Higher dimensional versions of the Folklore Theorem, I

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Abstract

We prove in this note that Markov transformations defined by infinitely many C^2 expanding branches $T_i : \Omega_i \subseteq \Omega \longrightarrow \Omega$ in $\mathbb{R}^m$ $(m \geq 1)$ satisfying a bounded distortion condition leaves invariant a unique ergodic smooth measure.

1 Introduction

This is the first of two notes on higher dimensional versions of the Folklore Theorem. In this issue we are aimed at proving the following result.

Folklore Theorem *Let Ω be a compact domain with piecewise smooth boundary in $\mathbb{R}^m$, an open and dense subset $\Omega_0 \subseteq \Omega$ and $T : \Omega_0 \longrightarrow int\,\Omega$ a C^2 expanding Markov transformation such that:*

1. *there is a sequence $\Omega_i \subseteq \Omega$ of subdomains with pairwise disjoint interiors such that $\Omega = \bigcup_i \Omega_i$ up to a Lebesgue measure zero set and $\Omega_0 = \bigcup_i int\,\Omega_i$;*
2. *for each $i > 0$ there is a 1-to-1 and onto transformation $T_i : \Omega_i \longrightarrow \Omega$ which has a C^2 extension $\hat{T}_i : U_i \longrightarrow V_i$ to neighborhoods U_i and V_i containing Ω_i and Ω, respectively. We define T as being $T|int\,\Omega_i = T_i|int\,\Omega_i$;*
3. *there is a constant λ such that $\inf_{x\in\Omega_i} \|DT_i(x)\| \geq \lambda > 1$ for every $i > 0$;*
4. *it holds the following* **bounded distortion condition**

$$\sup_{i>0} \frac{\sup_{z\in\Omega_i} \|D^2_{jk}T_i(z)\|}{\inf_{z,w\in\Omega_i} \|D_jT_i(z)\| \cdot \|D_jT_i(w)\|} = d < +\infty, \tag{1}$$

where D^2_{ij} and D_i denotes partial derivatives.

Then T preserves a unique asymptotic Borel probability μ ergodic which is absolutely continuous with respect to (wrt) the Lebesgue measure:

$$\mu(B) = \lim_{n\to+\infty} \frac{\mathcal{L}(T^{-n}B)}{\mathcal{L}(\Omega)},$$

where $\mathcal{L}$ is the Lebesgue measure in $\mathbb{R}^m$. In addition, μ has density bounded away from zero and infinite, i.e.

$$0 < \inf_{x \in \Omega} d\mu \,/\, d\mathcal{L}(x) \leq \inf_{x \in \Omega} d\mu \,/\, d\mathcal{L}(x) < +\infty.$$

$d\mu \,/\, d\mathcal{L}$ is the Radon-Nykodim derivative of μ respect to the Lebesgue measure $\mathcal{L}$ of $\mathbb{R}^m$. Consequently, the system $T : (\Omega, \mu) \circlearrowleft$ is exact.

By a domain we shall mean a *compact convex subset of* $\mathbb{R}^m$. T is exact if for every $B \in \bigcap_{n \geq 0} T^{-n}\mathcal{B}$ it holds $\mu(B)\mu(B^c) = 0$, where $\mathcal{B}$ are the Borel subsets. This is a strong ergodic property and it implies, in particular, that T is mixing. Cf. [9].

In some sense μ is the unique physically relevant equilibrium state for T. Also μ satisfies the entropy formula $h_\mu(T) = \int \log\, Jac\,T(x) d\mu(x)$, so it can be seen a an extremal of certain variational principle, where Jac denotes the Jacobian with respect to the Lesbesgue measure: $Jac(f)(x) = |\det\, Df(x)|$.

We say a few words about this kind of results and then we outline the main point in the proof.

Lasota and Yorke proved in [8] that certain expanding, piecewise C^2 transformations of the interval leaves invariant an absolutely continuous invariant measure (acim) using 'bounded variation' methods. Lasota-Yorke's theorem is part of a large list of results on the existence of acim for a variety of uniformly expanding systems. Compare Adler's foreword to Bowen's posthumous article [2] for a discussion on the origins of this nowadays widely known Folklore Theorem. In spite of suggestions made at the end of [8], higher dimensional generalizations of Lasota-Yorke's 'bounded variation' approach has taken much longer than expected, due in part to the difficulty in finding the appropriate definition of bounded variation in higher dimensions. Major attempts to prove the existence of acim for higher dimensional expanding endomorphisms with finitely many branches were made in [4], [5], [7], [13] using a notion of bounded variation as an integral of a generalized derivative and geometrical conditions on boundary domains of the expanding branches. These results are for finitely many expanding branches. In the case of infinite branches T_i their derivatives grow with i and relations between the first and second derivatives become crucial, so new difficulties arise in this case in the 'bounded variation' scenario. It is worth mention also the non Markovian case, where new phenomena arises as shows recent works by Buzzi and Tsuji.

In this work we use bounded distortion of the volume and Markov property in a setting of dimension theory to prove Folklore Theorem showing that the asymptotic measure μ is indeed a limit of a sequence of Gibbs measures μ_n supported on a sequence of hyperbolic repellors Λ_n which coalesces into a subset of positive Lebesgue measure. Actually μ_n maximizes a dimensional characteristic and satisfies a Frostman-like distribution property provided by certain dynamically defined Carathéodory's structure. We shall follow Pesin's monography [11] as main reference for details on definitions, statements and proofs for this dimension theory setting.

We borrow the above approach on fractal approximation of smooth ergodic measures from [10]. Let us explain the idea in a simplified framework.

Consider a sequence of Cantor sets in the unit interval $\Lambda_n \subseteq [0,1]$ with the following regularity properties:
(*i*) $\dim_{\mathcal{H}}(\Lambda_n) \longrightarrow 1$;
(*ii*) $C^{-1} \leq \mathcal{H}_{\alpha(n)}(\Lambda_n) \leq C$;
(*iii*) $C^{-1} \leq \mathcal{H}_{\alpha(n)}(\Lambda_n \cap B(x,r)) \,/\, r^{\alpha(n)} \leq C$,
for every $n > 0$, $0 < r < 1$, $x \in \Lambda_n$ and for a suitable constant $C > 1$, where $\alpha(n) = \dim_{\mathcal{H}}(\Lambda_n)$.

We affirm that $\Lambda = \bigcap_{n>0} \overline{\bigcup_{k \geq n} \Lambda_k}$ has positive Lebesgue measure. Indeed, let μ_n be the Hausdorff measure restricted to Λ_n normalized to a probability. By hypotheses μ_n has the following strong Frostman's uniform distribution property (cf. [11, Chapter II, pp.43]):

$$\mu_n(B(x,r)) \asymp r^{\dim_{\mathcal{H}}(\Lambda_n)} \quad \textit{uniformly for } x \in \Lambda_n \textit{ and } 0 < r < 1, \tag{2}$$

where we adopt the following notation: given ϕ and ψ two set functions $\phi(P) \asymp \psi(P)$ shall mean that $\phi(P) \,/\, \psi(P) \in [C^{-1}, C]$ for a uniform constant not depending on P. μ_n converges weakly, passing to a subsequence if necessary, to a measure μ whose support is contained in Λ such that $\mu(B(x,r)) \asymp r$. Using Frostman's uniform mass distribution principle we conclude that μ is equivalent to $\mathcal{L} \mid \mathit{supp}\ \mu$ and then that $\mathcal{L}(\Lambda) > 0$.

Given infinitely many expanding branches T_i which covers a single domain $\Omega \subseteq \mathbb{R}^m$ one choose, for each $n > 0$ finitely many branches, say $T_1, \cdots, T_n$ to form a sequence of (in general non conformal) hyperbolic repellors Λ_n. Each Λ_n is endowed with an ergodic measure $\mu_n = \mu_{\phi_n}$, the Gibbs measure of $\phi_n = -\alpha(n) \log \mathit{Jac}\, T$, where $0 < \alpha(n) < 1$ is the solution to the Bowen equation: $P(T \mid \Lambda_n, -\alpha(n) \log \mathit{Jac}\, T) = 0$ (cf. [11, pp. 106]). $\alpha(n)$ is a dimensional characteristic of a dynamically defined Carathéodory's structure (cf.[11, Chapter I] and [11, Appendix III]). Further, each Λ_n has a family $\wp(\Lambda_n)$ obtained by pulling back the initial Markov partition $\{\Omega_j\}_{j=1,\cdots,n}$ through the inverse contracting branches of T_i (cf. [11, pp. 194]). $\wp(\Lambda_n)$ which generates the Borel subsets of Λ_n. In this setting, the following weak form of Frostman's distribution property for the volume holds: $\mu_n(P) \asymp \mathrm{Vol}\,(P)^{\alpha(n)}$ for every $P \in \wp(\Lambda_n)$, bounded by a constant not depending on $n > 0$ neither P. Passing to a subsequence if necessary μ_n converges to an ergodic measure μ such that

$$\mu(P) \asymp \mathrm{Vol}\,(P)^{\alpha} \qquad \forall\ P \in \wp = \bigcup_{n>0} \wp_n, \tag{3}$$

where α is the limit of the sequence $\alpha(n)$. $\wp$ generates the limit set Λ obtained from the inverse contracting branches of T_i. The above property 3) suggests to construct a fractal measure $\mathcal{D}_{\alpha,\wp}$ with aid of Carathéodory's construction, using coverings by $\wp$-sets and $\mathrm{Vol}\,(P)^{\alpha}$ as set function, where $\alpha = \dim_{\mathcal{D}}(\Lambda)$ is the corresponding dimensional characteristic. We call $\mathcal{D}_{\alpha,\wp}$ *the dynamical measure and* $\dim_{\mathcal{D}}$ *the dynamical dimension* respectively.

We use Ruelle's theorem (cf. [9, Chapter III] or [14, Theorem 8]) to show that μ is absolutely continuous wrt to $\mathcal{D}_{\alpha,\wp}$. Weak distribution property (3)shows that $\dim_{\mathcal{D}}(\Lambda) = 1$ if and only if μ is absolutely continuous wrt Lebesgue measure $\mathcal{L}$. Further, in this case μ will be equivalent to $\mathcal{L}$ restricted to Λ, the limit set, since it has a density bounded away from zero and infinite.

The dynamical measure class generalizes to higher dimensions several well known features of the Hausdorff measure of conformal hyperbolic repellors. Compare [11, Chapter 7]. Even though, it is worth to mention that it does not reflect the geometry, but a dimensional property of the dynamics supported on these Cantor sets. In relation with this question we would like to address the following conjecture, raised by Pesin, as communicated to me by R.O. Ruggiero (compare [11, Appendix IV]):

Conjecture
The dynamical measure and the Hausdorff measure of a hyperbolic repellor Λ are equivalent if and only if Λ is conformal or quasi-conformally conjugated to a conformal hyperbolic repellor.

Nonetheless, not being a true geometrical invariant, the dynamical measure provides a useful tool as we shall see later.

Previous arguments were made for general, possibly non conformal, repellors. As a consequence of the present approach a continuity property of the Hausdorff dimension for conformal hyperbolic repellors, namely: let $T_i : \Omega_i \longrightarrow \Omega$ be a countable infinite collection of *conformal expanding branches* with bounded distortion and Λ the limit set generated by its contracting inverse branches. Then, there exists sequence $\Lambda_n \subseteq \Lambda$ of Cantor sets such that:

- $\dim_{\mathcal{H}}(\Lambda_n) \longrightarrow \dim_{\mathcal{H}}(\Lambda)$,
- $\mathcal{H}_{\alpha(n)}(\Lambda_n) \asymp 1$, where $\alpha(n) = \dim_{\mathcal{H}}(\Lambda_n)$ and
- $\mathcal{H}_{\alpha(n)}(\Lambda_n \cap B(x,r)) \asymp r^{\alpha(n)}$ uniformly for a suitable constant $C > 1$ not depending on $n > 0$ neither $x \in \Lambda$ nor $r > 0$.

This follows from properties of Moran's covers and our bounded distortion conditions. Compare [11, Chapter 7, pp. 200-201]

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2 Statement of results and proof of the Folklore Theorem

Let $T_i : \Omega_i \longrightarrow \Omega$ be a countable family of *expanding branches with bounded distortion* like those introduced in Theorem A. From now on we will allow $\bigcup_i \Omega_i$

to not cover completely Ω so it might be $\mathcal{L}(\Omega - \bigcup_i \Omega_i) > 0$.

The inverse maps $T_i^{-1} = \tau_i : \Omega \longrightarrow \Omega_i$ forms a collection of contracting branches defining an *iterated function system* (IFS). Let Λ be its limit set. We use τ_i to form a sequence of decompositions $\wp_n$ such that :

1. for each $P \in \wp_{n+1}$ there exists a unique $Q \in \wp_n$ such that $P \subseteq Q$;
2. elements of $\wp_n$ have pairwise disjoints interiors, i.e. $int\, P \cap int\, Q = \emptyset$, whenever $P \neq Q$;
3. diam $P \leq$ diam $\Omega\, \lambda^{-(n+1)}$ for every $P \in \wp_n$m $n \geq 0$.

In fact, for each $P \in \wp_n$ there is a unique sequence $i_0, \cdots, i_{n-1}$ of positive integers such that $P = \tau_{i_{n-1}} \circ \cdots \circ \tau_{i_0}(\Omega)$. The family of subsets set $\wp = \{P \in \wp_n : n \geq 0\}$ is a generating net of subsets for Λ meaning that $\wp$ is a directed family of sets (totally ordered by inclusion) and that for every $x \in \Lambda$ there is a (*not necessarily unique*) nested sequence $\wp_n(x)$ of $\wp$-sets such that $\{x\} = \bigcap_{n \geq 0} \wp_n(x)$, where $\wp_n(x)$ is the atom of $\wp_n$ containing x. The limit set Λ is a perfect compact subset of $\mathbb{R}^m$. If the subsets Ω_i are pairwise disjoint then Λ will be totally disconnected, so it is a Cantor set.

Given a family of expanding branches $T_i : \Omega_i \longrightarrow \Omega$ we may define a *Markov transformation* T by setting $T|int\, \Omega_i = T_i|int\, \Omega_i$ and extended it to a measurable transformation $\hat{T}$ defined on $\bigcup_i \Omega_i$, which is C^2 smooth in a dense subset of its domain. Λ will be the maximal invariant subset of the extension $\hat{T}$. The set $\mathcal{S} = \{x \in \bigcup_i \partial\, \Omega_i :\ \exists i, j > 0\ \ x \in \partial\, \Omega_i \cap \partial\, \Omega_j\ \}$ is the *singular set of* $\hat{T}$. Notice that extensions $\hat{T}$ are indeed quite arbitrary and not unique. However, as long as their singular set is negligible for all the purposes of the ergodic theory we can omit this arbitrariness and we will continue denoting the extensions $\hat{T}$ by T to avoid annoying notations.

Theorem A *Let $T : \bigcup_{i>0} \Omega_i \longrightarrow \Omega$ be the Markov transformation defined by countably many expanding branches $T_i : \Omega_i \longrightarrow \Omega$, $i > 0$ satisfying the bounded distortion condition (1)*

Then, there is a unique $0 < \alpha \leq 1$ which is the largest real number for which there exists an ergodic Borel T-invariant probability $\mu = \mu_\Lambda$ supported on Λ with the Frostman-like property $\mu(P) \asymp Vol(P)^\alpha$ for every atom $P \in \wp$, bounded by constants which only depend on the no linear distortion $d > 0$, the expansion coefficient λ and $Vol(\Omega)$ the volume covered by the expanding branches.

We can compute α in terms of the entropy and Lyapunov exponents of μ. For this we calculate $\lim_{n\to+\infty} \log \mu(\wp_n(x)) / \log \mathrm{Vol}\,(\wp_n(x))$. Indeed, by Shannon-MacMillan-Breiman's theorem $\lim_{n\to+\infty} -1/n \log \mu(\wp_n(x)) = h_\mu(T)$. On the other hand, $\mathrm{Vol}\,(\wp_n(x)) \asymp (Jac\, T^n(x))^{-1}$ bounded by constants which do not depend on x and n. See Section 5 below. Using this estimate we compute

$$\lim_{n\to+\infty} -\frac{1}{n} \log \mathrm{Vol}\,(\wp_n(x)) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log\ Jac\, T(x),$$

and this last term equals $\int \log \, JacT(z)d\mu(z)$ almost surely, by Birkhoff's theorem and the ergodicity of μ. Hence,

$$\alpha = \frac{h_\mu(T)}{\int \log \, JacT(z)d\mu(z)}. \tag{4}$$

Compare [11, pp. 112]. To the best of my knowledge previous computations connecting dimensional characteristic numbers to quantities such as entropy and Lyapunov exponents were first performed by P. Billingsley, so the above constructions are sometimes refered to as particular cases of a general Billingsley's measure. See [1, pp.].

α is in fact a dimension-like number. The uniqueness of this number and its relation to the asymptotic measure $\mu = \mu_\Lambda$ reflects this fact. We can make these relationships clearer in a dimension theory setting. For this we use the generating family $\wp$ to define a covering class and the Lebesgue measure in $\mathbb{R}^m$ as set function within the Carathéodory construction to define a fractal measure with its corresponding dimensional characteristic. Namely, let $a > 0$, $\delta > 0$ be real numbers. The following defines an outer measure on Λ:

$$\mathcal{D}_{a,\delta,\wp}(B) = \inf\{ \sum_{i=1}^{+\infty} \text{Vol}\ (U_i)^a : \mathcal{U} = \{U_i\}_{i>0} \text{ diam}\,(U_i) < \delta \ \} \quad \text{ and}$$

$$\mathcal{D}_{a,\wp}(B) = \lim_{\delta \to 0+} \mathcal{D}_{a,\delta,\wp}(B),$$

where infimun were taken over all δ-coverings $\mathcal{U}$ by $\wp$-sets. The outer measure $\mathcal{D}_{a,\wp}$ restricts to a Borel regular metric measure on Λ. The following is the Carathéodory dimensional characteristic associated to this construction:

$$\dim_{\mathcal{D},\wp}(X) = \inf\{a > 0 : \mathcal{D}_{a,\wp}(X) = 0\} = \sup\{a > 0 : \mathcal{D}_{a,\wp}(X) = +\infty\}.$$

We shall call this the dynamical measure and the dynamical dimension of Λ *with respect to* $\wp$ and denote it by $\mathcal{D}_{a,\wp}$ and $\dim_{\mathcal{D}}$, respectively. As $\wp$ will be fixed throughout we shall omit, when this is not confusing, its dependence on the generating net of subsets $\wp$. This characteristic has the following standard properties:

1. $\dim_{\mathcal{D}}(X) \leq \dim_{\mathcal{D}}(Y)$ if $X \subseteq Y$;

2. $\dim_{\mathcal{D}}(point) = 0$;

3. $\dim_{\mathcal{D}}(\bigcup_{i=0}^{+\infty} X_i) = \sup \dim_{\mathcal{D}}(X_i)$. Cf. [11, Theorem 1.1].

If we use the volume given by any other riemannian metric in $\mathbb{R}^m$ we get an equivalent measure. That give us a well defined equivalence class of measures which we continue denoting $\mathcal{D}_{a,\wp}$ or, shortly, $\mathcal{D}_a$. We are allowed to enlarge a our covering families by considering other generating nets $\wp^*$ which gives us equivalent presentations of the dynamical measure. We refer to next section for details. Theorem A says that $\alpha = \dim_{\mathcal{D}}(\Lambda)$ and $0 < \mathcal{D}_\alpha(\Lambda) < +\infty$. We shall use below Ruelle's theorem (cf.[14, Theorem 8]) to prove the following

Theorem B *Let Λ be a hyperbolic repellor defined by infinitely many C^2 expanding branches $T_i : \Omega_i \longrightarrow \Omega$, $i > 0$ with bounded distortion and $\wp$ the standard generating net. Then there exists a unique ergodic asymptotic Borel probability μ_Λ, absolutely continuous wrt the dynamical measure class such that*

$$\mu_\Lambda(B) = \lim_{n\to+\infty} \frac{\mathcal{D}_\alpha(T^{-n}B)}{\mathcal{D}_\alpha(\Lambda)}.$$

Further, $\dim_{\mathcal{D}}(\Lambda) = h_\mu(T) / \int \log\, JacT(x)d\mu(x)$ *and* $\mu_\Lambda(P) \asymp Vol(P)^\alpha$ *for* $P \in \wp$*, bounded by some constant* $C = C(d, \lambda, Vol(\Omega)) > 1$*. Moreover, its density is uniformly bounded away from zero and infinite, that is:*

$$C^{-1} \;\leq\; \inf_{x\in\Lambda} \frac{d\,\mu_\Lambda}{d\,\mathcal{D}_\alpha}(x) \;\leq\; \sup_{x\in\Lambda} \frac{d\,\mu_\Lambda}{d\,\mathcal{D}_\alpha}(x) \;\leq\; C.$$

Theorem B says, in particular, that μ_Λ is equivalent to $\mathcal{D}_\alpha$. This will be used to prove Theorem A. The following result contains Folklore Theorem as a Corollary.

Theorem C *Let $T : \Lambda \circlearrowleft$ be a hyperbolic repellor produced by infinitely many C^2 expanding branches $T_i : \Omega_i \longrightarrow \Omega$ with bounded distortion and $\mu = \mu_\Lambda$ the asymptotic measure provided by Theorem B. Then the following statements are equivalent:*

1. $h_\mu(T) = \int \log\, JacT(x)d\mu(x)$;
2. *Λ has positive Lebesgue measure;*
3. *μ_Λ is absolutely continuous wrt Lebesgue;*
4. *μ_Λ is equivalent to to Lebesgue measure restricted to Λ and*

$$C^{-1} \leq \inf_{x\in\Lambda} \frac{d\,\mu}{d\,\mathcal{L}}(x) \;\leq\; \sup_{x\in\Lambda} \frac{d\,\mu}{d\,\mathcal{L}}(x) \;\leq\; C.$$

In particular μ_Λ is the unique T-invariant measure with these properties, by ergodicity.

Proof. We prove first that $\mathcal{D}_1$ is equivalent to the Lebesgue measure restricted to Λ. In fact, let $\{U_i\}$ be a δ-covering of $B \subseteq \Lambda$ by $\wp$-sets, then, by Vitali's lemma we can find a covering of Λ by balls $B(x_i, r_i)$ such that $\sum_{j\in J_i} \mathrm{Vol}\,(B(x_j, r_j))$ equals $\mathrm{Vol}\,(U_i)$ for some index subfamily $J_i = J(U_i)$. Thus,

$$\mathcal{L}_\delta(B) = \inf_{\mathcal{B}} \sum_i \mathrm{Vol}\,(B(x_i, r_i)) \leq \sum_i \mathrm{Vol}\,(U_i),$$

infimum taken over δ-coverings of B by open balls. So, $\mathcal{L}_\delta(B) \leq \mathcal{D}_{1,\delta,\wp}(B)$ for every $\delta > 0$ and then $\mathcal{L}(B) \leq \mathcal{D}_{1,\wp}(B)$ for every Borel subset $B \subseteq \Lambda$.

Reciprocally, given $B \subseteq \Lambda$ we cover it with a δ-covering by balls such that $\sum_i \mathrm{Vol}\,(B(x_i, r_i)) < \mathcal{L}(B) + \epsilon$. By the net property we can find a δ-covering $\{U_i\}$ by $\wp$-sets subordinated to the covering by balls and such that $\sum_i \mathrm{Vol}\,(U_i) \leq \sum_i \mathrm{Vol}\,(B(x_i, r_i)) < \mathcal{L}(B) + \epsilon$ and thus $\mathcal{D}_{1,\delta}(B) \leq \mathcal{L}(B) + \epsilon$ for every $\epsilon > 0$ and $\delta > 0$. This prove the claim. As consequence, $\mathcal{L}(B) > 0$ implies $\dim_{\mathcal{D}}(B) = 1$.

Now, μ satisfies the Rochlin entropy formula $h_\mu(T) = \int \log\, Jac\,T(x) d\mu(x)$ if and only if $\dim_{\mathcal{D}}(\Lambda) = 1$, by (4). Also, by Theorem B $\dim_{\mathcal{D}}(\lambda) = 1$ if and only if μ_Λ is equivalent to the Lebesgue measure restricted to Λ. In this case we conclude that $\mathrm{Vol}\,(\Lambda) > 0$. Conversely, if $\mathrm{Vol}\,(\Lambda) > 0$ then $\dim_{\mathcal{D}}(\Lambda) = 1$ proving that μ_Λ is absolutely continuous and equivalent to $\mathcal{L}|\Lambda$, by our previous argument. □

Remark 2.1. If the singular set $\mathcal{S}$ of T has zero n-volume then it has zero μ-measure, so the asymptotic measure $\mu = \mu_\Lambda$ shall not depend on the extension $T = \hat{T}$ that we choose at the beginning. Notice however that, in a more general setting, geometric and measure theoretical properties of the singular set may have some influence on asymptotic measures pr.

For conformal repellors we have the stronger

Theorem D *Let $T_i : \Omega_i \longrightarrow \Omega$, $(i > 0)$ be a countable family of expanding branches with bounded distortion as in Theorem B and suppose in addition that they are conformal. Then, there exists a unique ergodic Borel probability $\mu = \mu_\Lambda$, absolutely continuous wrt the Hausdorff measure such that*

1. $\dim_{\mathcal{H}}(\Lambda) = m \dim_{\mathcal{D}}(\Lambda)$*;*

2. *μ satisfies Frostman's uniform distribution property*

$$C^{-1} \leq \frac{\mu(B(x,r))}{r^{\dim_{\mathcal{H}}(\Lambda)}} \leq C \tag{5}$$

   for every $0 < r < 1$ and $x \in \Lambda$.

In addition, μ_Λ has a density wrt $\mathcal{H}_\alpha|\Lambda$ bounded away from zero and infinite. Therefore, μ_Λ is equivalent to Hausdorff measure and $T : (\Lambda, \mu) \circlearrowleft$ is exact. Furthermore, $\dim_{\mathcal{H}}(\mu) = \dim_{\mathcal{H}}(\Lambda)$ and

$$\dim_{\mathcal{H}}(\mu) = \lim_{r \to 0^+} \frac{\log\, \mu(B(x,r))}{\log\, r} \qquad \mu - a.e. x \in \Lambda,$$

that is, μ is dimensionally exact.

Actually, under the above hypotheses we can find a sequence of ordinary Λ_n of hyperbolic repellors contained in Λ such that:

1. $\dim_{\mathcal{H}}(\Lambda_n) \longrightarrow \dim_{\mathcal{H}}(\Lambda)$

2. $\mathcal{H}_{\alpha(n)}(\Lambda_n) \asymp 1$;

3. $\mathcal{H}_{\alpha(n)}(\Lambda_n \cap B(x,r)) \asymp r^{\alpha(n)}$, uniformly in $x \in \Lambda_n$, $0 < r < 1$.

For this it is sufficient to define Λ_n as being the hyperbolic repellor generated by the first $n > 0$ branches $T_1, \cdots, T_n$. Notice that $\Lambda = \overline{\bigcup_n \Lambda_n}$. Then we use that conformality allows to view these as Moran-like constructions and that Frostman's uniform distribution property (5) hold for each μ_n, bounded by some uniform constant not depending on $n > 0$. Actually, μ_n maximizes dimension, therefore $\dim_{\mathcal{H}}(\Lambda_n) = \dim_{\mathcal{H}}(\mu_n)$ and $\dim_{\mathcal{H}}(\mu_n) \longrightarrow \dim_{\mathcal{H}}(\mu)$.

3 Bounded distortion and related facts

The following is a consequence of the bounded distortion property (1). Compare [9, Lemma 1.4, Chapter III]

Lemma 3.1. *There is a constant $C = C(d, \lambda) > 0$ only depending on distortion d and the expansion coefficient $\lambda > 1$ such that, for every $n \geq 0$*

$$\frac{Jac\,T^n(x)}{Jac\,T^n(y)} \leq \exp(C\|T^n(x) - T^n(y)\|) \quad \textit{whenever} \quad \wp_n(x) = \wp_n(y). \tag{6}$$

and such that

$$\left| \frac{Jac\,T^n(x)}{Jac\,T^n(y)} - 1 \right| \leq C\|T^n(x) - T^n(y)\|. \tag{7}$$

Proof. Using bounded distortion condition (1) we get the following estimate:

$$\sup_{i>0} \frac{\sup_{z\in\Omega_i} \|\nabla \log Jac(T_i)(z)\|}{\inf_{w\in\Omega_i} \|DT_i(w)\|} \leq d < +\infty. \tag{8}$$

Now, we follow an standard argument to prove (6) and (7) (cf.[14, Theorem 22]:

$$\log \frac{Jac\,T^n(x)}{Jac\,T^n(y)} \leq \sum_{i=0}^{n-1} |\log\, Jac\,T(T^i(x)) - \log\, Jac\,T(T^i(y))|$$
$$\leq \sum_{i=0}^{n-1} \sup_{z\in\wp_0(T^i(x))} \|\nabla \log\, Jac\,T(z)\| \cdot \|T^i(x) - T^i(y)\|$$
$$\leq \sum_{i=0}^{n-1} \frac{\sup_{z\in\wp_0(T^i(x))} \|\nabla \log\, Jac\,T(z)\|}{\inf_{w\in\wp_0(T^i(x))} \|DT^{n-i}(w)\|} \|T^n(x) - T^n(y)\|$$
$$\leq d \sum_{i=0}^{n-1} \lambda^{-(n-i-1)} \|T^n(x) - T^n(y)\|$$
$$\leq d \sum_{n=0}^{+\infty} \lambda^{-n} \|T^n(x) - T^n(y)\| = d(1 - \lambda^{-1})\|T^n(x) - T^n(y)\|, \tag{9}$$

using (8). This implies the inequality (6). (7) follows inmediatly. $\square$

We may want to use a somewhat stronger bounded distortion property than (1) which can be verified in all the interesting examples that we have in mind:

$$\sup_{i>0} \frac{\sup_{z\in\overline{U}_i} \|D^2_{jk}\hat{T}_i(z)\|}{\inf_{z,w\in\overline{U}_i} \|D_j\hat{T}_i(z)\| \cdot \|D_j\hat{T}_i(w)\|} = C < +\infty. \tag{10}$$

By hypotheses each expanding branch T_i as an extension $\hat{T}_i : U_i \longrightarrow V_i$ to some neighborhoods of Ω_i and Ω, respectively. Let $\hat{\tau}_i$ the inverse of $\hat{T}_i$. Given $\mathbf{i} = (i_0, \cdots, i_n) \in \mathbb{N}^n$ and a compact set $K \subseteq V_i$ we define $\hat{\tau}_{\mathbf{i}}(K) = \hat{\tau}_{i_n} \circ \cdots \circ \hat{\tau}_{i_0}(K)$. The following is a useful consequence of condition (10) and is proved following the same arguments as the above Lemma.

Lemma 3.2. *$Jac\,\hat{T}^n(x) \asymp Jac\,\hat{T}^n(y)$ uniformly on x , $y \in \hat{\tau}_{\mathbf{i}}(K)$ for every $n \geq 0$, $\mathbf{i} \in \mathbb{N}^n$, $K \subseteq V_{i_0}$, where $\hat{T}^n := \hat{T}_{i_n} \circ \cdots \circ \hat{T}_{i_0}$.*

By Lemma 3.2 $\mathrm{Vol}\,(\hat{\tau}_{\mathbf{i}}(K)) \asymp \mathrm{Vol}\,(\tau_{\mathbf{i}}(\Omega))$ for every $\mathbf{i} \in \mathbb{N}^n$, $n \geq 0$ and $K \subseteq V_{i_0}$, whenever $\mathrm{Vol}\,(K) \asymp \mathrm{Vol}\,(V_{i_0})$. One use this property to enlarge somewhat our covering families producing equivalent measures.

Namely, take compact subsets $K_i \subseteq V_i$ and suppose that $\mathrm{Vol}\,(V_i)\,/\,\mathrm{Vol}\,(K_i)$ is bounded uniformly for $i > 0$. Then, we define a decomposition $\wp^*_n$ by setting $P = \hat{\tau}_{i_n} \circ \cdots \circ \hat{\tau}_{i_0}(K_{i_0})$ and letting $i_0, \cdots, i_n$ run in $\mathbb{N}$, forming a generating net $\wp^*$. Previous discussion shows that the volume of the atoms of $\wp$ and $\wp^*$ will maintain a bounded relation producing equivalent Borel measures, $\mathcal{D}_{a,\wp}$ and $\mathcal{D}_{a,\wp'}$.

We close this Section proving that the dynamical measure is conformal. That will be used below.

Lemma 3.3. *The dynamical measure is conformal, in the following sense: for any $C^{1+\epsilon}$ map $h : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ it holds*

$$\mathcal{D}_\alpha(h(B)) = \int_B (Jac\, h(x))^\alpha \, d\mathcal{D}_\alpha(x)$$

for every Borel subset $B \subseteq \mathbb{R}^m$

Let $x \in P$ a $\wp$-set. Then

$$\frac{Vol(h(P))}{\mathrm{Vol}\,(P)} \approx Jh(x) \quad \text{if } \mathrm{diam}(P) \ll 1.$$

Now we choose a covering $\{U_i\}$ of P by $\wp$-sets such that

$$\sum_i \mathrm{Vol}\,(U_i)^\alpha \approx \mathcal{D}_\alpha(P) \quad \text{and} \quad \sum_i \mathrm{Vol}\,(h(U_i))^\alpha \approx \mathcal{D}_\alpha(h(P))$$

and points $x_i \in U_i$ such that

$$\frac{\mathrm{Vol}\,(U_i)}{\mathrm{Vol}\,(U_i)} \approx Jh(x_i) \quad \text{and} \quad Jh(x_i) \approx Jh(x),$$

choosing P smaller if necessary. Here $\mathcal{D}_\alpha$ is generated by $\{h(Q) : Q \in \wp\}$ and some riemannian volume. Then,

$$\begin{aligned}
\mathcal{D}_\alpha(h(P)) &\approx \sum_i \text{Vol } (h(U_i))^\alpha \\
&\approx \sum_i Jh(x_i)^\alpha \text{Vol } (U_i)^\alpha \\
&\approx Jh(x)^\alpha \sum_i \text{Vol } (U_i)^\alpha \\
&\approx Jh(x)^\alpha \, \mathcal{D}_\alpha(P).
\end{aligned}$$

We shall denote below $J_\alpha(h)$ the Jacobian of a map $h : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ with respect to the measure class $\mathcal{D}_\alpha$.

4 The existence of an asymptotic measure

Lemma 4.1. *Let $\Lambda \subseteq \mathbb{R}^m$ be a C^2 hyperbolic repellor produced by countably many (possibly infinite) expanding branches. Suppose in addition that the dynamical measure class is non trivial, that is $0 < \mathcal{D}_{\alpha,\wp}(\Lambda) < +\infty$. Then, there is a unique ergodic measure $\mu = \mu_\Lambda$ absolutely continuous wrt the measure class such that*

$$C^{-1} \leq \inf_{x\in\Lambda} \frac{d\,\mu_\Lambda}{d\,\mathcal{D}_\alpha}(x) \;\leq\; \sup_{x\in\Lambda} \frac{d\,\mu_\Lambda}{d\,\mathcal{D}_\alpha}(x) \;\leq\; C.$$

Further, μ_Λ is obtained by redistributing the mass of $\mathcal{D}_\alpha$ according to the dynamics, that is,

$$\mu(B) = \lim_{n\to+\infty} \frac{\mathcal{D}_\alpha(T^{-n}B)}{\mathcal{D}_\alpha(\Lambda)},$$

for every Borel subset $B \subseteq \Lambda$.

To prove this we follow Walters's proof of Ruelle's theorem. Compare [14, Theorem 8] and [9, Chapter III]. Let us put the setting.

We introduce $\phi = -\dim_{\mathcal{D}}(\Lambda) \log JacT$. Observe that, by conformality, $J_\alpha T = [JacT]^\alpha$ is the Jacobian of T respect to the dynamical measure class, since $0 < \mathcal{D}_{\alpha,\wp}(\Lambda) < +\infty$, by hypotheses. Let $\mathcal{L}_\phi$ be the Ruelle operator acting on the space Banach continuous functions $E = C^0(\Lambda)$ with supremum norm and $\mathcal{L}^*_\phi$ be its adjoint acting at the dual space of Radon measures. It is easily proved that $\mathcal{L}^*_\phi \, \mathcal{D}_\alpha = \mathcal{D}_\alpha$. We seek for a continuous function h such that $\mathcal{L}_\phi h = h$. h will be the unique positive eigenfunction of $\mathcal{L}_\phi$ up to normalization. $\mu_\Lambda(x) = h(x)\, d\,\mathcal{D}_\alpha(x) \,/\, \mathcal{D}_\alpha(\Lambda)$ is the asymptotic measure we are looking for. Further, h will be uniformly bounded by a constant $C = C(d)$ only depending on the distortion $C^{-1} \leq \inf_{x\in\Lambda} |h(x)| \leq \sup_{x\in\Lambda} |h(x)| \leq C$. To follow this program we need to show inequalities

$$\sum_{y\in T^{-n}x} e^{S_n\phi(y)} \leq K \quad \forall\, x \in \Lambda, \tag{11}$$

for some constant $K > 0$ and that

$$C_\phi(x,y) = \sup_{n>0} \sup_{y \in T^{-n}x} \sum_{i=0}^{n-1} \left[\phi(T^i(x)) - \phi(T^i(y))\right] \tag{12}$$

exists, is bounded by a constant C_ϕ, and $C_\phi(x,y) \longrightarrow 0$ as $d(x,x') \longrightarrow 0$. Cf. [14, Theorem 8]. In fact, if we let $\mu_n = 1\,/\,n \sum_{k=0}^{n-1}(T^k)^*\mathcal{D}_\alpha\,/\,\mathcal{D}_\alpha(\Lambda)$ then μ_n is absolutely continuous wrt $\mathcal{D}_\alpha$ and has a Jacobian wrt $\mathcal{D}_\alpha$, namely $h_n(x) = 1\,/\,n \sum_{k=0}^{n-1} \sum_{y \in \tau^{-n}x} e^{S_n\phi(y)}$, where $S_n\phi(x) = \sum_{i=0}^{n-1}\ \psi(T^i(x))$. Condition (11) and (11) imply that the sequence $\{h_n\}$ is uniformly bounded and equicontinuous so h is a limit point provided by Arzelá-Ascoli theorem. Standard arguments show that h is the unique fixed point of $\mathcal{L}_\psi$ up to normalization. Cf. [9] and [14].

Inequality (11) follows from Lemma 3.1 and an standard argument, using that, for every $x\,,\,y \in \Lambda$, $T^{-n}x = \{x_1, \cdots, x_n\}$ and $T^{-n}y = \{y_1, \cdots, y_n\}$, where $\wp_n(x_i) = \wp(y_i)$ so $C_\phi(x,y) \le C\,\|x-y\|$. Compare [9, Chapter III, Lemma 1.4]. On the other hand, $\int_\Lambda\ h_n(x)d\mathcal{D}_\alpha(x) = \mathcal{D}_\alpha(\Lambda)$ so $|h(x)| \in \mathcal{D}_\alpha(\Lambda)\cdot[e^{-\delta}, e^{\delta}]$, where $\delta = C(\lambda, d)\text{diam}\,(\Omega)$.

T has a Jacobian J_μ with respect to μ_Λ, namely $J_\mu T = J_\alpha T\, h \circ T\,/\,h$, so $J_\mu T^n = J_\alpha T^n\, h \circ T^n\,/\,h$. Hence,

$$\frac{J_\mu T^n(x)}{J_\mu T^n(y)} \le e^{2\delta} \frac{\sup_{x\in\Lambda}|h(x)|}{\inf_{x\in\Lambda}|h(x)|} \le e^{3\delta}$$

whenever $\wp_n(x) = \wp_n(y)$. Thus, $e^{-3\delta} \le\ \mu(\wp_n(x)))J_\mu T^n(x)\ \le\ e^{3\delta}$, using and standard argument. We also have the following standard

Volume Estimate *For every $x \in \Lambda$ and $n \ge 0$ the following inequalities holds:*

$$e^{-\delta}\, Vol(\Omega) \le Vol(\wp_n(x)) \cdot Jac\, T^n(x) \le e^{\delta}\, Vol(\Omega) \tag{13}$$

Notice that $\exp\, S_n\phi(x) = [Jac\,T^n(x)]^{-\alpha}$. So, $\text{Vol}\,(\wp_n(x))^\alpha$ is comparable with $Jac\,T^n(x))^{-\dim_{\mathcal{D}}(\Lambda)}$ bounded by uniform constants. Therefore,

$$\left[e^{-\delta}\text{Vol}\,(\Omega)\right]^\alpha \le \frac{\text{Vol}\,(\wp_n(x))^\alpha}{\exp\, S_n\phi(x)} \le \left[e^{\delta}\text{Vol}\,(\Omega)\right]^\alpha. \tag{14}$$

Now, $J_\mu T^n(x) = \exp(-S_n\phi(x))\, h(T^n(x))\,/\,h(x)$ so

$$\frac{(J_\mu T^n(x))^{-1}}{exp(S_n\phi(x))} \;=\; \frac{h(x)}{h(T^n(x))} \;\in\; [e^{-3\delta}, e^{3\delta}].$$

Hence, combining (13) and (13) we conclude that for every $n \ge 0$ and for every $x \in \Lambda$

$$e^{-5\delta}\text{Vol}\,(\Omega)^2 \le \frac{\mu_\Lambda(\wp_n(x))}{\text{Vol}\,(\wp_n(x))^\alpha} \le e^{5\delta}\text{Vol}\,(\Omega)^2.$$

So the lower and upper bounds for the dynamical measure only depend on the non linear distortion d, the expansion coefficient δ and the volume covered by the expanding branches $T_i : \Omega_i \longrightarrow \Omega$.

5 Proof of main results

In view of Lemma 4.1 in previous Section we can prove Theorem A showing that its dynamical class is non trivial. To prove this we show first that ordinary hyperbolic repellors, defined by finitely many expanding branches have non trivial dynamical measure and then we prove that any hyperbolic repellor defined by infinitely many expanding branches can be seen as a limit of a suitable sequence of ordinary hyperbolic repellors.

Lemma 5.1. *Let $\Lambda \subseteq \mathbb{R}^m$ be a C^2 hyperbolic repellor produced by finitely many expanding branches and $\mathcal{D}_{a,\wp}$ the dynamical measure class. Then, the following holds true:*

1. *$\alpha = \dim_{\mathcal{D}}(\Lambda)$ is the unique solution to $P(T|\Lambda, -\alpha \log JacT) = 0$, where $JacT(x) = |det\, DT(x)|$ is the Jacobian of T wrt the standard riemannian volume of $\mathbb{R}^m$;*
2. *$0 < \mathcal{D}_\alpha(\Lambda) < +\infty$;*
3. *the asymptotic measure μ_Λ provided by Lemma 4.1 is, in fact, the Gibbs measure for the potential $\phi = -\dim_{\mathcal{D}}(\Lambda) \log JacT$.*

Proof. Properties (1) and (2) in Lemma 5.1 are weel known and they were proved elsewhere. Compare [11, Theorem 20.1] and references therein. So we simply outline the main points in the argument. Indeed, $\mathrm{Vol}\,(\wp_n(x)) \asymp (JacT^n(x))^{-1}$ bounded by uniform constants. Now introduce a sequence $\alpha(n)$ is defined by the equation $\sum_{P\in\wp_n} \mathrm{Vol}\,(P)^{\alpha(n)} = 1$. It holds that $\alpha(n)$ converges to $\dim_{\mathcal{D}}(\Lambda$ and that using bounded distortion condition there is a constant $b > 0$ such that $|\alpha(n) - \dim_{\mathcal{D}}(\Lambda)| < b/n$. With this estimate we can prove

$$e^{-d}\mathrm{Vol}\,(\Omega)A^{-b} \leq \sum_{P\in\wp_n} \mathrm{Vol}\,(P)^\alpha \leq e^d \mathrm{Vol}\,(\Omega)A^b,$$

where $A = \sup_{x\in\Omega} JacT(x)$. This implies $P(T, -\alpha \log JacT) = 0$, since sums $\sum_{x\in Per_n(T)} \exp S_n\phi(x)$ are comparable with $\sum_{P\in\wp_n} \mathrm{Vol}\,(P)^\alpha$, bounded by uniform constants, since $\mathrm{Vol}\,(\wp_n(x))^\alpha \asymp [JacT^n(x)]^{-\alpha} = S_n\phi(x)$.

With a little more care we prove that $0 < \mathcal{D}_\alpha(\Lambda) < +\infty$. For this it is enough to prove that there is a constant $\kappa > 0$ such that

$$\mathrm{Vol}\,(P)^\alpha \geq \kappa^{-1} \sum_{P'\subseteq P:P'\in\wp(m)} Vol(P')^\alpha \tag{15}$$

for every $P \in \wp_n$ and $m \geq n \geq 0$. For, if this happens we would have for every $0 < \delta < 1$ a suitable $n = n(\delta)$ such that

$$\inf_{\mathcal{U}} \sum_{i=1}^{+\infty} \mathrm{Vol}\,(U_i)^\alpha \leq \inf_{m\geq n(\delta)} \sum_{P\in\wp(m)} Vol(P)^\alpha \leq \kappa \inf_{\mathcal{U}} \sum_{i=1}^{+\infty} \mathrm{Vol}\,(U_i)^\alpha, \tag{16}$$

where infimum is taken over all the δ-coverings by $\wp$-sets. To prove (15) we let $m \geq n \geq 0$ and $P' \in \wp_m$ contained in $P \in \wp_n$. Then, by the bounded distortion principle,

$$\frac{\mathrm{Vol}\,(P')^\alpha}{\mathrm{Vol}\,(P)^\alpha} \leq e^{d\alpha} \frac{\mathrm{Vol}\,(T^n(P'))^\alpha}{\mathrm{Vol}\,(T^n(P))^\alpha}.$$

Then, we get

$$\sum_{P' \in \wp_m : P' \subseteq P} \frac{\mathrm{Vol}\,(P')^\alpha}{\mathrm{Vol}\,(P)^\alpha} \leq \frac{e^{d\alpha}}{\min_{P \in \wp_0} \mathrm{Vol}\ (P))^\alpha} \sum_{P \in \wp(m-n)} \mathrm{Vol}\,(P)^\alpha$$
$$\leq \left(\frac{e^d}{\min_{P \in \wp_0} \mathrm{Vol}\ (P))} \right)^\alpha e^d\, A^b \max_{P \in \wp_0} \mathrm{Vol}\ (P)^\alpha,$$

and then, for every $P \in \wp_n$ and $m \geq n$

$$\sum_{P' \in \wp(m) : P' \subseteq P} \mathrm{Vol}\,(P')^\alpha \leq \kappa\, \mathrm{Vol}\,(P)^\alpha$$

where

$$\kappa := \frac{e^{2d} A^b \max_{P \in \wp_0} \mathrm{Vol}\ (P)}{(\min_{P \in \wp_0} \mathrm{Vol}\ (P))^\alpha}.$$

The Gibbs measure μ_ϕ associated to the potential $\phi = -\dim_{\mathcal{D}}(\Lambda)\, Jac T$ satisfies

$$C^{-1} \leq \frac{\mu(\wp_n(x))}{\exp(-Pn + S_n\phi(x))} \leq C \tag{17}$$

for every $x \in \Omega$, $n \geq 0$, for some constant $C > 1$ where $P = P(T, \phi)$ is the topological pressure and $S_n\phi(x) = \sum_{k=0}^{n-1} \phi(T^k(x))$. μ_ϕ is an equilibrium state for ϕ in the following sense $P(T, \phi) = h_\mu(T) + \int \phi(x) d\mu(x)$. As $P(T, \phi) = 0$ we conclude that $\mu_\phi(\wp_n(x)) \asymp S_n\phi(x)) \asymp \mathrm{Vol}\,(\wp_n(x))^{\dim_{\mathcal{D}}(\Lambda)} \asymp \mu_\Lambda(\wp_n(x))$, so $\mu_\phi = \mu_\Lambda$, because they are ergodic and T-invariant and since $\wp$ generates the Borel subsets of Λ. $\square$

Lemma 5.2. *Let $T : \bigcup_{i>0} \Omega_i \longrightarrow \Omega$ be a Markov transformation defined by infinitely many expanding branches T_i with bounded distortion and Λ be its limit set. Then, there is an increasing sequence of hyperbolic repellors $\Lambda_n \subseteq \Lambda$ and ergodic measures μ_n supported on Λ_n such that*

1. $\Lambda = \overline{\bigcup_{n>0} \Lambda_n}$

2. $\mu_n(P) \asymp Vol(P)^{\dim_{\mathcal{D}}(\Lambda_n)}$ *bounded by constants which only depends on the non linear distortion $d > 0$, the expanding coefficient and the volume of Ω for every $n > 0$ and $P \in \wp(\Lambda_n)$. $\wp(\Lambda_n)$ is a subfamily of $\wp$ which generates Λ_n and $\wp(\Lambda_n) \subseteq \wp(\Lambda_{n+1})$.*

Proof. Given $0<\delta<1$ we can find $N=N(\delta)>0$, the first integer such that

$$\sum_{i=1}^{N}\mathrm{Vol}\,(\Omega_i)\geq(1-\delta)\mathrm{Vol}\,(\bigcup_{i=1}^{+\infty}\Omega_i)$$

Choose $K=K(\delta)\subseteq\Omega$ a proper compact subdomain such that $(i):\bigcup_{i=1}^{N}\Omega_i\subseteq K$ and (ii) :Vol $(K)\geq(1-\delta/2)\mathrm{Vol}\,(\Omega)$. Take compact pairwise disjoint subdomains $K_i\subseteq\Omega_i$ such that $T_i:K_i\longrightarrow K$ is one to one and onto. $T_i|K_i:K_i\longrightarrow K$ is an expanding branch with bounded distortion not depending on δ neither K or $i>0$. Let us define $T_\delta|K_i=T_i$. $T_\delta:\bigcup_{P\in\mathcal{P}_0}P\longrightarrow K$ defines a topologically mixing hyperbolic repellor Λ_δ which is topologically conjugated to a full-shift in $N=N(\delta)$ symbols. $\wp_0(\delta)=\{K_i(\delta)\}_{i=1,\cdots,N(\delta)}$ is the initial Markov partition for Λ_δ which yields, in turn, a generating Markov net $\wp(\delta)$ pulling back $\wp_0(\delta)$.

Now fix $0<\delta_0<1$ and a sequence $0<\delta_n<\delta_0$ converging to zero. For each such δ_n we choose a compact domain $K_n=K(\delta_n)$ and $N_n=N(\delta_n)$ satisfying (i) and (ii) above and such that Vol $(K_n)\geq(1-\delta_0/2)\mathrm{Vol}\,(\Omega)$ for every n. μ_n is the equilibrium state for the potential $\phi=-\dim_{\mathcal{D}}(\Lambda_n)\log JacT$. Clearly, $\Lambda=\overline{\bigcup_n\Lambda_n}$. This proves (1). Also (2) holds true in view of Lemma 4.1 and Lemma 5.1, since the volume covered by the expanding branches forming Λ_n is uniformly bounded from above a below: $(1-\delta_0)\mathrm{Vol}\,(\Omega)\leq\mathrm{Vol}\,(K(\delta_n)))\leq\mathrm{Vol}\,(\Omega)$ for every $P\in\wp(\Lambda_n)$ and $n>0$.

Now we explain how to choose the generating subfamilies $\wp(\Lambda_n)$.

We associate to $\wp(\delta_n)$ a subfamily of $\wp$ which is also a generating net for Λ_n using the symbolic dynamics: for each $Q\in\wp(\delta_n)$ there exists unique sequence $i_0,\cdots,i_m$ such that $Q=\tau_{i_m}\circ\cdots\circ\tau_{i_0}(K)$. We associate to Q the $\wp$ cylinder $P(i_0,\cdots,i_m)=\tau_{i_m}\circ\cdots\circ\tau_{i_0}(\Omega)$. $Q(i_0,\cdots,i_m)\subseteq P(i_0,\cdots,i_m)$ and Vol $(Q(i_0,\cdots,i_m))\asymp\mathrm{Vol}\,(P(i_0,\cdots,i_m))$, uniformly so $\mathcal{D}_{\alpha(n),\wp(\Lambda_n)}\asymp\mathcal{D}_{\alpha,\wp(\delta_n)}$, bounded by constants which do not depend on n, in view of remarks of previous Section. □

Now we are ready to conclude Theorem A and Theorem B proofs. Notice first that $\wp(\Lambda_n)\subseteq\wp(\Lambda_{n+1})$ implies that, given $N>0$

$$C^{-1}\leq\frac{\mu_n(P)}{\mathrm{Vol}\,(P)^{\dim_{\mathcal{D}}(\Lambda_n)}}\leq C\quad\forall\ P\in\wp(\Lambda_N)\quad n\geq N.$$

Let Λ be the maximal invariant subset of $T:\bigcup_i\Omega_i\longrightarrow\Omega$. Then $\Lambda=\overline{\bigcup_n\Lambda_n}$. Observe that $\alpha=\lim_{n\to+\infty}\dim_{\mathcal{D}}(\Lambda_n)$ exists since $\dim_{\mathcal{D}}(\Lambda_n)$ is monotone and bounded. Let μ^* be a limit point of μ_n. μ^* is ergodic since it is a limit of ergodic measures and for every $P\in\wp$ we have

$$\mu^*(P)\geq\limsup_{n\to+\infty}\mu_n(P)\geq C^{-1}\limsup_{n\to+\infty}\mathrm{Vol}\,(P)^{\dim_{\mathcal{D}}(\Lambda_n)}$$

since $P\in\wp$ is closed. Thus $\mu^*(P)\geq C^{-1}\mathrm{Vol}\,(P)^\alpha$ for every $P\in\wp$, concluding that $\mathcal{D}_{\alpha,\wp}(\Lambda)<+\infty$, using Frostman's lemma argument. Furthermore, $\mu^*(B)\geq$

$C^{-1} \cdot \mathcal{D}_\alpha(B)$ for every Borel subset $B \subseteq \Lambda$ so it is absolutely continuous respect to the dynamical measure class. Similarly so,

$$\mu^*(int\, P) \leq \liminf_{n \to +\infty} \mu_n(int\, P) \leq C \liminf_{n \to +\infty} \mathrm{Vol}\,(P)^{\dim_{\mathcal{D}}(\Lambda_n)},$$

since $\mathrm{Vol}\,(\partial P) = 0$ for every $P \in \wp$. Then, $\mu^*(int\, P) \leq C\, \mathrm{Vol}\,(P)^\alpha$ and this imply $\mathcal{D}_\alpha(\Lambda) > 0$.

Therefore $\alpha = \dim_{\mathcal{D}}(\Lambda)$ and $0 < \mathcal{D}_{\alpha,\wp}(\Lambda) < +\infty$. By Lemma 4.1 there is a unique asymptotic measure μ_Λ and we conclude that $\mu^* = \mu_\Lambda$ since both are ergodic and absolutely continuous wrt $\mathcal{D}_{\alpha,\wp}$, completing the proof of Theorem B. On the other hand, for every $P \in \wp_n$ we have

$$\frac{\mathcal{D}_\alpha(\Lambda)}{\mathrm{Vol}\,(\Omega)^\alpha} = \frac{\mathcal{D}_\alpha(T^n(P \cap \Lambda))}{\mathrm{Vol}\,(T^n(P))^\alpha} \leq e^{d\alpha} \frac{\mathcal{D}_\alpha(P \cap \Lambda)}{\mathrm{Vol}\,(P)^\alpha}.$$

Similarly so $\mathcal{D}_\alpha(P \cap \Lambda) \,/\, \mathrm{Vol}\,(P)^\alpha \leq e^{d\alpha} \mathcal{D}_\alpha(\Lambda) \,/\, \mathrm{Vol}\,(\Omega)^\alpha$. Thus, for every $P \in \wp$ it holds

$$\mu_\Lambda(P) \asymp \mathcal{D}_\alpha(P \cap \Lambda) \asymp \mathrm{Vol}\,(P)^\alpha$$

bounded by some constant $C = C(d, \lambda, \mathrm{Vol}\,(\Omega))$, proving Theorem B.

6 Conformal hyperbolic repellors

Now, we come up to the conformal case. So, let us suppose that the expanding branches $T_i : \Omega_i \longrightarrow \Omega$ are conformal. This implies $Jac\, T_i(x) \asymp \|DT_i(x)\|^m$ for a constant $C > 1$, only depending on the euclidean metric. Thus, a bounded distortion property for the Jacobian implies a similar property for $\|DT\|$: $\mathrm{Vol}\,(P) \asymp \mathrm{diam}\,(P)^m$. Indeed, bounded distortion for $\|DT_i\|$ implies that $\mathrm{diam}\,(\wp_n(x)) \asymp \|DT^n(x)\|$ and $\mathrm{Vol}\,(\wp_n(x)) \asymp Jac\, T^n(x)$ bounded uniformly by constants not depending on $x \in \Lambda$ and $n \geq 0$, showing $\mathrm{Vol}\,(P) \asymp \mathrm{diam}\,(P)^m$ as claimed. Now we introduce Moran covers which relates dynamical measure to the Hausdorff measure. Namely, given $r > 0$ and $x \in \Lambda$ define $n = n(x, r) > 0$ as the minimum positive integer satisfying $(Jac\, T^n(x))^{-1} \leq r$ and $(Jac\, T^{n-1} f(x))^{-1} > r$. So, for each $0 < r < 1$ we extract a disjoint maximal sub-family $\wp_r \subseteq \{\wp_{n(x,r)}(x) : x \in \Lambda\}$ such that $\bigcup_{P \in \wp_r} P = \bigcup_{x \in \Lambda} \wp_{n(x,r)}(x)$. $\wp_r$ satisfy the following property:

(*) *there exists a universal constant $M > 1$ such that $B(x, r)$ intersects at most M atoms in the family $\wp_r$, for every $x \in \Lambda$ and $0 < r < 1$*

This number is the Moran multiplicity of the family $\mathcal{P} = \{P \in \wp_n : n \geq 0\}$. Moran's property (*) in view of contentions $B(x, C^{-1} r) \subseteq \wp_r(x) \subseteq B(x, C\, r)$, which hold for every $0 < r < 1$ and $x \in \Lambda$ and some universal constant $C > 1$ only depending on the distortion and since $\wp_r$-atoms are disjoint. $\wp_r$ are called *Moran's covers associated to the generating net $\wp$*. Compare [11, pp. 124] and [11, pp. 199].

Now, Theorem B implies that there exists an ergodic T-invariant probability $\mu = \mu_\Lambda$ such that $\mu(\wp_r(x)) \,/\, \mathrm{Vol}\,(\wp_r(x))^{\dim_{\mathcal{D}}(\Lambda)}$ is bounded away from zero and infinite for every $x \in \Lambda$ and $0 < r < 1$ and we conclude from the above remarks that $\mu(B(x,r)) \asymp r^{m \dim_{\mathcal{D}}(\Lambda)}$ for every $x \in \Lambda$, $0 < r < 1$ bounded by a suitable universal constant $C > 1$. This shows $\dim_{\mathcal{H}}(\Lambda) = m \dim_{\mathcal{D}}(\Lambda)$, so that μ_Λ, the dynamical measure and the Hausdorff measure are equivalent.

By the same arguments we show that the sequence Λ_n provided by Main Lemma and their Gibbs measures μ_n satisfies $\mu_n(B(x,r)) \asymp r^{\dim_{\mathcal{H}}(\Lambda_n)}$ for every $n > 0$, $0 < r < 1$ and $x \in \Lambda_n$. Therefore, $\dim_{\mathcal{H}}(\Lambda_n) \longrightarrow \dim_{\mathcal{H}}(\Lambda)$ and

$$\mathcal{H}_{\alpha(n)}(\Lambda_n) \quad \text{and} \quad \frac{\mathcal{H}_{\alpha(n)}(\Lambda_n \cap B(x,r))}{r^{\dim_{\mathcal{H}}(\Lambda_n)}} \quad \in [C^{-1}, C]$$

uniformly in $x \in \Lambda_n$, $0 < r < 1$ and $n > 0$, for a suitable constant $C > 1$.

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