# POSITIVE SOLUTIONS TO SINGULAR SECOND AND THIRD ORDER DIFFERENTIAL EQUATIONS FOR QUANTUM FLUIDS

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Abstract

A steady-state hydrodynamic model for quantum fluids is analyzed. The momentum equation can be written as a dispersive third-order equation for the particle density where viscous effects can be incorporated. The phenomena that admit positivity of the solutions are studied. The cases: dispersive or non-dispersive, viscous or non-viscous are thoroughly analyzed with respect to positivity and existence or non-existence of solutions. It is proven that in the dispersive, non-viscous model, a classical positive solution only exists for "small" data and no weak solution can exist for "large" data, whereas the dispersive, viscous problem admits a classical positive solution for all data. The viscous term is shown to correspond to hyper-viscosity or ultra-diffusion. The proofs are based on a reformulation of the equations as a singular elliptic second-order problem and on a variant of the Stampacchia truncation technique. The results are extended to general third-order equations in any space dimension.

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## 1. Introduction

The present paper is concerned with hydrodynamic models for quantum fluids. The evolution of the quantum fluid is governed by the conservation laws of mass and momentum for the particle density $n$ and the particle current density $J$:

$$\frac{\partial n}{\partial t} + \mathrm{div}\ J = 0, \tag{1.1}$$

$$\frac{\partial J}{\partial t} + \mathrm{div}\Big(\frac{J \otimes J}{n} + P\Big) - nF = W - \nu B, \tag{1.2}$$

where $P = (P_{ik})$ denotes the pressure term, $F$ the sum of the external forces, $W$ the momentum relaxation term, and $\nu B$ the viscous term with viscosity $\nu \geq 0$. The tensor product $J \otimes J$ is given by the components $J_i J_k$ with $i, k = 1, \ldots, d$.

We consider an isothermal or isentropic quantum fluid of charged particles. Then, the pressure tensor is assumed to be of the form $P = (Tp(n)\delta_{ik})$ where $\delta_{ik}$ is the Kronecker symbol. The pressure function $p$ is given by the relation $p(n) = n$ in the isothermal case and $p(n) = n^\alpha$ with $\alpha > 1$ in the isentropic case, and $T$ is a (scaled) temperature constant. We assume that the external force is the gradient of the sum of the electric potential $V$ and the quantum Bohm potential

$$Q = \delta^2 \frac{1}{\sqrt{n}} \Delta \sqrt{n},$$

$\delta > 0$ being the scaled Planck constant. Equations (1.1)–(1.2) are coupled to Poisson's equation for the electric potential,

$$\lambda^2 \Delta V = n - C, \tag{1.3}$$

where $\lambda > 0$ denotes the scaled Debye length, and $C = C(x)$ models fixed charged background ions. The relaxation term is given by $W = -J/\tau$, $\tau = \tau(x) > 0$ being the relaxation time. The choice of the viscous term $\nu B$ will be precised below. We show that accounts for an ultra-diffusive effect, (also called hyper-viscosity.)

With these assumptions, the stationary quantum hydrodynamic equations with viscosity can be formulated as

$$\mathrm{div}\ J = 0, \tag{1.4}$$

$$\mathrm{div}\Big(\frac{J\otimes J}{n}\Big) + T\nabla p(n) - n\nabla V - \delta^2 n\nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big) = -\frac{J}{\tau} - \nu B. \tag{1.5}$$

The equations (1.3)-(1.5) are solved in a bounded domain $\Omega\subset\mathbb{R}^d$ $(d\geq 1)$ occupied by the fluid. Eqs. (1.4)–(1.5) are in scaled form; we refer to [Jün97] for the choice of the scaling.

In the case $\delta = 0$ and $\nu = 0$, we get the classical hydrodynamic equations which are considered, for instance, by Degond and Markowich [DM93,Mar91] in two and three space dimensions for subsonic flow, and by Gamba [Gam92] in one space dimension for transonic flow. The two-dimensional viscous hydrodynamic equations $\delta = 0$ and $\nu > 0$ are studied, in the potential flow formulation, by Gamba and Morawetz [Gam97,GaMo96].

The quantum hydrodynamic equations $\delta > 0$ and $\nu = 0$ arise in semiconductor modeling where it has been used for analyzing the flow of electrons in quantum semiconductor devices, like resonant tunneling diodes [Gar94]. Very similar model equations have been employed in other areas of physics, e.g. in super fluidity [LM93] and in superconductivity [Fey72]. We refer to [Gar94,GaMa97,GMR96] for a justification and derivation of the quantum hydrodynamic equations. Mathematically, these models have been studied by Jerome and Zhang [ZJ96] and Gyi and Jüngel [GJ98] in one space dimension and by Jüngel [Jün98] in several space dimensions. The existence of strong solutions of the boundary-value problem (with Dirichlet or Neumann boundary conditions) could be proved under a smallness condition on the data which corresponds essentially to a subsonic condition of the underlying classical hydrodynamic problem. However, no results are available for 'large' data.

Viscous or diffusive terms in the quantum hydrodynamic equations are recently derived by Arnold *et al.* [ALMS98]. We will show that for a class of special viscosities, a positive solution exists for all data.

Consider the one-dimensional equations (1.3)–(1.5)

$$\left(\frac{J^2}{n}+Tp(n)\right)_x - nV_x - \delta^2 n\left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)_x = -\frac{J}{\tau(x)} - \nu n(\beta(n))_{xx}, \tag{1.6}$$

$$\lambda^2 V_{xx} = n - C(x), \tag{1.7}$$

in the interval $\Omega = (0,1)$ for prescribed $J > 0$, subject to the boundary conditions

$$n(0) = n_0, \quad n(1) = n_1, \quad V(0) = V_0, \quad V_x(0) = -E_0. \tag{1.8}$$

We choose

$$\beta(n) = -\frac{1}{\gamma-1}\frac{1}{n^{(\gamma-1)/2}} \quad \text{with } \gamma > 4.$$

If $\delta > 0$, a fifth boundary condition is needed for well-posedeness; see (1.9). The main objective of this paper is to understand the phenomena that admit existence and positivity of the solutions to the *third-order* boundary value problem (1.6)–(1.8). It is well known that the positivity of solutions to higher order equations is a delicate problem since maximum principle arguments generally do not apply. We refer to [BP98,BLS94,PGG98] for recent studies of the positivity of solutions to *fourth-order* elliptic equations where similar difficulties arise. It turns out that for the problem (1.6)–(1.8), the *ultra-diffusive* term given by $\beta$ prevents the solution from cavitating. More precisely, let us consider the following cases:

**Case $\delta = 0$, $\nu > 0$.** This problem has been studied by Gamba [Gam92] with $B = n_{xx}$. It is shown that there exists a positive solution to the one-dimensional equations (1.6)–(1.8). Moreover, the lower and upper bounds for this solution do not depend on the viscosity $\nu$, and it is possible to perform rigorously the limit $\nu \to 0$.

**Case $\delta > 0$, $\nu > 0$.** For any given $J > 0$, there exists a classical solution $(n, V)$ to (1.6)–(1.8) satisfying the additional boundary condition

$$\delta^2\frac{(\sqrt{n})_{xx}(0)}{\sqrt{n_0}} - \nu\beta'(n)n_x(0) = \frac{J^2}{2n_0^2} + Th(n_0) - V_0 + K, \tag{1.9}$$

where $h(s)$ is the enthalpy function defined by $h'(s) = sp'(s)$, $s > 0$, and $h(1) = 0$, and $K > 0$ is a constant whose value is given below (see (1.13)). Furthermore, the particle density satisfies

$$0 < m(\nu) \le n(x) \le M \quad \text{for all } x \in \Omega.$$

The constants $m(\nu)$ and $M$ do not depend on $\delta > 0$, but $m(\nu)$ depends on $\nu$ such that $m(\nu) \to 0$ as $\nu \to 0$ (Theorem 2.1). In order to get an explicit lower bound for $n$, the *ultra-diffusive* term $\nu n\beta(n)_{xx}$ is necessary. This term is used to control the convective term $J^2/n$.

**Case $\delta > 0$, $\nu = 0$.** Without viscosity, we only get the existence of 'subsonic' solutions. We call a solution $(n, V)$ to (1.6)–(1.8) 'subsonic' if the density $n$ is positive and satisfies the condition

$$J/n < \sqrt{Tp'(n)} \qquad \text{in } \Omega.$$

More precisely, under some conditions on the pressure $p$, there exists a constant $J_0 > 0$ such that for all $0 < J \leq J_0$ there exists a classical solution $(n, V)$ to (1.6)–(1.9) with positive lower and upper bounds for $n$ not depending on $\delta$ or $\nu$. This solution is 'subsonic'. Moreover, there exists a constant $J_1 > 0$ such that for all $J \geq J_1$ the problem (1.6)–(1.9) cannot have a weak solution with positive $n$ (Corollary 4.5). Finally, we can prove uniqueness in the class of 'subsonic' solutions (Theorem 5.1).

We emphasize the fact that the above results can be extended to *multi-dimensional* problems. In this situation, the current density $J$ is no longer constant and a different formulation, for instance a potential-flow formulation as in [Jün98], has to be employed (see [GaJü99]). In the final section of this paper we give some details how to extend our techniques to general multi-dimensional third-order problems of the type

$$\nabla(A(u)\Delta u) = \lambda\nabla(F(u)) + \nabla(G(u)) + \nu\nabla(B(u)(1\cdot\nabla)u) \quad \text{in } \Omega \subset \mathbb{R}^d,$$
$$u = u_D \quad \text{on } \partial\Omega,$$

where $(1 \cdot \nabla)u = \sum_j \partial_j u$. The nonlinear functions $A$, $B$, $F$ and $G$ satisfy some conditions. This problem is singular since $F(u) = u^{-\alpha}$ with $\alpha > 0$ is admissible. We notice that third-order equations are also used in the modeling of long water waves in channels with small depths (Korteweg-de-Vries equation; see, e.g., [KPV93]) and of light waves guided in an optical fiber (third-order Schrödinger equation; see [Lau97]). In fact, we can prove for this problem that in case $\nu = 0$ there exists a solution for sufficiently small $\lambda > 0$, whereas there is no solution for sufficiently large $\lambda > 0$ (Theorem 6.2). If $\nu > 0$ then a solution exists for all $\lambda > 0$ (Theorem 6.1).

Notice that the quantum hydrodynamic Eq. (1.5) can be written via the change of variable $u = \sqrt{n}$ in the form

$$\nabla(A(u)\Delta u) = f(u, \nabla F(u), \nu B),$$

where $A(u) = \delta^2 u^{-1}$ and $F(u) = u^{-2}$. Thus the study of the above general third-order problem is an important step in the existence analysis of the *multi-dimensional* quantum hydrodynamic equations (see [GaJü99]).

For the proofs of the above results we combine the techniques developed by the authors [Gam92,GaMo96,GyJü98,Jün98]. The main idea is to integrate Eq. (1.6) once in order to get an elliptic *singular* second-order problem for which maximum principle arguments apply. The explicit lower and upper bounds for the particle density are obtained by using a variant of the Stampacchia truncation method [Sta66]. For the existence results we employ the Leray-Schauder fixed point theorem.

In order to derive the system of second-order equations, we rewrite Eq. (1.6):

$$n\left[\frac{J^2}{2n^2} + Th(n) - V - \delta^2\frac{(\sqrt{n})_{xx}}{\sqrt{n}} + J\int_0^x \frac{ds}{\tau n} + \nu\beta(n)_x\right]_x = 0.$$

This implies, if $n > 0$,

$$\frac{J^2}{2n^2} + Th(n) - V - \delta^2\frac{(\sqrt{n})_{xx}}{\sqrt{n}} + J\int_0^x \frac{ds}{\tau n} + \nu\beta(n)_x = -K,$$

where $K$ is a constant. Observing that $\beta(n) = -n^{-(\gamma-1)/2}/(\gamma-1)$ and setting $w = \sqrt{n}$ gives

$$\delta^2 w_{xx} = \frac{J^2}{2w^3} + Twh(w^2) - Vw + Kw + Jw\int_0^x \frac{ds}{\tau w^2} + \nu\frac{w_x}{w^\gamma}, \tag{1.10}$$

$$\lambda^2 V_{xx} = w^2 - C(x). \tag{1.11}$$

These equations have to be solved in $\Omega = (0,1)$ subject to the boundary conditions

$$w(0) = w_0\ , \quad w(1) = w_1, \quad V(0) = V_0, \quad V_x(0) = -E_0, \tag{1.12}$$

where $w_0 = \sqrt{n_0}$ and $w_1 = \sqrt{n_1}$. For the constant $K$ we choose

$$K \stackrel{\text{def}}{=} V_0 + \max(-E_0, 0) + \lambda^{-2}M^2, \text{ where} \tag{1.13}$$

$$M \stackrel{\text{def}}{=} \max(w_0, w_1, M_0), \tag{1.14}$$

and $M_0$ is such that $h(M_0^2) \geq 0$. The constant $K$ is taken in such a way that $-V(x)+K \geq 0$ holds (see Section 2). Notice that every solution $(w, V)$ of (1.10)–(1.12) satisfying $w > 0$ in $\Omega$ gives a solution $(n, V)$ to (1.6)–(1.7) subject to the boundary conditions (1.8)–(1.9).

In this paper we impose the following assumptions:

(H1) $h \in C^1(0,\infty)$ and $p'$ (defined by $p'(s) = h'(s)/s$, $s > 0$) are non-decreasing, and $h$ satisfies

$$\lim_{s\to\infty} h(s) > 0 \ , \quad \lim_{s\to 0+} h(s) < 0 \ , \quad \lim_{s\to 0+} \sqrt{s}\, h(s) > -\infty \ . \tag{1.15}$$

(H2) $C \in L^2(\Omega)$, $C \geq 0$ in $\Omega$; $\tau \in L^\infty(\Omega)$, $\tau(x) \geq \tau_0 > 0$ in $\Omega$.

(H3) $J, w_0, w_1, \delta, \lambda, T > 0$ ; $\nu \geq 0$ ; $V_0, E_0 \in \mathbb{R}$,

Typical examples for $h$ are $h(s) = \log(s)$, $s > 0$, or $h(s) = s^{\alpha-1} - 1$, $s \geq 0$, with $\alpha > 1$.

The outline of the paper is as follows. In Section 2 we prove the existence of solutions to (1.10)–(1.12) for $\nu > 0$. Section 3 is devoted to the existence analysis for the case $\nu = 0$. The non-existence of solutions to (1.10)–(1.12) for $\nu = 0$ is shown in Section 4. In Section 5 the uniqueness of 'subsonic' solutions is proved. Finally, in Section 6 we extend our methods to third-order equations in several space dimensions.

## 2. Existence of solutions for $\nu > 0$

In this section we prove the following theorem:

**Theorem 2.1.** *Let the hypotheses* (H1)–(H3) *hold and let* $\nu > 0$. *Then, for any* $J > 0$, *there exists a classical solution* $(w, V) \in (C^2(\overline{\Omega}))^2$ *to* (1.10)–(1.12) *satisfying*

$$0 < m(\nu) \leq w(x) \leq M, \quad \text{for all } \ x \in \Omega \ .$$

*Remark.* The constant $m(\nu)$ is defined by

$$m(\nu) = \min(w_0, w_1, m_1, m_2) \ ,$$

where

$$h(4m_1^2) \leq 0 \ ,$$

$$m_2 \leq \left( \frac{1}{2^{\gamma+1}} \frac{\nu}{J^2/2 + J/\tau_0 + \max(0, K-k)} \right)^{1/(\gamma-4)} \ , \quad \text{and}$$

$$k = V_0 - \max(E_0, 0) + \lambda^{-2} \|C\|_{L^1(\Omega)} \ .$$

The constant $M$ is defined in (1.14).

In order to prove Theorem 2.1, define the function

$$f(x) = \varepsilon(2-x), \quad x \in [0,1], \quad 0 < \varepsilon < \min(1, M/2),$$

and consider the truncated problem

$$\delta^2 w_{xx} = \frac{J^2 w}{2t_\varepsilon(w)^4} + Twh(w^2) - Vw + Kw + Jw\int_0^x \frac{ds}{\tau t_\varepsilon(w)^2} + \nu\frac{(t_f(w_M))_x w}{t_f(w_M)^{\gamma+1}}\ , \tag{2.1}$$

$$\lambda^2 V_{xx} = w^2 - C(x) \ \text{ in } \ \Omega\ , \tag{2.2}$$

where $t_\varepsilon(w) = \max(\varepsilon, w)$ and $t_f(w_M) = \max\big(f(\cdot), \min(M, w(\cdot))\big)$. We can show the existence of solutions to (2.1)–(2.2), (1.12) for any $J > 0$ and any $\nu \geq 0$.

**Proposition 2.2.** *Let the assumptions* (H1)–(H3) *hold and let $\nu \geq 0$ and $\varepsilon > 0$. Then, for any $J > 0$, there exists a solution $(w, V) \in (H^2(\Omega))^2$ to* (2.1), (2.2), (1.12) *satisfying $0 \leq w(x) \leq M$ in $\Omega$.*

For the proof of Proposition 2.2, we consider the approximate problem

$$\begin{aligned}\delta^2 w_{xx} &= \frac{J^2 w^+}{2t_\varepsilon(w)^4} + Tw^+h(w^2) - Vw^+ + Kw^+ \\ &\quad + Jw^+\int_0^x \frac{ds}{\tau t_\varepsilon(w)^2} + \nu\frac{(t_f(w_M))_x w^+}{t_f(w_M)^{\gamma+1}}\ ,\end{aligned} \tag{2.3}$$

$$\lambda^2 V_{xx} = w_M^2 - C(x) \ \text{ in } \ \Omega\ , \tag{2.4}$$

where $w^+ = \max(0, w)$, $w_M = \min(M, w)$. Let $(w, V)$ be a weak solution to (2.3), (2.4), (1.12). Then we have the following a priori estimates:

**Lemma 2.3.** ($L^\infty$ estimates) *It holds for all $x \in \Omega$*

$$0 \leq w(x) \leq M\ , \qquad k \leq V(x) \leq K\ , \tag{2.5}$$

*where*

$$k = V_0 - \max(E_0, 0) + \lambda^{-2}\|C\|_{L^1(\Omega)}\ .$$

*Proof.* The problem (2.4), (1.12) is equivalent to

$$V(x) = V_0 - E_0 x + \lambda^{-2}\int_0^x\int_0^y (w(z)_M^2 - C(z))\, dz\, dy\ , \tag{2.6}$$

for $x \in [0,1]$, which implies

$$V_0 - \max(E_0,0) + \lambda^{-2}\|C\|_{L^1(\Omega)} \le V(x) \le V_0 + \max(-E_0,0) + \lambda^{-2}M^2 \ .$$

This shows the second chain of inequalities of (2.5).

Using $w^- = \min(0,w)$ as test function in (2.3) we get immediately $w \ge 0$ in $\Omega$. Finally, with the test function $(w-M)^+ = \max(0, w-M)$ in (2.3), we obtain

$$\int_\Omega \left((w-M)_x^+\right)^2 \le \int_\Omega (w-M)^+ w\left[-\frac{J^2}{2t_\varepsilon(w)^4} - Th(M^2) + (V-K) - J\int_0^x \frac{ds}{\tau t_\varepsilon(w)^2}\right] dx - \nu\int_\Omega \frac{(t_f(w_M))_x w(w-M)^+}{t_f(w_M)^{\gamma+1}}\, dx \ ,$$

taking into account the monotonicity of $h$. The first integral on the right-hand side is non positive since $V \le K$ and $h(M^2) \ge h(M_0^2) \ge 0$. Therefore

$$\int_\Omega \left((w-M)_x^+\right)^2 \le -\nu\int_\Omega \frac{(t_f(w_M))_x w(w-M)^+}{t_f(w_M)^{\gamma+1}} \le 0.$$

Hence $w \le M$ in $\Omega$.

**Lemma 2.4.** ($H^1$ estimates) *There exist constants $c_1, c_2 > 0$ only depending on given data and on $\delta, \varepsilon$ and $M$ (but not on $w$ and $V$) such that*

$$\|w\|_{H^1(\Omega)} \le c_1 \ , \quad \|V\|_{H^1(\Omega)} \le c_2 \ .$$

*Proof.* The second assertion follows from Lemma 2.3 and

$$V_x(x) = -E_0 + \lambda^{-2}\int_0^x (w(y)_M^2 - C(y))\, dy \ .$$

The first assertion follows from Lemma 2.3 and Eq. (2.3), after employing the test function $w - w_D$, where $w_D(x) = (1-x)w_0 + xw_1$. Indeed, we have

$$\delta^2\int_\Omega w_x^2\, dx = \delta^2\int_\Omega w_x w_{Dx}\, dx - \nu\int_\Omega \frac{(t_f(w_M))_x w(w-w_D)}{t_f(w_M)^{\gamma+1}}\, dx - \int_\Omega (w-w_D)w\left[\frac{J^2}{2t_\varepsilon(w)^4} + Th(w^2) - V + K + J\int_0^x \frac{ds}{\tau t_\varepsilon(w)^2}\right] dx \ .$$

Using Young's inequality for the first two integrals on the right-hand side and Lemma 2.3 for the last integral, we easily get

$$\frac{\delta^2}{2}\int_\Omega w_x^2 \le c \ .$$

**Lemma 2.5.** ($H^2$ estimate) *There exists a constant $c_3 > 0$ not depending on $w$ such that*

$$\|w\|_{H^2(\Omega)} \leq c_3 \ .$$

*Proof.* The lemma follows immediately from Eq. (2.3), Lemma 2.4 and the embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$.

*Proof of Proposition 2.2.* We apply the Leray-Schauder fixed point theorem (see, e.g., [GT83]). Let $u \in H^1(\Omega)$ and let $V \in H^2(\Omega)$ be the unique solution of

$$\lambda^2 V_{xx} = u_M^2 - C(x) \quad \text{ in } \Omega \ , \qquad V(0) = V_0 \ , \ V_x(0) = -E_0 \ .$$

Let $w \in H^2(\Omega)$ be the unique solution of

$$\begin{aligned} \delta^2 w_{xx} = \sigma \bigg[ &\frac{J^2 u^+}{2t_\varepsilon(u)^4} + T u^+ h(u^2) - V u^+ + K u^+ \\ &+ J u^+ \int_0^x \frac{ds}{\tau t_\varepsilon(u)^2} \bigg] + \sigma\nu \frac{(t_f(u_M))_x u^+}{t_f(u_M)^{\gamma+1}} \ , \\ w(0) = \sigma w_0, \ &w(1) = \sigma w_1, \ \text{where } \sigma \in [0,1]. \end{aligned}$$

This defines the fixed-point operator $S : H^1(\Omega) \times [0,1] \to H^1(\Omega)$, $(u, \sigma) \mapsto w$. It holds $S(u, 0) = 0$ for all $u \in H^1(\Omega)$. Similarly as in the proofs of Lemmas 2.3–2.5 we can show that there exists a constant $c > 0$ independent of $w$ and $\sigma$ such that

$$\|w\|_{H^2(\Omega)} \leq c$$

for all $w \in H^1(\Omega)$ satisfying $S(w, \sigma) = w$. Standard arguments show that $S$ is continuous and compact, noting that the embedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$ is compact. Thus, the fixed-point theorem applies.

For the proof of Theorem 2.1 we only have to show that $w$ is strictly positive in $\Omega$.

*Proof of Theorem 2.1.* Taking $(w - f)^- \in H_0^1(\Omega)$ as test function in Eq. (2.3) gives:

$$\begin{aligned} \delta^2 \int_\Omega \left((w-f)_x^-\right)^2 dx &= \int_\Omega (-(w-f)^-) w \bigg[ \frac{J^2}{2t_\varepsilon(w)^4} + T h(w^2) - V + K \\ &\qquad + J \int_0^x \frac{ds}{\tau t_\varepsilon(w)^2} + \frac{\nu (t_f(w))_x}{t_f(w)^{\gamma+1}} \bigg] dx \\ &\leq \int_\Omega (-(w-f)^-) w \left[ \frac{J^2}{2\varepsilon^4} + T h(f^2) - k + K + \frac{J}{\tau_0 \varepsilon^2} + \frac{\nu f_x}{f^{\gamma+1}} \right] dx \\ &\leq \int_\Omega (-(w-f)^-) w \bigg[ \frac{J^2}{2\varepsilon^4} + T h(f^2) - k + K + \frac{J}{\tau_0 \varepsilon^2} \\ &\qquad - \frac{\nu\varepsilon}{(2\varepsilon)^{\gamma+1}} \bigg] dx, \end{aligned}$$

using the monotonicity of $h$ and Lemma 2.3. Therefore

$$\delta^2 \int_\Omega \left((w-f)_x^-\right)^2 dx \le \int_\Omega (-(w-f)^-)w\left[\frac{J^2}{2\varepsilon^4}+Th(4\varepsilon^2)-k+K+\frac{J}{\tau_0\varepsilon^2}-\frac{\nu}{2^{\gamma+1}\varepsilon^\gamma}\right] dx \,.$$

We claim that for sufficiently small $\varepsilon > 0$, the expression in the brackets is non positive, which implies $w(x) \ge f(x) \ge \varepsilon > 0$ in $\Omega$, i.e., we get the assertion of the theorem after taking $m(\nu) = \varepsilon$. Now choose $\varepsilon \in (0,1)$ such that (see (H1))

$$\varepsilon \le \left(\frac{1}{2^{\gamma+1}}\,\frac{\nu}{J^2/2+J/\tau_0+\max(K-k,0)}\right)^{1/(\gamma-4)} \quad \text{and} \quad h(4\varepsilon^2) \le 0 \,.$$

Notice that $\gamma > 4$. Then, since $\varepsilon \le 1$,

$$\begin{aligned} &\frac{J^2}{2\varepsilon^4}+Th(4\varepsilon^2)+K-k+\frac{J}{\tau_0\varepsilon^2}-\frac{\nu}{2^{\gamma+1}\varepsilon^\gamma} \\ &\quad \le \frac{1}{\varepsilon^4}\left(\frac{J^2}{2}+\frac{J}{\tau_0}+\max(0,K-k)-\frac{\nu}{2^{\gamma+1}\varepsilon^{\gamma-4}}\right) \\ &\quad \le 0 \,. \end{aligned}$$

## 3. Existence of solutions for $\nu = 0$

In the case of vanishing viscosity we can only expect to get existence of solutions for sufficiently small $J > 0$, corresponding to a subsonic condition for the hydrodynamic equations ($\delta = 0$).

In this section we need the following assumption. Assume that there exists $m_0 > 0$ such that

$$\frac{1}{2}Tp'(m_0^2)+Th(m_0^2)+\frac{1}{\tau_0}\sqrt{Tp'(m_0^2)}+K-k \le 0 \,. \tag{3.1}$$

Since $K-k = -E_0+\lambda^{-2}(M^2-\|C\|_{L^1(\Omega)})$, this assumption is satisfied if, for instance,

(i) $\lim_{s\to 0+} h(s) = -\infty$, or

(ii) $E_0 > 0$ is sufficiently large.

Indeed, in case (i), the condition (3.1) becomes true for sufficiently small $m_0 > 0$, observing that $p'$ is non-decreasing such that $p'(m_0^2)$ cannot be arbitrarily large near zero. In case (ii) we can choose, for instance, $m_0 > 0$ such that $h(m_0^2) \le 0$, and then take $E_0 > 0$ large enough such that

$$E_0 \ge \frac{1}{2}Tp'(m_0^2)+\frac{1}{\tau_0}\sqrt{Tp'(m_0^2)}+\lambda^{-2}(M^2-\|C\|_{L^1(\Omega)}) \,.$$

Now, define

$$m = \min\{w_0, w_1, m_0\} \,.$$

**Theorem 3.1.** *Let the assumptions* (H1)–(H3) *hold and let* $\nu = 0$. *Furthermore, let* $J > 0$ *be such that*

$$J \le J_0 \stackrel{\text{def}}{=} m^2\sqrt{Tp'(m^2)}\ . \tag{3.2}$$

*Then there exists a classical solution* $(w, V) \in (C^2(\overline{\Omega}))^2$ *to* (1.10)–(1.12) *satisfying*

$$0 < m \le w(x) \le M \quad in \ \ \Omega\ .$$

*Remark.* The condition (3.2) can be interpreted as a "subsonic condition" since it implies that the velocity $J/n$ satisfies

$$\frac{J}{n} = \frac{J}{w^2} \le \frac{J}{m^2} \le \sqrt{Tp'(m^2)} \le \sqrt{Tp'(n)}\ .$$

*Proof.* By Proposition 2.2, there exists a solution $(w, V)$ to the truncated problem (2.3)–(2.4), (1.12) with $\varepsilon = m > 0$. It remains to show that $w \ge m$ in $\Omega$. Using $(w-m)^-$ as test function gives

$$\begin{aligned}
\delta^2 &\int_\Omega \left((w-m)^-_x\right)^2 dx \\
&= \int_\Omega (-(w-m)^-)w\left[\frac{J^2}{2t_m(w)^4} + Th(w^2) - V + K + J\int_0^x \frac{ds}{\tau t_m(w)^2}\right] dx \\
&\le \int_\Omega (-(w-m)^-)w\left[\frac{J_0^2}{2m^4} + Th(m^2) - k + K + \frac{J_0}{\tau_0 m^2}\right] dx \\
&= \int_\Omega (-(w-m)^-)w\left[\frac{1}{2}Tp'(m^2) + Th(m^2) - k + K + \frac{1}{\tau_0}\sqrt{Tp'(m^2)}\right] dx \\
&\le \int_\Omega (-(w-m)^-)w\left[\frac{1}{2}Tp'(m_0^2) + Th(m_0^2) - k + K + \frac{1}{\tau_0}\sqrt{Tp'(m_0^2)}\right] dx \\
&\le 0,
\end{aligned}$$

in view of the definition (3.1) of $m_0$. This implies that $w \ge m$ in $\Omega$.

Next, we prove that every weak solution is necessarily strictly positive:

**Proposition 3.2.** *Let* $(w, V) \in (H^1(\Omega))^2$ *be a weak solution to* (1.10)–(1.12) *with* $\nu \ge 0$ *and* $w^{-3} \in L^1(\Omega)$. *Then there exists* $m > 0$ *such that* $w(x) \ge m > 0$ *for all* $x \in \Omega$.

*Proof.* Since $1/w^3 \in L^1(\Omega)$, it holds $w_{xx} \in L^1(\Omega)$, which implies $w \in W^{2,1}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$. Suppose that there exists $x_0 \in \Omega$ such that $w(x_0) = 0$. Then

$$w(x) = (x - x_0)\int_0^1 w_x(\lambda x + (1-\lambda)x_0)\,dx\ , \qquad x \in \Omega\ ,$$

and

$$\int_0^1 \frac{dx}{|w(x)|^3} \geq \int_0^1 \left( \int_0^1 |w_x(s)|\, ds \right)^{-3} |x - x_0|^{-3}\, dx = \infty \ ,$$

contradicting the integrability of $1/w^3$. Hence $w > 0$ in $\Omega$, and since $w$ is continuous in $\overline{\Omega}$, there exists $m > 0$ such that $w(x) \geq m$ for all $x \in \Omega$.

## 4. Non-existence of solutions for $\nu = 0$

We show that, under some condition on $h$, a weak solution to (1.10)–(1.12) cannot exist if $J > 0$ is large enough. For this, we have to prove that $w$ is bounded from above independently of $J$.

**Lemma 4.1.** *Let $(w, V)$ be any weak solution to (1.10)–(1.12) with $\nu = 0$, $K \in \mathbb{R}$ and $J > 0$. Furthermore, let*

$$h(s) \geq c_0(s^{\alpha-1} - 1) \quad \text{for} \quad s \geq 0 \ , \quad \text{with} \quad \alpha > 2. \tag{4.1}$$

*Then there exists $M_0 > 0$, independent of $J$, such that $w(x) \leq M_0$ for all $x \in \Omega$.*

*Remark.* The bound $M_0$ does not depend on $K$ if $K > 0$.

*Proof.* Take $(w - \mu)^+$ with $\mu \geq \max(w_0, w_1)$ as test function in (1.10) to get

$$\begin{aligned} \delta^2 \int_\Omega (w - \mu)_x^{+2}\, dx = \int_\Omega (w - \mu)^+ w \left[ -\frac{J^2}{2w^4} - J \int_0^x \frac{ds}{\tau w^2} - h(w^2) - K \right] dx \\ + \int_\Omega (w - \mu)^+ w V\, dx \ . \end{aligned} \tag{4.2}$$

The main difficulty is to estimate the last integral. From (2.6) it follows that

$$V(x) \leq V_0 + \max(-E_0, 0) + \lambda^{-2} \int_\Omega w^2\, dx \ .$$

Since

$$w^2 = (w + \mu)(w - \mu) + \mu^2 \leq (w + \mu)(w - \mu)^+ + \mu^2 \leq 2w(w - \mu)^+ + \mu^2,$$

we get

$$V(x) \leq V_0 + \max(-E_0, 0) + 2\lambda^{-2} \int_\Omega w(w - \mu)^+\, dx + \mu^2 \lambda^{-2}$$

and, setting $V_1 \stackrel{\text{def}}{=} V_0 + \max(-E_0, 0)$,

$$\begin{aligned}\int_\Omega (w-\mu)^+ wV\,dx &\le \int_\Omega (w-\mu)^+ w(V_1 + \mu^2\lambda^{-2})\,dx \\ &\qquad + 2\lambda^{-2}\Big(\int_\Omega (w-\mu)^+ w\,dx\Big)^2 \\ &\le V_1 \int_\Omega (w-\mu)^+ w\,dx + \lambda^{-2}\int_\Omega (w-\mu)^+ w^3\,dx \\ &\qquad + 2\lambda^{-2}\int_\Omega (w-\mu)^{+2} w^2\,dx \\ &\le V_1 \int_\Omega (w-\mu)^+ w\,dx + 3\lambda^{-2}\int_\Omega (w-\mu)^+ w^3\,dx\ ,\end{aligned}$$

since $\mu \le w$ on $\{x : w(x) \ge \mu\}$ and $(w-\mu)^+ \le w$. Therefore we get from (4.2)

$$\delta^2 \int_\Omega (w-\mu)_x^{+2}\,dx \le \int_\Omega (w-\mu)^+ w[V_1 + 3\lambda^{-2}w^2 - K - c_0(w^{2\alpha-2} - 1)]\,dx\ .$$

Since $2\alpha - 2 > 2$, there exists $M_0 > 0$ such that

$$c_0 M_0^{2\alpha-2} - 3\lambda^{-2}M_0^2 - V_1 + K - c_0 \ge 0\ , \tag{4.3}$$

which implies, after taking $\mu = M_0$, that $w \le M_0$ in $\Omega$.

**Theorem 4.2.** *Let* (H1)–(H3) *and* (4.1) *holds and let* $K \in \mathbb{R}$ *and* $\nu = 0$. *Then there exists* $J_1 > 0$ *such that if* $J \ge J_1$ *then the problem* (1.10)–(1.12) *cannot have a weak solution.*

*Remark.* The constant $J_1 > 0$ is defined by

$$\frac{J_1^2}{2M_0^3} = 8\delta^2(\max(w_0, w_1) + 1) - Th_0 + M_0(|V_0| + |E_0| + \lambda^{-2}M_0^2 - \min(0, K))\ , \tag{4.4}$$

and $M_0 > 0$ is defined by (4.3).

*Proof.* Suppose that there exists a weak solution $(w, V)$ to (1.10)–(1.12) for all $J > 0$ for some $K \in \mathbb{R}$. Since $w \in H^1(\Omega) \hookrightarrow C^0(\overline{\Omega})$ and $w > 0$ in $\Omega$, by Proposition 3.2, we get $w_{xx} \in C^0(\overline{\Omega})$. Thus $w$ is a classical solution. Set $h_0 = \inf\{sh(s^2) : 0 < s \le M_0\} > -\infty$. Let $J \ge J_1$, where $J_1$ is defined in (4.4). Then

$$\begin{aligned}\delta^2 w_{xx} &\ge \frac{J^2}{2M_0^3} + Th_0 - w(\sup_\Omega V - \min(0, K)) \\ &\ge \frac{J_1^2}{2M_0^3} + Th_0 - M_0(|V_0| + |E_0| + \lambda^{-2}M_0^2 - \min(0, K)) \\ &= 8\delta^2(\max(w_0, w_1) + 1)\ .\end{aligned}$$

Introduce $p(x) = 4(\max(w_0, w_1) + 1)(x - \frac{1}{2})^2 - 1$, $x \in [0, 1]$. Then

$$p(0) = p(1) = \max(w_0, w_1) \ ,$$
$$p_{xx}(x) = 8(\max(w_0, w_1) + 1) \ ,$$

for $x \in \Omega$, which implies

$$w - p \le 0 \ \text{ on } \ \partial\Omega, \quad (w - p)_{xx} \ge 0 \ \text{ in } \ \Omega \ .$$

The maximum principle gives $w - p \le 0$ in $\overline{\Omega}$. In particular, $w(\frac{1}{2}) \le p(\frac{1}{2}) = -1 < 0$, which contradicts the positivity of $w$.

**Corollary 4.5.** *Let* (H1)–(H3) *hold and let* $\nu = 0$.

(i) *There exists* $J_0 > 0$ *such that for all* $J \le J_0$ *there exists a solution* $(n, V) \in (C^3(\overline{\Omega}))^2$ *to* (1.6)–(1.9) *with strictly positive* $n$.

(ii) *Let in addition* (4.1) *hold. Then there exists* $J_1 > 0$ *such that for all* $J \ge J_1$, *the problem* (1.6)–(1.9) *cannot have a weak solution.*

*Proof.* The first part follows from Theorem 3.1 after setting $n = w^2$ and differentiating the equation (1.10) with $\nu = 0$. The second part follows from Theorem 4.2 since the constant $K \in \mathbb{R}$ is fixed in (1.9). Thus the problems (1.6)–(1.9) and (1.10)–(1.12) are equivalent and the non-existence of solutions to (1.10)–(1.12) implies the non-existence of solutions to (1.6)–(1.9).

## 5. Uniqueness of solutions for $\nu = 0$

We can prove the uniqueness of 'subsonic' solutions to (1.6)–(1.9). Our main result is the following theorem.

**Theorem 5.1.** *Let the assumptions* (H1)–(H3) *hold and let* $1/\tau \equiv 0$. *Suppose that the function* $h$ *satisfies, for some* $h_0 > 0$,

$$(h(x) - h(y))(x - y) \ge h_0(x - y)^2 \quad \textit{for all } x, y > 0. \tag{5.1}$$

*Furthermore, suppose that* $Th_0\varepsilon \ge 1$ *for some given* $0 < \varepsilon < 1$. *Then there is uniqueness of weak solutions to* (1.10)–(1.12) *(and to* (1.6)–(1.9)*) in the class of positive densities satisfying the 'subsonic' condition*

$$J/w(x)^2 \le \sqrt{(1 - \varepsilon)Tp'(w(x)^2)} \qquad \textit{for all } x \in \Omega. \tag{5.2}$$

*Proof.* Let $(w_1, V_1)$ and $(w_2, V_2)$ be two weak solutions to (1.10)–(1.12) satisfying the condition (5.2). Then $w_1$, $w_2$ are positive in $\Omega$, by Proposition 3.2, and classical solutions of (1.10). Introduce $f(w) = J^2/2w^4 + (1-\varepsilon)Th(w^2)$. Then, proceeding similarly as in [BO86], we write

$$-\delta^2\frac{w_{1,xx}}{w_1} + \delta^2\frac{w_{2,xx}}{w_2} = -\frac{f(w_1)}{w_1} + \frac{f(w_2)}{w_2} - \varepsilon T(h(w_1^2) - h(w_2^2)) + V_1 - V_2.$$

Multiplying this equation by $w_1^2 - w_2^2$, integrating over $\Omega$ and integrating by parts, we obtain, after elementary computations,

$$\begin{aligned}\int_\Omega (w_1^2 + w_2^2)(\ln w_1 - \ln w_2)_x^2 dx &= \int_\Omega \Big[\Big(w_{1,x} - \frac{w_1}{w_2}w_{2,x}\Big)^2 + \Big(w_{2,x} - \frac{w_2}{w_1}w_{1,x}\Big)^2\Big]dx \\ &= -\int_\Omega \Big(\frac{f(w_1)}{w_1} - \frac{f(w_2)}{w_2}\Big)(w_1^2 - w_2^2)dx \qquad (5.3) \\ &\quad - \varepsilon T\int_\Omega (h(w_1^2) - h(w_2^2))(w_1^2 - w_2^2)dx \\ &\quad + \int_\Omega (V_1 - V_2)(w_1^2 - w_2^2)dx.\end{aligned}$$

Now multiply the difference of Eq. (1.11) for $V_1$ and $V_2$ by $V_1 - V_2$, integrate over $\Omega$ and integrate by parts to get

$$\begin{aligned}\int_\Omega (V_1 - V_2)(w_1^2 - w_2^2)dx &= -\lambda^2\int_\Omega (V_1 - V_2)_x^2 dx + \lambda^2(V_1 - V_2)(1)(V_1 - V_2)_x(1) \\ &\le -\lambda^2\int_\Omega (V_1 - V_2)_x^2 dx + \Big(\int_0^1 (w_1^2 - w_2^2)dx\Big)^2 \\ &\le -\lambda^2\int_\Omega (V_1 - V_2)_x^2 dx + \int_0^1 (w_1^2 - w_2^2)^2 dx,\end{aligned}$$

where we used the expression (2.6) for $V_1$ and $V_2$ and Jensen's inequality. Furthermore, the function $s \mapsto f(s)/s$ is non-decreasing if and only if

$$\frac{d}{ds}\frac{f(s)}{s} = -\frac{2J^2}{s^5} + 2(1-\varepsilon)Tsh'(s^2) = \frac{2}{s}\Big(-\frac{J^2}{s^4} + T(1-\varepsilon)p'(s^2)\Big) \ge 0.$$

In view of condition (5.2) this implies that

$$-\int_\Omega \Big(\frac{f(w_1)}{w_1} - \frac{f(w_2)}{w_2}\Big)(w_1^2 - w_2^2)dx \le 0.$$

Hence, we obtain from Eq. (5.3) and condition (5.1)

$$\begin{aligned}&\int_\Omega (w_1^2 + w_2^2)(\ln w_1 - \ln w_2)_x^2 dx \\ &\quad \le -\lambda^2\int_\Omega (V_1 - V_2)_x^2 dx + (1 - \varepsilon Th_0)\int_0^1 (w_1^2 - w_2^2)^2 dx \\ &\quad \le 0,\end{aligned}$$

since $\varepsilon Th_0 \ge 1$ by assumption. Hence $w_1 = w_2$ and $V_1 = V_2$ in $\Omega$.

## 6. Third-order equations in several dimensions

In this section we show how the methods of the previous sections can be extended to third-order equations in any space dimension. This is the first important step for performing the existence analysis of the multi-dimensional quantum hydrodynamic equations. Since the proofs are similar to those of the previous sections, we only sketch the proofs. Let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain and consider

$$\nabla(A(u)\Delta u) = \lambda\nabla(F(u)) + \nabla(G(u)) + \nu\nabla(B(u)(1\cdot\nabla)u) \quad \text{in } \Omega, \tag{6.1}$$

$$Bu = u_D \quad \text{on } \partial\Omega, \tag{6.2}$$

where $\nu \geq 0$ and $\lambda > 0$. Recall that $(1\cdot\nabla)u = \sum_j \partial_j u$. When the boundary conditions are such that the integration of Eq. (6.1) along streamlines is well defined, we obtain

$$\Delta u = \lambda f(u) + g(u) + Ka(u) + \nu b(u)(1\cdot\nabla)u, \tag{6.3}$$

where $K \in \mathbb{R}$ is a constant and

$$f(u) = \frac{F(u)}{A(u)}, \quad g(u) = \frac{G(u)}{A(u)}, \quad a(u) = \frac{1}{A(u)}, \quad b(u) = \frac{B(u)}{A(u)}.$$

Any solution $u \in H^2(\Omega)$ to (6.3), (6.2) solves the problem (6.1), (6.2) and vice versa. We assume that

$$\partial\Omega \in C^{1,1},\ u_D \in H^2(\Omega) \cap L^\infty(\Omega);\ u_D \geq u_0 > 0 \text{ on } \partial\Omega; \tag{6.4}$$

$$a, b, f, g \in C(0,\infty),\ a, g \text{ are non-decreasing},\ a, b \text{ are non-negative}; \tag{6.5}$$

$$g(0+) < 0;\ g(+\infty) > 0;\ \lim_{s\to 0+} a(s)/sb(s) = 0; \tag{6.6}$$

$$\inf_{0<s<M} f(s) > 0\ \forall M > 0;\ \lim_{s\to 0+} f(s)/sb(s) = 0. \tag{6.7}$$

Admissible functions are, for instance,

$$f(u) = u^{-\alpha}, \quad g(u) = u^\beta - 1, \quad a(u) = u^\gamma, \quad b(u) = u^{-\delta}$$

with $\alpha, \beta, \gamma > 0$, $\gamma > \delta - 1$, and $\delta > 1 + \alpha$.

**Theorem 6.1.** *Let $\nu > 0$ and $K > 0$. Then there exists a solution $u \in H^2(\Omega)$ to (6.1)–(6.2) for all $\lambda > 0$.*

*Remark.* The assumption $K > 0$ can be weakened by choosing appropriate assumptions on the functions $a$, $f$, and $g$.

*Proof.* Since $\Omega$ is bounded, there exists $R>0$ such that $\Omega$ is contained in the ball $B_R(0)$ of radius $R$ and center 0. Introduce the comparison function $\phi(x)=\phi(x_1,\ldots,x_d)=\frac{\varepsilon}{R}(2R-x_1)$ and $0<\varepsilon\le\min(u_0,M/3)$. Set $t_\phi(u_M)=\max(\phi(\cdot),\min(M,u(\cdot)))$ and consider the truncated problem

$$\begin{aligned}\Delta u &= \lambda\frac{f(t_\phi(u_M))}{t_\phi(u)}u^+ + \frac{g(t_\phi(u_M))}{t_\phi(u)}u^+ + K\frac{a(t_\phi(u_M))}{t_\phi(u_M)}u^+ \\ &\quad + \nu\frac{b(t_\phi(u_M))}{t_\phi(u_M)}u^+(1\cdot\nabla)(t_\phi(u_M)) \qquad \text{in } \Omega, \end{aligned} \tag{6.8}$$

$$u = u_D \qquad \text{on } \partial\Omega, \tag{6.9}$$

where $u^+=\max(0,u)$. Using the methods of the proof of Proposition 2.2, we easily get the existence of a solution $u\in H^2(\Omega)$ to (6.8)–(6.9) for any $\nu\ge 0$ (here we use that $Ka(s)\ge 0$). It remains to show that $\varepsilon\le u(x)\le M$ for $x\in\Omega$.

First we observe that by using $u^-$ as test function in (6.8), we immediately conclude that $u(x)\ge 0$ in $\Omega$. In order to prove that $u\le M$ in $\Omega$ for some $M>0$ we use $(u-M)^+$ with $M\ge\|u_D\|_{L^\infty(\Omega)}$ as test function in (6.8). Since there exists a constant $M>0$ such that $g(M)\ge 0$, we can show as in the proof of Lemma 2.3 that $(u-M)^+=0$ in $\Omega$.

For the lower bound we use $(u-\phi)^-$ as test function in (6.8), observing that $\varepsilon\le\phi(x)\le 3\varepsilon$ in $\Omega$:

$$\begin{aligned}&\int_\Omega |\nabla(u-\phi)^-|^2dx \\ &\quad = \int_\Omega\big(-(u-\phi)^-\big)u\Big[\lambda\frac{f(t_\phi(u))}{t_\phi(u)} + \frac{g(t_\phi(u))}{t_\phi(u)} + K\frac{a(t_\phi(u))}{t_\phi(u)} \\ &\qquad + \nu\frac{b(t_\phi(u))}{t_\phi(u)}(1\cdot\nabla)(t_\phi(u))\Big]dx \\ &\quad = \int_\Omega\big(-(u-\phi)^-\big)u\Big[\lambda\frac{f(\phi)}{\phi} + \frac{g(\phi)}{\phi} + K\frac{a(\phi)}{\phi} + \nu\frac{b(\phi)}{\phi}(1\cdot\nabla)\phi\Big]dx \\ &\quad \le \int_\Omega\big(-(u-\phi)^-\big)ub(\phi)\Big[\lambda\frac{f(\phi)}{\phi b(\phi)} + \frac{g(\phi)}{\phi b(\phi)} + \frac{Ka(\phi)}{\phi b(\phi)} - \frac{\nu}{3R}\Big]dx \\ &\quad \le 0,\end{aligned}$$

by choosing $\varepsilon$ small enough such that the inequalities $g(\phi)\le 0$, $f(\phi)/\phi b(\phi)\le\nu/6\lambda R$, and $a(\phi)/\phi b(\phi)\le\nu/6KR$ are satisfied in $\Omega$. This is possible in view of the conditions (6.6)–(6.7). We conclude that $u\ge\phi\ge\varepsilon$ in $\Omega$.

**Theorem 6.2.** *Let $\nu = 0$.*

(i) *Let $g(0+) < 0$. Furthermore, assume that $K > 0$ and $\lim_{s\to 0+} a(s)/s = 0$. Then there exists $\lambda_0 > 0$ such that for all $0 < \lambda \le \lambda_0$ there exists a solution $u \in H^2(\Omega)$ to (6.1)–(6.2) satisfying*

$$\Delta u = \lambda f(u_D) + g(u_D) + Ka(u_D) \qquad \text{on } \partial\Omega. \tag{6.10}$$

(ii) *Let $g(0+) > -\infty$. Then there exists $\lambda_1 > 0$ such that for all $\lambda \ge \lambda_1$ there is no weak solution to (6.1)–(6.2), (6.10).*

*Proof.* For the first part of the theorem we only have to show that the solution $u$ to (6.8)–(6.9) is strictly positive for sufficiently small $\lambda > 0$. Take $\varepsilon = m > 0$ and $\delta > 0$ such that $m \le u_0$, $g(\phi)/\phi \le -\delta < 0$, and $a(\phi)/\phi \le \delta/2K$ (if $K > 0$) in $\Omega$. The existence of $m$ and $\delta$ is ensured by condition (6.6) and the assumption on $a$. Let $f_m = \sup_{m<s<3m} f(s)/s$. Then $f_m > 0$ in view of assumption (6.7). Choose $0 < \lambda_0 \le \delta/2f_m$ and let $0 < \lambda \le \lambda_0$. Using $(u-m)^-$ as test function in Eq. (6.7) yields

$$\begin{aligned}
\int_\Omega |\nabla(u-m)^-|^2 dx &= \int_\Omega \big(-(u-m)^-\big)u\Big[\lambda\frac{f(t_\phi(u))}{t_\phi(u)} + \frac{g(t_\phi(u))}{t_\phi(u)} + K\frac{a(t_\phi(u))}{t_\phi(u)}\Big]dx \\
&\le \int_\Omega \big(-(u-m)^-\big)u\Big[\lambda_0\frac{f(\phi)}{\phi} + \frac{g(\phi)}{\phi} + K\frac{a(\phi)}{\phi}\Big]dx \\
&\le \int_\Omega \big(-(u-m)^-\big)u\big[\lambda_0 f_m - \delta/2\big]dx \\
&\le 0.
\end{aligned}$$

Thus $u \ge m > 0$ in $\Omega$.

To prove the second part of the theorem, let $u \in H^1(\Omega)$ be a solution to (6.2)–(6.3) with $u \le M$ in $\Omega$ and $K \in \mathbb{R}$. Using the positivity of $f$, we can easily see that the constant $M$ does not depend on $\lambda$. Let $x_0 \in \Omega$. Since $\Omega$ is open, there exists a ball $B_r(x_0)$ of radius $r$ with center $x_0$ contained in $\Omega$. Set $f_0 = \inf\{f(s) : 0 < s < M\} > 0$ and $a_0 = \inf\{a(s) : 0 < s < M\} \ge 0$ and choose $\lambda_1 > 0$ such that $L \stackrel{\text{def}}{=} \lambda_1 f_0 + g(0+) + \max(0, Ka_0) \ge 2d(M+1)/r^2$ and $L \ge M$. Notice that we assumed $g(0+) > -\infty$. We show that $u(x_0) < 0$ which is a contradiction to the non-negativity of $u$. It holds (in the sense of distributions)

$$\Delta u \ge \lambda f_0 + g(0+) + \max(0, Ka_0) \ge \lambda_1 f_0 + g(0+) + \max(0, Ka_0) = L \quad \text{ in } \Omega.$$

Now define

$$p(x) = \frac{L}{2d}|x - x_0|^2 - 1 \quad \text{for } x \in B_r(x_0).$$

Then

$$\Delta p = L \quad \text{in } B_r(x_0), \qquad p = r^2 L/2d - 1 \geq 0 \quad \text{on } \partial B_r(x_0).$$

This implies

$$\Delta(u - p) \geq 0 \quad \text{in } B_r(x_0), \qquad u - p \leq M - r^2 L/2d + 1 \leq 0 \quad \text{on } \partial B_r(x_0).$$

By the maximum principle, we conclude that $u - p \leq 0$ in $B_r(x_0)$. In particular, $u(x_0) \leq p(x_0) = -1 < 0$, contradiction.

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