CONCAVITY OF EIGENVALUE SUMS AND THE SPECTRAL SHIFT FUNCTION

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Dedicated to Robert Schrader on the occasion of his 60th birthday

ABSTRACT. It is well known that the sum of negative (positive) eigenvalues of some finite Hermitian matrix V is concave (convex) with respect to V. Using the theory of the spectral shift function we generalize this property to self-adjoint operators on separable Hilbert space with an arbitrary spectrum. More precisely, we prove that the spectral shift function integrated with respect to the spectral parameter from $-\infty$ to λ (from λ to $+\infty$) is concave (convex) with respect to trace class perturbations. The case of relative trace class perturbations is also considered.

1. INTRODUCTION AND MAIN RESULTS

Consider an arbitrary Hermitian matrix V. Let $\lambda_i(V)$, $i \in \mathbb{N}$ denote its (repeated) eigenvalues enumerated in the increasing order and consider the following eigenvalues sums

$$S_\lambda^{(-)}(V) = \sum_{i:\lambda_i(V)\leq\lambda} (\lambda_i(V) - \lambda), \qquad S_\lambda^{(+)}(V) = \sum_{i:\lambda_i(V)\geq\lambda} (\lambda_i(V) - \lambda).$$

Equivalently, these sums can be written in the form

$$S_\lambda^{(-)}(V) = -\int_{-\infty}^{\lambda} N^{(-)}(\lambda';V)d\lambda', \qquad S_\lambda^{(+)}(V) = \int_{\lambda}^{\infty} N^{(+)}(\lambda';V)d\lambda',$$

where $N^{(\pm)}$ denote the counting functions, i.e. $N^{(\pm)}(\lambda;V) = \#\{\pm\lambda_i(V) \geq \pm\lambda\}$. By means of the min-max principle it can be easily proved (see e.g. [26, 17]) that $S^{(-)}(V)$ is concave and $S^{(+)}(V)$ is convex with respect to V, i.e. for any Hermitian matrices V_1 and V_2 and any $\alpha \in [0,1]$

$$\pm\, S_\lambda^{(\pm)}(\alpha V_1 + (1-\alpha)V_2) \leq \pm\left(\alpha\, S_\lambda^{(\pm)}(V_1) + (1-\alpha)\, S_\lambda^{(\pm)}(V_2)\right). \tag{1}$$

These inequalities play an important role in several problems of quantum and statistical physics (see e.g. [17] and references therein).

In the present note we show that for a wide class of self-adjoint operators on the separable Hilbert space $\mathfrak{H}$, which need not be finite dimensional or have purely discrete spectrum, the properties (1) are valid for properly regularized $S_\lambda^{(\pm)}$. More precisely, for an arbitrary self-adjoint operator A_0 and any self-adjoint trace class operator V we consider

$$\zeta^{(-)}(\lambda; A_0+V, A_0) := \int_{-\infty}^{\lambda} \xi(\lambda'; A_0+V, A_0)d\lambda' \tag{2}$$

and

$$\zeta^{(+)}(\lambda; A_0+V, A_0) := \int_{\lambda}^{\infty} \xi(\lambda'; A_0+V, A_0)d\lambda', \tag{3}$$

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where $\xi(\lambda; A_0+V, A_0)$ is the spectral shift function for the pair of operators (A_0+V, A_0). A review on the spectral shift function is published by Birman and Yafaev [3] (see also the book [27] and [19, 5, 6, 7] for recent results). Most of the results of the present note have appeared previously in [13] in a slightly less general form.

In the sequel we use the notation $\mathcal{J}_p$, $p \geq 1$ for the von Neumann - Schatten ideals of compact operators such that in particular $\mathcal{J}_1$ denotes the set of the trace class operators (see e.g. [9]). $\operatorname{spec}(A)$ denotes the spectrum of the operator A. Under $\mathcal{Q}(A)$ we understand the domain of the quadratic form associated with the self-adjoint operator A.

Recall that for an arbitrary self-adjoint operator A_0 and any self-adjoint trace class operator V the spectral shift function $\xi(\lambda; A_0+V, A_0)$ exists such that $\xi(\cdot; A_0+V, A_0) \in L^1(\mathbb{R})$. Let $F \in C^1_{\mathrm{loc}}(\mathbb{R})$ be such that its derivative F' belongs to the Wiener class $W(\mathbb{R})$, i.e. $F'(\lambda)$ is representable in the form

$$F'(\lambda) = \int_{\mathbb{R}} e^{-i\lambda t} d\sigma(t),$$

where $\sigma(\cdot)$ is a finite complex-valued Borel measure on $\mathbb{R}$, $|\sigma(\mathbb{R})| < \infty$. Then

$$\operatorname{tr}\left(F(A_0+V) - F(A_0)\right) = \int_{\mathbb{R}} F'(\lambda)\xi(\lambda; A_0+V, A_0)d\lambda. \tag{4}$$

A wider functional class for which the trace formula (4) remains valid is discussed in [2] (see also [3]).

If in some open interval (a,b) the spectrum of A_0 is purely discrete then $\xi(b-0; A_0+V, A_0) - \xi(a+0; A_0+V, A_0)$ equals a.e. the difference of the total multiplicities of the spectra of A_0 and A_0+V lying in (a,b). Thus if we take $A_0 = \lambda_+ I$ with some $\lambda_+ > \sup\operatorname{spec}(V)$, then $\zeta^{(-)}(\lambda; A_0+V, A_0) = S^{(-)}_\lambda(V)$ for all $\lambda < \lambda_+$. Similarly $A_0 = \lambda_- I$ with some $\lambda_- < \inf\operatorname{spec}(V)$ leads to $\zeta^{(+)}(\lambda; A_0+V, A_0) = S^{(+)}_\lambda(V)$ for all $\lambda > \lambda_-$.

Theorem 1. *For an arbitrary real-valued nonincreasing f of bounded total variation the functional*

$$g(V) = \int_{\mathbb{R}} f(\lambda)\xi(\lambda; A_0+V, A_0)d\lambda \tag{5}$$

is concave with respect to the perturbation V, i.e. for arbitrary $V_1,\ V_2 \in \mathcal{J}_1$ the inequality

$$g(\alpha V_1 + (1-\alpha)V_2) \geq \alpha g(V_1) + (1-\alpha)g(V_2) \tag{6}$$

holds for arbitrary $\alpha \in [0,1]$.

In particular we can take $f(\lambda) = \chi_{(-\infty,\lambda_0]}(\lambda)$, the characteristic function of $(-\infty,\lambda_0]$ with arbitrary $\lambda_0 \in \mathbb{R}$, such that $g(V) = \zeta^{(-)}(\lambda_0; A_0+V, A_0)$, the integrated spectral shift function (2). From Theorem 1 it follows that $\zeta^{(-)}(\lambda; A_0+V, A_0)$ is concave with respect to V.

It is known that

$$\int_{\mathbb{R}} \xi(\lambda; A_0+V, A_0)d\lambda = \operatorname{tr} V,$$

which is obviously linear in V. Since an arbitrary nondecreasing function $\widetilde{f}$ of bounded total variation can be represented as a difference of a constant and a nonincreasing f of bounded total variation we obtain

Corollary 2. *For arbitrary real-valued nondecreasing $\widetilde{f}$ of bounded total variation the functional*

$$\widetilde{g}(V) = \int_{\mathbb{R}} \widetilde{f}(\lambda)\xi(\lambda; A_0+V, A_0)d\lambda$$

is convex with respect to the perturbation V, i.e. for arbitrary $V_1,\ V_2 \in \mathcal{J}_1$ the inequality

$$\widetilde{g}(\alpha V_1 + (1-\alpha)V_2) \leq \alpha\widetilde{g}(V_1) + (1-\alpha)\widetilde{g}(V_2)$$

holds for arbitrary $\alpha \in [0,1]$.

In particular $\widetilde{f}(\lambda) = \chi_{[\lambda_0,+\infty)}(\lambda)$ satisfies the conditions of the corollary and thus $\zeta^{(+)}(\lambda; A_0 + V, A_0)$ (3) is convex with respect to V.

Corollary 3. *Let the function f satisfy the conditions of Theorem 1. Let $V(\alpha)$ be a $\mathcal{J}_1$-valued operator family concave (in the operator sense) with respect to α. Then the real-valued function $\alpha \mapsto g(V(\alpha))$ is concave. Similarly, if f satisfies the conditions of Corollary 2 and $V(\alpha)$ is convex, then $\alpha \mapsto \widetilde{g}(V(\alpha))$ is also convex.*

We note that a special case of this result was proved recently by Gesztesy, Makarov, and Motovilov in [8, Corollary 1.9] by different methods.

In the article [4] written by the present author in collaboration with R. Geisler and R. Schrader we have proven that the integrated spectral shift function for the pair of Schrödinger operators is concave with respect to the perturbation potential. Here we will prove that this property holds for an arbitrary pair $(A,\ A_0)$ of self-adjoint semibounded operators on the separable Hilbert space $\mathfrak{H}$ such that their difference is a relative trace class perturbation of A_0.

More precisely we suppose that A_0 is a self-adjoint operator, semibounded from below, and V is also self-adjoint and A_0-compact in the form sense, i.e. $(A_0 + a)^{-1/2} V (A_0 + a)^{-1/2}$ is compact. Then the operator $A_V = A_0 + V$, defined in the form sense, is self-adjoint with $\mathcal{Q}(A_V) = \mathcal{Q}(A_0)$ and also semibounded from below. Suppose that for some $p \geq 1$ and for all sufficiently large a

$$(A_V + a)^{-p} - (A_0 + a)^{-p} \in \mathcal{J}_1. \tag{7}$$

By the invariance principle (see e.g. [3, 27]) the spectral shift function $\xi(\lambda; A_V, A_0)$ is defined by the relation

$$\xi(\lambda; A_V, A_0) = -\xi((\lambda + a)^{-p}; (A_V + a)^{-p}, (A_0 + a)^{-p}). \tag{8}$$

It vanishes for $\lambda < \inf\{\mathrm{spec}(A_V), \mathrm{spec}(A_0)\}$ and satisfies the trace formula (4).

In special case $A_0 = -\Delta$ in $L^2(\mathbb{R}^\nu)$ and V being the multiplication operator by a real-valued measurable function $V(x)$ the conditions above are satisfied with any $p > (\nu - 1)/2$ for $\nu \geq 4$ and with $p = 1$ for $\nu \leq 3$ provided that

$$\begin{array}{lll} V \in L^{\nu/2}(\mathbb{R}^\nu) \cap l^1(L^2(\mathbb{R}^\nu)) & \text{for} & \nu \geq 5, \\ V \in L^r(\mathbb{R}^\nu) \cap l^1(L^2(\mathbb{R}^\nu)) & \text{for} & \nu = 4 \quad \text{and some } r > \nu/2, \\ V \in L^2(\mathbb{R}^\nu) \cap L^1(\mathbb{R}^\nu) & \text{for} & \nu = 2, 3, \\ V \in L^1(\mathbb{R}) & \text{for} & \nu = 1. \end{array}$$

For the definition of the Birman - Solomyak classes $l^p(L^q)$ see e.g. [24].

Let $\mathcal{C}(A_0, a_0)$ denote a set of self-adjoint operators on the separable Hilbert space $\mathfrak{H}$ satisfying the following properties:

(i) every $V \in \mathcal{C}(A_0, a_0)$ is A_0-compact in the form sense;

(ii) there are $a_0 \in \mathbb{R}$ such that $a_0 > -\inf\mathrm{spec}(A_V)$ for all $V \in \mathcal{C}(A_0, a_0)$ and some $p \geq 1$ such that the condition (7) is satisfied for all $V \in \mathcal{C}(A_0, a_0)$ and all $a \geq a_0$;

(iii) the set $\mathcal{C}(A_0, a_0)$ is convex, i.e. $V_1, V_2 \in \mathcal{C}(A_0, a_0)$ implies that $\alpha V_1 + (1-\alpha)V_2 \in \mathcal{C}(A_0, a_0)$ for all $\alpha \in [0,1]$.

We will say that a set possessing these properties is A_0-*convex*. Obviously $\mathcal{C}(A_0, a_0)$ is also A-convex for any operator A such that $A - A_0 \in \mathcal{C}(A_0, a_0)$.

As an example consider two arbitrary A_0-compact in the form sense self-adjoint operators such that

$$(A_0+a)^{-1/2}V_j(A_0+a)^{-p-1/2}\in\mathcal{J}_1,\qquad j=1,2$$

for some $a>-\inf\operatorname{spec}(A_0)$. Any operator lying in the convex hull $\{\alpha V_1+(1-\alpha)V_2,\ \alpha\in[0,1]\}$ of $\{V_1,V_2\}$ is obviously also A_0-compact in the form sense. Take $a_0>a$ such that

$$\|(A_0+a_0)^{-1/2}V_j(A_0+a_0)^{-1/2}\|<1$$

for both $j=1,2$. By Theorem XI.12 of [22] we obtain that the condition (7) is satisfied for all $V=\alpha V_1+(1-\alpha)V_2$ with $\alpha\in[0,1]$ and arbitrary $a\geq a_0$. Thus, the convex hull of $\{V_1,V_2\}$ is A_0-convex.

Theorem 4. *Let A_0 be a self-adjoint operator semibounded from below and $\mathcal{C}(A_0,a_0)$ be some A_0-convex set. Let q equal p if $p=1$ and the smallest odd integer larger than p if $p>1$. For an arbitrary real-valued nonnegative nonincreasing f of bounded total variation on $[-a_0,+\infty)$ such that*

$$\sup_{\lambda\in[-a_0,+\infty)}(1+|\lambda|)^{q+1}|f(\lambda)|<\infty$$

the functional

$$g(V)=\int_{\mathbb{R}}f(\lambda)\xi(\lambda;A_V,A_0)d\lambda$$

is concave on $\mathcal{C}(A_0,a_0)$, i.e. for arbitrary $V_1,\ V_2\in\mathcal{C}(A_0,a_0)$ the inequality

$$g\left(\alpha V_1+(1-\alpha)V_2\right)\geq\alpha g(V_1)+(1-\alpha)g(V_2)\tag{9}$$

holds for arbitrary $\alpha\in[0,1]$.

As discussed in [4] (see also Proposition 3.4 below) the concavity (convexity) of $g(V)$ ($\widetilde{g}(V)$, respectively) implies that $g(\alpha V)$ is subadditive and $\widetilde{g}(\alpha V)$ is superadditive with respect to $\alpha\in\mathbb{R}_+$. Subadditivity and superadditivity properties with respect to the *perturbation* (rather than with respect to the coupling constant) do not hold generally. In the special case of the Schrödinger operators this was observed in [4, 14, 15]. Subadditivity and superadditivity properties of the spectral shift function play an important role in some problems related to the random Schrödinger operators [14, 15]. Also they allow one to study the strong coupling limit. In particular, we have

Corollary 5. *Let A_0 be an arbitrary self-adjoint operator and $V\geq 0$. Assume that either*

(i) V is trace class

or

(ii) A_0 is semibounded from below, V is A_0-compact in the form sense and $(A_0+a)^{-1/2}V\cdot(A_0+a)^{-p-1/2}\in\mathcal{J}_1$ for some $p\geq 1$ and some $a>-\inf\operatorname{spec}(A_0)$.

Then for any nonincreasing function f of bounded total variation, which in the case (ii) satisfies additionally the conditions of Theorem 4, the limit

$$\lim_{\alpha\to\infty}\frac{1}{\alpha}\int_{\mathbb{R}}f(\lambda)\xi(\lambda;A_0+\alpha V,A_0)d\lambda$$

exists and is finite.

For other results related to the strong coupling limit we refer to [20, 21, 23].

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2. Trace Class Perturbations

The proof of Theorem 1 relies on the following result of Birman and Solomyak [2]:

Lemma 2.1. *Let $f \geq 0$ be a nonincreasing function with bounded total variation. Then for any self-adjoint operators A_0 and V on $\mathfrak{H}$, $V \in \mathcal{J}_1$*

(i) the real-valued function

$$\alpha \mapsto \operatorname{tr}[f(A_0 + \alpha V)V]$$

is nonincreasing, i.e. for $\alpha_1 \leq \alpha_2$ the inequality

$$\operatorname{tr}[f(A_0 + \alpha_1 V)V] \geq \operatorname{tr}[f(A_0 + \alpha_2 V)V]$$

holds,

(ii)

$$\int_{\mathbb{R}} f(\lambda)\xi(\lambda; A_0 + \alpha V, A_0)d\lambda = \int_0^\alpha \operatorname{tr}[f(A_0 + sV)V]\, ds.$$

Remark 2.2. *The proof in* [2] *of the part (i) relies on the theory of the double Stieltjes operator integral. An alternative proof not using this formalism is given by Gesztesy, Makarov, and Motovilov in* [8]. *Part (ii) of the lemma is proven in* [2] *for the case $f(\lambda) = \chi_{(-\infty,\lambda_0]}(\lambda)$. The present extension is immediate. Alternative proofs of (ii) have appeared in* [25, 5]. *An operator-valued version of this formula for sign-definite perturbations is given in* [5].

From Lemma 2.1 (i) it follows that

$$G: \ \alpha \mapsto \int_0^\alpha \operatorname{tr}[f(A_0 + sV)V]\, ds$$

is concave. Indeed a necessary and sufficient condition for $G(\cdot)$ to be concave is

$$2G(\alpha) - G(\alpha + h) - G(\alpha - h) \geq 0 \tag{2.1}$$

for all $\alpha \in \mathbb{R}$, $h \geq 0$. Since $\alpha \mapsto \operatorname{tr}[f(A_0 + \alpha V)V]$ is nonincreasing we have

$$\int_\alpha^{\alpha+h} \operatorname{tr}[f(A_0 + sV)V]\, ds - \int_{\alpha-h}^{\alpha} \operatorname{tr}[f(A_0 + sV)V]\, ds \leq 0,$$

which is equivalent to (2.1). Now by the claim (ii) of Lemma 2.1 it follows that the functional $g(V)$ (5) is concave with respect to the coupling constant.

By the chain rule for the spectral shift function (see e.g. [3])

$$\xi(\lambda; A_1 + \alpha V, A_1) = \xi(\lambda; A_1 + \alpha V, A_0) + \xi(\lambda; A_0, A_1),$$

we have that

$$\int_{\mathbb{R}} f(\lambda)\xi(\lambda; A_1 + \alpha V, A_0)d\lambda$$

is also concave with respect to α for arbitrary A_0 and A_1 such that $A_1 - A_0 \in \mathcal{J}_1$. Thus for arbitrary $t_1, t_2 \in \mathbb{R}$ and arbitrary $V \in \mathcal{J}_1$ we have

$$\begin{aligned}&\int_{\mathbb{R}} f(\lambda)\xi(\lambda; A_1 + \alpha t_1 V + (1-\alpha)t_2 V, A_0)d\lambda \\ &\qquad \geq \alpha \int_{\mathbb{R}} f(\lambda)\xi(\lambda; A_1 + t_1 V, A_0)d\lambda + (1-\alpha)\int_{\mathbb{R}} f(\lambda)\xi(\lambda; A_1 + t_2 V, A_0)d\lambda\end{aligned}$$

for all $\alpha \in [0,1]$. Taking $t_1 = 0$, $t_2 = 1$, $A_1 = A_0 + V_1$, and $V = V_2 - V_1$ we obtain

$$g(\alpha V_1 + (1-\alpha)V_2) \geq \alpha g(V_1) + (1-\alpha) g(V_2),$$

thus proving the claim of Theorem 1, however, under the additional requirement that $f \geq 0$. To eliminate this requirement let us consider the function f_1, which differs from $f \geq 0$ by a negative constant c. Since $\operatorname{tr} V$ is linear in V, the induced functional

$$g_1(V) = \int_{\mathbb{R}} f_1(\lambda)\xi(\lambda; A_0 + V, A_0)d\lambda = g(V) + c \operatorname{tr} V.$$

is also concave in V. This completes the proof of Theorem 1.

Proof of Corollary 3. By Theorem 1

$$g(\alpha V_1 + (1-\alpha)V_2) \geq \alpha g(V_1) + (1-\alpha)g(V_2) \tag{2.2}$$

for all $\alpha \in [0, 1]$. By the monotonicity of the spectral shift function with respect to the perturbation $g(V)$ is nondecreasing with respect to V, i.e. $g(V_1) \geq g(V_2)$ for $V_1 \geq V_2$.

Let now $V_1 = V(\beta_1)$ and $V_2 = V(\beta_2)$. By the concavity of $V(\alpha)$, i.e. by

$$V(\alpha\beta_1 + (1-\alpha)\beta_2) \geq \alpha V(\beta_1) + (1-\alpha)V(\beta_2),$$

and by the monotonicity of $g(V)$, from (2.2) it follows that

$$g(V(\alpha\beta_1 + (1-\alpha)\beta_2)) \geq \alpha g(V(\beta_1)) + (1-\alpha)g(V(\beta_2)).$$

The second part of the claim can be proved similarly. □

3. Relative Trace Class Perturbations

We turn to the case of relative trace class perturbations of A_0 and prove Theorem 4. The conditions of Theorem 4 imply that

$$\sup_{\lambda\in[-a_0,+\infty)} (\lambda + a_0)^{q+1}|f(\lambda)| < \infty \tag{3.1}$$

with q being defined as in Theorem 4. Choose arbitrary $V_1, V_2 \in \mathcal{C}(A_0, a_0)$. Obviously, $\{(1-\alpha)(V_2 - V_1),\ \alpha \in [0, 1]\} \subseteq \mathcal{C}(A_0, a_0) - V_1$ and the set $\mathcal{C}_1(A_0, a_0) := \mathcal{C}(A_0, a_0) - V_1$ is A_0-convex. Note that $0 \in \mathcal{C}_1(A_0, a_0)$. Thus, as in the case of trace class perturbations it suffices to prove that for any $V \in \mathcal{C}_1(A_0, a_0)$ the function $g(\alpha V)$ is concave with respect to $\alpha \in [0, 1]$.

We start with the simplest case $p = 1$ in the condition (7). For all $a \geq a_0$ the resolvents $(A_{\alpha V}+a)^{-1}$ and $(A_0+a)^{-1}$ are bounded nonnegative operators. By assumption their difference is trace class and therefore the spectral shift function $\xi(\lambda; A_{\alpha V}, A_0)$ can be defined by means of the invariance principle as given by (8). For all $a \geq a_0$ it satisfies the inequality

$$\int_{\mathbb{R}} \frac{|\xi(\lambda; A_{\alpha V}, A_0)|}{(\lambda + a)^2} d\lambda \leq \|(A_{\alpha V} + a)^{-1} - (A_0 + a)^{-1}\|_{\mathcal{J}_1}. \tag{3.2}$$

Lemma 3.1. *Let $f(\lambda) \geq 0$ be a nonincreasing function of bounded total variation. Then for all $a \geq a_0$*

$$g_a(\alpha) = \int_{\mathbb{R}} \frac{f(\lambda)}{(\lambda + a)^2}\xi(\lambda; A_{\alpha V}, A_0)d\lambda$$

is concave with respect to α.

Proof. We change the integration variable $\lambda \to t = (\lambda + a)^{-1}$ and use the invariance principle (8) to obtain

$$g_a(\alpha) = -\int_0^\infty f\left(\frac{1-at}{t}\right)\xi(t; (A_{\alpha V} + a)^{-1}, (A_0 + a)^{-1})dt. \tag{3.3}$$

It is easy to see that $f((1 - at)/t)$ is nondecreasing with respect to t. It is well known (see e.g. [16, 1], [18, Proposition 1.3.11]) that the function $x \mapsto x^{-1}$ is concave on the set of invertible positive operators, i.e. for arbitrary invertible positive operators X and Y the inequality $(\beta X + (1-\beta)Y)^{-1} \leq \beta X^{-1} + (1-\beta)Y^{-1}$ holds in the operator sense for all $\beta \in [0,1]$. Taking $X = A_{\alpha_1 V} + a$ and $Y = A_{\alpha_2 V} + a$ with an arbitrary $a \geq a_0$ and using the fact that $\beta X + (1-\beta)Y = A_{(\beta\alpha_1+(1-\beta)\alpha_2)V} + a$ we obtain that

$$(A_{(\beta\alpha_1+(1-\beta)\alpha_2)V} + a)^{-1} \leq \beta(A_{\alpha_1 V} + a)^{-1} + (1-\beta)(A_{\alpha_2 V} + a)^{-1}$$

for all $\beta \in [0,1]$, i.e. the operator $(A_{\alpha V} + a)^{-1}$ is convex with respect to $\alpha \in \mathbb{R}$. Therefore by Corollary 3 the integral in (3.3) is convex with respect to α and thus $g_a(\alpha)$ is concave. $\square$

From (3.1) it follows that the function f satisfies the condition

$$\sup_{\lambda\in[-a_0,+\infty)} (\lambda + a_0)^2 |f(\lambda)| < \infty.$$

Since $\xi(\lambda; A_{\alpha V}, A_0) = 0$ for all $\lambda \leq -a_0$ we may suppose that $\lambda \geq -a_0$. Thus for all $a \geq 2a_0$ we have $a^2(a+\lambda)^{-2} \leq 4$. Obviously,

$$\begin{aligned} &\left| \frac{a^2}{(\lambda+a)^2} f(\lambda)\xi(\lambda; A_{\alpha V}, A_0) \right| \\ &\quad \leq \; 4\frac{|\xi(\lambda; A_{\alpha V}, A_0)|}{(\lambda + a_0)^2} \sup_{\lambda\in[-a_0,+\infty)} (\lambda + a_0)^2 |f(\lambda)|. \end{aligned}$$

Therefore, by (3.2) and by the Lebesgue dominated convergence theorem we have

$$\lim_{a\to+\infty} a^2 g_a(\alpha) = \int_{\mathbb{R}} f(\lambda)\xi(\lambda; A_{\alpha V}, A_0)d\lambda.$$

From Lemma 3.1 it follows now that the integral on the r.h.s. is concave with respect to α. As noted above the concavity with respect to the coupling constant implies the concavity with respect to the perturbation. This remark completes the proof of Theorem 4 in the case $p = 1$.

We turn to the case $p > 1$ in the condition (7) and note that the operator $(A_{\alpha V} + a_0)^{-p}$ is neither convex nor concave with respect to α [16, 1]. To treat this case we need the following

Lemma 3.2. *(*[12, Theorem 1], [27, Theorem 8.10.4]*) Assume that $A^p - A_0^p \in \mathcal{J}_1$ for some $p > 1$. Then $A - A_0 \in \mathcal{J}_q$ for any $q > p$. Let $\{P_n\}_{n\in\mathbb{N}}$ be a strictly monotone family of finite dimensional orthogonal projections converging strongly to the identity operator I. Then*

$$\xi(\lambda; A, A_0) = \lim_{n\to\infty} \xi(\lambda; A_0 + P_n(A - A_0)P_n, A_0)$$

in $L^1(\mathbb{R}; \lambda^{q-1}d\lambda)$, where q is the smallest odd integer greater than p.

The following lemma generalizes Lemma 3.1 to the case $p > 1$:

Lemma 3.3. *Let $f(\lambda) \geq 0$ be a nonincreasing function of bounded total variation. Then for all $a \geq a_0$*

$$g_a(\alpha) = \int_{\mathbb{R}} \frac{f(\lambda)}{(\lambda + a)^{q+1}} \xi(\lambda; A_{\alpha V}, A_0)d\lambda$$

is concave with respect to α.

Proof. By the invariance principle the spectral shift function $\xi(\lambda; A_V, A_0)$ can be represented in the form

$$-\xi((\lambda + a)^{-1}; (A_V + a)^{-1}, (A_0 + a)^{-1}).$$

We introduce the operator $W(\alpha) = (A_{\alpha V}+a)^{-1}-(A_0+a)^{-1}$. Let $\{P_n\}_{n\in\mathbb{N}}$ be a family of finite dimensional orthogonal projections as in Lemma 3.2. Consider $W_n(\alpha) = P_n W(\alpha) P_n \in \mathcal{J}_1$ and define

$$\begin{aligned} g_a^{(n)}(\alpha) &= -\int_{\mathbb{R}} \frac{f(\lambda)}{(\lambda+a)^{q+1}}\xi((\lambda+a)^{-1}; (A_0+a)^{-1} + W_n(\alpha), (A_0+a)^{-1})d\lambda \\ &= -\int_0^\infty f\left(\frac{1-at}{t}\right) t^{q-1}\xi(t; (A_0+a)^{-1} + W_n(\alpha), (A_0+a)^{-1})dt \end{aligned}$$

with q being defined as in Theorem 4. Recall that $W(\alpha)$ is convex with respect to α and therefore $P_n W(\alpha) P_n$ is also convex. Thus by Corollary 3 the function $g_a^{(n)}(\alpha)$ is concave for every $n \in \mathbb{N}$. To prove that

$$\lim_{n\to\infty} g_a^{(n)}(\alpha) = g_a(\alpha) \tag{3.4}$$

we estimate as follows

$$\begin{aligned} |g_a(\alpha) - g_a^{(n)}(\alpha)| \leq \sup_{\lambda\in[-a_0,+\infty)} |f(\lambda)| \int_0^\infty t^{q-1}\Big|\xi(t; (A_0+a)^{-1} + W(\alpha), (A_0+a)^{-1}) \\ -\xi(t; (A_0+a)^{-1} + W_n(\alpha), (A_0+a)^{-1})\Big| dt. \end{aligned}$$

By Lemma 3.2 the r.h.s. tends to zero thus proving (3.4) and completing the proof of the lemma. □

To complete the proof of Theorem 4 as in the case $p = 1$ we consider the limit of $a^{q+1}g_a(\alpha)$ as $a \to +\infty$. By the inequality

$$\begin{aligned} \int_{\mathbb{R}} \frac{|\xi(\lambda; A_{\alpha V}, A_0)|}{(\lambda+a)^{q+1}} d\lambda &\leq \int_{\mathbb{R}} \frac{|\xi(\lambda; A_{\alpha V}, A_0)|}{(\lambda+a)^{p+1}} d\lambda \\ &\leq \left\|(A_{\alpha V}+a)^{-p} - (A_0+a)^{-p}\right\|_{\mathcal{J}_1} \end{aligned}$$

valid for all $a \geq a_0 + 1$ and again by the Lebesgue dominated convergence theorem we obtain

$$\lim_{a\to+\infty} a^{q+1} g_a(\alpha) = \int f(\lambda)\xi(\lambda; A_{\alpha V}, A_0)d\lambda.$$

Now from Lemma 3.3 it follows that the integral on the r.h.s. is concave with respect to α. This completes the proof of Theorem 4.

We turn to the proof of Corollary 5.

Proposition 3.4. *Under the assumptions of Corollary 5 (but without the restriction $V \geq 0$) the functional $g(V)$ is subadditive in the coupling constant in the sense that for arbitrary $\alpha_1, \alpha_2 \geq 0$*

$$g((\alpha_1+\alpha_2)V) \leq g(\alpha_1 V) + g(\alpha_2 V). \tag{3.5}$$

Moreover, for arbitrary $\alpha_1, \alpha_2 \geq 0$ the inequality

$$g((\alpha_1-\alpha_2)V) \geq g(\alpha_1 V) + g(-\alpha_2 V) \tag{3.6}$$

holds.

Proof. The assumption (ii) of the Corollary 5 and the proof of Theorem XI.12 in [22] imply that for an arbitrary finite interval $[a, b] \subset \mathbb{R}$ there is finite $a_0 \in \mathbb{R}$ such that $(A_{\alpha V}+a)^{-p} - (A_0+a)^{-p} \in \mathcal{J}_1$ for all $a \geq a_0$. Thus we may set $\mathcal{C}(A_0, a_0) = \{\alpha V,\ \alpha \in [a, b]\}$. By Theorem 4 we obtain that $g(\alpha V)$ is concave with respect to $\alpha \in [a, b]$. Since a and b are arbitrary the function $g(\alpha V)$ is concave on $\mathbb{R}$. In the case of assumption (i) the concavity of $g(\alpha V)$ for all $\alpha \in \mathbb{R}$ is guaranteed directly by Theorem 1.

Recall that the necessary and sufficient condition [11, Theorem 6.2.4] for a measurable concave function $\phi(\alpha)$ to be subadditive on $\mathbb{R}_+$ is that $\phi(+0) \geq 0$. This proves (3.5). To prove (3.6) we use the fact (see e.g. [10, Theorem 110]) that any continuous concave function $\phi(x)$ satisfies the inequality

$$\phi(x-h') + \phi(x+h') \geq \phi(x-h) + \phi(x+h) \tag{3.7}$$

provided that $|h| \geq |h'|$. We set $x = (\alpha_1 - \alpha_2)/2$, $h' = (\alpha_2 - \alpha_1)/2$, $h = (\alpha_2 + \alpha_1)/2$ and apply the inequality (3.7) to the function $g(\alpha V)$. Since $g(0) = 0$ we arrive at the claim (3.6). □

Proof of Corollary 5. Let $\gamma = \inf_{\alpha>0} \alpha^{-1}\phi(\alpha)$. Recall (see e.g. [11, Theorem 6.6.1]) that if $\phi(\alpha)$ is a measurable subadditive function, which is finite for all finite α, then $-\infty \leq \gamma < \infty$ and

$$\lim_{\alpha\to+\infty} \frac{\phi(\alpha)}{\alpha} = \gamma.$$

We take $\phi(\alpha) = g(\alpha V)$. By Proposition 3.4 it is subadditive on $\mathbb{R}_+$. By the monotonicity propertiy of the spectral shift function the condition $V \geq 0$ implies that $\phi(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}_+$. Therefore $\gamma \geq 0$, thus proving the corollary. □

References

[1] J. Bendat and S. Sherman, *Monotone and convex operator functions*, Trans. Amer. Math. Soc. **79**, 58 – 71 (1955).

[2] M.Sh. Birman and M. Z. Solomyak, *Remarks on the spectral shift function*, J. Sov. Math. **3**, 408 – 419 (1975).

[3] M.Sh. Birman and D.R. Yafaev, *The spectral shift function. Work by M.G.Krein and its further development*, St. Petersburg Math. J. **4**, 833 – 870 (1993).

[4] R. Geisler, V. Kostrykin, and R. Schrader, *Concavity properties of Krein's spectral shift function*, Rev. Math. Phys. **7**, 161 – 181 (1995).

[5] F. Gesztesy, K.A. Makarov, and S.N. Naboko, *The spectral shift operator*, p. 59 – 90 in J. Dittrich, P. Exner, and M. Tater (Eds.) *Mathematical Results in Quantum Mechanics*, Operator Theory: Advances and Applications, Vol. 108, Birkhäuser, Basel, 1999.

[6] F. Gesztesy and K.A. Makarov, *Some applications of the spectral shift operator*, in A.G. Ramm, P.N. Shivakumar, and A.V. Strauss (Eds.) Fields Institute Communication Series, Amer. Math. Soc., Providence, RI (to appear).

[7] F. Gesztesy and K.A. Makarov, *The Ξ operator and its relation to Krein's spectral shift operator*, preprint (1999); available from `http://www.ma.utexas.edu/mp_arc/` as 99-112.

[8] F. Gesztesy, K.A. Makarov, and A.K. Motovilov, *Monotonicity and concavity properties of the spectral shift function*, preprint (1999); available from `http://www.ma.utexas.edu/mp_arc/` as 99-339.

[9] I.C. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space*, Amer. Math. Soc. Transl. of Math. Monographs 18, Providence, RI, 1969.

[10] G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Univ. Press, 1934.

[11] E. Hille, *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloquim Publications, Vol.31, New York, 1948.

[12] L.S. Koplienko, *On the theory of the spectral shift function*, Problemy Mat. Fiz. **5**, 62 – 79 (1971) (Russian).

[13] V. Kostrykin, *The Spectral Shift Function and Its Applications to Random Schrödinger Operators*, Habilitationsschrift, RWTH Aachen, 1999 (unpublished).

[14] V. Kostrykin and R. Schrader, *Scattering theory approach to random Schrödinger operators in one dimension*, Rev. Math. Phys. **11**, 187 – 242 (1999).

[15] V. Kostrykin and R. Schrader, *The density of states and the spectral shift density of random Schrödinger operators*, Rev. Math. Phys. (to appear); available from `http://www.ma.utexas.edu/mp_arc/` as 99-239.

[16] F. Kraus, *Über konvexe Matrixfunktionen*, Math. Z. **41**, 18 – 42 (1936).

[17] E.H. Lieb and H. Siedentop, *Convexity and concavity of eigenvalue sums*, J. Stat. Phys. **63**, 811 – 816 (1991).

[18] G.K. Pedersen, *C^*-Algebras and Their Automorphism Groups*, Academic Press, London, 1979.

[19] A.B. Pushnitski, *Representation of the spectral shift function for perturbations of fixed signs*, St. Petersburg Math. J. **9**, 1181 – 1194 (1998).

[20] A.B. Pushnitski, *Integral estimates for the spectral shift function*, St. Petersburg Math. J. **10** (1999) (to appear); available from `http://www.ma.utexas.edu/mp_arc/` as 98-415.

[21] A.B. Pushnitski, *Spectral shift function of the Schrödinger operator in the large coupling constant limit*, Commun. in PDE (to appear); available from `http://www.ma.utexas.edu/mp_arc/` as 98-417.

[22] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, III: Scattering Theory*, Academic Press, New York, 1979.

[23] O. Safronov, *Spectral shift function in the large coupling constant limit*, preprint (1999); available from `http://www.ma.utexas.edu/mp_arc/` as 99-332.

[24] B. Simon, *Trace Ideals and Their Applications*, Cambridge University Press, New York, 1979.

[25] B. Simon, *Spectral averaging and the Krein spectral shift*, Proc. Amer. Math. Soc. **126**, 1409 – 1413 (1998).

[26] W. Thirring, *Lehrbuch der mathematischen Physik, Vol. III: Quantenmechanik von Atomen und Molekulen*, Springer, Vienna, 1979.

[27] D.R. Yafaev, *Mathematical Scattering Theory. General Theory*, Amer. Math. Soc. Transl. of Math. Monographs Vol. 105, Providence, RI, 1992.

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