Correlations, Spectral gap and Log-Sobolev inequalities for unbounded spins systems.

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Abstract. In this article, we would like to review recent results concerning the links between the decay of correlations, the spectral gap and the Log-Sobolev inequalities. This was motivated by various papers by Antoniouk&Antoniouk, B. Zegarlinski and N. Yoshida. We are mainly reporting on contributions by Helffer-Sjöstrand, Helffer, Yoshida and Bodineau-Helffer but also present some new results.

1. Introduction and Preliminaries

In this paper we are going to study some relations between equilibrium properties and dynamics for the Gibbs measures. The results are expressed in terms of finite size conditions for Gibbs measures defined on finite domains $\Lambda \subset \mathbb{R}^d$

$$\forall X = (x_i)_{i\in\Lambda} \in \mathbb{R}^\Lambda , \quad E^\Lambda(dX) = \frac{1}{Z^\Lambda} \exp\Big(-\sum_j \phi(x_j) - W(X)\Big)\, dX \ ,$$

where ϕ is the single spin phase and W is an interaction potential depending on boundary conditions (see Section 2 for the precise definition).

The interplay between dynamics and equilibrium properties of the spins systems has been thoroughly investigated during the past years. On the one hand, some dynamics relax exponentially fast towards their equilibrium measures and therefore, their behavior can be used to describe equilibrium properties. On the other hand, the speed of convergence of the dynamics is related to equilibrium quantities, like the correlation length.

During the last decade, the decay of correlations, the Poincaré inequalities and the Log-Sobolev inequalities were singled out as relevant tools to describe the links between the relaxation of the dynamics and the equilibrium measures. In this paper we review some of the recent developments for unbounded spins systems. We mainly focus on an analytical approach of the Glauber dynamics based on the use

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of the Witten Laplacian. For an overview of the wide range of the dynamics and of the different relaxation modes we refer the reader to the accounts of Liggett [**Li**] and Martinelli [**Ma**]. In the latter, the phase transition regime is also investigated.

In the theory of particles systems, finite size conditions play a key role. In particular, this feature was emphasized by Dobrushin and Shlosman's Theory of *complete analyticity* [**DS1, DS2**] : they showed that the strong mixing condition, i.e. the exponential decay of correlations uniformly over the boundary conditions and the domains, implies that the associated Gibbs measure is well behaved. Aizenman and Holley [**AH**] proved in a discrete setting that the strong mixing property implies that the operators associated to the Glauber dynamics have uniformly bounded spectral gaps and that the dynamics relax exponentially fast to equilibrium. A more accurate description of the behavior of the dynamics can be obtained by the hypercontractivity property and by its functional counterpart, the Log-Sobolev inequality.

The Log-Sobolev inequality can be derived by different methods. For Gibbs measures with continuous spins, it was proved by Bakry and Emery [**BaEm**] that a sufficient condition for Log-Sobolev inequality is the uniform strict convexity of the hamiltonian. Nevertheless, there is a counter-example of a non-convex hamiltonian such that the scheme of their proof fail even if the Log-Sobolev inequality holds (see subsection 4.1). In this case, a more systematic method can be implemented, provided one has a precise description of the Gibbs measure properties.

Log-Sobolev inequality can be easily deduced on product spaces by tensorizing one-dimensional inequalities; this is no longer the case for Gibbs measures. Nevertheless, in absence of phase transition, Gibbs measures can be considered as a perturbation of product measures and the decay of correlations can play an analogous role to the independence property. Such an observation was first implemented by Zegarlinski to derive Log-Sobolev inequalities for one-dimensional spin systems [**Ze1**]. Following a similar approach, Stroock and Zegarlinski extended this result in a series of papers [**StZe1, StZe2, StZe3**]. They showed in the case of compact phase spaces (discrete and continuous) that the strong mixing is equivalent to the Poincaré and to the Log-Sobolev inequalities uniformly over the boundary conditions and the domains. A step further was made by Martinelli and Olivieri [**MO1**], [**MO2**] who introduced the *regular complete analyticity* and considered the notions previously described only on cubes. This restriction enables them to provide results in a region of the phase diagram where the strong mixing does not hold. In a discrete setting, alternative proofs by a martingale method were derived by Lu and Yau [**LuYa**] for Glauber and Kawasaki dynamics.

In the same spirit, Zegarlinski showed in [**Ze3**] that Log-Sobolev inequalities in the case of non-compact spin systems can be obtained from the decay of correlations. This was further investigated by Yoshida [**Yo1, Yo3**], who proved the equivalence between the exponential decay, the Poincaré inequalities and the Log-Sobolev inequalities (on finite size domains uniformly with respect to the boundary conditions).

Finally, let us note that Log-Sobolev inequalities can be applied to derive fine L^∞-controls of the relaxation of unbounded spin systems to equilibrium (see [**Ze3, Yo2**]).

In this paper, we focus on a more analytical approach of the above results, which is based mainly on a representation of the correlations via the Witten Laplacian. This formula was introduced by Helffer and Sjöstrand in [**HeSj**] and further developed in [**Sj**]. As a consequence of this representation, exponential decay of the correlations has been proven under various assumptions on the hamiltonian [**BaJeSj, He1, He2, He3**]. This representation also triggered several studies on the massless free field, which we will not discuss here. In fact the approach used in this context does not rely on the spectral properties of the Witten Laplacian but on homogenization procedures for an infinite dimensional PDE [**NS**] or for an alternative representation in terms of a random walk in a random environment [**DGI**].

The method developed in [**He3**] in order to control the decay of correlations by the Witten Laplacian, differs from the recursive procedure based on the Dobrushin criterion (see [**COPP**] and [**AA2**]). Since this method is based on spectral estimates, it enables us to control the correlations in terms of L^2 estimates. This may be viewed as a generalization of the results of [**AA2**] which were dependent only on L^∞-norm : in the case of unbounded spin systems the test functions might be bounded in L^2 and not in L^∞. The analysis of [**AA2**] also holds for super-quadratic interaction potential. In Section 3, we show that the techniques developed in [**He3**] can be generalized to non-quadratic interactions and we derive a control of the correlations in L^2-weighted spaces. These results hold in a perturbative regime under mild conditions on the potentials. Non-perturbative results on the decay of correlations can be related to Poincaré inequalities for finite size domains. This was proved in [**Li, StZe1, Yo3**] by combining the exponential relaxation of the dynamics with an argument of finite speed of propagation. Here, we propose an alternative proof of similar results based on spectral theory and L^2 estimates.

Finally, in Section 4, we discuss the equivalence between the decay of correlations, the Poincaré inequalities and the Log-Sobolev inequalities proven by Yoshida [**Yo3**] (see [**StZe3**] for compact phase spaces). The control on the decay of correlations previously mentioned enables us (see [**BoHe**]) to recover, in a perturbative regime, the Log-Sobolev inequalities under weak assumptions on the potentials.

2. Notations

2.1. Gibbs measures. For any domain Λ of $\mathbb{Z}^d$, we consider the following ferromagnetic Hamiltonian on the phase space $\mathbb{R}^\Lambda$ defined as follows

(2.1)
$$\Phi^{\Lambda,\omega}(X) = \sum_{j\in\Lambda} \phi(x_j) + \frac{\mathcal{J}}{2} \sum_{i,j\in\Lambda\ ,\ j\sim i} V(x_j - x_i) + \mathcal{J} \sum_{i\in\Lambda\ ,\ j\notin\Lambda\ ,\ j\sim i} V(\omega_j - x_i),$$

where $X = (x_i)_{i\in\Lambda}$ and $\omega = (\omega_j)_{j\in\Lambda^c}$ are the boundary conditions. Furthermore the one particle phase ϕ is a C^∞ function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \exp(-\phi(t))dt < +\infty \ , \tag{2.2}$$

the interaction coefficient $\mathcal{J}$ is positive and the interaction potential V is an even, convex, positive C^∞ function satisfying, for a suitable constant $C > 0$,

$$\forall (s,t) \in \mathbb{R}^2, \qquad |V''(t-s)| \leq C(1+\phi''(t)^2)^{\frac{1}{4}}(1+\phi''(s)^2)^{\frac{1}{4}} \ . \tag{2.3}$$

For simplicity, we restrict to nearest neighbor interaction (denoted by $i \sim j$); nevertheless the results hold for finite range potentials.

The Gibbs measure associated to the previous Hamiltonian is defined by

$$dE^{\Lambda,\omega} := \frac{1}{Z^{\Lambda,\omega}} \exp\big(- \Phi^{\Lambda,\omega}(X)\big)\, dX \ , \tag{2.4}$$

where $Z^{\Lambda,\omega}$ is a normalization factor. We are going to analyze finite size conditions and their implications on the thermodynamic behavior of the above Gibbs measures. Our main assumption is an assumption of convexity at ∞ of the single spin phase ϕ. We assume that there exists a bounded C^∞ function s such that $\tilde{\phi} := \phi + s$ is strictly convex. More precisely, there exists $\rho > 0$ such that

$$(\phi + s)''(t) \geq \rho > 0 \ , \qquad \forall t \in \mathbb{R} \ . \tag{2.5}$$

The main model comes from the field theory and corresponds to the choice of

$$V(u) = u^2 \quad \text{and} \quad \phi(x) = \lambda x^4 + \nu x^2 \ , \tag{2.6}$$

where the parameters λ and ν satisfy

$$\lambda > 0 \ , \ \nu \in \mathbb{R} \ . \tag{2.7}$$

We will also consider non-quadratic interaction. A typical example (which was proposed by Antoniouk & Antoniouk [**AA2**]) is given by :

$$\phi(x) = \lambda x^8 + \nu x^2 \ . \tag{2.8}$$

where the parameter λ is positive and ν may be negative, and

$$V(u) = u^4 \ . \tag{2.9}$$

Then the condition (2.3) is satisfied; there exists $C > 0$ such that :

$$(t-s)^2 \leq C(1 + t^{12})^{\frac{1}{4}} (1 + s^{12})^{\frac{1}{4}} \ .$$

2.2. Laplacians. Let Λ be a finite domain of $\mathbb{R}^\Lambda$ and ω be the boundary conditions outside Λ. We set

$$j \in \Lambda, \qquad X_j = \nabla_j = (\partial_{x_{j1}}, \cdots, \partial_{x_{jN}}) \ , \tag{2.10}$$

and denote by X_j^* its adjoint in $L^2(\mathbb{R}^\Lambda, E^{\Lambda,\omega})$

$$X_j^* = -\nabla_j + (\nabla_j \Phi^{\Lambda,\omega}) \ . \tag{2.11}$$

The operator $\Delta^{(0)}_{\Phi^{\Lambda,\omega}}$ is defined as the unique selfadjoint extension of

$$\forall f \in C_0^2(\mathbb{R}^\Lambda), \qquad \Delta^{(0)}_{\Phi^{\Lambda,\omega}} f = -\Delta_\Lambda f + \nabla_\Lambda \Phi^{\Lambda,\omega} \cdot \nabla_\Lambda f = \sum_{j\in\Lambda} X_j^* X_j f \ .$$

Notice that $\Delta^{(0)}_{\Phi^{\Lambda,\omega}}$ is the Laplacian associated to the Dirichlet form on $L^2(\mathbb{R}^\Lambda, E^{\Lambda,\omega})$. The spectral gap of $\Delta^{(0)}_{\Phi^{\Lambda,\omega}}$ is defined as the gap between the two first eigenvalues of $\Delta^{(0)}_{\Phi^{\Lambda,\omega}}$ and is denoted by $\mathcal{S}(\Lambda,\omega)$. The dynamics associated to the semigroup $\big(P_t = \exp(-t\Delta^{(0)}_{\Phi^{\Lambda,\omega}})\big)_{t\geq 0}$ generated by $\Delta^{(0)}_{\Phi^{\Lambda,\omega}}$ is called the Glauber dynamics.

Let us denote by $\Delta^{(1)}_{\Phi^{\Lambda,\omega}}$ the corresponding natural Laplacian on 1-forms attached to the variables x_Λ. It can also be identified to a Witten Laplacian. We recall that if $F = (f_i)_{i\in\Lambda}$ (identified with the 1-form $F = \sum_i f_i dx_i$) then

$$(\Delta^{(1)}_{\Phi^{\Lambda,\omega}} F)_i = \sum_{j\in\Lambda} X_j^* \, X_j \, f_i + \frac{1}{2}\sum_{j\in\Lambda} \frac{\partial^2 \Phi^{\Lambda,\omega}}{\partial x_i \partial x_j} f_j \; . \tag{2.12}$$

In the following, we identify the notations df and ∇f. In particular, the L^2-norm associated to 1-forms will be denoted by

$$\|d_\Lambda f\|^2_{\Omega^{1,2}_\Phi} = \langle d_\Lambda f \cdot d_\Lambda f\rangle_{\Lambda,\omega} = \sum_{i\in\Lambda} \langle (\partial_i f)^2\rangle_{\Lambda,\omega} \; ,$$

where $\langle \cdot \rangle_{\Lambda,\omega}$ denotes the mean value with respect to the measure $dE^{\Lambda,\omega}$. Notice also that

$$d_\Lambda (\Delta^{(0)}_{\Phi^{\Lambda,\omega}} f) = \Delta^{(1)}_{\Phi^{\Lambda,\omega}} d_\Lambda f = \Delta^{(1)}_{\Phi^{\Lambda,\omega}} \nabla_\Lambda f \; .$$

3. Decay of correlations

3.1. The representation formula for the correlations. The covariance associated to $f, g \in C^\infty_{temp}(\mathbb{R}^\Lambda)$ is defined by

$$E^{\Lambda,\omega}(f;g) := \mathrm{Cov}_{\,\Lambda,\omega}(f,g) = \langle (f - \langle f\rangle_{\Lambda,\omega})(g - \langle g\rangle_{\Lambda,\omega})\rangle_{\Lambda,\omega} \; , \tag{3.1}$$

where $C^\infty_{temp}(\mathbb{R}^\Lambda)$ is the space of C^∞ functions with polynomial growth.

The following representation of the correlations was introduced by Helffer and Sjöstrand in [**HeSj, Sj**]

$$E^{\Lambda,\omega}(f;g) = \langle (\Delta^{(1)}_{\Phi^{\Lambda,\omega}})^{-1} d_\Lambda f \, \cdot \, d_\Lambda g\rangle_{\Lambda,\omega} \; . \tag{3.2}$$

Several inequalities can be derived from (3.2), in particular generalizations of the Brascamp-Lieb inequality [**BL**] were studied in [**He1, He2**]. By using this formula, Helffer proved in [**He3**] the exponential decay of the correlations

Theorem 3.1. *If the previous assumptions are satisfied and if $|V''|$ is bounded, then there exists $\mathcal{J}_0 > 0$, $c > 0$ and $\lambda > 0$ such that the following inequality holds, for all functions f and g in $C^\infty_{temp}(\mathbb{R}^\Lambda)$,*

$$|E^{\Lambda,\omega}(f;g)| \le \lambda \exp\big(-c\, d(S_f, S_g)\big) \|d_\Lambda f\|_{\Omega^{1,2}_\Phi} \, \|d_\Lambda g\|_{\Omega^{1,2}_\Phi} \; , \tag{3.3}$$

uniformly with respect to the other parameters $\Lambda \subset \mathbb{Z}^d$, $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, $\mathcal{J} \in [0, \mathcal{J}_0]$.

Here S_f is the support of the function f defined as the smallest set Σ in Λ such that f can be written as a function of the variables x_ℓ, $\ell \in \Sigma$.

We stress the fact that (3.3) involves only L^2 estimates. So that (3.3) implies that the spectral gap is greater than some constant $\frac{1}{\lambda}$ uniformly with respect to Λ, ω (take $f = g$).

3.2. Non quadratic interactions. In this subsection, we are mainly interested in the perturbative regime ($\mathcal{J}$ small). If V'' is unbounded, we shall prove the following extension of Theorem 3.1.

THEOREM 3.2. *Under the general assumptions, there exists $C > 0$, $\mathcal{J}_0 > 0$ such that, for any $\Lambda \subset \mathbb{Z}^d$, any $\mathcal{J} \in [0, \mathcal{J}_0]$, any $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, and any tempered functions f and g on $\mathbb{R}^\Lambda$,*

$$|E^{\Lambda,\omega}(f;g)| \leq C \ \exp\left(-\frac{1}{C} d(S_f, S_g)\right) \ \|\Theta \cdot d_\Lambda f\|_{\Omega^{1,2}_\Phi} \ \|\Theta \cdot d_\Lambda g\|_{\Omega^{1,2}_\Phi} \ , \tag{3.4}$$

with

$$(\Theta(X))_{jk} = (\phi''(x_j) + C)^{-\frac{1}{2}} \delta_{jk} \ . \tag{3.5}$$

Here the choice of C is in particular determined by the condition that

$$\phi''(x) + C \geq 1 \ . \tag{3.6}$$

Note that it is not a new assumption, but only a weak consequence of our assumption of convexity at ∞.

REMARK 3.3. When $f = g$, we recover some weak form of the Brascamp-Lieb inequality [**BL**].

REMARK 3.4. This theorem is inspired by the study in [**AA2**] of uniformly strictly convex function ϕ. These authors propose an estimate with an L^∞-norm and thus assume that f and g have bounded derivatives. The constant C before the exponential in (3.4) becomes in this case dependent of the size of the support. Note also that an unpublished result by J.-D. Deuschel [**De**] leads to a proof of the decay with also L^∞-norms.

Proof.
We follow [**He3**] and focus mainly on what has to be modified, if V'' is unbounded. All the proof was detailed in [**He4**], for uniformly bounded V''. This is essentially an a priori estimate, which is reminiscent of the proof of Brascamp-Lieb inequality in [**He1**].

We define the matrix $\tilde{\Theta}(X)$ by

$$(\tilde{\Theta}(X)) = \Theta^{-1}(X) \ . \tag{3.7}$$

Let u be the solution of

$$f - \langle f \rangle_{\Lambda,\omega} = \Delta^{(0)}_{\Phi^{\Lambda,\omega}} u \ , \qquad \langle u \rangle_{\Lambda,\omega} = 0 \ . \tag{3.8}$$

Then

$$d_\Lambda f = \Delta^{(1)}_{\Phi^{\Lambda,\omega}} d_\Lambda u \ . \tag{3.9}$$

We define also

$$M_{jk} = \delta_{jk} \ \rho(j) \ , \tag{3.10}$$

where

$$\rho(\ell) := \exp\big(- \kappa d(\ell, S_f)\big) \ . \tag{3.11}$$

Here $\kappa > 0$ will be chosen independently of any parameter Λ, ω and f.

A direct consequence of the representation formula (3.2) is the following inequality

$$|E^{\Lambda,\omega}(f;g)| = |\langle d_\Lambda u \cdot d_\Lambda g\rangle_{\Lambda,\omega}| \leq \|\tilde{\Theta} M^{-1} d_\Lambda u\|_{\Omega^{1,2}_\Phi} \cdot \|\Theta M d_\Lambda g\|_{\Omega^{1,2}_\Phi} \ . \tag{3.12}$$

Thus, the proof of (3.4) is reduced to the proof of

$$\|\tilde{\Theta} \cdot M^{-1} d_\Lambda u\|_{\Omega^{1,2}_\Phi} \leq C \|\Theta \cdot M^{-1} d_\Lambda f\|_{\Omega^{1,2}_\Phi} \ . \tag{3.13}$$

By computing the diagonal coefficients, one gets the exponential decay.

In fact (3.13) will be a straightforward consequence of

$$\langle (M^{-1} d_\Lambda f)\cdot\sigma\rangle_{\Lambda,\omega} \geq \frac{1}{\tilde{C}} ||\, \tilde{\Theta}\, \sigma\, ||^2_{\Omega^{1,2}_\Phi} - \tilde{C}\mathcal{J} ||\tilde{\Theta}\sigma||^2_{\Omega^{1,2}_\Phi} \ , \tag{3.14}$$

with $\sigma = M^{-1} d_\Lambda u$. In order to see this, we apply (3.14) with $\mathcal{J}$ small and get

$$\|\tilde{\Theta}\sigma\|_{\Omega^{1,2}_\Phi} \cdot \|\Theta M^{-1} d_\Lambda f\|_{\Omega^{1,2}_\Phi} \geq \left|\langle (M^{-1} d_\Lambda f)\cdot \sigma\rangle_{\Lambda,\omega}\right| \geq \frac{1}{2\tilde{C}} \|\, \tilde{\Theta}\, \sigma\, \|^2_{\Omega^{1,2}_\Phi} \ , \tag{3.15}$$

Therefore the bound (3.13) holds.

We are now going to prove (3.14). The first step is to write

$$M^{-1}\, d_\Lambda f = \Delta^{(1)}_{\Phi^{\Lambda,\omega}}\sigma + (M^{-1}\Delta^{(1)}_{\Phi^{\Lambda,\omega}} M - \Delta^{(1)}_{\Phi^{\Lambda,\omega}})\sigma \ . \tag{3.16}$$

The last term in the RHS depends only on the off-diagonal part of $\Delta^{(1)}_{\Phi^{\Lambda,\omega}}$, namely on the $|\Lambda|\times|\Lambda|$ matrix $\mathrm{Hess}\,\Phi^i$ defined by

$$\mathrm{Hess}\,\Phi^i_{jk} = \begin{cases} -V''(x_j - x_k)\ , & \text{if } j,k\in\Lambda\ ,\ j\sim k\ ,\\ 0, & \text{otherwise}\ .\end{cases}$$

We would like to show that, there exists a constant C such that, for any 1-form v, we have

$$\langle \mathrm{Hess}\,\Phi^i v \ \cdot\ v\rangle_{\Lambda,\omega} \leq C \|\tilde{\Theta} v\|^2_{\Omega^{1,2}_\Phi} \ . \tag{3.17}$$

Using (2.3), we observe that, for any X, any $v = \sum_j v_j dx_j$ and for $(j,k)\in\Lambda^2$ such that $j\sim k$,

$$\begin{aligned} |V''(x_j - x_k) v_j(x) v_k(x)| &\leq C(1+|\phi''(x_j)|^2)^{\frac14}(1+|\phi''(x_k)|^2)^{\frac14}|v_j(x)|\cdot|v_k(x)| \\ &\leq \tfrac{C}{2}\left((1+|\phi''(x_j)|^2)^{\frac12}|v_j(x)|^2 + (1+|\phi''(x_k)|^2)^{\frac12}|v_k(x)|^2\right) \\ &\leq \tilde{C}\left((\tilde{\Theta}v)_j(x)^2 + (\tilde{\Theta}v)_k(x)^2\right) \ . \end{aligned}$$

This permits to show (3.17) easily. Choosing κ small enough, we see that

$$\left|\langle (M^{-1}\Delta^{(1)}_{\Phi^{\Lambda,\omega}} M - \Delta^{(1)}_{\Phi^{\Lambda,\omega}})\sigma\cdot\sigma\rangle_{\Lambda,\omega}\right| \leq \mathcal{J} C\, \|\tilde{\Theta}\sigma\|^2_{\Omega^{1,2}_\Phi} \ . \tag{3.18}$$

REMARK 3.5. In [**BaJeSj**], similar results are obtained but the control of the interaction involves $(1+|\phi'|^2)$ instead of $(1+|\phi''|^2)$.

The second step is to decompose $\Delta^{(1)}_{\Phi^{\Lambda,\omega}}$ as follows

$$\langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}}\sigma\cdot\sigma\rangle_{\Lambda,\omega} = \sum_{j,k}\langle (X_j\sigma_k)^2\rangle_{\Lambda,\omega} + \sum_j \langle \phi''_j\,\sigma_j^2\rangle_{\Lambda,\omega} + \mathcal{J}\langle \mathrm{Hess}\,\Phi^i\sigma\cdot\sigma\rangle_{\Lambda,\omega} \ , \tag{3.19}$$

where ϕ_j is the effective spin phase at the site $j \in \Lambda$

$$\phi_j(t) = \phi(t) + \mathcal{J} \sum_{k \sim j} V(t - x_k) \ . \tag{3.20}$$

Here we observe that due to the convexity of V and the assumption that $\mathcal{J} \geq 0$, the family of phases ϕ_j satisfies (2.5) uniformly. Moreover, we have the uniform inequality

$$\phi_j''(t) \geq \phi''(t) \ . \tag{3.21}$$

From (3.19) and the assumption on convexity (2.5), we get the following lower bound for some $\mathcal{J}$ small enough

(3.22)
$$\langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}} \sigma \cdot \sigma \rangle_{\Lambda,\omega} \geq \sum_{j \in \Lambda} \langle \phi_j'' \sigma_j^2 \rangle_{\Lambda,\omega} + \mathcal{J} \langle \text{Hess}\, \Phi^i \sigma \cdot \sigma \rangle_{\Lambda,\omega} \geq C \|\tilde{\Theta}\sigma\|^2_{\Omega^{1,2}_\Phi} - C' \|\sigma\|^2_{\Omega^{1,2}_\Phi} \ ,$$

where C, C' are constants.

On the other hand, using the spectral gap of single phase Laplacian (see [**He3**]), we obtain from (3.18) and (3.19) that

$$\langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}} \sigma \cdot \sigma \rangle_{\Lambda,\omega} \geq \lambda^{(1)} \|\sigma\|^2_{\Lambda,\omega} - C\mathcal{J} \|\tilde{\Theta}\sigma\|^2_{\Omega^{1,2}_\Phi} \ , \tag{3.23}$$

where $\lambda^{(1)}$ is a lower bound of the first eigenvalue of the operators $(\Delta^{(1)}_{\Phi^{\{j\}}})_{j \in \Lambda}$.

Combining (3.22) and (3.23), we see that

$$\begin{aligned} C \|\tilde{\Theta}\sigma\|^2_{\Omega^{1,2}_\Phi} &\leq \langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}} \sigma \cdot \sigma \rangle_{\Lambda,\omega} + C' \|\sigma\|^2_{\Omega^{1,2}_\Phi} \ , \\ &\leq \left(1 + \frac{C'}{\lambda^{(1)}}\right) \langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}} \sigma \cdot \sigma \rangle_{\Lambda,\omega} + \frac{C}{\lambda^{(1)}} \mathcal{J} \|\tilde{\Theta}\sigma\|^2_{\Omega^{1,2}_\Phi} \ . \end{aligned}$$

Choosing $\mathcal{J}$ small enough, this concludes the proof.

3.3. Equivalence between spectral gap and decay. Throughout this section, we suppose that $\|V''\|_\infty$ is finite. In the proof of Theorems 3.1 and 3.2, estimates on the Witten Laplacian $\Delta^{(1)}_{\Phi^{\Lambda,\omega}}$ were obtained by using only spectral quantities related to 1-point Witten Laplacians. In order to derive non perturbative results, we decompose the Witten Laplacian on Λ into Witten Laplacians with a large number of variables. This enables us to replace the microscopic estimates by a control on mesoscopic scales which is supposed to be valid in a wider range of temperatures. Such a procedure is reminiscent of the *regular complete analyticity* which was described in the introduction.

Let us first introduce some notation. Let $\mathcal{N} = (n_1, \dots, n_d)$ be a collection of integers and denote by $B_\mathcal{N}(x)$ the box centered in x and of side lengths $\mathcal{N}$

$$B_\mathcal{N}(x) = \left\{ (y_1, \dots, y_d) \in \mathbb{Z}^d \mid \quad -n_i < y_i - x_i \leq n_i \right\}.$$

If $\mathcal{N} = (N, \dots, N)$, we simply write B_N. Let $\mathcal{B}(N_0)$ be the set of boxes with minimum side length N_0 (see also [**Yo1**]). The spectral gap (see Subsection 2.2) on boxes in $\mathcal{B}(N_0)$ is related to uniform decay of correlations.

THEOREM 3.6. *If there exists $c_0 > 0$ and N_0 such that uniformly for $N \geq N_0$*

$$\forall \Lambda \in \mathcal{B}(N), \qquad \mathcal{S}(\Lambda) = \inf_\omega \mathcal{S}(\Lambda, \omega) \geq c_0,$$

then there exists $\lambda, c > 0$ and N_1 such that the following inequality holds for all functions f and g in $C^\infty_{temp}(\mathbb{R}^\Lambda)$

$$|E^{\Lambda,\omega}(f;g)| \leq \lambda \exp\big(- c\, d(S_f, S_g)\big)\ \|d_\Lambda f\|_{\Omega^{1,2}_\Phi}\ \|d_\Lambda g\|_{\Omega^{1,2}_\Phi}\ , \tag{3.24}$$

uniformly with respect to the other parameters $N \geq N_1$, $\Lambda \in \mathcal{B}(N)$ and $\omega \in \mathbb{R}^{\mathbb{Z}^d}$.

In fact this result can be strengthen and it will be clear from the proof that a softer assumption allowing a slow decay of the spectral gap with respect to the size of the cubes would lead to the same conclusion (see [**StZe4**] for similar results on Log-Sobolev constants). Namely,

THEOREM 3.7. *We assume that V'' is bounded. Let $S_N = \inf_{\Lambda\in\mathcal{B}(N),\omega} \mathcal{S}(\Lambda,\omega)$. If*

$$\lim_{N\to\infty} S_N\, N = \infty\ , \tag{3.25}$$

then there is $c_0 > 0$ and N_0 such that

$$\forall N \geq N_0, \qquad S_N \geq c_0\ .$$

REMARK 3.8. Notice that a stronger assumption on positivity of the spectral gap uniformly over every domains would lead to a decay of correlations uniformly over every domains.

Before proving the theorems, let us first introduce some more notations and give a preliminary important estimate. The sub-lattice $N\mathbb{Z}^d$ will be denoted by $\mathcal{L}_N$. Let Λ be a finite domain of $\mathcal{B}(M)$, where M is an integer which will be chosen very large. For a given $N \leq M$, we partition Λ into disjoint sub-domains

$$\bigcup_{k\in\mathcal{L}_N} B_N^\Lambda(x+k),$$

where $B_N^\Lambda(x+k) = \Lambda \cap B_N(x+k)$. The set of edges on the boundaries of $B_N^\Lambda(\cdot)$ is defined by

$$\partial B_N^\Lambda(x) = \big\{\{i,j\} \in \Lambda^2 \ |\quad i\sim j, \ \text{ s. t. } \exists k, \text{ with } \ i \in B_N(x+k),\ j \notin B_N(x+k)\big\}.$$

Let $F_\Lambda = (f_i)_{i\in\Lambda}$ be a function supported by Λ. By using (2.12), we can decompose $\langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}} F_\Lambda\ \cdot\ F_\Lambda\rangle_{\Lambda,\omega}$ as follows

$$\begin{aligned}\langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}} F_\Lambda\ \cdot\ F_\Lambda\rangle_{\Lambda,\omega} \quad &\geq \quad \sum_{k\in\mathcal{L}_N} \langle \Delta^{(1)}_{B_N^\Lambda(x+k)} F_{B_N^\Lambda(x+k)}\ \cdot\ F_{B_N^\Lambda(x+k)}\rangle_{\Lambda,\omega} \\ &\qquad + \sum_{\{i,j\}\in\partial B_N^\Lambda(x)} \langle \frac{\partial^2 \Phi^{\Lambda,\omega}}{\partial x_i \partial x_j}\, f_i\ f_j\rangle_{\Lambda,\omega}\ .\end{aligned}$$

Summing over all the x in $B_N(0)$, one obtains

$$\begin{aligned}\langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}} F_\Lambda\ \cdot\ F_\Lambda\rangle_{\Lambda,\omega} \quad &\geq \quad \frac{1}{N^d} \sum_{x\in B_N(0)}\sum_{k\in\mathcal{L}_N} \langle \Delta^{(1)}_{B_N^\Lambda(x+k)} F_{B_N^\Lambda(x+k)}\ \cdot\ F_{B_N^\Lambda(x+k)}\rangle_{\Lambda,\omega} \\ &\quad - \quad \frac{1}{N^d} \sum_{x\in B_N(0)}\ \sum_{\{i,j\}\in\partial B_N^\Lambda(x)} \left| \langle \frac{\partial^2 \Phi^{\Lambda,\omega}}{\partial x_i \partial x_j}\, f_i\ f_j\rangle_{\Lambda,\omega}\right|\ .\end{aligned}$$

As V'' is bounded, this leads to

$$\begin{aligned}\langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}} F_\Lambda \cdot F_\Lambda \rangle_{\Lambda,\omega} \quad &\geq \quad \frac{1}{N^d} \sum_{x \in B_N(0)} \sum_{k \in \mathcal{L}_N} \langle \Delta^{(1)}_{B_N^\Lambda(x+k)} F_{B_N^\Lambda(x+k)} \cdot F_{B_N^\Lambda(x+k)} \rangle_{\Lambda,\omega} \\ &\qquad -2\alpha_d\, \mathcal{J} \frac{N^{d-1}}{N^d} \sum_{\{i,j\} \in \Lambda} 1_{i\sim j} \, \langle \, |f_i \, f_j| \rangle_{\Lambda,\omega} \ , \end{aligned} \tag{3.26}$$

where α_d depends on the dimension d and on $\|V''\|_\infty$.

The above inequality will be useful to derive Theorem 3.7.

Proof of Theorem 3.7 : We first note that

$$\mathcal{S}(\Lambda,\omega) \langle d_\Lambda u \, \cdot \, d_\Lambda u \rangle_{\Lambda,\omega} \leq \langle (\Delta^{(0)}_{\Phi^{\Lambda,\omega}} u)^2 \rangle_{\Lambda,\omega} = \langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}} d_\Lambda u \, \cdot \, d_\Lambda u \rangle_{\Lambda,\omega} \ .$$

Our assumption relies only on regular sets and a translate $B_N^\Lambda(x+k)$ does not in general belong to $\cup_{n=[\frac{N}{2}]}^N \mathcal{B}(n)$; nevertheless for any vertex in Λ, there is at least $\frac{N^d}{2^d}$ translates of B_N^Λ which are in $\cup_{n=[\frac{N}{2}]}^N \mathcal{B}(n)$. Since a Witten Laplacian on any domain is a positive operator, we can neglect the terms in (3.26) with Witten Laplacians on boxes which are not in $\cup_{n=[\frac{N}{2}]}^N \mathcal{B}(n)$. The other terms are bounded from below by the assumption on the spectral gap. Noticing that $\mathcal{J}$ is finite, we get by replacing F_Λ by $d_\Lambda u$ in (3.26)

$$\langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}} d_\Lambda u \, \cdot \, d_\Lambda u \rangle_{\Lambda,\omega} \geq \big(\tfrac{S_{N/2}}{2^d} - 2\alpha_d \mathcal{J} \tfrac{N^{d-1}}{N^d} \big) \textstyle\sum_{k \in \Lambda} \langle (d_k u)^2 \rangle_{\Lambda,\omega} \ . \tag{3.27}$$

Thus, it remains to choose N large enough such that $S_{\frac{N}{2}} - \frac{2^{d+1}\alpha_d \mathcal{J}}{N}$ is positive. ∎

We are going now to derive Theorem 3.6.
Proof of Theorem 3.6 : Choosing the same notation as in Subsection 3.2, we can reduce the proof to

$$\langle (M^{-1} d_\Lambda f) \cdot \sigma \rangle_{\Lambda,\omega} \geq \tilde{C} \| \, \sigma \|^2_{\Omega^{1,2}_\Phi} \ , \tag{3.28}$$

where $\sigma = M^{-1} d_\Lambda u$ and $\tilde{C}$ is a constant. We start with the identity

$$\langle (M^{-1} d_\Lambda f) \cdot \sigma \rangle_{\Lambda,\omega} = \langle M^{-1} \Delta^{(1)}_{\Phi^{\Lambda,\omega}} M M^{-1} d_\Lambda u \, \cdot \, \sigma \rangle_{\Lambda,\omega} = \langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}} \, M\sigma \, \cdot \, M^{-1}\sigma \rangle_{\Lambda,\omega} \ . \tag{3.29}$$

We can rewrite (3.26) in the form

$$\begin{aligned}\langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}} M\sigma \quad \cdot \quad M^{-1}\sigma \rangle_{\Lambda,\omega} \geq& \\ &\frac{1}{N^d} \sum_{x \in B_N(0)} \sum_{k \in \mathcal{L}_N} \langle \Delta^{(1)}_{B_N^\Lambda(x+k)} (M\sigma)_{B_N^\Lambda(x+k)} \, \cdot \, (M^{-1}\sigma)_{B_N^\Lambda(x+k)} \rangle_{\Lambda,\omega} \\ &\quad - \frac{2\alpha_d \mathcal{J}}{N} \sum_{i \in \Lambda} \langle \, \sigma_i^2 \rangle_{\Lambda,\omega} \ . \end{aligned}$$

Since the coefficients of the matrix M are slowly varying, we obtain

$$\begin{aligned}\langle \Delta^{(1)}_{\Phi^{\Lambda,\omega}} M\sigma \ \cdot \ & M^{-1}\sigma\rangle_{\Lambda,\omega} \geq \\ & \frac{1}{N^d}\sum_{x\in B_N(0)}\sum_{k\in\mathcal{L}_N}\langle \Delta^{(1)}_{B_N^\Lambda(x+k)}\sigma_{B_N^\Lambda(x+k)} \ \cdot \ \sigma_{B_N^\Lambda(x+k)}\rangle_{\Lambda,\omega} \\ & - \Big(O(1-\exp(-\kappa)) + O(\frac{1}{N})\Big)\,\|\sigma\|^2_{\Omega^{1,2}_\Phi}\,.\end{aligned}$$

The bound $\mathcal{S}_{\frac{N}{2}}$ on the spectral gap cannot be used at this stage because σ is not an exact 1-form. Therefore, a technical Lemma (which we are going to prove later) is required

LEMMA 3.9. *Let $D \subset \Lambda$ be two finite domains in $\mathbb{Z}^d$. Then, there exists a constant $\alpha_d > 0$ such that for any matrix $\rho = (\rho_i)_{i\in D}$ with positive coefficients and any function $u \in C^\infty_{temp}(\mathbb{R}^\Lambda)$ the following holds*

$$\langle \Delta^{(1)}_D \rho d_D u \ \cdot \ \rho d_D u\rangle_{\Lambda,\omega} \geq \Big(\mathcal{S}(D)\frac{\bar\rho^2}{\hat\rho^2} - \alpha_d\big(\frac{\hat\rho^2}{\bar\rho^2} - 1\big)\Big)\,\|\rho d_D u\|^2_{\Omega^{1,2}_\Phi}\,.$$

where $\hat\rho = \sup_{i\in D}\rho_i$ and $\bar\rho = \inf_{i\in D}\rho_i$.

Since the matrix M has slowly varying coefficients (3.10), the previous Lemma enables us to conclude the proof of Theorem 3.6 by choosing first N large and then κ small enough such that

$$c_0\exp(-C_d\kappa N) - \alpha_d\big(\exp(C_d\kappa N) - 1\big) + O(1-\exp(-\kappa)) + O(\frac{1}{N}) > 0,$$

where C_d is a constant depending only on the dimension. ∎

Proof of Lemma 3.9 :

By using the identity (2.12), we see that

$$\langle \Delta^{(1)}_D \rho d_D u \ \cdot \ \rho d_D u\rangle_{\Lambda,\omega} = \sum_{i,j\in D}\rho_i^2\|X_j d_i u\|^2_{\Omega^{1,2}_\Phi} + \frac{1}{2}\sum_{i,j\in D}\rho_i\rho_j\langle\frac{\partial^2\Phi^{\Lambda,\omega}}{\partial x_i\partial x_j}d_j u\ d_i u\rangle_{\Lambda,\omega}\,.$$

This leads to

$$\begin{aligned}\langle \Delta^{(1)}_D \rho d_D u \ \cdot \ \rho d_D u\rangle_{\Lambda,\omega} \ \geq \ & \bar\rho^2\,\langle \Delta^{(1)}_D d_D u \ \cdot \ d_D u\rangle_{\Lambda,\omega} \\ & + \sum_{i,j\in D}\big(\rho_i\rho_j - \bar\rho^2\big)\langle\frac{\partial^2\Phi^{\Lambda,\omega}}{\partial x_i\partial x_j}d_j u\ d_i u\rangle_{\Lambda,\omega}\,.\end{aligned}$$

The lower bound on the spectral gap of $\Delta^{(0)}_D$ can be used now. There exists a constant $\alpha_d > 0$ depending only on $\|V''\|_\infty$ and the dimension d such that the following holds

$$\langle \Delta^{(1)}_D \rho d_D u \ \cdot \ \rho d_D u\rangle_{\Lambda,\omega} \geq \bar\rho^2\mathcal{S}(D)\,\|d_D u\|^2_{\Omega^{1,2}_\Phi} - \alpha_d\big(\hat\rho^2 - \bar\rho^2\big)\,\|d_D u\|^2_{\Omega^{1,2}_\Phi}\,.$$

This implies the Lemma. ∎

4. Log-Sobolev Inequalities

We consider a measure[1] $d\mu := \exp(-\Phi)\, dx$ on $\mathbf{M} = \mathbb{R}^N$ and want to analyze the following inequality known as Log-Sobolev inequality

$$\int_M |f|^2 \ln |f|^2 \, d\mu - \|f\|_\mu^2 \ln \|f\|_\mu^2 \leq C \int_M |\nabla f|^2 \, d\mu \,. \tag{4.1}$$

We shall assume that μ is a probability measure

$$\int_M d\mu = 1 \,. \tag{4.2}$$

The best constant such that (4.1) is satisfied (if it exists) will be denoted by $C_{L.S.}$. This inequality can be seen as a control of the entropy, which is defined, for a non positive function g such that $E_\mu(g \ln^+ g) < +\infty$, in the following way :

$$\mathrm{Ent}_\mu(g) = E_\mu(g \ln g) - E_\mu(g) \ln E_\mu(g) \,. \tag{4.3}$$

Here we have used the notation : $E_\mu(g) = \int g \; d\mu$. We note that the entropy has the following properties

$$\mathrm{Ent}_\mu(g) \geq 0 \quad \text{and} \quad \mathrm{Ent}_\mu(\alpha g) = \alpha \ \mathrm{Ent}_\mu(g) \ , \ \forall \alpha \geq 0 \,. \tag{4.4}$$

4.1. Log-Sobolev inequalities in the strictly convex case. In this section, we study the Log-Sobolev inequality under a condition of uniform strict convexity of the phase Φ defined on $\mathbf{M} = \mathbb{R}^N$. More precisely we assume the existence of $\lambda > 0$ such that

$$\mathrm{Hess}\, \Phi(x) \geq \lambda \ , \qquad \forall x \in \mathbf{M} \,. \tag{4.5}$$

This assumption is a particular case of the Bakry-Emery criterion [**BaEm**].

THEOREM 4.1. *Under the condition (4.5), the following Log-Sobolev inequality is satisfied :*

$$\int_M |f|^2 \ln |f|^2 d\mu - \|f\|_\mu^2 \ln \|f\|_\mu^2 \leq \frac{2}{\lambda} \int_M |\nabla f|^2 d\mu \ , \tag{4.6}$$

where $d\mu$ is the probability measure $\exp(-\Phi)\, dX$.

Consequently, the best Log-Sobolev constant satisfies :

$$C_{L.S.} \leq \frac{2}{\lambda} \,. \tag{4.7}$$

Theorem 4.1 will be obtained as a particular case of the following more general theorem inspired by [**AA1**] (see also [**He4**])

THEOREM 4.2. *Let Ψ be a function in $C^0([0,+\infty[) \cap C^\infty(]0,+\infty[)$ satisfying the conditions*

$$\Psi'' > 0 \qquad \text{and} \qquad (1/\Psi'')'' \leq 0 \ , \tag{4.8}$$

and suppose that there exists a positive constant C such that

$$\forall t \geq C, \qquad \Psi''(t) \leq C \,. \tag{4.9}$$

[1] On a compact riemannian manifold $\mathbf{M}$, this would be a measure of the type $\exp(-\Phi)\; d\sigma$ where $d\sigma$ is the riemannian measure on $\mathbf{M}$.

Then under assumption [2] *(4.5), the following holds*

$$\int_{\mathbf{M}} \Psi(f) d\mu - \Psi(\int_{\mathbf{M}} f) \leq \frac{1}{2\lambda} \int_{\mathbf{M}} \Psi''(f)\, |\nabla f|^2 d\mu\ , \tag{4.10}$$

for any $f > 0$ in the class of the C^1 functions with bounded derivatives.

The basic examples for which the above assumptions are satisfied are

- $\Psi(x) = x^2$ and this leads to Poincaré inequality, observing the inequality

$$|\nabla |f|\,| \leq |\nabla f| \ \text{ a. e } . \tag{4.11}$$

- $\Psi(x) = x \ln x$ and this leads (after a change of functions $f = g^2$) to the standard logarithmic Sobolev inequality,
- $\Psi(x) = x^p$, with $1 < p \leq 2$.

For $\Psi = x \ln x$, one can prove that the Log-Sobolev inequality is implied by the following inequality

$$\begin{aligned} \lambda \int_{\mathbf{M}} e^u ||\nabla u(x)||^2\, d\mu \quad &\leq \quad \int_{\mathbf{M}} e^u (\nabla u \ \cdot \ \mathrm{Hess}\, \Phi\ \nabla u)\, d\mu \\ &\qquad + \sum_{ij} \int_{\mathbf{M}} e^u\, |\partial_{ij} u|^2\, d\mu\ . \end{aligned} \tag{4.12}$$

If Φ is strictly convex, the above inequality clearly holds. In order to understand whether the condition (4.5) might be improved, one can start to study the one dimensional case. We have indeed the property that if (4.12) is true on $\mathbb{R}$ with $d\mu = \exp(-\phi(x))\, dx$, then the same inequality is true on $\mathbb{R}^N$ for the N-product measure $d\mu_N = \otimes_N d\mu$. So we are facing the simple question : Under which condition on ϕ do we have

$$\begin{aligned} \lambda \int_{\mathbb{R}} \exp(u(t) - \phi(t))\, u'(t)^2\, dt \quad &\leq \quad \int_{\mathbb{R}} \exp(u(t) - \phi(t)) \phi''(t)\, u'(t)^2\, dt \\ &\qquad + \quad \int_{\mathbb{R}} \exp(u(t) - \phi(t))\, u''(t)^2\, dt\ . \end{aligned} \tag{4.13}$$

The complete answer to this problem seems open outside the case when Φ is strictly convex. Let us just give here a counter-example[3]. We consider ϕ_0 an even C^∞ phase convex at ∞ but such that $\phi_0''(0) < 0$. Then there exists γ_0 such that Inequality (4.13) with $\phi = \gamma\phi_0$ is not true when $\gamma > \gamma_0$. We can indeed find u such that $u = -b\gamma t^2/2$ in the neighborhood of 0 and $\phi - u$ has a unique non degenerate minimum at the origin giving the contradiction. Here b has to satisfy the condition $-\phi_0''(0) < b < -2\phi_0''(0)$. The contradiction comes through the use of the Laplace integral method (with large parameter γ) which shows that the RHS of (4.13) becomes strictly negative as γ tends to infinity.

Nevertheless, as ϕ_γ is convex at ∞, the corresponding logarithmic inequality is satisfied. This property is implied by the lemma below (see for example [**DeSt**]).

LEMMA 4.3. *If Φ and $\tilde{\Phi}$ are two phases such that*

$$\Phi - \tilde{\Phi} = S\ , \tag{4.14}$$

[2]Our proof uses also other technical assumptions which will be mentioned later on the derivatives of Φ which are probably essentially technical.

[3]We thank M. Ledoux for motivating discussions around this problem.

with S bounded, then

$$C_{L.S}(\mathbf{M}, \exp -\Phi\ d\sigma) \leq \exp\left(2 \sup_{x\in\mathbf{M}} |S(x)|\right) C_{L.S}(\mathbf{M}, \exp -\tilde{\Phi}\ d\sigma)\ . \tag{4.15}$$

More generally, it was noticed by Herbst (see [**Le**]) that a necessary condition is the existence of $C > 0$, $D > 0$ such that

$$\int_{|x|\geq r} \exp(-\Phi(x))\ dx \leq C\ \exp(-Dr^2)\ , \tag{4.16}$$

as $r \to +\infty$.

Another problem would be to understand more precisely how the Witten Laplacian is related to the Log-Sobolev inequality. It will be explained in the next section that the uniform positivity of the first eigenvalue of the Witten Laplacian implies the decay of correlations and therefore the Log-Sobolev inequality for the spins systems. Such an assumption on the first eigenvalue is clearly satisfied under the strict convexity hypothesis (4.5). Therefore, we might wonder whether the formalism of the Witten Laplacian could provide an extension of the Bakry-Emery criterion. In fact the following example shows that is probably not the case. Take $\phi(x) = (1 + |x|^2)^{\frac{\theta}{2}}$; then, for $\theta > 1$, the Poincaré inequality is true and the Log-Sobolev inequality is false for $\theta \in [1, 2[$ (see [**DeSt**] exercise 6.2.47).

In this case, we do not have a uniform lower bound (with respect to $\mathcal{J}$ and α) for the family of Witten-Laplacian on 1-forms associated to the family of phases

$$\phi_{\mathcal{J},\alpha} = \phi + \frac{1}{2}\mathcal{J} d t^2 - \alpha t\ .$$

When $\mathcal{J} = 0$, one can easily get

$$\inf \sigma\left(-\frac{d^2}{dt^2} + (\phi'(t) - \alpha)^2 + \frac{1}{2}\phi''(t)\right) = \mathcal{O}(\alpha^{\frac{\theta-2}{\theta-1}})\ . \tag{4.17}$$

4.2. Finite size conditions. We are mainly concerned by a series of equivalences between strong mixing, uniform Poincaré inequalities and uniform Log-Sobolev inequalities.

THEOREM 4.4. *Under the assumption that the single spin phase is super-convex at ∞, that is that, for any $m > 0$, there exists a bounded C^2 function s such that,*

$$(\phi + s)''(t) \geq m\ , \tag{4.18}$$

and assuming that the interaction is convex with bounded second derivatives, then, for any $\mathcal{J} > 0$, the following conditions are equivalent :

1. *Correlations decay exponentially fast*

$$\left|E^{\Lambda,\omega}(f; g)\right| \leq C\ \exp\left(-\frac{1}{C} d(S_f, S_g)\right)\ \|d_\Lambda f\|_{\Omega^{1,2}_\Phi}\ \|d_\Lambda g\|_{\Omega^{1,2}_\Phi}\ , \tag{4.19}$$

   uniformly with respect to Λ, ω.
2. *The Poincaré inequalities hold uniformly with respect to Λ, ω, i.e. the spectral gaps of the Laplacians $\Delta^{(0)}_{\Phi^{\Lambda,\omega}}$ admit a uniform lower bound.*
3. *The Log-Sobolev inequalities hold uniformly.*

The above theorem is part of a more elaborate theorem by Yoshida [**Yo3**] (extending results of Stroock-Zegarlinski in the compact case [**StZe1, StZe2**], see also Liggett [**Li**]). Notice although that our assumptions differ slightly from those used by Yoshida [**Yo3**].

Proof of Theorem 4.4

Some of the implications are standard. The second condition is simply a particular case of the first one (take $f = g$). The Log-Sobolev inequality implies the Poincaré inequality with the same constant (see [**Ro**]).

From Theorem 3.6, we know that the second condition implies the first one. It has been proved in [**BoHe**] (see also [**Yo1**]) that condition 1 implies condition 3. ∎

As explained in the introduction, the previous theorem is standard in the theory of spins systems with compact phase spaces and was first extended to non-compact spaces by Yoshida [**Yo3**]. The scheme of our proof differs from the one developed in [**Yo3**]. Nevertheless, we stress the fact that the results in [**Yo3**] are stronger because they do not rely on an iterative procedure : if for a given domain Λ, the spectral gap is uniformly positive independently of the boundary conditions then it is proved in [**Yo3**] that the correlations decay in Λ. Our method requires uniformity of the spectral gap over the domains and the boundary conditions.

The equivalence of the two first points is true without using the convexity at ∞. Of course the existence of a spectral gap implies that the bottom of the essential spectrum of $\Delta^{(1)}_{\Phi^{\Lambda,\omega}}$ (which is always essentially selfadjoint when Φ is C^∞) is strictly positive.

The statement that the second assertion implies, when (2.5) is satisfied, the third assertion was given in [**Ze3**] without detailed proof when $d > 1$. A detailed proof was then proposed by Yoshida [**Yo1**] under the assumptions (4.18), quadraticity of the interaction (assumption appearing also in [**Ze3**]) and other restrictive technical assumptions on the single spin phase which were removed in [**BoHe**]. In [**BoHe**], it was proven in the perturbative case, that for $\mathcal{J} \geq 0$ small enough, the 3 conditions above hold under the weaker assumption (2.5) and the boundedness of $|V''|$. In fact, the first assertion can even be obtained (see [**He1, He2, He3**]) in the perturbative case under weaker conditions than (2.5); for example (4.19) is true as soon as

$$-\frac{d^2}{dt^2} + \frac{1}{2}(\phi'' + s'') > 0 \, . \tag{4.20}$$

Finally, we would like to mention that for $d = 1$, Zegarlinski proved in [**Ze3**] that the three assertions are true for any $\mathcal{J} > 0$ under the assumption (4.18). Our method do not enables us to take advantage of the dimension and we cannot prove the decay of correlations for an arbitrary intensity of the interaction in a one-dimensional model.

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