ and also its various derivatives are pointwise uniformly bounded operators in appropriate exponentially weighted space for $E \in I_\delta$ (I_δ defined in (1.13)), by using (4.18) and (3.5), it is easy to see that (4.19) generalizes to

$$\langle (\phi'''_{ijk})^2 \rangle = \mathbf{O}(\gamma^4) e^{-\tilde{\alpha} d(i',j',k')}, \tag{4.20}$$

for some $\tilde{\alpha} > 0$, where

$$i' = \pi i \in \Lambda, \qquad j' = \pi j \in \Lambda, \qquad k' = \pi k \in \Lambda,$$

and $d(i', j', k')$ is the lengh of the shortest path connecting i', j' and k'. We obtain

$$\| e^{-\phi} d\phi''_{ik} \| = \mathbf{O}(\gamma^2) e^{-\alpha|i'-k'|} \tag{4.11}$$

for some $\alpha > 0$ uniform in Λ, where

$$\begin{aligned} i' &= \pi i \in \Lambda \\ k' &= \pi k \in \Lambda. \end{aligned}$$

Note that by using supersymmetry we gained $\mathbf{O}(\gamma^{1/2})$ in (4.21); the naive estimate would have been $\mathbf{O}(\gamma^{3/2})$.

Using (4.21) in (4.10), we obtain

$$\|(\phi'' - \langle \phi'' \rangle) R_- w\|^2 = \mathbf{O}(\gamma^4) \sum_{j,k} w_j w_k e^{-\alpha|j-k|} = \mathbf{O}(\gamma^4) \|w\|^2.$$

Taking the square root on both sides, we obtain the lemma. □

Remark. Lemma 4.2 should be seen as a statement of the "small" variation of ϕ'', and indeed any function f whose derivative f is "small". We will see more general versions of (4.9) later in Lemma 4.8.

Proof of Proposition 4.1. Let

$$\mathcal{P}(z) \stackrel{\text{def}}{=} \begin{pmatrix} \Delta_\phi^{(1)} - z & R_- \\ R_+ & 0 \end{pmatrix}. \tag{4.22}$$

It is a self-adjoint operator with domain $\mathcal{D}(\Delta_\phi^{(1)}) \times \ell^2$. Hence it is enough to establish (4.8). We write out (4.8) separately :

$$\begin{cases} (\Delta_\phi^{(1)} - z)u + R_- u_- = v \\ \qquad\qquad\qquad R_+ u = v_+. \end{cases} \tag{4.23}$$

We first look at the special case $v_+ = 0$. (Later we reduce the general case to the special case).

Taking the inner product with u in the first equation in (4.23), we obtain

$$\big(u, (\Delta_\phi^{(1)} - z)u\big) + (u, R_- u_-) = (u, v). \tag{4.24}$$

Using the second equation in (4.23) with $v_+ = 0$ and the fact that $R_-^* = R_+$, we see that $(u, R_- u_-) = 0$ and we have

$$\big(u, (\Delta_\phi^{(1)} - z)u\big) = (u, v).$$

So $\frac{1}{\mathbf{O}(1)} \|u\|^2 \leq \big(u, (\Delta_\phi^{(1)} - z)u\big) = (u, v) \leq \|u\| \ \|v\|$ where we used (4.4) and the assumption on z in the Proposition. Hence

$$\|u\| \leq \mathbf{O}(1) \|v\|. \tag{4.25}$$

We now take the inner product with $R_- u_-$ in the first equation in (4.23) :

$$\big(R_- u_-, (\Delta_\phi^{(1)} - z)u\big) + (R_- u_-, R_- u_-) = (R_- u_-, v). \tag{4.26}$$

Since $(\Delta_\phi^{(0)} \otimes I) R_- u_- = 0$, $\|(\phi'' - \langle \phi'' \rangle) R_- u_-\| = \mathbf{O}(\gamma^2) \|u_-\|$ by using Lemma 4.2, we have

$$\begin{aligned} \big(R_- u_-, (\Delta_\phi^{(1)} - z)u\big) &= (\langle \phi'' \rangle R_- u_-, u) + \mathbf{O}(\gamma^2) \|u_-\| \ \|u\| \\ &= \mathbf{O}(\gamma^2) \|u_-\| \ \|u\| \end{aligned} \tag{4.27}$$

where we used the fact that $R_+ \langle \phi'' \rangle u = \langle \phi'' \rangle R_+ u = 0$. Hence (4.24) gives

$$\|u_-\|^2 - \mathbf{O}(\gamma^2) \|u_-\| \ \|u\| \leq \|u_-\| \ \|v\|$$

where we used $\|R_- u_-\|^2 = \|u_-\|^2$. So

$$\begin{aligned} \|u_-\| &\leq \|v\| + \mathcal{O}(\gamma^2) \|u\| \\ &= \mathbf{O}(1) \|v\| \end{aligned} \tag{4.28}$$

We now consider (4.23) in general. Since $R_+ R_- = I$, we obtain

$$\begin{aligned} (\Delta_\phi^{(1)} - z)(u - R_- v_+) + R_- u_- &= v + (z - \phi'') R_- v_+ \\ R_+(u - R_- v_+) &= 0 \end{aligned} \tag{4.29}$$

which is of the type (4.23) with $v_+ = 0$.

Applying the estimates (4.23) and (4.26) to (4.27), we have

$$\begin{aligned}\|u - R_- v_+\| + \|u_-\| &= \mathbf{O}(1)(\|v + (Z - \phi'')R_- v_+\|) \\ &= \mathbf{O}(1)(\|v\| + \|v_+\|).\end{aligned}$$

Since

$$\begin{aligned}\|u - R_- v_+\| &\geq \|u\| - \|R_- v_+\| \\ &\geq \|u\| - \|v_+\|,\end{aligned}$$

we obtain

$$\|u\| + \|u_-\| = \mathbf{O}(1)(\|v\| + \|v_+\|). \tag{4.30}$$

$\square$

The above proposition shows that the self-adjoint Grushin operator

$$\mathcal{P} = \begin{pmatrix} \Delta_\phi^{(1)} - z & R_- \\ R_+ & 0 \end{pmatrix}$$

introduced in (4.12) is invertible for such z, γ. Let $\mathcal{E}$ denote the inverse of $\mathcal{P}$:

$$\mathcal{E} \overset{\text{def}}{=} \mathcal{P}^{-1} \overset{\text{def}}{=} \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \tag{4.31}$$

where $E_- = E_+^*$, $\mathcal{E} \in \mathcal{L}\big(L^2(\mathbf{R}^{2\Lambda}, \wedge\mathbf{R}^{2\Lambda}); L^2(\mathbf{R}^{2\Lambda}, \wedge\mathbf{R}^{2\Lambda})\big)$.

We then have the following identity :

$$(\Delta_\phi^{(1)} - z)^{-1} = E - E_+ E_{-+}^{-1} E_-, \tag{4.32}$$

where E, E_+, E_- and E_{-+} are analytic in z. Therefore (4.32) implies that

$$z \in \sigma(\Delta_\phi^{(1)}) \ \text{ iff } \ 0 \in \sigma(E_{-+}).$$

E_{-+} is our effective operator.

Using Lemma 4.2, we have moreover the following approximations for E_{-+}, E_+, E_- :

Proposition 4.3. *For every $C > 0$, there exists $\gamma_0 > 0$, such that for all $z \leq 2\big(r_0 + r_E - \frac{1}{C}\big)$, $|z| = \mathbf{O}(1)$, all $0 \leq \gamma \leq \gamma_0$,*

$$\begin{aligned}&\|E_{-+} - (z - 2\langle\phi''\rangle)\|_{\mathcal{L}(\ell^2, \ell^2)} = \mathbf{O}(\gamma^2), \\ &\|E_+ - R_-\|_{\mathcal{L}(\ell^2, L^2)} = \mathbf{O}(\gamma^2), \\ &\|E_- - R_+\|_{\mathcal{L}(L^2, \ell^2)} = \mathbf{O}(\gamma^2).\end{aligned} \tag{4.33}$$

Proof. From (4.31)

$$u = Ev + E_+ v_+ \\ u_- = E_- v + E_{-+} v_+. \tag{4.34}$$

Take $v = 0$ and let

$$E_+^0 = R_- \\ E_-^0 = R_+ \\ E_{-+}^0 = z - 2\langle \phi'' \rangle. \tag{4.35}$$

Take as approximate solutions :

$$u^0 = E_+^0 v_+ = R_- v_+ \\ u_-^0 = E_{-+}^0 v_+ = (z - 2\langle \phi'' \rangle) v_+. \tag{4.36}$$

We then have

$$\begin{aligned}(\Delta_\phi^{(1)} - z)(u^0 - u) + R_-(u_-^0 - u_-) &= (\Delta_\phi^{(1)} - z) R_- v_+ + R_-(z - 2\langle \phi'' \rangle) v_+ \\ &= 2(\phi'' - \langle \phi'' \rangle) R_- v_+,\end{aligned} \tag{4.37}$$

where we used the fact that $v = 0$, and

$$R_+(u^0 - u) = R_+(R_- v_+ - u) = 0. \tag{4.38}$$

Applying Proposition 4.1 to the system of equations comprised of (4.37) and (4.38), we have

$$\begin{aligned}\|u^0 - u\| + \|u_-^0 - u_-\| &= \mathbf{O}(1)\|(\phi'' - \langle \phi'' \rangle) R_- v_+\| \\ &= \mathbf{O}(\gamma^2)\|v_+\|\end{aligned} \tag{4.39}$$

where we used Lemma 4.2 to obtain the last inequality. Combining (4.39) with (4.34) and (4.36), we then obtain the Lemma. □

2. A first Grushin problem for $\Delta_\phi^{(0)}$.

As mentioned earlier, in order to obtain an accuracy which is of $\mathbf{O}(\gamma^2)$, we need to also use (a side from $e^{-\phi} dx_j$) $z_k^* e^{-\phi} dx_j$, $z_k^* z_\ell^* e^{-\phi} dx_j$ $(k \geq \ell)$ (after orthonormalization) as our Grushin basis. We do this in two steps. We first add $z_k^* e^{-\phi} dx_j$ to our basis. In order to obtain the correct analogue of the lower bound in (4.4) for $u \perp e^{-\phi} dx_j$, $z_k^* e^{-\phi} dx_j$, for all j, $k \in \bar{\Lambda}$, we need to study more in depth the spectrum of $\Delta_\phi^{(0)}$. As before, we shall exploit in an essential way the relation between the spectra of $\Delta_\phi^{(0)}$ and $\Delta_\phi^{(1)}$.

We start by posing a variant of the Grushin problem in (4.7) :

$$\begin{pmatrix} \Delta_\phi^{(1)} - z & d_\phi d_\phi^* \langle 2\phi'' \rangle^{-1} R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{4.40}$$

where $\langle 2\phi''\rangle^{-1}$ means the inverse of the matrix $\langle 2\phi''\rangle$ (Recall also that we have identified 1-forms with $\mathbf{R}^{\bar{\Lambda}}$ valued functions.); $u \in \mathcal{D}(\Delta_\phi^{(1)})$, $v \in L^2$, $u_-, v_+ \in \ell^2$; R_+, R_- are as defined before in (4.5,4.6). Let $\mathcal{P}$ be the Grushin operator defined in the LHS of (4.40). The interest of the above Grushin problem is that once its well posedness is established, it implies the well posedness of a related Grushin problem for $\Delta_\phi^{(0)}$ (Proposition 4.5), which in turn gives us the desired spectral information on $\Delta_\phi^{(0)}$.

Proposition 4.4. *For every $C > 0$, there exists $\gamma_0 > 0$ such that the problem (4.40) is well posed for all $z \leq 2\big(r_0 + r_E - \frac{1}{C}\big)$, $0 \leq \gamma \leq \gamma_0$. Moreover*

$$\|u\| + \|u_-\| \leq \mathbf{O}(1)(\|v\| + \|v_+\|). \tag{4.41}$$

Proof. We prove the proposition by reference to the well-posedness of the Grushin problem in Proposition 4.1.

(4.40) can be written as

$$\begin{cases} (\Delta_\phi^{(1)} - z)u + R_-u_- = v + (1 - d_\phi d_\phi^* \langle 2\phi''\rangle^{-1})R_-u_- \\ \qquad\qquad R_+u = v_+. \end{cases} \tag{4.42}$$

From Proposition 4.1, we have

$$\|u\| + \|u_-\| \leq \mathbf{O}(1)\big(\|v\| + \|(1 - d_\phi d_\phi^* \langle 2\phi''\rangle^{-1})R_-u_-\| + \|v_+\|\big). \tag{4.43}$$

Since

$$\begin{aligned} d_\phi d_\phi^* \langle 2\phi''\rangle^{-1} R_-u_- &= d_\phi^* d_\phi \langle 2\phi''\rangle^{-1} R_-u_- + [d_\phi, d_\phi^*]\langle 2\phi''\rangle^{-1} R_-u_- \\ &= 2\phi'' \circ \langle 2\phi''\rangle^{-1} R_-u_-, \end{aligned} \tag{4.44}$$

the second term in the RHS of (4.43) can be written as

$$\begin{aligned} \|(1 - d_\phi d_\phi^* \langle 2\phi''\rangle^{-1})R_-u_-\| &= \|(\langle 2\phi''\rangle - 2\phi'')\langle 2\phi''\rangle^{-1} R_-u_-\| \\ &\leq \mathbf{O}(\gamma^2)\|\langle 2\phi''\rangle^{-1} u_-\| \\ &\leq \mathbf{O}(\gamma^2)\|u_-\| \end{aligned} \tag{4.45}$$

where we used Lemma 4.2 to reach the first inequality and (4.1) to reach the last inequality. Using (4.45) in (4.43), we have

$$\|u\| + \|u_-\| \leq \mathbf{O}(1)(\|v\| + \|v_+\|) + \mathbf{O}(\gamma^2)\|u_-\|.$$

So

$$\|u\| + \|u_-\| \leq \mathbf{O}(1)(\|u\| + \|v_+\|)$$

for $z \leq 2\big(r_0 + r_E - \frac{1}{C}\big)$ and small enough γ. This proves uniqueness, i.e.,

$$\mathcal{P}U = 0 \implies U = 0.$$

By a similar argument we prove

$$\mathcal{P}^*U = 0 \implies U = 0. \tag{4.46}$$

Using (4.5), $\mathrm{Ran}\,(R_+) \sim \ell^2(\bar\Lambda) \sim \mathbf{C}^{2\Lambda}$, so

$$\begin{aligned}\big(\mathrm{Ran}\,(\Delta_\phi^{(1)}) \oplus \{0\}\big) \times \ell^2(\bar\Lambda) \subseteq \mathrm{Ran}\,(\mathcal{P})& \\ &\subseteq \big(\mathrm{Ran}\,(\Delta_\phi^{(1)}) \oplus \mathrm{Ran}\,(d_\phi d_\phi^*\langle 2\phi''\rangle^{-1})R_-\big) \times \ell^2(\bar\Lambda)\\ &\subset L^2(\mathbf{R}^{2\Lambda}, \mathbf{C}^{2\Lambda}) \times \ell^2(\bar\Lambda).\end{aligned}$$

From (4.44), we see that $\mathrm{Ran}\,(d_\phi d_\phi^*\langle 2\phi''\rangle^{-1})R_-$ is of finite dimension. Using (4.1), we see that the hypothsis of Theorem 1.7 of [Jo] are satisfied, so $\Delta_\phi^{(1)}$ has closed range:

$$\mathrm{Ran}\,(\Delta_\phi^{(1)}) = \overline{\mathrm{Ran}\,(\Delta_\phi^{(1)})}.$$

Combinning the above arguments, we obtain that

$$\mathrm{Ran}\,(\mathcal{P}) = \overline{\mathrm{Ran}\,(\mathcal{P})} \qquad \text{in}\, L^2 \times \ell^2.$$

From Theorem 3.1 of [Jo], we then obtain that

$$\begin{aligned}(\mathrm{Ran}\,\mathcal{P})^\perp &= \mathrm{Ker}\,(\mathcal{P}^*)\\ &= \emptyset,\end{aligned}$$

where we also used (4.46). So $\mathcal{P}$ is onto. Therefore we have existence of solutions to (4.40). $\square$

We now come to the main reason why we posed (4.40) in the first place, which is to use (4.40) to establish the well-posedness of the Grushin problem for $\Delta_\phi^{(0)}$ in the following Proposition 4.5.

Let

$$e_0 = e^{-\phi}$$

$$\begin{pmatrix} e_1 \\ \vdots \\ e_{2|\Lambda|}\end{pmatrix} = (\langle 2\phi''\rangle)^{-1/2}\begin{pmatrix} z_1^* e^{-\phi} \\ \vdots \\ z_{2|\Lambda|}^* e^{-\phi}\end{pmatrix} \tag{4.47}$$

be the new Grushin basis. Note that $(e_j, e_k) = \delta_{jk}$, $j, k \in \{0\} \cup \bar\Lambda$. Then we have

Proposition 4.5. *For every $C > 0$, there exists $\gamma_0 > 0$ such that the following Grushin problem for $\Delta_\phi^{(0)}$:*

$$\begin{cases} (\Delta_\phi^{(0)} - z)u + \sum_{j=0}^{2\Lambda} u_-(j)e_j = v \\ (u, e_j) = v_+(j), \;\; j \in \{0\} \cup \bar\Lambda, \end{cases} \tag{4.48}$$

where $u \in \mathcal{D}(\Delta_\phi^0)$, $v \in L^2(\mathbf{R}^{2\Lambda})$, u_-, $v_+ \in \big(\mathbf{R} \oplus \ell^2(\bar\Lambda)\big)$ is well posed for all $z \le 2\big(r_0 + r_E - \frac{1}{C}\big)$, $0 \le \gamma \le \gamma_0$. Moreover

$$\|u\| + \|u_-\| \le \mathbf{O}(1)(\|v\| + \|v_+\|). \tag{4.49}$$

To prove Proposition 4.5, we specialize to the case where v is a closed form : $d_\phi v = 0$, $v \in \mathcal{S}$ in (4.40). We first need to prove :

Lemma 4.6. *For C, γ_0, z such that Proposition 4.4 holds, if $v \in \mathcal{S}$ is a closed form, i.e., $d_\phi v = 0$, then so is $u : d_\phi u = 0$.*

Proof. Applying d_ϕ to the first equation in (4.40), we obtain

$$(\Delta_\phi^{(2)} - z)d_\phi u = 0. \tag{4.50}$$

Since $v \in \mathcal{S}$, from (4.40), using an argument similar to that used in sect. III leading to (3.30), we see that $d_\phi u \in \mathcal{S} \subset \mathcal{D}(\Delta_\phi^{(2)})$. Since $(\Delta_\phi^{(2)} - z) > c > 0$ for z considered in Proposition 4.4, by using (4.1,3.22), we conclude that

$$d_\phi u = 0.$$

□

Proof of Proposition 4.5. Let $v \in \mathcal{S}$ be a closed 1−form in (4.40). Since $d_\phi \overset{\text{def}}{=} e^{-\phi} d e^{\phi}$ and we work in $\mathbf{R}^{2\Lambda}$, closed forms are exact. Therefore if $d_\phi v = 0$, then $v = d_\phi \tilde{v}$. for some $\tilde{v}$. To show that $\tilde{v} \in \mathcal{S}$, we proceed as follows. Write

$$v = \Delta_\phi^{(1)} w \tag{4.51}$$

for some $w \in \mathcal{S}$. Applying d_ϕ to (4.51), we obtain

$$d_\phi v = \Delta_\phi^{(2)} d_\phi w = 0 \tag{4.52}$$

as v is assumed to be closed. Since $\Delta_\phi^{(2)} > 0$, (4.52) implies that $d_\phi w = 0$. Using this in (4.51), we have that $v = d_\phi(d_\phi^* w)$. So $\tilde{v} = d_\phi^* w \in \mathcal{S}$. Lemma 4.6 then implies that $u = d_\phi \tilde{u}$ for some $\tilde{u} \in \mathcal{S}$. Note there is no uniqueness for $\tilde{u}$, if $\tilde{u}$ is a solution, then so is $\tilde{u} + Ce^{-\phi}$, for all C. Therefore in order to specify $\tilde{u}$ uniquely, we need to specify C.

Let $\tilde{v}$ be a given function in $\mathcal{S}(\mathbf{R}^{2\Lambda})$, v_+ a given vector in $\mathbf{R} \oplus \ell^2(\bar{\Lambda})$. We consider the Grushin problem as in (4.40) with $v = d_\phi \tilde{v}$. Let w be a 1−form $\in \mathcal{S}$. Taking the scalar product of the first equation in (4.40) with w, we obtain

$$(w, (\Delta_\phi^{(1)} - z)d_\phi \tilde{u}) + (w, d_\phi d_\phi^* \langle 2\phi'' \rangle^{-1} R_- u_-) = (w, d_\phi \tilde{v}). \tag{4.53}$$

Note that there is an ambiguity when we write the solution u as $d_\phi \tilde{u}$, which we clarify by imposing the condition

$$R_+ \tilde{u} = (e^{-\phi}, \tilde{u}) = v_+(0). \tag{4.54}$$

We have from (4.53)

$$(\Delta_\phi^{(1)} - z)d_\phi \tilde{u} + d_\phi d_\phi^* \langle 2\phi'' \rangle^{-1} R_- u_- = d_\phi \tilde{v}. \tag{4.55}$$

Since the kernel of d_ϕ, $\operatorname{Ker} d_\phi$ is $\mathbf{R}e^{-\phi}$, we obtain as an immediate consequence the following equation

$$(\Delta_\phi^{(0)} - z)\tilde{u} + d_\phi^* \langle 2\phi'' \rangle^{-1} R_- u_- + R_- u_-(0) = \tilde{v}. \tag{4.56}$$

The second equation in (4.40) gives

$$R_+ d_\phi \tilde{u} = v_+. \tag{4.57}$$

The well-posedness of the Grushin problem (4.40), then implies the well posedness of the Grushin problem composed of (4.56), (4.57) and (4.54). Let e_0 denote the function $e^{-\phi}$, then

$$(d_\phi^* \langle 2\phi'' \rangle^{-1}) R_- u_- + R_- u_-(0) = \sum_{j,k} \langle 2\phi'' \rangle_{jk}^{-1} z_k^* e_0 u_-(j) + e_0 u_-(0). \tag{4.58}$$

Define $\begin{pmatrix} e_1 \\ \vdots \\ e_{2|\Lambda|} \end{pmatrix} = [\langle 2\phi'' \rangle^{-1/2}] \begin{pmatrix} z_1^* e_0 \\ \vdots \\ z_{2|\Lambda|}^* e_0 \end{pmatrix}$, from (4.56), (4.57) (4.58), the fact that the Grushin operator defined by (4.48) has closed range (by using the same arguments leading to the closed-rangeness of $\mathcal{P}$ defined in (4.40)) and that $\mathcal{S}$ is dense in L^2, we then obtain the well-posedness of the Grushin problem (4.38) in $L^2 \times (\mathbf{R} \oplus \ell^2)$ with the $v_+(0)$ there equals to the $v_+(0)$ in (4.54) and the $v_+(j)$, $(j \neq 0)$ there equals to $\sum_k \langle 2\phi'' \rangle_{jk}^{-1/2} v_+(k)$ of the $v_+(k)$ defined in (4.40). □

Remark. Although we did not state it that way, the earlier remark that the second eigenvalue of $\Delta_\phi^{(0)} \geq 2r_0$ implies the well-posedness of

$$\begin{cases} (\Delta_\phi^{(0)} - z)u + u_-(0)e_0 = v \\ \qquad\qquad (u, e_0) = v_+(0) \end{cases} \tag{4.59}$$

for $z \leq 2\big(r_0 - \frac{1}{\mathbf{O}(1)}\big)$ and sufficiently small γ. This leads to the well-posedness of the Grushin problem for $\Delta_\phi^{(1)}$ in (4.7) for z in *the larger domain* : $z \leq 2\big(r_0 + r_E - \frac{1}{\mathbf{O}(1)}\big)$ (Proposition 4.1), which in turn implies the well-posedness of the Grushin problem for $\Delta_\phi^{(0)}$ in (4.48) in the larger domain : $z \leq 2\big(r_0 + r_E - \frac{1}{\mathbf{O}(1)}\big)$. Clearly, we have started an iteration process.

3. The second Grushin problem for $\Delta_\phi^{(1)}$.

Bearing in mind the preceeding remark, the proposition below should come as no surprise.

Let $e_\alpha dx_j$, (all $\alpha \in \{0\} \cup \bar{\Lambda}$, $j \in \bar{\Lambda}$) be our Grushin basis. (From now on, to avoid confusion we use latin letters only to denote an element of $\bar{\Lambda}$. For elements of other larger sets, we generally use greek letters). Define $R_+ : L^2(\mathbf{R}^{2\Lambda}) \to \big(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\big) \otimes \ell^2(\bar{\Lambda})$ by

$$(R_+ u)(j, \alpha) = (u, e_\alpha dx_j) = (u_j, e_\alpha), \;\; j \in \bar{\Lambda}, \;\; \alpha \in \{0\} \cup \bar{\Lambda}, \tag{4.60}$$

where $u = \sum_{j \in \bar{\Lambda}} u_j dx_j$. Let $R_- = R_+^*$ as usual. Note that $\big(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\big) \otimes \ell^2(\bar{\Lambda})$ is of dimension $2|\Lambda|(2|\Lambda| + 1)$. We have

Proposition 4.7. *For every $C>0$, there exists $\gamma_0>0$ such that the Grushin problem*

$$\begin{pmatrix} \Delta_\phi^{(1)}-z & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{4.61}$$

where $u\in\mathcal{D}(\Delta_\phi^{(1)})$, $v\in L^2(\mathbf{R}^{2\Lambda};\wedge\mathbf{R}^{2\Lambda})$, $u_-,v_+\in\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda)$ is well posed for all $z\le 2\big(r_0+2r_E-\frac{1}{C}\big)$, $0\le\gamma\le\gamma_0$. Moreover

$$\|u\|+\|u_-\|\le\mathbf{O}(1)(\|v\|+\|v_+\|). \tag{4.62}$$

In order to prove the above Proposition, we need the following lemma on small variations (It is instructive to compare with Lemma 4.2) :

Lemma 4.8. *Let $u_-\in\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda)$. Define*

$$\overline{\phi}''\in\mathcal{L}\left(\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda);\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda)\right)$$

to be

$$\overline{\phi}''_{\beta j;\alpha k}=(e_\beta,\phi''_{jk}e_\alpha),\ \ j,k\in\bar\Lambda,\ \ \alpha,\beta\in\{0\}\cup\bar\Lambda. \tag{4.63}$$

Define $M\in\mathcal{L}\big(\mathbf{R}\oplus\ell^2(\bar\Lambda);\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)$ to be

$$M_{\alpha\beta}=(e_\alpha,\sum_{j\in\bar\Lambda}z_j^*z_je_\beta),\ \ \alpha,\beta\in\{0\}\cup\bar\Lambda \tag{4.64}$$

(Note that $M_{\alpha\beta}=0$ if $\alpha=0$ or $\beta=0$ or both.)

We have

$$\|\phi''R_-u_- - R_-\overline{\phi}''u_-\|=\mathbf{O}(\gamma^2)\|u_-\| \tag{4.65}$$

where $R_-=R_+^*$, R_+^* is as defined in (4.50) and

$$\Big\|\Big(\sum z_j^*z_j\Big)e_\alpha-\sum_{\beta\in\{0,\bar\Lambda\}}M_{\alpha\beta}e_\beta\Big\|=\mathbf{O}(\gamma^2) \tag{4.66}$$

Proof. Define $\tilde\phi''\in\mathcal{L}\left(\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda);\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda)\right)$ to be

$$\tilde\phi''_{\beta j;\alpha k}=\langle\phi''_{jk}\rangle\delta_{\beta\alpha}. \tag{4.67}$$

We first show that

$$\|R_-(\overline{\phi}''-\tilde\phi'')u_-\|=\|(\overline{\phi}''-\tilde\phi'')u_-\|=\mathbf{O}(\gamma^2)\|u_-\|. \tag{4.68}$$

Assume $\alpha \neq 0$, $\beta \neq 0$,

$$\begin{aligned}
\overline{\phi}''_{\beta j;\alpha k} &= (e_\beta, \phi''_{jk} e_\alpha) \\
&= \sum_{\omega,\kappa} \langle 2\phi''\rangle^{-1/2}_{\beta\omega} \langle 2\phi''\rangle^{-1/2}_{\alpha\kappa} (z^*_\omega e_0, \phi''_{jk} z^*_\kappa e_0) \\
&= \sum_{\omega,\kappa} \langle 2\phi''\rangle^{-1/2}_{\beta\omega} \langle 2\phi''\rangle^{-1/2}_{\alpha\kappa} (e_0, z_\omega \phi''_{jk} z^*_\kappa e_0) \\
&= \sum_{\omega,\kappa} \langle \phi''\rangle^{-1/2}_{\beta\omega} \langle \phi''\rangle^{-1/2}_{\alpha\kappa} \big(\langle \phi''_{jk}\phi''_{\omega\kappa}\rangle + \frac{1}{2}\langle \phi^{(4)}_{jk\omega\kappa}\rangle\big).
\end{aligned} \tag{4.69}$$

So

$$\begin{aligned}
(\overline{\phi}'' - \tilde{\phi}'')_{\beta j;\alpha k} =& \sum_{\omega,\kappa} \langle \phi''\rangle^{-1/2}_{\beta\omega} \langle \phi''\rangle^{-1/2}_{\alpha\kappa} \big(\langle \phi''_{jk}\phi''_{\omega\kappa}\rangle - \langle \phi''_{jk}\rangle\langle \phi''_{\omega\kappa}\rangle\big) \\
&+ \frac{1}{2}\sum_{\omega,\kappa} \langle \phi''\rangle^{-1/2}_{\beta\omega} \langle \phi''\rangle^{-1/2}_{\alpha\kappa} \langle \phi^{(4)}_{jk\omega\kappa}\rangle \qquad (\alpha \neq 0,\ \beta \neq 0),
\end{aligned} \tag{4.70}$$

$$(\overline{\phi}'' - \tilde{\phi}'')_{0j;0k} = 0, \ \ (\overline{\phi}'' - \tilde{\phi}'')_{0j;\ell k} = (\overline{\phi}'' - \tilde{\phi}'')_{\ell j;0k} = \sum_m \langle 2\phi''\rangle^{-1/2}_{\ell m} \langle \phi'''_{jmk}\rangle = 0, \tag{4.71}$$

where to reach the last equality we used the fact that $\phi(x) = \phi(-x)$ and hence $\phi'''(x) = -\phi'''(-x)$. Using Lemma 4.2 to approximate the first term in (4.70) and a straight forward computation to approximate the second term in (4.70), and since $\phi^{(4)} = \mathbf{O}(\gamma^2)$ pointwise, we obtain (4.68).

Using (4.67) (4.68) in (4.65), (4.6), we see that it is sufficient to prove

$$\|(\phi'' - \langle \phi''\rangle)R_- u_-\| = \mathbf{O}(\gamma^2)\|u_-\|. \tag{4.72}$$

Write

$$\|(\phi'' - \langle \phi''\rangle)R_- u_-\|^2 = \big((\phi'' - \langle \phi''\rangle)R_- u_-, (\phi'' - \langle \phi''\rangle)R_- u_-\big). \tag{4.73}$$

Once again, we write

$$(\phi'' - \langle \phi''\rangle)R_- u_- - (e_0, (\phi'' - \langle \phi''\rangle)R_- u_-)e_0 = \Delta^{(0)}_\phi \tilde{u} \tag{4.74}$$

for some $\tilde{u}$. Let A denote the LHS of (4.74). Since the second term in A is $\mathbf{O}(\gamma^2)\|u_-\|$, in order to prove (4.72), it is sufficient to prove

$$\|A\| = \mathbf{O}(\gamma^2)\|u_-\|. \tag{4.75}$$

Using our usual strategy, we write

$$\begin{aligned}
\|A\|^2 &= \sum_{j,k,\ell} \Big(\sum_\alpha u_-(k,\alpha) d_\phi[(\phi''_{jk} - \langle \phi''_{jk}\rangle)e_\alpha], (\Delta^{(1)}_\phi)^{-1}\Big(\sum_\beta u_-(\ell,\beta) d_\phi[(\phi''_{j\ell} - \langle \phi''_{j\ell}\rangle)e_\beta]\Big)\Big) \\
&\leq \mathbf{O}(1) \sum_{j,k,\ell} \Big\| \sum_\alpha u_-(k,\alpha) d_\phi[(\phi''_{jk} - \langle \phi''_{jk}\rangle)e_\alpha]\Big\| \ \Big\| \sum_\beta u_-(\ell,\beta) d_\phi[\phi''_{j\ell} - \langle \phi''_{j\ell}\rangle e_\beta]\Big\|
\end{aligned} \tag{4.76}$$

where we used the fact that $\|(\Delta_\phi^{(1)})^{-1}\| = \mathbf{O}(1)$.

Define $\omega_\alpha \stackrel{\text{def}}{=} d_\phi[(\phi''_{jk} - \langle\phi''_{jk}\rangle)e_\alpha]$ $(\alpha \in \{0\} \cup \bar\Lambda)$. Then

$$\begin{aligned}
\omega_\alpha &= \sum_i z_i[(\phi''_{jk} - \langle\phi''_{jk}\rangle)e_\alpha]dx_i \\
&= \sum_i \sum_{\alpha'}(\phi''_{jk} - \langle\phi''_{jk}\rangle)\langle\phi''\rangle^{-1}_{\alpha\alpha'}\phi''_{i\alpha'}e_0 dx_i + \sum_i \phi''_{ijk}e_\alpha dx_i \ \text{ if } \ \alpha \neq 0 \\
&= \sum_i \phi'''_{ijk}e_0 dx_i \ \text{ if } \ \alpha = 0.
\end{aligned} \tag{4.77}$$

Define $W_{\alpha\beta} = (\omega_\alpha, \omega_\beta)$, $\alpha, \beta \in \{0\}\cup\bar\Lambda$. Then using our usual "integration by parts" similar to (3.25)–(3.31) and using (4.12) and (4.18) to estimate $\langle(\phi^{(4)})^2\rangle$, we obtain

$$\|W\| = \mathbf{O}(\gamma^4)e^{-2\delta|j-k|} \ \text{ on } \ (\mathbf{R} \oplus \ell^2(\bar\Lambda)) \tag{4.78}$$

for some $\delta > 0$ (uniformly in Λ). Let $\tilde u_-(k) = \sqrt{\sum_\alpha u_-^2(k,\alpha)}$, $(k \in \bar\Lambda)$. Then substituting (4.78) and (4.77) in (4.76), we obtain

$$\begin{aligned}
\|A\|^2 &\leq \mathbf{O}(\gamma^4)\sum_{j,k,\ell}\tilde u_-(k)\tilde u_-(\ell)e^{-\delta|j-k|}e^{-\delta|j-\ell|} \\
&\leq \mathbf{O}(\gamma^4)\|u_-\|^2.
\end{aligned} \tag{4.79}$$

Taking the square root on both side, we obtain (4.75) and hence (4.65).

We now prove (4.66). The case $\alpha = 0$ is trivially true. We therefore assume $\alpha \neq 0$. We first compute $M_{\alpha\beta}$ for $\alpha \neq 0$, $\beta \neq 0$.

$$\begin{aligned}
M_{\alpha\beta} &= (e_\alpha, \sum z_j^* z_j e_\beta) \\
&= \sum_{\alpha',\beta'}\langle 2\phi''\rangle^{-1/2}_{\alpha\alpha'}\langle 2\phi''\rangle^{-1/2}_{\beta\beta'}\sum_j(z^*_{\alpha'}e_0, \ \ z_j^* z_j z^*_{\beta'}e_0) \\
&= 2\sum_{\alpha',\beta'}\langle 2\phi''\rangle^{-1/2}_{\alpha\alpha'}\langle 2\phi''\rangle^{-1/2}_{\beta\beta'}\sum_j\langle\phi''_{j\alpha'}\phi''_{j\beta'}\rangle \\
&= 2\sum_{\alpha',\beta',j}\langle\phi''\rangle^{-1/2}_{\alpha\alpha'}\langle\phi''\rangle^{-1/2}_{\beta\beta'}\langle\phi''_{j\alpha'}\rangle\langle\phi''_{j\beta'}\rangle + \mathbf{O}(\gamma^2)e^{-\delta|\alpha-\beta|} \\
&= \langle 2\phi''\rangle_{\alpha\beta} + \mathbf{O}(\gamma^2)e^{-\delta|\alpha-\beta|}, \ \ (\alpha \neq 0, \ \ \beta \neq 0),
\end{aligned} \tag{4.80}$$

where to reach the fourth line from the third, we used Lemma 4.2. Let $F = (\sum z_j^* z_j)e_\alpha - \sum M_{\alpha\beta}e_\beta$. We first note that $(F, e_0) = 0$. Therefore $F = \Delta_\phi^{(0)}\tilde u$ for some $\tilde u$. Proceeding as usual, we see that

$$\begin{aligned}
(F,F) &= (\Delta_\phi^{(0)}\tilde u, F) \\
&= (d_\phi\tilde u, d_\phi F) \\
&= ((\Delta_\phi^{(1)})^{-1}d_\phi F, d_\phi F).
\end{aligned}$$

So

$$\|F\| \leq \mathbf{O}(1)\|d_\phi F\|, \tag{4.81}$$

where

$$\begin{aligned} d_\phi F =& \sum z_i \Big(\Big(\sum z_j^* z_j \Big) e_\alpha - \sum M_{\alpha\beta} e_\beta \Big) dx_i \\ \stackrel{\text{def}}{=}& \sum g_i dx_i. \end{aligned} \tag{4.82}$$

Since

$$\begin{aligned} \Big(e_0, z_i \Big(\sum z_j^* z_j \Big) e_\alpha \Big) &= 2^{3/2} \sum_{\alpha' \neq 0, j} \langle \phi'' \rangle_{\alpha\alpha'}^{-1/2} (e_0, \phi''_{ij} \phi''_{j\alpha'} e_0) \\ &= 2^{3/2} \sum_{j, \alpha' \neq 0} \langle \phi'' \rangle_{\alpha\alpha'}^{-1/2} \langle \phi''_{ij} \rangle \langle \phi''_{j\alpha'} \rangle + \mathbf{O}(\gamma^2) e^{-\delta|i-\alpha|} \\ &= \langle 2\phi'' \rangle_{i\alpha}^{3/2} + \mathbf{O}(\gamma^2) e^{-\delta|i-\alpha|} \qquad (\alpha \neq 0), \end{aligned} \tag{4.83}$$

and

$$\begin{aligned} z_i \sum_{\beta \neq 0} M_{\alpha\beta} e_\beta &= \sum_{\beta\beta' \neq 0} M_{\alpha\beta} z_i \langle 2\phi'' \rangle_{\beta\beta'}^{-1/2} z_{\beta'}^* e_0 \\ &= \langle 2\phi'' \rangle_{i\alpha}^{3/2} e_0 + \mathbf{O}(\gamma^2) e^{-\delta|i-\alpha|} \qquad (\alpha \neq 0), \end{aligned} \tag{4.84}$$

where we used (4.64) to reach the last equality, we obtain that

$$(g_i, e_0) = \mathbf{O}(\gamma^2) e^{-\delta|i-\alpha|}.$$

So

$$\begin{aligned} (g_i, g_i) &= ((\Delta_\phi^{(1)})^{-1} d_\phi g_i, d_\phi g_i) + \mathbf{O}(\gamma^4) e^{-2\delta|i-\alpha|} \\ &= \mathbf{O}(\gamma^4) e^{-2\delta|i-\alpha|}, \\ \|g_i\| &= \mathbf{O}(\gamma^2) e^{-\delta|i-\alpha|}, \end{aligned} \tag{4.85}$$

by proceeding in a similar way as in the proof of Lemma 4.2. Hence $\|d_\phi F\| = \mathbf{O}(\gamma^2)$. Using (4.82),(4.85) and summing over i, we obtain (4.66). □

Proof of Proposition 4.7. We proceed exactly as in the proof of Proposition 4.1, Proposition 4.5 in place of (4.4) and Lemma 4.8 in place of Lemma 4.2. □

4. The second Grushin problem for $\Delta_\phi^{(0)}$.

We proceed as in subsect. 2 by first posing a variant of the Grushin problem in (4.61). Let

$$\mathcal{M}(\beta, j; \alpha, k) \stackrel{\text{def}}{=} (e_\beta, z_j z_k^* e_\alpha), \tag{4.86}$$

$$\mathcal{M} \in \mathcal{L}\left(\left(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\right) \otimes \ell^2(\bar{\Lambda}); \left(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\right) \otimes \ell^2(\bar{\Lambda})\right).$$

Let

$$\tilde{\mathcal{M}}(\beta, j; \alpha, k) = \begin{cases} 2\langle \phi'' \rangle_{jk}, & \alpha = \beta = 0 \\ 0 & \alpha = 0 \text{ or } \beta = 0 \text{ but not both} \\ 2(\langle \phi'' \rangle_{\beta k}^{1/2} \langle \phi'' \rangle_{\alpha j}^{1/2} + \delta_{\alpha\beta} \langle \phi''_{jk} \rangle), & \alpha \neq 0, \ \beta \neq 0. \end{cases} \tag{4.87}$$

Then $\tilde{\mathcal{M}} \in \mathcal{L}\left(\left(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\right) \otimes \ell^2(\bar{\Lambda}); \left(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\right) \otimes \ell^2(\bar{\Lambda})\right)$ and $\|\mathcal{M} - \tilde{\mathcal{M}}\| = \mathbf{O}(\gamma^2)$. Clearly $\tilde{\mathcal{M}}^{-1}$ exists and $\tilde{\mathcal{M}}^{-1} \in \mathcal{L}\left(\left(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\right) \otimes \ell^2(\bar{\Lambda}); \left(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\right) \otimes \ell^2(\bar{\Lambda})\right)$. So $\mathcal{M}^{-1}$ exists and $\mathcal{M}^{-1} \in \mathcal{L}\left(\left(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\right) \otimes \ell^2(\bar{\Lambda}); \left(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\right) \otimes \ell^2(\bar{\Lambda})\right)$.

Proposition 4.9. *For every $C > 0$, there exists $\gamma_0 > 0$ such that the Grushin problem*

$$\begin{pmatrix} \Delta_\phi^{(1)} - z & d_\phi d_\phi^* R_- \mathcal{M}^{-1} \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{4.88}$$

where $u \in \mathcal{D}(\Delta_\phi^{(1)})$, $v \in L^2(\mathbf{R}^{2\Lambda}; \wedge \mathbf{R}^{2\Lambda})$, u_-, $v_+ \in \left(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\right) \otimes \ell^2(\bar{\Lambda})$ is well posed for all $z \leq 2\left(r_0 + 2r_E - \frac{1}{C}\right)$, $0 \leq \gamma \leq \gamma_0$. Moreover

$$\|u\| + \|u_-\| \leq \mathbf{O}(1)(\|v\| + \|v_+\|). \tag{4.89}$$

Proof. Similar to the proof of Proposition 4.4, it is enough to prove that for all u_-,

$$\|R_- u_- - d_\phi d_\phi^* R_- \mathcal{M}^{-1} u_-\| = \mathbf{O}(\gamma^2) \|u_-\|. \tag{4.90}$$

Since both $\mathcal{M}$, $\mathcal{M}^{-1} \in \mathcal{L}\left(\left(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\right) \otimes \ell^2(\bar{\Lambda}); \left(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\right) \otimes \ell^2(\bar{\Lambda})\right)$, this is equivalent to prove

$$\|\mathcal{M} R_- u_- - d_\phi d_\phi^* R_- u_-\| = \mathbf{O}(\gamma^2) \|u_-\|. \tag{4.91}$$

The LHS of (4.91) without the norm sign, equals to

$$\sum_{j,k,\alpha,\beta} \mathcal{M}(\beta, j; \alpha, k) u_-(\alpha, k) e_\beta dx_j - \sum_{j,k,\alpha,\beta} z_j z_k^* (u_-(\alpha, k) e_\alpha) dx_j \overset{\text{def}}{=} \sum_{j \in \bar{\Lambda}} \psi_j dx_j. \tag{4.92}$$

Using our basic "integration by parts" as in (3.25)–(3.31), we proceed essentially as in the proof of Lemma 4.8, which we summarize as follows :

$$\begin{aligned} \sum_{j \in \bar{\Lambda}} \|\psi_j\| \leq & \mathbf{O}(1) \sum_{j \in \bar{\Lambda}} \|d\psi_j\| + \mathbf{O}(\gamma^2) \|u_-\| \\ \overset{\text{def}}{=} & \mathbf{O}(1) \sum_{j \in \bar{\Lambda}} \left\| \sum \overline{\psi}_{j\ell} dx_\ell \right\| + \mathbf{O}(\gamma^2) \|u_-\| \\ = & \mathbf{O}(1) \sum_{j, \ell \in \bar{\Lambda}} \|\overline{\psi}_{j\ell}\| + \mathbf{O}(\gamma^2) \|u_-\| \\ \leq & \mathbf{O}(1) \sum_{j, \ell \in \bar{\Lambda}} \|d\overline{\psi}_{j\ell}\| + \mathbf{O}(\gamma^2) \|u_-\| \\ \leq & \mathbf{O}(\gamma^2) \|u_-\| \end{aligned} \tag{4.93}$$

the proposition then follows immediately by repeating the arguments in the proof of Proposition 4.4. $\square$

Let $u_j = z_j^* e_0$, $j \in \bar{\Lambda}$, $u_0 = e_0$. Define

$$\bar{\mathcal{M}}(k,\alpha;j,\beta) \stackrel{\text{def}}{=} \frac{1}{2}(z_k^* u_\alpha, z_j^* u_\beta), \qquad k,\, j \in \bar{\Lambda}, \quad \alpha,\, \beta \in \{0\} \cup \bar{\Lambda}. \tag{4.94}$$

We consider $\bar{\mathcal{M}}$ as an element of

$$\big(\ell^2(\bar{\Lambda}) \oplus [\ell^2(\bar{\Lambda}) \otimes_s \ell^2(\bar{\Lambda})]\big)^* \otimes \big(\ell^2(\bar{\Lambda}) \oplus [\ell^2(\bar{\Lambda}) \otimes_s \ell^2(\bar{\Lambda})]\big),$$

which in turn can be seen as a tensor in $\big(\ell^2(\bar{\Lambda}) \oplus [\ell^2(\bar{\Lambda}) \otimes \ell^2(\bar{\Lambda})]\big)^{\otimes^2}$, (separately) symmetric in the first set of indices and the last set of indices (cf. sect III).

$\bar{\mathcal{M}}$ is the restriction to the symmetric subspace: $\ell^2(\bar{\Lambda}) \oplus [\ell^2(\bar{\Lambda}) \otimes_s \ell^2(\bar{\Lambda})]$ of an operator, which (with a slight abuse of notation) we also denote by

$$\bar{\mathcal{M}} \in \mathcal{L}\big(\ell^2(\bar{\Lambda}) \oplus [\ell^2(\bar{\Lambda}) \otimes \ell^2(\bar{\Lambda})]; \ell^2(\bar{\Lambda}) \oplus [\ell^2(\bar{\Lambda}) \otimes \ell^2(\bar{\Lambda})]\big). \tag{4.95}$$

It is easy to see by perturbation theory that $\bar{\mathcal{M}}^{-1}$ exists satisfying

$$\bar{\mathcal{M}} \circ \bar{\mathcal{M}}^{-1} = \bar{\mathcal{M}}^{-1} \circ \bar{\mathcal{M}} = I \oplus [I \otimes I], \tag{4.96}$$

where $I_k^i = \delta_{ik}$, $(I \otimes I)_{k\ell}^{ij} = \delta_{ik}\delta_{j\ell}$. When restricted to the symmetric subspace: $\ell^2(\bar{\Lambda}) \oplus [\ell^2(\bar{\Lambda}) \otimes_s \ell^2(\bar{\Lambda})]$, we have that (with the same abuse of notation)

$$\bar{\mathcal{M}} \circ \bar{\mathcal{M}}^{-1} = \bar{\mathcal{M}}^{-1} \circ \bar{\mathcal{M}} = I \oplus [I \otimes_s I], \tag{4.97}$$

where

$$(I \otimes_s I)_{k\ell}^{ij} = \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}. \tag{4.98}$$

Since $\bar{\mathcal{M}}^{-1} > 0$ as an operator and moreover $\bar{\mathcal{M}}^{-1}$ is a symmetric operator, $\bar{\mathcal{M}}^{-1/2}$ is also well defined:

$$\bar{\mathcal{M}}^{-1/2} \circ \bar{\mathcal{M}}^{-1/2} = \bar{\mathcal{M}}^{-1}. \tag{4.99}$$

Define

(4.100)

$$e_{k\alpha} \stackrel{\text{def}}{=} \sum_{j,\beta} 2\bar{\mathcal{M}}^{-1/2}(k,\alpha;j,\beta) z_j^* u_\beta, \ \ (k,j = 1,\dots,2|\Lambda|, \ \ \alpha,\beta = 0,1,\dots,2|\Lambda|,\, k \neq \alpha),$$
$$e_{kk} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \sum_{j,\beta} 2\bar{\mathcal{M}}^{-1/2}(k,k;j,\beta) z_j^* u_\beta, \ \ (k = 1,\dots,2|\Lambda|).$$

$e_{k\alpha}$ $(k \geq \alpha)$ are orthonormalized by using (4.94,4.99). Let $e_{k\alpha}$ $(k \in \bar{\Lambda}, \alpha \in \{0\} \cup \bar{\Lambda}, k \geq \alpha)$, e_0 be our Grushin basis. Similar to Proposition 4.5, we arrive at the main conclusion of this subsection :

Proposition 4.10. *For every $C > 0$, there exists $\gamma_0 > 0$ such that the following Grushin problem for $\Delta_\phi^{(0)}$:*

$$
(4.101) \qquad \begin{cases} (\Delta_\phi^{(0)} - z)u + \sum\limits_{k\in\bar\Lambda,\alpha\in\{0,\bar\Lambda\, k\geq\alpha\}} u_-(k,\alpha)e_{k\alpha} + u_-(0)e_0 = v \\ \qquad\qquad (u, e_{k\alpha}) = v_+(k,\alpha) \\ \qquad\qquad (u, e_0) = v_+(0) \end{cases}
$$

where $u \in \mathcal{D}(\Delta_\phi^{(0)})$, $v \in L^2(\mathbf{R}^{2\Lambda})$, $u_-, v_+ \in \mathbf{R} \oplus \ell^2(\bar\Lambda) \oplus [\ell^2(\bar\Lambda) \otimes_s \ell^2(\bar\Lambda)]$,which is of dimension $1 + 2|\Lambda| + |\Lambda|(2|\Lambda| + 1)$, is well posed for all $z \leq 2\big(r_0 + 2r_E - \frac{1}{C}\big)$, $0 \leq \gamma \leq \gamma_0$. Moreover

$$
(4.102) \qquad \|u\| + \|u_-\| \leq \mathbf{O}(1)(\|v\| + \|v_+\|).
$$

(The proof is the same as the proof for Proposition 4.5, so we omit it).

5. The third Grushin problem for $\Delta_\phi^{(1)}$ and approximations to $\mathbf{O}(\gamma^4)$.

Let $e_{k\alpha}dx_j$, e_0dx_j, ($\alpha \in \{0\}\cup\bar\Lambda$, $k, j \in \bar\Lambda$, $k \geq \alpha$) defined in (4.100) be our Grushin basis. Define $R_+ : L^2(\mathbf{R}^{2\Lambda}; \mathbf{R}^{2\Lambda}) \to \big(\mathbf{R} \oplus \ell^2(\bar\Lambda) \oplus [\ell^2(\bar\Lambda) \otimes_s \ell^2(\bar\Lambda)]\big) \otimes \ell^2(\bar\Lambda)$ by

$$
(4.103) \qquad \begin{gathered} (R_+u)(j,k,\alpha) = (u, e_{k\alpha}dx_j) = (u_j, e_{k\alpha}) \\ (R_+u)(j,0) = (u, e_0dx_j) = (u_j, e_0) \end{gathered}
$$

where $u = \sum\limits_{j\in\bar\Lambda} u_j dx_j$. Let $R_- = R_+^*$ as usual. As a direct consequence of Propositon 4.10, we have

Proposition 4.11. *For every $C > 0$, there exists $\gamma_0 > 0$ such that the Grushin problem*

$$
(4.104) \qquad \begin{pmatrix} \Delta_\phi^{(1)} - z & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}
$$

where $u \in \mathcal{D}(\Delta_\phi^{(1)})$, $v \in L^2(\mathbf{R}^{2\Lambda}; \mathbf{R}^{2\Lambda})$ $u_-, v_+ \in \big(\mathbf{R} \oplus \ell^2(\bar\Lambda) \oplus [\ell^2(\bar\Lambda) \otimes_s \ell^2(\bar\Lambda)]\big) \otimes \ell^2(\bar\Lambda)$, is well posed for all $z \leq 2\big(r_0 + 3r_E - \frac{1}{C}\big)$, $0 \leq \gamma \leq \gamma_0$. Moreover

$$
(4.105) \qquad \|u\| + \|u_-\| \leq \mathbf{O}(1)(\|v\| + \|v_+\|).
$$

As in the proof of Proposition 4.7, we need the following

Lemma 4.12. *Let $u_- \in \big(\mathbf{R} \oplus \ell^2(\bar\Lambda) \oplus [\ell^2(\bar\Lambda) \otimes_s \ell^2(\bar\Lambda)]\big) \otimes \ell^2(\bar\Lambda)$. Define $\overline{\phi}''$:*

$\mathcal{L}\Big(\big(\mathbf{R} \oplus \ell^2(\bar\Lambda) \oplus [\ell^2(\bar\Lambda) \otimes_s \ell^2(\bar\Lambda)]\big) \otimes \ell^2(\bar\Lambda); \big(\mathbf{R} \oplus \ell^2(\bar\Lambda) \oplus [\ell^2(\bar\Lambda) \otimes_s \ell^2(\bar\Lambda)]\big) \otimes \ell^2(\bar\Lambda)\Big)$

to be

$$\begin{aligned}
\overline{\phi}''_{\ell\beta j;m\alpha k} &= (e_{\ell\beta}, \phi''_{jk} e_{m\alpha}), \quad (j,k,\ell,m \in \bar\Lambda, \alpha,\beta \in \{0\}\cup\bar\Lambda) \\
\overline{\phi}''_{0j;m\alpha k} &= (e_0, \phi''_{jk} e_{m\alpha}) \\
\overline{\phi}''_{\ell\beta j;0k} &= (e_{\ell\beta}, \phi''_{jk} e_0) \\
\overline{\phi}''_{0j;0k} &= (e_0 \phi''_{jk} e_0) = \langle \phi''\rangle_{jk}.
\end{aligned} \tag{4.106}$$

Define

$$M_{k\alpha;j\beta} = \Big(e_{k\alpha} \sum_{\ell\in\bar\Lambda} z^*_\ell z_\ell e_{j\beta}\Big). \tag{4.107}$$

Note that

$$M_{k\alpha,0} = M_{0,k\alpha} = M_{0,0} = 0. \tag{4.108}$$

We have

$$\|\phi'' R_- u_- - R_- \overline{\phi}'' u_-\| = \mathbf{O}(\gamma^2)\|u_-\| \tag{4.109}$$

and

$$\Big\|\Big(\sum z^*_\ell z_\ell\Big) e_{k\alpha} - \sum_{\substack{j\in\bar\Lambda \\ \beta\in\{0,\bar\Lambda\}}} M_{k\alpha,j\beta} e_{j,\beta}\Big\| = \mathbf{O}(\gamma^2). \tag{4.110}$$

Proof. The proof proceeds exactly as in the proof of Lemma 4.8. We note that

$$\begin{aligned}
\overline{\phi}''_{\ell\beta j;m\alpha k} &= (e_{\ell\beta} \phi''_{jk} e_{m\alpha}) \\
&= \bar{\mathcal{M}}^{-1/2}(\ell\beta,\ell'\beta')\bar{\mathcal{M}}^{-1/2}(m\alpha,m'\alpha')(z^*_{\ell'} u_{\beta'}, \phi''_{jk} z^*_{m'} u_{\alpha'}) \\
&= \bar{\mathcal{M}}^{-1/2}(\ell\beta,\ell'\beta')\bar{\mathcal{M}}^{-1/2}(m\alpha,m'\alpha')(u_{\beta'}, z_{\ell'} \phi''_{jk} z^*_{m'} u_{\alpha'}) \\
&= \langle\phi''_{jk}\rangle(\delta_{\alpha\beta}\delta_{\ell m} + \delta_{\alpha\ell}\delta_{\beta m}) + \bar{\mathcal{M}}^{-1/2}(\ell\beta,\ell'\beta')\bar{\mathcal{M}}^{-1/2}(m\alpha,m'\alpha') \\
&\qquad (z^*_{\ell'} e_{\beta'}, (\phi''_{jk} - \langle\phi''_{jk}\rangle) z^*_{m'} e_{\alpha'})
\end{aligned} \tag{4.111}$$

where we used (4.100,4.106).

Using (4.12) and (4.18), it can be easily shown that the second term in the last line of (4.111) correspond to an operator which is of $\mathbf{O}(\gamma^2)$ on $\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes_s \ell^2(\bar\Lambda)]\big)\otimes\ell^2(\bar\Lambda)$.

Let

$$\begin{aligned}
\tilde\phi_{\ell\beta j;m\alpha k} &= \langle\phi''\rangle_{jk}(\delta_{\ell m}\delta_{\alpha\beta} + \delta_{\ell\alpha}\delta_{\beta m}) = \langle\phi''\rangle_{jk}(I\otimes_s I)_{\ell\beta m\alpha} \\
\tilde\phi_{\ell\beta j;0k} &= 0 \\
\tilde\phi_{0j;m\alpha k} &= 0 \\
\tilde\phi_{0j;0k} &= \langle\phi''_{jk}\rangle.
\end{aligned} \tag{4.112}$$

Then

$$\|R_-(\overline{\phi}'' - \tilde{\phi}'')u_-\| = \|(\overline{\phi}'' - \tilde{\phi}'')u_-\|$$
$$= \mathbf{O}(\gamma^2)\|u_-\|,$$

where we used the fact that the operators defined in the second and third equations of (4.106) are of $\mathbf{O}(\gamma^2)$. We may now follow the proof of Lemma 4.8 to obtain (4.109). Similarly we obtain (4.110), by using the same type of arguments that we used to prove (4.66) of Lemma 4.8. □

The proof of Proposition 4.11 now follows immediately by applying Lemma 4.12, in the same way that the proof of Proposition 4.1 follows from Lemma 4.2.

Remark. It is worth noticing that in Lemma 4.2, 4.8 and 4.12, the estimates on the RHS's are all of $\mathbf{O}(\gamma^2)\ldots$. This is fundamentally due to the fact that $[z_j, z_k^*] = 2\phi''_{jk}$ and $\sqrt{\langle(\phi''')^2\rangle} = \mathbf{O}(\gamma^2)$ and hence the variation of ϕ''_{jk} is of $\mathbf{O}(\gamma^2)$ i.e., $\sqrt{\langle(\phi''_{jk})^2\rangle - \langle\phi''_{jk}\rangle^2} = \sqrt{\langle(\phi''_{jk} - \langle\phi''_{jk}\rangle)^2\rangle} = \mathbf{O}(\gamma^2)$. (This is of course the content of Lemma 4.2). If ϕ'' were a constant : $\phi''' = 0$ then all the estimates on the RHS would be replaced by 0 this corresponds to a high-dimentional harmonic oscillator. The point of this paper is to do perturbation theory around the harmonic oscillator as the dimension goes to infinity.

Using Lemma 4.12 and proceeding exactly as in the proof of Proposition 4.3, we obtain the following approximations :

Proposition 4.13. *For every $C > 0$, there exists $\gamma_0 > 0$, such that for all $z \leq 2\big(r_0 + 3r_E - \frac{1}{C}\big)$, $|z| = \mathbf{O}(1)$, all $0 \leq \gamma \leq \gamma_0$,*

$$\|E_{-+} - (z - M - 2\overline{\phi}'')\|_{\mathcal{L}(\ell^2,\ell^2)} = \mathbf{O}(\gamma^2)$$

$$\|E_+ - R_-\|_{\mathcal{L}(\ell^2,L^2)} = \mathbf{O}(\gamma^2)$$

$$\|E_- - R_+\|_{\mathcal{L}(L^2,\ell^2)} = \mathbf{O}(\gamma^2)$$

where $M, \overline{\phi}''$ are defined in (4.107,4.106), ℓ^2 is for $\big(\mathbf{R} \oplus \ell^2(\bar{\Lambda}) \oplus [\ell^2(\bar{\Lambda}) \otimes_s \ell^2(\bar{\Lambda})]\big) \otimes \ell^2(\bar{\Lambda})$ and L^2 for $L^2(\mathbf{R}^{2\Lambda}; \mathbf{R}^{2\Lambda})$.

Let $\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}\begin{pmatrix} v \\ v_+ \end{pmatrix}$ be the solution to the Grushin problem in Proposition 4.11. We specialize to the case $v = 0$ and $v_+ \in \ell^2(\bar{\Lambda})$, i.e. $v_+ = \{v_+(j, 0), j \in \bar{\Lambda})\}$. Then

$$u = E_+ v_+$$
$$u_- = E_{-+} v_+.$$

Define Λ_+ to be the projection :

$$\Lambda_+ : \big(\mathbf{R} \oplus \ell^2(\bar{\Lambda}) \oplus [\ell^2(\bar{\Lambda}) \otimes_s \ell^2(\bar{\Lambda})]\big) \otimes \ell^2(\bar{\Lambda}) \longrightarrow \mathbf{R} \otimes \ell^2(\bar{\Lambda}) \sim \ell^2(\bar{\Lambda}), \tag{4.113}$$

$\forall u : (\Lambda_+ u)(j, 0) = u(j, 0)$, $(\Lambda_+ u)(j, k, \alpha) = 0$ for all $k \in \bar{\Lambda}$, $\alpha \in \{0\} \cup \bar{\Lambda}$. Assume γ, z are such that Propositions 4.11, 4.13 hold and $|z| = \mathbf{O}(1)$. We then have the following improved approximations to E_+, E_- and E_{-+} in this special case.

Proposition 4.14. *For $|z| = \mathbf{O}(1)$, $z \leq 2\big(r_0 + 3r_E - \frac{1}{\mathbf{O}(1)}\big)$*

$$\|(E_{-+} - (z - 2\overline{\phi}''))\Lambda_+\|_{\mathcal{L}(\ell^2,\ell^2)} = \mathbf{O}(\gamma^4)$$

$$\|(E_+ - R_-)\Lambda_+\|_{\mathcal{L}(\ell^2,L^2)} = \mathbf{O}(\gamma^4)$$

$$\|\Lambda_+(E_- - R_+)\|_{\mathcal{L}(L^2,\ell^2)} = \mathbf{O}(\gamma^4) \tag{4.114}$$

where $\overline{\phi}''$ is as defined in (4.106) in Lemma 4.12, ℓ^2 is short for $\big(\mathbf{R} \oplus \ell^2(\bar{\Lambda}) \oplus [\ell^2(\bar{\Lambda}) \otimes_s \ell^2(\bar{\Lambda})]\big) \otimes \ell^2(\bar{\Lambda})$ and L^2 for $L^2(\mathbf{R}^{2\Lambda}; \mathbf{R}^{2\Lambda})$.

Proof. Let

$$u^0 = R_- v_+$$
$$u^0_- = (z - 2\overline{\phi}'')v_+$$

(where $v_+ \in \ell^2(\bar{\Lambda})$) be approximations to u, u_- in (). We then obtain that

$$(\Delta^{(1)}_\phi - z)(u^0 - u) + R_-(u^0_- - u_-) = 2(\phi'' - \overline{\phi}'')R_- v_+$$
$$R_+(u^0 - u) = 0,$$

where we used the fact that $(\Delta^{(0)}_\phi \otimes I)(R_- v_+) = 0$ for $v_+ \in \ell^2(\bar{\Lambda})$. Applying Proposition 4.11 to the Grushin problem in (), we obtain

$$\|u^0 - u\| + \|u^0_- - u_-\| \leq 2\|(\phi'' - \overline{\phi}'')R_- v_+\|.$$

Therefore in order to prove the proposition, it is sufficient to prove that

$$\|(\phi'' - \overline{\phi}'')R_- v_+\| = \mathbf{O}(\gamma^4)\|v_+\| \tag{4.115}$$

for $v_+ \in \ell^2(\bar{\Lambda})$. (Compare (4.115) with (4.109) and (4.110))

$$\|(\phi'' - \overline{\phi}'')R_- v_+\|^2 = \sum_{j,k,p} \big[(\phi''_{jk} e_0 - \langle\phi''_{jk}\rangle e_0 - \sum_{\ell,\beta,\ell\geq\beta} (e_{\ell\beta}, \phi''_{jk} e_0) e_{\ell\beta}\big] v_+(k)$$
$$\big[(\phi''_{jp} e_0 - \langle\phi''_{jp}\rangle e_0 - \sum_{u,\omega,u\geq\omega} (e_{u\omega}, \phi''_{jp} e_0) e_{u\omega}\big] v_+(p)$$

where $e_{\ell\beta}$ is as defined earlier in (4.100).

Let $A_1 \overset{\text{def}}{=} \phi''_{jk} e_0 - \langle\phi''_{jk}\rangle e_0 - \sum_{\ell\geq\beta} (e_{\ell\beta}\phi''_{jk} e_0) e_{\ell\beta}$. Since $(A_1, e_0) = 0$, $A_1 = \Delta^{(0)}_\phi \tilde{u}_1$. By using our usual mechanism, we need to estimate $\|dA_1\|$. Hence we need to estimate

$$
\begin{aligned}
A_2 \stackrel{\text{def}}{=} z_n A_1 &= z_n(\phi''_{jk}e_0 - \langle \phi''_{jk}\rangle e_0) - \sum_{\ell\geq\beta}(e_{\ell\beta}\phi''_{jk}e_0)e_{\ell\beta} \\
&= \phi'''_{jkn}e_0 - \sum_{\ell,0}(e_{\ell 0}\phi''_{jk}e_0)z_n e_{\ell 0} - \sum_{\ell\geq\beta>0}(e_{\ell\beta}\phi''_{jk}e_0)z_n e_{\ell\beta} \\
&= \phi'''_{jkn}e_0 - \langle 2\phi''\rangle^{-1/2}_{\ell\ell'}(z^*_{\ell'}e_0, \phi''_{jk}e_0)z_n\langle 2\phi''\rangle^{-1/2}_{\ell\ell''}z^*_{\ell''}e_0 \\
&\quad - \sum_{\ell\geq\beta>0}(e_{\ell\beta}\phi''_{jk}e_0)z_n e_{\ell\beta} \\
&= \phi'''_{jkn}e_0 - \langle 2\phi''\rangle^{-1}_{\ell'\ell''}\langle \phi'''_{jk\ell'}\rangle(2\phi''_{n\ell''})e_0 \\
&\quad - \sum_{\ell\geq\beta>0}(e_{\ell\beta}\phi''_{jk}e_0)z_n e_{\ell\beta} \\
&= \phi'''_{jkn}e_0 - \sum_{\ell\geq\beta>0}(e_{\ell\beta}\phi''_{jk}e_0)z_n e_{\ell\beta},
\end{aligned}
$$

where we used the fact that $\langle \phi'''_{jk\ell}\rangle = 0$ to reach the last equality. Since $(A_2, e_0) = 0$, $A_2 = \Delta^{(0)}_\phi \tilde{u}_2$. We need to estimate $\|dA_2\|$. Hence we define

$$
A_3 = z_m A_2 = \phi^{(4)}_{jknm}e_0 - \sum_{\ell\geq\beta>0}(e_{\ell\beta}\phi''_{jk}e_0)z_m z_n e_{\ell\beta}
$$

$$
(A_3, e_0) = \langle \phi^{(4)}_{jknm}\rangle - \sum_{\ell\geq\beta>0}(e_{\ell\beta}, \phi''_{jk}e_0)(e_0, z_m z_n e_{\ell\beta}).
$$

Since $(e_0, z^*_m z^*_n e_0) = 0$ and

$$
\|\phi''_{jk}e_0 - \sum_{\ell\geq\beta}(e_{\ell\beta}, \phi''_{jk}e_0)e_{\ell\beta} - \langle\phi''_{jk}\rangle e_0\| = \mathbf{O}_{jk}(\gamma^2)
$$

$$
\|z^*_m z^*_n e_0 - \sum_{\ell\geq\beta}(e_{\ell\beta}, z^*_m z^*_n e_0)e_{\ell\beta}\| = \mathbf{O}_{mn}(\gamma^2),
$$

we obtain

$$
(A_3, e_0) = \langle \phi^{(4)}_{jknm}\rangle - (e_0, z_m z_n \phi''_{jk}e_0) + \mathbf{O}_{jknm}(\gamma^4) = \mathbf{O}_{jknm}(\gamma^4).
$$

Therefore A_3 can be written as

$$
A_3 \stackrel{\text{def}}{=} A_4 + \mathbf{O}_{jknm}(\gamma^4)
$$

where $A_4 = \Delta^{(0)}_\phi \tilde{u}_3$. We are only left to estimate $\|dA_4\|$; to that end, we define

$$
A_5 = z_s A_4 = \phi^{(5)}_{jknms}e_0 + \mathbf{O}_{jknms}(\gamma^4).
$$

Since $\|\phi^{(5)}_{jknms}e_0\| = \mathbf{O}_{jknms}(\gamma^4)$ by using (4.12) and (4.18) (cf. sect VII for more of such computations.), we obtain that $\|A_5\| = \mathbf{O}_{jknms}(\gamma^4)$. Using now the fact that ϕ'' and all higher Hessians (obtained by repeated differentiation of (4.2)) are bounded operators in appropriately exponentially weighted spaces, we obtain (4.115) by combining the estimates on $A_1, A_2, \ldots A_5$. We therefore obtain the Proposition. □

V. Grushin Problems in Weighted Spaces

Beginning in this section, we study the Grushin problem in Proposition 4.11 in appropriate weighted spaces. This is the necessary final step toward obtaining the asymptotics of $\langle G(E;\mu,\nu)\rangle_g$ as $|\mu-\nu|\to\infty$. In this section, we study the general classes of weights that we are allowed to take. In sect. VI, we further specify a class of such weights, using the convolution structure that is inherent in the problem. Propositions 4.11, 4.13 as they stand are not sufficient for obtaining the asymptotics. However, from (3.31), we see that we are only interested at $z=0$. When $z=0$, using Propositions 4.10, 4.11, 4.13, we deduce our desired weighted estimates. But first we need to specify a class $\mathcal{W}$ of weights.

$\mathcal{W}$ consists of weights ρ: $\bar{\Lambda}\to(0,\infty)$ such that if $\rho\in\mathcal{W}$, then

(W1) $\rho\phi''(x)\rho^{-1}$, $\rho^{-1}\phi''(x)\rho=\mathbf{O}(1)$ in $\mathcal{L}\big(\ell^2(\bar{\Lambda});\ell^2(\bar{\Lambda})\big)$ for all x, where $\mathbf{O}(1)$ is uniform in ρ, x. Further ρ is such that $\rho_{i^{(1)}}=\rho_{i^{(2)}}=\rho_i$, where $i^{(1)}\in\Lambda\times\{1,2\}=\bar{\Lambda}$, $i^{(2)}\in\Lambda\times\{1,2\}=\bar{\Lambda}$, $i\in\Lambda$.

(W2) If $\rho\in\mathcal{W}$, then $\rho^{-1}\in\mathcal{W}$, and ρ satisfies

$$(\rho\phi''(x)\rho^{-1}t,t)\geq(\frac{1}{\mathbf{O}(1)}-r_0-2r_E)\|t\|^2,\quad t\in\ell^2(\bar{\Lambda}),\tag{5.1}$$

for all x, where $\mathbf{O}(1)$ is some positive constant uniform in ρ, x; r_0 is taken to be $r_0=$ the smallest eigenvalue of $\langle\phi''\rangle-\frac{1}{C}$ and $r_E=E-2d-\frac{1}{C}$ with arbitrarily large $C>0$ and small enough γ depending on C.

Remark. $\mathcal{W}$ becomes useful only when $\mathbf{O}(1)$ is such that the RHS of (5.1) is negative, i.e., there exists $\rho\in\mathcal{W}$ satisfying (W1), such that

$$(\frac{1}{\mathbf{O}(1)}-r_0-2r_E)\|t\|^2=-|\frac{1}{\mathbf{O}(1)}-r_0-2r_E|\|t\|^2\leq(\rho\phi''(x)\rho^{-1}t,t)<-c\|t\|^2<0.\tag{5.2}$$

(See (5.6).) From the expression for ϕ'' in (4.2), we see that in order for ρ to verify (W1), we only need to check the last term, the $\mathbf{O}(\gamma^4)$ term. To $\mathbf{O}(\gamma)$, that term decays in $|i-j|$ at approximately *twice* the rate of $(E-\Delta)^{-1}$ and hence of $(\phi''(x))^{-1}$. So for sufficiently small γ, this does not cause any problems, and there exist $\rho\in\mathcal{W}$ such that (5.2) is satisfied. For more precise statement, see (6.17,6.18).

We now use the weights in $\mathcal{W}$ to obtain the weighted analogues of Propositions 4.11, 4.13 at $z=0$. We first establish that the Grushin problem in Proposition 4.11 at $z=0$ is well posed for $\rho\in\mathcal{W}$. We proceed as in the proof of Proposition 4.11. We first look at the special case $v_+=0$ and write

$$\begin{cases}\Delta_\phi^{(1)}u+R_-u_-=v\\ \qquad\qquad R_+u=0.\end{cases}\tag{5.3}$$

Multiplying (5.3) by ρ, we obtain

$$\begin{cases}\rho\Delta_\phi^{(1)}u+\rho R_-u_-=\rho v\\ \qquad\qquad \rho R_+u=0,\end{cases}\tag{5.4}$$

where ρv for v a 1-form is defined to be $\rho v = \sum \rho_j v_j dx_j$, $j \in \bar{\Lambda}$; ρR_+ is defined to be: $\rho R_+ = (R_-\rho)^* = (\rho R_-)^*$. Taking the scalar product with ρu in the first equation of (5.4), we have

$$(\rho u, \rho\Delta_\phi^{(1)} u) + (\rho u, \rho R_- u_-) = (\rho u, \rho v). \tag{5.5}$$

Using the second equation of (5.4) and the fact that ρ and R_- commute, we have

$$\begin{aligned}
&(\rho u, \rho\Delta_\phi^{(1)} u) + (\rho u, \rho R_- u_-)\\
=&(\rho u, (\Delta_\phi^{(0)} \otimes I)\rho u) + 2(\rho u, \rho\phi''\rho^{-1}\rho u)\\
\geq&2(r_0 + 2r_E)\|\rho u\|^2 + 2(\frac{1}{\mathbf{O}(1)} - r_0 - 2r_E)\|\rho u\|^2\\
\geq&\frac{1}{\mathbf{O}(1)}\|\rho u\|^2,
\end{aligned} \tag{5.6}$$

where r_0, r_E are as in (4.1,4.3) and we used Proposition 4.10. Combining (5.6) and (5.5), we obtain

$$\|\rho u\| \leq \mathbf{O}(1)\|\rho v\|. \tag{5.7}$$

We next take the scalar product of the first equation in (5.4) with $\rho R_- u_-$:

$$(\rho R_- u_-, \rho\Delta_\phi^{(1)} u) + (\rho R_- u_-, \rho R_- u_-) = (\rho R_- u_-, \rho v). \tag{5.8}$$

So

$$(\rho(\Delta_\phi^{(0)} \otimes I)R_- u_-, \rho u) + 2(\rho\phi''\rho^{-1}\rho R_- u_-, \rho u) + \|\rho u_-\|^2 \leq \|\rho u_-\|\|\rho v\|. \tag{5.9}$$

Using the weighted analogue of Lemma 4.12, which can be proved similarly using in addition (W1), and the second equation of (5.4), we obtain

$$\mathbf{O}(\gamma^2)\|\rho u_-\|\,\|\rho u\| + \|\rho u_-\|^2 \leq \|\rho u_-\|\,\|\rho v\|.$$

Therefore

$$\|\rho u_-\| \leq \mathbf{O}(1)\|\rho v\|. \tag{5.10}$$

We now consider (5.4) in general. Since $R_+R_- = I$, we have from (5.4):

$$\begin{aligned}
\rho\Delta_\phi^{(1)}(u - R_- v_+) + \rho R_- u_- &= \rho(v - \Delta_\phi^{(1)} R_- v_+)\\
\rho R_+(u - R_- v_+) &= 0.
\end{aligned} \tag{5.11}$$

Applying the estimates (5.7) and (5.10) to (5.11), we obtain

$$\begin{aligned}
\|\rho(u - R_- v_+)\| + \|\rho u_-\| &\leq \mathbf{O}(1)\|\rho(v - \Delta_\phi^{(1)} R_- v_+)\|\\
&\leq \mathbf{O}(1)(\|\rho v\| + \|\rho v_+\|),
\end{aligned} \tag{5.12}$$

where we used the weighted analogue of Lemma 4.12. Since

$$\|\rho(u - R_- v_+)\| \geq \|\rho u\| - \|\rho v_+\|, \tag{5.13}$$

we obtain from (5.12) that

$$\|\rho u\| + \|\rho u_-\| \leq \mathbf{O}(1)(\|\rho v\| + \|\rho v_+\|), \tag{5.14}$$

which implies that
(5.15)

$$\|\rho E(0)\rho^{-1}\|_{\mathcal{L}(L^2,L^2)},\ \|\rho E_+(0)\rho^{-1}\|_{\mathcal{L}(\ell^2,L^2)},\ \|\rho E_-(0)\rho^{-1}\|_{\mathcal{L}(L^2,\ell^2)},\ \|\rho E_{-+}(0)\rho^{-1}\|_{\mathcal{L}(\ell^2,\ell^2)} = \mathbf{O}(1),$$

where L^2 is shorthand for $L^2(\mathbf{R}^{2\Lambda};\mathbf{R}^{2\Lambda})$ and ℓ^2 is $\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)]\big)\otimes\ell^2(\bar\Lambda)$.

It is also easy to see that all the steps in the proof of Propositions 4.13, 4.14 go through in the weighted space. We hence have the following weighted analogues:

(5.16)

$$\begin{aligned}
&\|\rho(E_{-+}(0)+(M+2\bar\phi''))\rho^{-1}\|_{\mathcal{L}(\ell^2,\ell^2)} = \mathbf{O}(\gamma^2) && \|\rho(E_{-+}(0)+2\bar\phi'')\rho^{-1}\Lambda_+\|_{\mathcal{L}(\ell^2,\ell^2)} = \mathbf{O}(\gamma^4)\\
&\|\rho(E_+(0)-R_-)\rho^{-1}\|_{\mathcal{L}(\ell^2,L^2)} = \mathbf{O}(\gamma^2) && \|\rho(E_+(0)-R_-)\rho^{-1}\Lambda_+\|_{\mathcal{L}(\ell^2,L^2)} = \mathbf{O}(\gamma^4)\\
&\|\rho^{-1}(E_-(0)-R_+)\rho\|_{\mathcal{L}(L^2,\ell^2)} = \mathbf{O}(\gamma^2) && \|\Lambda_+\rho^{-1}(E_-(0)-R_+)\rho\|_{\mathcal{L}(L^2,L^2)} = \mathbf{O}(\gamma^4),
\end{aligned}$$

where Λ_+ is the projection as defined in (4.113); M, $\bar\phi''$ are as in Propositions 4.12, 4.13.

We end this section by summarizing the above results as

Theorem 5.1. *At $z = 0$, the Grushin problem in Proposition 4.11 is well posed for $\rho\in\mathcal{W}$, with the solution:*

$$\begin{aligned} u &= E(0)v + E_+(0)v_+\\ u_- &= E_-(0)v + E_{-+}(0)v_+, \end{aligned}$$

where $E(0)$, $E_+(0)$, $E_-(0)$, $E_{-+}(0)$ satisfy (5.15) and (5.16), uniformly in $\rho\in\mathcal{W}$.

VI. The Inverse of E_{-+} and Convolution Equations

Starting from this section, we use the symmetry in the problem to facilitate some of the computations. Naturally similar results hold in the general case. Recall from (2.9) that $\phi(x) = \phi(-x)$. Therefore $\langle\phi^{(n)}(x)\rangle = 0$ for n odd. (This was already used earlier in the proof of Proposition 4.14 for $n = 3$.) $\Delta_\phi^{(1)}$ is parity conserving.

Using this reflection symmetry, the Grushin basis for $\Delta_\phi^{(0)}$ constructed earlier in (4.47) and (4.100) simplifies into:

$$
\begin{aligned}
e_0 &= e^{-\phi}\\
e_k &= \sum_j \langle 2\phi''\rangle^{-1/2}{}_{kj}(z_j^* e^{-\phi}) \quad (k,j\in\bar\Lambda)\\
e_{kk'} &= \sum_{jj'} (2\bar{\mathcal{M}})^{-1/2}{}_{kk';jj'}(z_j^* z_{j'}^* e^{-\phi}) \quad (k,k',j,j'\in\bar\Lambda, k\geq k')\\
e_{kk} &= \frac{1}{\sqrt{2}}\sum_{jj'} (2\bar{\mathcal{M}})^{-1/2}{}_{kk;jj'}(z_j^* z_{j'}^* e^{-\phi}) \quad (k,j,j'\in\bar\Lambda)
\end{aligned}
\tag{6.1}
$$

where

$$
\bar{\mathcal{M}}_{kk';jj'} = 2\langle\phi''\rangle_{kj}\langle\phi''\rangle_{k'j'} + 2\langle\phi''\rangle_{kj'}\langle\phi''\rangle_{k'j} + \langle\phi^{(4)}\rangle_{kk'jj'}.
\tag{6.2}
$$

(The simplification comes from the fact that $\langle\phi'''(x)\rangle = 0$.) In Proposition 4.13, using $e_0 dx_j$, $e_k dx_j$, $e_{kk'}dx_j$ ($k,k',j\in\bar\Lambda$; $k\geq k'$) as the Grushin basis, we obtained the effective operator $E_{-+}(z)$ of $\Delta_\phi^{(1)} - z$. Recall the identifications:

$$
\begin{aligned}
&\sum_j e_0 dx_j \in \mathbf{R}^{2\Lambda} \sim \ell^2(\bar\Lambda)\\
&\sum_{k,j} e_k dx_j \in \mathbf{R}^{2\Lambda}\otimes\mathbf{R}^{2\Lambda} \sim \ell^2(\bar\Lambda)\otimes\ell^2(\bar\Lambda)\\
&\sum_{k\geq k',j} e_{kk'}dx_j \in \big(\mathbf{R}^{2\Lambda}\otimes_s\mathbf{R}^{2\Lambda}\big)\otimes\mathbf{R}^{2\Lambda} \sim \big(\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda);
\end{aligned}
\tag{6.3}
$$

and $E_{-+}(z)$ is therefore a bounded operator in

$$
\begin{aligned}
&\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)]\big)\otimes\ell^2(\bar\Lambda)\\
=&\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes\ell^2(\bar\Lambda)]\oplus\big[\big(\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda)\big]\\
\overset{\text{def}}{=}&F^1\oplus F^2\oplus F^3 = F.
\end{aligned}
\tag{6.4}
$$

Furthermore, Theorem 5.1 implies that at $z=0$, $E_{-+}(z=0)$ is a bounded operator in

$$
\ell^2_\rho(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes\ell^2_\rho(\bar\Lambda)]\oplus\big[\big(\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)\big)\otimes\ell^2_\rho(\bar\Lambda)\big] \overset{\text{def}}{=} F^1_\rho\oplus F^2_\rho\oplus F^3_\rho = F_\rho,
\tag{6.5}
$$

where $\rho\in\mathcal{W}$ satisfies (W1,2).

We now look more closely at (W1,2), taking into account the special form of ϕ''. For the sake of convenience, we repeat the expression for $2\phi''$ from (4.2):

(6.6)

$$
\begin{aligned}
(2\phi''(x))_{i^{(\alpha)}j^{(\beta)}} =&2E\delta_{ij}\delta_{\alpha\beta} - 2\Delta_{ij}\delta_{\alpha\beta} - 2\gamma k'(\gamma x_i\cdot x_i)\delta_{ij}\delta_{\alpha\beta} - 4\gamma^2 x_i^{(\alpha)}x_i^{(\beta)}k''(\gamma x_i\cdot x_i)\delta_{ij}\\
&+ 2\gamma^2 k''(\gamma x_i\cdot x_i)(E-\Delta-\gamma\operatorname{diag}k')^{-1}_{ii}\delta_{ij}\delta_{\alpha\beta}\\
&+ 4\gamma^3 x_i^{(\alpha)}x_i^{(\beta)}k'''(\gamma x_i\cdot x_i)(E-\Delta-\gamma\operatorname{diag}k')^{-1}_{ii}\delta_{ij}\\
&+ 4\gamma^4 x_i^{(\alpha)}x_j^{(\beta)}k''(\gamma x_i\cdot x_i)k''(\gamma x_j\cdot x_j)[(E-\Delta-\gamma\operatorname{diag}k')^{-1}_{ij}]^2,
\end{aligned}
$$

where $k = \log \hat{g}$ and $\hat{g}$ is the Laplace transform of g; and we used the point of view of $(\mathbf{R}^2)^\Lambda$, i.e., $i,\ j \in \Lambda$, $\alpha,\ \beta \in \{1,2\}$, $x_i,\ x_j \in \mathbf{R}^2$. Δ_{ij} is as defined in (1.1), for $i,\ j \in \Lambda$ and $\Delta = \Delta_\Lambda$ is the matrix of Δ_{ij}. To avoid confusion, we now write Δ_Λ for the finite volume Laplacian and reserve Δ for the corresponding one on $\ell^2(\mathbf{Z}^d)$. The spectrum of Δ_Λ is contained in $[-2d, 2d]$.

At this point it is good to recall the problem that we started with: We study

$$H = \Delta + \gamma V, \text{ on } \ell^2(\mathbf{Z}^d), \tag{6.7}$$

as defined in (1.1). For all $\delta > 0$, let $I_\delta = (2d + \delta, \infty)$ or $I_\delta = (-\infty, -2d - \delta)$ depending on the support of dg as in (1.11). We are interested in the asymptotics of $\langle G(E, \mu, \nu)\rangle_g$ as $|\mu - \nu| \to \infty$ for $E \in I_\delta$. We proceed by using finite volume approximations. We take $\Lambda \ni \mu, \nu$ to be a finite subset of $\mathbf{Z}^d$ and impose the Dirichlet boudary conditions. For given μ, ν, we take Λ sufficiently large, so that μ, ν is "well" in the interior of Λ. Using the resolvent series and spectral theory, the limit as $\Lambda \nearrow \mathbf{Z}^d$ exists, is independent of the boundary conditions and we have moreover

$$\lim_{\Lambda \nearrow \mathbf{Z}^d} \langle G_\Lambda(E; \mu, \nu)\rangle_g = \langle G(E; \mu, \nu)\rangle_g. \tag{6.8}$$

Therefore we further specify Λ to be a cube of side length $N >> 1$.

We now make it more precise the class of weights $\mathcal{W}$ that we shall take, using the fact that $R - \Delta$ ($R \in \mathbf{R}$) is a convolution matrix on $\ell^2(\mathbf{Z}^d)$:

$$\mathcal{F}(R - \Delta)(\xi) = R - \sum_{i=1}^{d} \cos \xi_i, \quad \xi = \{\xi_i\} \in \mathbf{T}^{d^*}.$$

(We abused slightly the notation and wrote $\Delta(j - k)$ for Δ_{jk}.) Let $R > 2d$, $y = \{y_i\} \in \mathbf{R}^{d^*} \sim \mathbf{R}^d$. Define

$$D_R = \{y_i | \sum_{i=1}^{d} \cosh y_i < R\} \subset \mathbf{R}^d; \tag{6.9}$$

D_R is a strictly convex, compact domain in $\mathbf{R}^d$. Let

$$F(y) = 2 \sum_{i=1}^{d} \cosh y_i.$$

We define the support function of D_R:

$$H_{D_R}(k) = \sup_{y \in \bar{D}_R} y \cdot k$$

for $k \in \mathbf{Z}^d/\{0\}$. $H_{D_R}(k)$ is even, convex and homogeneous of degree 1. Therefore $H_{D_R}(k)$ defines a distance on $\mathbf{Z}^d$: $d_R(k, \ell) = H_{D_R}(k - \ell)$ for $k,\ \ell \in \mathbf{Z}^d$.

Let ρ: $\mathbf{Z}^d \to (0,\infty)$ be a class of weights such that

$$\frac{\rho(k)}{\rho(\ell)} \leq e^{d_R(k,\ell)}. \tag{6.10}$$

(This is always possible by a standard modification of the weight at infinity.)

Let $2d < r \leq R$ and let ρ satisfy (W1,2) and

$$\max(|\log\frac{\rho(k)}{\rho(\ell)} - y(k)\cdot(k-\ell)|, |\log\frac{\rho(k)}{\rho(\ell)} - y(\ell)\cdot(k-\ell)|) \leq \epsilon \qquad (\epsilon << 1) \tag{6.11}$$

for $|k-\ell| \leq 1$, $y(k)$, $y(\ell) \in D_r$. Then we have

Proposition 6.1. *If ρ satisfies (6.10), (6.11), then*

$$(\rho(R-\Delta)t, \rho t) \geq (R - r - \tilde{\epsilon}(\epsilon))\|\rho t\|_{\ell^2}^2, \qquad t \in \ell^2(\mathbf{Z}^d;\mathbf{R}) \tag{6.12}$$

where $\tilde{\epsilon} \to 0$ as $\epsilon \to 0$.

proof. It is equivalent to prove that

$$(\rho\Delta t, \rho t) \leq (r + \tilde{\epsilon}(\epsilon))\|\rho t\|_{\ell^2}^2.$$

Therefore it suffices to prove

$$\|\frac{1}{2}(\rho\Delta\rho^{-1} + \rho^{-1}\Delta\rho)\|_{\ell^2} \leq r + \tilde{\epsilon}(\epsilon). \tag{6.13}$$

To prove (6.13), we estimate

$$\begin{aligned}
&\sup_k \sum_\ell \frac{1}{2}\left(\frac{\rho(k)}{\rho(\ell)} + \frac{\rho(\ell)}{\rho(k)}\right)\Delta(k-\ell) \\
\leq &\sup_k \sum_{|k-\ell|=1} e^\epsilon \cosh(y(k)\cdot(k-\ell)) \\
= &e^\epsilon F(y(k)) \\
\leq &e^\epsilon r.
\end{aligned}$$

We have the same estimate for $\sup_\ell \sum_k \cdots$ and the proposition follows. $\square$

Remark. From now on we will have the need to distinguish operators on $\ell^2(\bar{\Lambda})$ from their counterparts on $\ell^2(\mathbf{Z}^d)$. So we systematically put back the subscript Λ, when we refer to the former.

From (6.6), (6.7), $2\phi''_\Lambda$ (previously denoted $2\phi''$) is a bounded operator on $\ell^2(\bar{\Lambda})$. Denote the last term in the RHS of (6.6) by S_Λ and the first six terms by T_Λ, we have

$$2\phi''_\Lambda = T_\Lambda + S_\Lambda, \tag{6.14}$$

where T_Λ can be further written as

$$T_\Lambda = (E - \Delta_\Lambda) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathbf{O}(\gamma). \tag{6.15}$$

Let ρ_E: $\mathbf{Z}^d \to (0, \infty)$ be a class of weights satisfying (6.10) (6.11) with R replaced by E and r strictly less than E. Let $\rho_{\Lambda,E}$: $\bar{\Lambda} \to (0, \infty)$ be the restriction of ρ_E to $\bar{\Lambda}$ such that

$$\rho_{\Lambda,E}(i^{(\alpha)}) = \rho_{\Lambda,E}(i^{(\beta)}) = \rho_{\Lambda,E}(i) \tag{6.16}$$

for $i \in \Lambda$, α, $\beta \in \{1, 2\}$ and $i^{(\alpha)}$, $i^{(\beta)} \in \bar{\Lambda}$. Proposition 6.1 implies that for γ sufficiently small

$$T_\Lambda^{-1} \in \mathcal{L}(\ell^2_{\rho_{\Lambda,E}}(\bar{\Lambda}); \ell^2_{\rho_{\Lambda,E}}(\bar{\Lambda})) \tag{6.17}$$

and that

$$S_\Lambda \in \mathcal{L}(\ell^2_{(\rho_{\Lambda,E})^2}(\bar{\Lambda}); \ell^2_{(\rho_{\Lambda,E})^2}(\bar{\Lambda})), \tag{6.18}$$

where we used the fact that $\Delta(k) \geq 0$ for all $k \in \mathbf{Z}^d$. Notice that it is the weight $(\rho_{\Lambda,E})^2$ that appears in (6.18). In view of (6.18), (6.9) and the elementary fact that $\cosh 2x = 2\cosh^2 x - 1$, we have

$$\begin{aligned} \cosh 2x - 1 &= 2\cosh^2 x - 2 \\ &= 2(\cosh x - 1)(\cosh x + 1) \\ &\geq 4(\cosh x - 1). \end{aligned}$$

Therefore we may replace $(\rho_{\Lambda,E})^2$ by $\rho_{\Lambda,\bar{E}}$ where

$$\bar{E} - 2d = 4(E - 2d) - \frac{1}{\mathbf{O}(1)}$$

for γ sufficiently small. Using (6.18), (W1,2), the class of weights $\rho_{\bar{E}}$: $\mathbf{Z}^d \to (0, \infty)$ that we shall take can then be specified further as satisfying (6.10) and (6.11) with R replaced by

$$\bar{E} = E + 3(E - 2d) - \frac{1}{\mathbf{O}(1)} \quad (E > 2d). \tag{6.19}$$

Let

$$H_\Lambda = M_\Lambda + 2\overline{\phi''_\Lambda}, \tag{6.20}$$

where M_Λ, $\overline{\phi''_\Lambda}$ are as M, $\overline{\phi''}$ in the earlier notations in (4.64) and (4.63). Then $H_\Lambda \in \mathcal{L}(F_{\rho_\Lambda}; F_{\rho_\Lambda})$ for $\rho \in \mathcal{W}$ satisfying (6.10) (6.11) and is an approximation to $E_{-+,\Lambda}(0)$ satisfying (5.15) and (5.16). Let Λ^j_+ ($j = 1, 2, 3.$) be the projections:

$$\begin{aligned} \Lambda^1_+ F &= F^1 = \ell^2(\bar{\Lambda}) \\ \Lambda^2_+ F &= F^2 = \ell^2(\bar{\Lambda}) \otimes \ell^2(\bar{\Lambda}) \\ \Lambda^3_+ F &= F^3 = [\ell^2(\bar{\Lambda}) \otimes_s \ell^2(\bar{\Lambda})] \otimes \ell^2(\bar{\Lambda}). \end{aligned} \tag{6.21}$$

(Λ_+^1 was denoted earlier by Λ_+ in (4.113).) Let

$$
(6.22)\qquad
\begin{aligned}
H_{j,\Lambda} &= \Lambda_+^j H_\Lambda \Lambda_+^j \quad (j=1,2,3.),\\
H_{jk,\Lambda} &= \Lambda_+^j H_\Lambda \Lambda_+^k \quad (j\neq k,\ j,k=1,2,3.);\\
Q_{j,\Lambda} &= \Lambda_+^j E_{-+,\Lambda}(0)\Lambda_+^j \quad (j=1,2,3.),\\
Q_{jk,\Lambda} &= \Lambda_+^j E_{-+,\Lambda}(0)\Lambda_+^k \quad (j\neq k,\ j,k=1,2,3.);\\
D_\Lambda &= \oplus_{j=1}^3 Q_{j,\Lambda}\\
R_\Lambda &= E_{-+,\Lambda}(0) - D_\Lambda.
\end{aligned}
$$

Since parity is conserved,

$$H_{12,\Lambda} = H_{21,\Lambda} = H_{23,\Lambda} = H_{32,\Lambda} = 0;$$

similarly,

$$Q_{12,\Lambda} = Q_{21,\Lambda} = Q_{23,\Lambda} = Q_{32,\Lambda} = 0.$$

Then by using the resolvent identity of $[E_{-+,\Lambda}(0)]^{-1}$, we have

$$
(6.23)\qquad
\begin{aligned}
&\Lambda_+^1[E_{-+,\Lambda}(0)]^{-1}\Lambda_+^1\\
=&\Lambda_+^1 D_\Lambda^{-1}\Lambda_+^1 + \Lambda_+^1 D_\Lambda^{-1} R_\Lambda D_\Lambda^{-1}\Lambda_+^1 + \Lambda_+^1 D_\Lambda^{-1} R_\Lambda [E_{-+,\Lambda}(0)]^{-1}\Lambda_+^1\\
=&(Q_{1,\Lambda} - Q_{13,\Lambda} Q_{3,\Lambda}^{-1} Q_{31,\Lambda})^{-1}\\
\stackrel{\text{def}}{=}&\tilde{Q}_\Lambda^{-1}.
\end{aligned}
$$

Furthermore, from Theorem 5.1, we have

$$
(6.24)\qquad
\begin{aligned}
\|Q_{1,\Lambda} - H_{1,\Lambda}\|_{\mathcal{L}(F^1_{\rho_\Lambda},F^1_{\rho_\Lambda})} &= \mathbf{O}(\gamma^4)\\
\|Q_{13,\Lambda} - H_{13,\Lambda}\|_{\mathcal{L}(F^3_{\rho_\Lambda},F^1_{\rho_\Lambda})} &= \mathbf{O}(\gamma^4)\\
\|Q_{31,\Lambda} - H_{31,\Lambda}\|_{\mathcal{L}(F^1_{\rho_\Lambda},F^3_{\rho_\Lambda})} &= \mathbf{O}(\gamma^4)\\
\|Q_{3,\Lambda} - H_{3,\Lambda}\|_{\mathcal{L}(F^3_{\rho_\Lambda},F^3_{\rho_\Lambda})} &= \mathbf{O}(\gamma^2).
\end{aligned}
$$

From the definition of $\tilde{Q}_\Lambda$ in the RHS of (6.23), the only operator that we need to look into is $Q_{3,\Lambda}^{-1}$. Using (6.24), we see that to order $\mathbf{O}(\gamma^4)$, it is sufficient to study $H_{13,\Lambda}H_{3,\Lambda}^{-1}H_{31,\Lambda}$. To that end, we compute $H_{3,\Lambda}$ as defined by (6.20), (6.21). To gain experience and also to see the structure, we start by computing $H_{1,\Lambda}$, $H_{2,\Lambda}$.

By definition:

$$
(6.25)\qquad
\begin{aligned}
(H_{1,\Lambda})_{ij} &= \langle e^{-\phi}, (\Delta_\phi^{(0)}\delta_{ij} + 2\phi''_{\Lambda,ij})e^{-\phi}\rangle\\
&= \langle 2\phi''_\Lambda\rangle_{ij},\\
\text{so}\quad H_{1,\Lambda} &= \langle 2\phi''_\Lambda\rangle.
\end{aligned}
$$

(The ϕ's that appear below should all be understood as ϕ_Λ's. For notational simplicity, we usually only put the subscript back on the end results.)
(6.26)

$$\begin{aligned}
(H_{2,\Lambda})_{ii';jj'} &= (e_i, (\Delta_\phi^{(0)}\delta_{i'j'} + 2\phi''_{i'j'})e_j) \\
&= (2\langle\phi''\rangle)^{-1/2}_{ik}(2\langle\phi''\rangle)^{-1/2}_{j\ell}\Big((e_0, z_k(\sum z_n^* z_n)z_\ell^* e_0)\delta_{i'j'} + 2(e_0, z_k\phi''_{i'j'}z_\ell^* e_0)\Big) \\
&= (2\langle\phi''\rangle)^{-1/2}_{ik}(2\langle\phi''\rangle)^{-1/2}_{j\ell} \\
&\Big(4(e_0, (\sum_n \phi''_{kn}\phi''_{n\ell})e_0)\delta_{i'j'} + 4(e_0, \phi''_{i'j'}\phi''_{k\ell}e_0) + 2(e_0, \phi^{(4)}_{i'j'k\ell}e_0)\Big) \\
&= (2\langle\phi''\rangle)^{-1/2}_{ik}(2\langle\phi''\rangle)^{-1/2}_{j\ell}[4(\sum_n(\langle\phi''\rangle_{kn}\langle\phi''\rangle_{n\ell})\delta_{i'j'} + 4\langle\phi''\rangle_{i'j'}\langle\phi''\rangle_{k\ell} \\
&\quad + 2\langle\phi^{(4)}_{i'j'k\ell}\rangle + 4\sum_n(e^{-\phi}d\phi''_{kn}, (\Delta_\phi^{(1)})^{-1}(e^{-\phi}d\phi''_{n\ell}))\delta_{i'j'} \\
&\quad + 4(e^{-\phi}d\phi''_{i'j'}, (\Delta_\phi^{(1)})^{-1}(e^{-\phi}d\phi''_{k\ell}))].
\end{aligned}$$

So

$$H_{2,\Lambda} = \langle 2\phi''_\Lambda\rangle \otimes I + I \otimes \langle 2\phi''_\Lambda\rangle + \mathbf{O}(\gamma^2), \tag{6.27}$$

where $\mathbf{O}(\gamma^2)$ is in the sense of bounded operators in $F^2_{\rho_\Lambda}$. Similarly

$$H_{3,\Lambda} = [\langle 2\phi''_\Lambda\rangle \otimes_s I] \otimes I + [I \otimes_s \langle 2\phi''_\Lambda\rangle] \otimes I + [I \otimes_s I] \otimes \langle 2\phi''_\Lambda\rangle + \mathbf{O}(\gamma^2), \tag{6.28}$$

where $\mathbf{O}(\gamma^2)$ is in the sense of bounded operators in $F^3_{\rho_\Lambda}$.

Let

$$P = [\langle 2\phi''_\Lambda\rangle \otimes_s I] \otimes I + [I \otimes_s \langle 2\phi''_\Lambda\rangle] \otimes I + [I \otimes_s I] \otimes \langle 2\phi''_\Lambda\rangle. \tag{6.29}$$

Then $P^{-1} \in \mathcal{L}(F^3_{\rho_\Lambda}; F^3_{\rho_\Lambda})$, where $\rho \in \mathcal{W}$ satisfies (6.10) (6.11) with $\bar{E}$ in (6.19) further lowered to

$$E + 2(E - 2d) - 1/\mathbf{O}(1). \tag{6.30}$$

Using (6.28), we have that $H^{-1}_{3,\Lambda} \in \mathcal{L}(F^3_{\rho_\Lambda}; F^3_{\rho_\Lambda})$. Using (6.24), we therefore have that

$$Q^{-1}_{3,\Lambda} \in \mathcal{L}(F^3_{\rho_\Lambda}; F^3_{\rho_\Lambda}). \tag{6.31}$$

We now compute H_{13}:

$$\begin{aligned}
(H_{13})_{jkmn} &= 2(e_0, \phi''_{jk}e_{mn}) \\
&= \sum_{m',n'}(2\mathcal{M})^{-1/2}_{mnm'n'}(e_0, \phi''_{jk}z^*_{m'}z^*_{n'}e_0) \\
&= \sum_{m',n'}(2\mathcal{M})^{-1/2}_{mnm'n'}\langle\phi^{(4)}_{jkm'n'}\rangle \\
&= \mathbf{O}(\gamma^2),
\end{aligned} \tag{6.32}$$

where we used (4.12) and (4.18) (Recall the proof of Lemma 4.2.) to reach the last estimate. (cf. sect. VII for more of such computations.) Therefore by the usual argument, we obtain that $H_{13} = \mathbf{O}(\gamma^2) \in \mathcal{L}(F^3_{\rho_\Lambda}; F^3_{\rho_\Lambda})$. Using (6.24) (6.31), we obtain that

$$\|Q_{13}Q_3^{-1}Q_{31}\|_{\mathcal{L}(\ell^2_\rho(\bar\Lambda),\ell^2_\rho(\bar\Lambda))} = \mathbf{O}(\gamma^4). \tag{6.33}$$

Combinning (6.33) with (6.24), we then obtain that

$$\|\tilde{Q}_\Lambda - H_{1,\Lambda}\|_{\mathcal{L}(\ell^2_\rho(\bar\Lambda);\ell^2_\rho(\bar\Lambda))} = \mathbf{O}(\gamma^4), \tag{6.34}$$

where $\tilde{Q}_\Lambda$ is as defined in (6.23). Therefore to $\mathbf{O}(\gamma^4)$, in order to study $\Lambda^1_+ E^{-1}_{-+}(0)\Lambda^1_+$, we only need to study $H_{1,\Lambda} = \langle 2\phi''_\Lambda\rangle$.

VII. Asymptotics for the Averaged Green's Function

Recall from sect. I that $\sigma(H) = [-2d, 2d] + \gamma \operatorname{supp} dg$ almost surely and that $E \in I_\delta \not\subseteq \sigma(H)$. Recall also from sect. VI that for such E's and for all given μ, ν, we take $\mu, \nu \ni \Lambda \subset \mathbf{Z}^d$ to be a cube of side length N. We have that

$$\lim_{\Lambda \nearrow \mathbf{Z}^d} \langle (E - H_\Lambda)^{-1}(\mu,\nu)\rangle_g = \lim_{\Lambda \nearrow \mathbf{Z}^d} \langle G_\Lambda(E;\mu,\nu)\rangle_g = \langle G(E;\mu,\nu)\rangle_g.$$

For the computations in this section, it is more convenient to revert back to the point of view of associating $(\mathbf{R}^2)^\Lambda$ to Λ. The indices i, j, μ, $\cdots$ etc. below are once again in Λ and $x_i, \cdots \in \mathbf{R}^2$.

For the cubes mentioned above, using (3.31), we have

(7.1)

$$\begin{aligned}\langle G_\Lambda(E;\mu,\nu)\rangle_g &= \sum_{\alpha=1,2} (e^{-\phi}dx^{(\alpha)}_\mu, (\Delta^{(1)}_\phi)^{-1} e^{-\phi} dx^{(\alpha)}_\nu) \\ &= \sum_{\alpha=1,2} (e^{-\phi}dx^{(\alpha)}_\mu, E_\Lambda e^{-\phi}dx^{(\alpha)}_\nu) + (e^{-\phi}dx^{(\alpha)}_\mu, (E_{+,\Lambda}E^{-1}_{-+,\Lambda}E_{-,\Lambda})e^{-\phi}dx^{(\alpha)}_\nu).\end{aligned}$$

From Theorem 5.1, $\|\rho_\Lambda E_\Lambda \rho_\Lambda^{-1}\|_{\mathcal{L}(L^2,L^2)} = \mathbf{O}(1)$ uniformly in Λ, where as before, ρ_Λ is the restriction of ρ to Λ and ρ satisfies (6.10) (6.11) with

$$R = E + 3(E - 2d) - \frac{1}{\mathbf{O}(1)}.$$

We therefore have

$$|(e^{-\phi}dx^{(\alpha)}_\mu, E_\Lambda e^{-\phi}dx^{(\alpha)}_\nu)| = \mathbf{O}(1)e^{-d_r(\mu,\nu)}$$

uniformly in Λ, where

$$r = R - \frac{1}{\mathbf{O}(1)} = E + 3(E - 2d) - \frac{1}{\mathbf{O}(1)}. \tag{7.2}$$

(Recall that this is approximately twice the decay rate that we expect from $\langle G_\Lambda(E;\mu,\nu)\rangle_g$.) As a consequence, we obtain that

$$\overline{\lim}_{\Lambda\nearrow\mathbf{Z}^d}(e^{-\phi}dx_\mu^{(\alpha)}, E_\Lambda e^{-\phi}dx_\nu^{(\alpha)}) = \mathbf{O}(1)e^{-d_r(\mu,\nu)}. \tag{7.3}$$

To study the second term in the second equality of (7.1), we use Theorem 5.1, (5.15) and (5.16) we obtain

$$\begin{aligned}&(e^{-\phi}dx_\mu^{(\alpha)}, (E_{+,\Lambda}E_{-+,\Lambda}^{-1}E_{-,\Lambda})(e^{-\phi}dx_\nu^{(\alpha)}))\\ =&(R_{+,\Lambda}(e^{-\phi}dx_\mu^{(\alpha)}), E_{-+,\Lambda}^{-1}R_{+,\Lambda}(e^{-\phi}dx_\nu^{(\alpha)}))(1+\mathbf{O}(\gamma^4))\\ =&(1+\mathbf{O}(\gamma^4))(\Lambda_+^1(E_{-+,\Lambda}^{-1})\Lambda_+^1)(\mu,\nu),\end{aligned} \tag{7.4}$$

where $\mathbf{O}(\gamma^4)$ is uniform in Λ. Using (6.34), (6.23) we have

$$(\Lambda_+^1(E_{-+,\Lambda}^{-1})\Lambda_+^1)(\mu,\nu) = (\langle 2\phi_\Lambda''\rangle + \mathbf{O}(\gamma^4))^{-1}(\mu,\nu) \tag{7.5}$$

where $\mathbf{O}(\gamma^4)$ (uniformly in Λ) is in the sense of $\mathcal{L}(\ell^2_{\rho_\Lambda}(\bar\Lambda);\ell^2_{\rho_\Lambda}(\bar\Lambda))$ and ρ_Λ is the restriction of ρ to Λ with R lowered to

$$R = E + 2(E-2d) - \frac{1}{\mathbf{O}(1)}.$$

To simplify things further, we use

Lemma 7.1.

$$\begin{aligned}\langle\phi_\Lambda''\rangle = \big(E-\Delta_\Lambda-&\gamma\langle v\rangle_g - \gamma^2\,\mathrm{diag}\{\langle(v-\langle v\rangle_g)^2\rangle_g[(E-\Delta_\Lambda-\gamma\langle v\rangle_g)^{-1}(j,j)]\}\\ &-\gamma^3\,\mathrm{diag}\{\langle(v-\langle v\rangle_g)^3\rangle_g[(E-\Delta_\Lambda-\gamma\langle v\rangle_g)^{-1}(j,j)]^2\}\big)\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}+\mathbf{O}(\gamma^4),\end{aligned}$$

where the operator in the parenthesis acts on $\ell^2(\Lambda)$, $\mathbf{O}(\gamma^4)$ (uniformly in Λ) is in the sense of $\mathcal{L}(\ell^2_{\rho_\Lambda}(\bar\Lambda);\ell^2_{\rho_\Lambda}(\bar\Lambda))$ with

$$R = E + 2(E-2d) - \frac{1}{\mathbf{O}(1)}$$

as in (6.30)

Proof. We use Proposition 2.2 to estimate (6.6). We estimate term by term. To compute the integral, it is convenient to go back to the supersymmetric representation. We have

(7.6)

$$\begin{aligned}\langle k'(\gamma x_i\cdot x_i)\rangle &= \int k'(\gamma x_i\cdot x_i)e^{-2\phi}\prod_{j\in\Lambda}d^2x_j\\ &= \int k'(\gamma x_i\cdot x_i)e^{-(\sum_{j\in\Lambda}EX_j\cdot X_j - \sum_{j,k\in\Lambda,|j-k|_1=1}X_j\cdot X_k - \sum_{j\in\Lambda}k(\gamma X_j\cdot X_j))}\prod_{j\in\Lambda}d^2X_j.\end{aligned}$$

Since
(7.7)
$$\int k'(\gamma X_i\cdot X_i)e^{-(\sum_{j\in\Lambda} EX_j\cdot X_j-\sum_{j,k\in\Lambda,|j-k|_1=1} X_j\cdot X_k-\sum_{j\in\Lambda} k(\gamma X_j\cdot X_j))}\prod_{j\in\Lambda} d^2X_j = k'(0)=\langle v\rangle_g$$

by using Proposition 2.2, and

$$k'(\gamma X_i\cdot X_i)=k'(\gamma x_i\cdot x_i)+\xi_i\eta_i\gamma k''(\gamma x_i\cdot x_i); \tag{7.8}$$

we have
(7.9)
$$\begin{aligned}\langle k'(\gamma x_i\cdot x_i)\rangle = \langle v\rangle_g+\gamma\int \eta_i\xi_i k''(\gamma x_i\cdot x_i)e^{-(\sum_{j\in\Lambda} EX_j\cdot X_j-\sum_{j,k\in\Lambda,|j-k|_1=1} X_j\cdot X_k-\sum_{j\in\Lambda} k(\gamma X_j\cdot X_j))}\\ \prod_{j\in\Lambda} d\xi_j d\eta_j\prod_{j\in\Lambda} d^2x_j\\ =\langle v\rangle_g+\gamma\int k''(\gamma x_i\cdot x_i)(E-\Delta_\Lambda-\gamma\,\mathrm{diag}\,k'(\gamma x_j\cdot x_j))^{-1}_{ii}e^{-2\phi}\prod_{j\in\Lambda} d^2x_j,\end{aligned}$$

where we used (2.12) and integrated over the anticommutative variables. (cf. also (2.17).) Hence the sum of the average of the third and the fifth term in the RHS of (6.6) gives $-\gamma\langle v\rangle_g$.

We now look at the fourth term in (6.6). Using similar arguments as in (7.6)–(7.9), we have
(7.10)
$$\begin{aligned}&\langle x_i^{(\alpha)}x_i^{(\beta)}k''(\gamma x_i\cdot x_i)\rangle\\ =&\frac{1}{2}\delta_{\alpha\beta}\langle x_i\cdot x_i k''(\gamma x_i\cdot x_i)\rangle\\ =&\frac{1}{2}\delta_{\alpha\beta}\int \eta_i\xi_i k''(\gamma x_i\cdot x_i)\\ &\qquad e^{-(\sum_{j\in\Lambda} EX_j\cdot X_j-\sum_{j,k\in\Lambda,|j-k|_1=1} X_j\cdot X_k-\sum_{j\in\Lambda} k(\gamma X_j\cdot X_j))}\prod_{j\in\Lambda} d^2X_j\\ &+\frac{1}{2}\delta_{\alpha\beta}\gamma\langle x_i\cdot x_i k'''(\gamma x_i\cdot x_i)(E-\Delta_\Lambda-\gamma\,\mathrm{diag}\,k'(\gamma x_j\cdot x_j))^{-1}_{ii}\rangle.\end{aligned}$$

After multiplying by $-4\gamma^2$, the second term in the RHS of (7.10) cancelles the average of the sixth term. So we only need to estimate the first term in the RHS of (7.10). After integrating out the anticommutative variables, we have

(7.11)

$$\int \eta_i\xi_i k''(\gamma x_i\cdot x_i)e^{-(\sum_{j\in\Lambda}EX_j\cdot X_j-\sum_{j,k\in\Lambda,|j-k|_1=1}X_j\cdot X_k-\sum_{j\in\Lambda}k(\gamma X_j\cdot X_j))}\prod_{j\in\Lambda}d^2X_j$$

$$=\int k''(\gamma x_i\cdot x_i)(E-\Delta_\Lambda-\gamma\,\mathrm{diag}\,k'(\gamma x_j\cdot x_j))^{-1}_{ii}e^{-2\phi}\prod_{j\in\Lambda}d^2x_j$$

$$=\int k''(\gamma x_i\cdot x_i)\{(E-\Delta_\Lambda-\gamma k'(0))^{-1}_{ii}$$

$$+\gamma[(E-\Delta_\Lambda-\gamma k'(0))^{-1}(\mathrm{diag}(k'-k'(0)))(E-\Delta_\Lambda-k'(0))^{-1}]_{ii}\}e^{-2\phi}\prod_{j\in\Lambda}d^2x_j+\mathbf{O}(\gamma^2)$$

$$=k''(0)(E-\Delta_\Lambda-\gamma\langle v\rangle_g)^{-1}_{ii}+\gamma k'''(0)[(E-\Delta_\Lambda-\gamma\langle v\rangle_g)^{-1}_{ii}]^2+\mathbf{O}(\gamma^2)$$

$$=\langle(v-\langle v\rangle_g)^2\rangle_g(E-\Delta_\Lambda-\gamma\langle v\rangle_g)^{-1}_{ii}+\gamma\langle(v-\langle v\rangle_g)^3\rangle_g[(E-\Delta_\Lambda-\gamma\langle v\rangle_g)^{-1}_{ii}]^2+\mathbf{O}(\gamma^2),$$

where we used Proposition 2.2 and resolvent series for

$$(E-\Delta_\Lambda-\gamma\,\mathrm{diag}\,k'(\gamma x_j\cdot x_j))^{-1}_{ii}$$

about

$$E-\Delta_\Lambda-\gamma k'(0)=E-\Delta_\Lambda-\gamma\langle v\rangle_g.$$

Since the last term of the RHS of (6.6) is of order $\mathbf{O}(\gamma^4)$ by using a similar argument as in (7.10), (7.11), we obtain the Lemma by using (2.8,2.9). □

Remark. We note the rather remarkable cancellations that occured in the computation of $\langle\phi''\rangle$. We suspect that this continues to higher orders.

For all $\delta>0$, let I_δ be defined as in (1.11). Assuming dg satisfies (H1-3), we then have the principal result of the paper:

Theorem 7.2. *For all $\delta>0$, there exists $\gamma_0>0$, such that for all $0<\gamma<\gamma_0$, all $E\in I_\delta$, all μ, $\nu\in\mathbf{Z}^d$,*

(7.12)

$$\begin{aligned}\log\langle G(E,\mu,\nu)\rangle_g &=\log\langle G(E,\mu-\nu,0)\rangle_g\\ &=\log\big(E-\Delta-\gamma\langle v\rangle_g-\gamma^2\langle(v-\langle v\rangle_g)^2\rangle_g[(E-\Delta-\gamma\langle v\rangle_g)^{-1}(0,0)]\\ &\qquad-\gamma^3\langle(v-\langle v\rangle_g)^3\rangle_g[(E-\Delta-\gamma\langle v\rangle_g)^{-1}(0,0)]^2\big)^{-1}(\mu-\nu,0)\\ &\quad+\mathbf{O}(\gamma^4)(|\mu-\nu|+1)\\ &=\log(\widetilde{E}-\Delta)^{-1}(\mu-\nu,0)+\mathbf{O}(\gamma^4)(|\mu-\nu|+1),\end{aligned}$$

where

$$\begin{aligned}\widetilde{E}\overset{\mathrm{def}}{=}&E-\gamma\langle v\rangle_g-\gamma^2\langle(v-\langle v\rangle_g)^2\rangle_g[(E-\Delta-\gamma\langle v\rangle_g)^{-1}(0,0)]\\ &-\gamma^3\langle(v-\langle v\rangle_g)^3\rangle_g[(E-\Delta-\gamma\langle v\rangle_g)^{-1}(0,0)]^2\\ \notin&[-2d,2d].\end{aligned}\tag{7.13}$$

Proof. Since all the estimates are uniform in Λ and the limit $\Lambda\nearrow\mathbf{Z}^d$ exists, combinning (7.1)–(7.5), Lemma 7.1, (6.23) and taking the limit $\Lambda\nearrow\mathbf{Z}^d$, we obtain the Theorem. □

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