An ergodic theoretic approach to mean field coupled maps

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December 2, 1998

Abstract

We study infinite systems of globally coupled maps with permutation invariant interaction as limits of large finite-dimensional systems. Because of the symmetry of the interaction the interesting invariant measures are the exchangeable ones. For infinite systems this means in view of de Finetti's theorem that we must look for time invariant measures within the class of mixtures of spatial i.i.d. processes. If we consider only those invariant measures in that class as physically relevant which are weak limits of SRB-measures of the finite-dimensional approximations, we find for systems of piecewise expanding interval maps that the limit measures are in fact mixtures of absolutely continuous measures on the interval which have densities of uniformly bounded variation.

The law of large numbers is violated (in the sense of Kaneko) if a nontrivial mixture of i.i.d. processes can occur as a weak limit of finite-dimensional SRB-measures. We prove that this does neither happen for C^3-expanding maps of the circle (extending slightly a result of Järvenpää) nor for mixing tent maps for which the critical orbit finally hits a fixed point (making rigorous a result of Chawanya and Morita).

1 Introduction

The purpose of these notes is to provide a precise mathematical framework within which it is possible to describe the dynamics of finite and infinite systems of *mean field coupled maps* as well as the limiting transition from large finite to infinite systems. Mean field coupled maps (also called *globally coupled maps* in the literature) have the potential to show a surprising dynamical behaviour which is sometimes referred to as *violation of the law of large numbers*, a term which conveys only insufficiently the rich structure of these dynamics. The following observation is reported in the literature: If the action of an infinite (for practical purposes: extremely large finite) system of very simple chaotic dynamical units, e.g. of identical tent maps, is perturbed by its own mean field, then, in many cases, the mean field itself shows a highly complex and structured dynamical behaviour, although it is just the average state of a family of chaotic units which, given the value of the mean field, act independently of each other. In particular, there exists a "component" of the dynamics which cannot be located, not even partially, in any finite subcollection of the family of chaotic units. This effect is investigated, both numerically [9, 10, 5, 4, 18, 19] and theoretically [8, 5, 4], but on a mathematical level much remains to be done in order to achieve a deeper understanding.

I would like to advocate the following point of view: Since an infinite system of mean field coupled maps is completely invariant under arbitrary permutations of its components, its statistical properties should be described by *exchangeable distributions* (in the sense of probability theory). From the general theory of these distributions [1] it follows at once that the mean field fluctuations of the system reflect that part of the dynamics which can be described in terms of the tail field of the infinite collection of units. So the term *violation of the* 0-1-*law* describes the situation much better than *violation of the law of large numbers.* The second aspect I want to underline is the importance to understand how this violation of the 0-1-law shines up in the behaviour of

*Research partially supported by the Deutsche Forschungsgemeinschaft (DFG).

large finite systems. In fact, it is known that for moderate coupling strengths the dynamics of an arbitrary finite number of mixing coupled tent maps is governed by a unique mixing absolutely continuous invariant measure [13], and only a good knowledge of the infinite system can reveal something about the important dynamical fine structure of this measure, i.e. structure below the very coarse concepts of ergodicity, mixing, etc. Needless to say that such ideas are common knowledge in the stochastic theory of many particle systems.

At the end of this introduction I want to stress that some of the ideas in these notes appear, although often only implicitly and without mathematical rigour, in the existing literature, most noteworthy in [5] and [4].

2 Systems with continuous units

In this section (and, in fact, in the whole paper) we assume that the individual maps are continuously acting on a compact metric space and that also the coupling mechanism is continuous. This allows us to study exchangeable distributions as the natural class of measures that is left invariant by the action of the system and to describe the transition from finite to infinite systems in two complementary ways. We finish this section with a precise definition of what we mean by *violation of the* 0-1-*law*.

2.1 The uncoupled system

X is a compact metric space with metric d and $\tau : X \to X$ is continuous. The pair (X, τ) is our basic dynamical unit. For a finite or countable index set I let $\Omega := X^I$ and denote by $T : \Omega \to \Omega$ the map defined by

$$(T\omega)_i = \tau\omega_i \quad (i \in I) \ .$$

T describes the uncoupled dynamics of a family of copies of (X, τ) indexed over I. If necessary we write Ω_I and T_I instead of Ω and T. In order to avoid double subscripts i_j, we assume from now on that $I = I_n = \{1, \ldots, n\}$ if I is finite and that $I = \mathbb{N}$ if I is infinite.

Before we describe a general form of a mean field interaction which allows us to define coupled maps $T_\epsilon : \Omega \to \Omega$, where ϵ is a parameter that reflects the coupling strength, we discuss two particular examples which motivated this paper.

2.2 Two examples of mean field couplings

Let $I = \{1, \ldots, n\}$.

Example 1 [**Tent maps**]
$X = [0, 1]$, τ is piecewise monotone, e.g. τ might be a tent map, and

$$(T_\epsilon\omega)_i = (1-\epsilon)\tau\omega_i + \frac{\epsilon}{n}\sum_{j=1}^{n} \tau\omega_j \quad \text{for all } i \in I_n. \tag{1}$$

Note that the coupling term $\epsilon\,(-\tau\omega_i + \frac{1}{n}\sum_j \tau\omega_j)$ at position i can be written as a continuous function of $\tau\omega_i$ and of the empirical distribution $p_\omega := \frac{1}{n}\sum_{j=1}^{n} \delta_{\omega_j}$, namely as $\epsilon\,(-\tau\omega_i + \int_X \tau(y)\,dp_\omega(y))$. Since this term can be rewritten as $\epsilon\,(-\tau\omega_i + \int_X y\,dp_{T\omega}(y))$ in this example (and in the next one as well), T_ϵ is the composition of two maps, namely

$$T_\epsilon = \Phi_\epsilon \circ T \quad \text{where} \quad (\Phi_\epsilon\omega)_i = \omega_i + \epsilon\,(-\omega_i + \int_X y\,dp_\omega(y)) \ .$$

This is the kind of systems studied in [8, 5, 4, 18, 19] for which the phenomenon of violation of the law of large numbers is reported. Observe that $\Phi_\epsilon : \Omega \to \Omega$ is not a diffeomorphism (it is not onto). This, so far, prevents all attempts to study these examples using techniques from statistical physics.

Example 2 [Expanding circle maps]
$X = \mathbb{T}^1$ (where $\mathbb{T} = [0, 2\pi)$ with addition $\mod 2\pi$), $\tau : X \to X$ of class C^2 (not only piecewise C^2), and

$$(T_\epsilon\omega)_i = \tau\omega_i + \frac{\epsilon}{n}\sum_{j=1}^{n} \sin(\tau\omega_i - \tau\omega_j) \quad \text{for all } i \in I_n. \tag{2}$$

As $\sum_j \sin(\tau\omega_i - \tau\omega_j) = \sin\tau\omega_i \sum_j \cos\tau\omega_j - \cos\tau\omega_i \sum_j \sin\tau\omega_j$, the coupling term at position i is again a continuous function of $\tau\omega_i$ and of the empirical distribution $p_{T\omega}$. In this case the coupling transformation $\Phi_\epsilon : \Omega \to \Omega$, $(\Phi_\epsilon\omega)_i = \omega_i + \frac{\epsilon}{n}\sum_j \sin(\omega_i - \omega_j)$ is an analytic diffeomorphism of the torus $\mathbb{T}^n$. This fact plays a crucial role in [7] where, among many other results, it is proved that no violation of the law of large numbers occurs provided that τ extends analytically to a complex annulus around $\mathbb{T}^1$. That proof relies heavily on techniques from statistical physics. [1] For related work see also [8].

2.3 A general model for mean field couplings

Next we describe a general form of a mean field interaction motivated by the preceding examples. The only requirement is that for finite I the coupling term at position i is a continuous function of ω_i and of the empirical distribution $p(\omega) := \frac{1}{n}\sum_{j=1}^{n} \delta_{\omega_j}$.

More precisely, denote by $\mathcal{P}$ the set of all Borel probabilities on X. $\mathcal{P}$ is a compact metrizable space with respect to the topology of weak convergence. Then each continuous map $S_\epsilon : X \times \mathcal{P} \to X$ defines a coupled map

$$T_\epsilon : \Omega \to \Omega, \quad (T_\epsilon\omega)_i := S_\epsilon(\tau\omega_i, p(\omega)) \ .$$

Sometimes it is convenient to consider the map $x \mapsto S_\epsilon(x, \nu)$ for some fixed $\nu \in \mathcal{P}$. Then we write $S_{\epsilon,\nu}(x) := S_\epsilon(x, \nu)$. In Example 1 we used

$$S_\epsilon(x, \nu) = x + \epsilon \cdot \left(-x + \int_X \tau(y)\, d\nu(y)\right) \ ,$$

in Example 2 it was

$$S_\epsilon(x, \nu) = x + \epsilon \cdot \left(\sin(x) \int_X \cos(\tau y)\, d\nu(y) - \cos(x) \int_X \sin(\tau y)\, d\nu(y)\right)$$

Remark 1 Note that in typical examples the family of maps $(S_\epsilon)_\epsilon$ will be defined only for a certain parameter interval including $\epsilon = 0$, that in these cases $S_{0,\nu} = \mathrm{Id}_X$ for all $\nu \in \mathcal{P}$ and that the maps S_ϵ typically depend continuously on the parameter ϵ. Most of our further analysis does not depend on these properties, however.

The above chosen set-up extends naturally to the case of infinite I. If $I = \mathbb{N}$, $p : \Omega_\mathbb{N} \to \mathcal{P}$ can be partially defined by

$$p(\omega) := p_\omega := \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \delta_{\omega_i} \quad \text{if this limit exists in the weak topology.}$$

Denote the set of ω where p is defined by $\Omega_\mathbb{N}^{\mathrm{reg}}$. Then $\Omega_\mathbb{N}^{\mathrm{reg}}$ has full measure for each shift invariant Borel probability on $\Omega_\mathbb{N}$ and hence *a fortiori* also for each exchangeable Borel probability on $\Omega_\mathbb{N}$. Exchangeability will be further discussed in Section 2.5.

In order to describe the essential symmetry property of T_ϵ denote by $\Sigma(I)$ the set of all permutations σ of I such that $\sigma(i) = i$ for all but finitely many $i \in I$. For $\sigma \in \Sigma(I)$ define $\sigma^* : \Omega_I \to \Omega_I$ by $(\sigma^*\omega)_i := \omega_{\sigma(i)}$ $(i \in I)$. It is easily checked that

$$T_\epsilon \circ \sigma^* = \sigma^* \circ T_\epsilon \ .$$

[1] The author studies more general analytic couplings, indeed.

Remark 2 If I is finite, then $T_\epsilon : \Omega_I \to \Omega_I$ is continuous with respect to the usual Euclidean metric on Ω_I. If $I = \mathbb{N}$, we equip Ω_I with the metric $d_I(\omega, \omega') := \sum_{i=1}^{\infty} 2^{-i} d(\omega_i, \omega'_i)$. With this metric T_ϵ is not continuous, but it is Borel measurable.

2.4 The action of T_ϵ on probability measures on X

The action of T and of T_ϵ can be transferred completely to $\mathcal{P}$. Observe first that for finite and infinite I holds

▷ $p(T\omega) = \tau p_\omega$ for all $\omega \in \Omega$,

and if $I = I_n$ is finite also the following is true:

▷ $p : \Omega_I \to \mathcal{P}$ is continuous; so all $\mathcal{P}_n := p(\Omega_{I_n}) \subseteq \mathcal{P}$ are compact.

▷ $p \circ \sigma^* = p$ for all $\sigma \in \Sigma(I)$.

▷ If μ is a Borel probability on Ω_I, then $p\mu$ is a Borel probability on $\mathcal{P}$ supported by $\mathcal{P}_n$.

▷ $$p(T_\epsilon \omega) = \frac{1}{n} \sum_{i=1}^{n} \delta_{S_{\epsilon, p(\omega)}(\tau \omega_i)} = \frac{1}{n} \sum_{i=1}^{n} (S_{\epsilon, p_\omega} \circ \tau) \delta_{\omega_i} = (S_{\epsilon, p_\omega} \circ \tau) p_\omega$$

This motivates to define $\mathcal{T}_\epsilon : \mathcal{P} \to \mathcal{P}$ by

$$\mathcal{T}_\epsilon(\nu) := (S_{\epsilon,\nu} \circ \tau)\nu \ .$$

As τ and S_ϵ are continuous, also $\mathcal{T}_\epsilon$ is continuous.

For the subsequent discussion it is important to be aware of the following notational subtlety: If we write $\mathcal{T}_\epsilon(\nu)$, then $\nu \in \mathcal{P}$, i.e. ν is a probability measure on X and so is $\mathcal{T}_\epsilon(\nu)$. If we write $\mathcal{T}_\epsilon P$, i.e. if the argument is not put into brackets, then P is a probability measure on $\mathcal{P}$, and $\mathcal{T}_\epsilon P$ is the measure on $\mathcal{P}$ defined by $(\mathcal{T}_\epsilon P)(\mathcal{A}) = P(\mathcal{T}_\epsilon^{-1} \mathcal{A})$ for measurable $\mathcal{A} \subseteq \mathcal{P}$. Analogous conventions are used for other maps. Observe now that for finite I holds:

▷ $p(T_\epsilon \omega) = \mathcal{T}_\epsilon(p_\omega)$ for all $\omega \in \Omega$. As $\bigcup_n \mathcal{P}_n$ is dense in $\mathcal{P}$, this means that $\mathcal{T}_\epsilon$ is the unique continuous extension of the action induced by T_ϵ on $\bigcup_n \mathcal{P}_n$ to all of $\mathcal{P}$. In a slightly different notation this is the same as $p \circ T_\epsilon = \mathcal{T}_\epsilon \circ p$.

▷ If $T_\epsilon \mu = \mu$ for a Borel probability μ on Ω_I, then $\mathcal{T}_\epsilon\, p\mu = (\mathcal{T}_\epsilon \circ p)\mu = (p \circ T_\epsilon)\mu = p\mu$.

Finally we note that for each $\nu \in \mathcal{P}$ and $T_\epsilon : \Omega \to \Omega$ holds:

$$T_\epsilon \nu^{\mathbb{N}} = ((S_{\epsilon,\nu} \circ \tau)\nu)^{\mathbb{N}} = (\mathcal{T}_\epsilon(\nu))^{\mathbb{N}} \ . \tag{3}$$

2.5 The action of T_ϵ on exchangeable distributions on Ω

Denote by $\mathcal{M} = \mathcal{M}_I$ the set of all Borel probabilities on $\Omega = \Omega_I$ and let

$$\mathcal{M}^{ex} = \mathcal{M}_I^{ex} := \{\mu \in \mathcal{M} : \sigma^* \mu = \mu \ \forall \sigma \in \Sigma(I)\}$$

be the set of all *exchangeable* distributions on Ω. For $I = \mathbb{N}$, a classical theorem of de Finetti asserts, that each exchangeable distribution is a mixture of product distributions in the following sense:

Theorem 1 (de Finetti, first version; see [1]) *To each $\mu \in \mathcal{M}_{\mathbb{N}}^{ex}$ there exists a unique Borel probability P on $\mathcal{P}$ such that*

$$\mu(A) = \int_{\mathcal{P}} \nu^{\mathbb{N}}(A)\, dP(\nu) \quad \textit{for all measurable } A \subseteq \Omega. \tag{4}$$

In short-hand: $\mu = \int \nu^{\mathbb{N}}\, dP(\nu)$. Here $\nu^{\mathbb{N}}$ denotes the infinite product measure on Ω with all factors equal to ν. (Vice versa, each $\mu \in \mathcal{M}$ that can be represented as in (4) belongs to $\mathcal{M}_{\mathbb{N}}^{ex}$.)

In particular there is a one-to-one correspondence given by (4) between measures $\mu \in \mathcal{M}_{\mathbb{N}}^{ex}$ and Borel probabilities P on $\mathcal{P}$.

There is another way to represent $\mu \in \mathcal{M}_{\mathbb{N}}^{ex}$, which conveys more of the probabilistic structure of the coordinate process on Ω under the distribution μ. Denote by $\mathcal{F}$ the Borel σ-algebra and by $\mathcal{F}^{tail} \subseteq \mathcal{F}$ the tail-field of Ω.

Theorem 2 (de Finetti, second version; see [1]) *To each $\mu \in \mathcal{M}_{\mathbb{N}}^{ex}$ there exists a unique $\mathcal{F}^{tail}$-measurable stochastic kernel $Q : \Omega \times \mathcal{F} \to [0,1]$ such that*

1. *$\mu(A) = \int_\Omega Q(\omega, A)\, d\mu(\omega)$ for each $A \in \mathcal{F}$ and*
2. *for each $\omega \in \Omega$ there is some $\nu_\omega \in \mathcal{P}$ such that $Q(\omega, .) = \nu_\omega^{\mathbb{N}}$.*

Remark 3 Given $\mu = \int \nu_\omega^{\mathbb{N}}\, d\mu(\omega)$, define $\psi : \Omega \to \mathcal{P}$ by $\psi(\omega) := \nu_\omega$. Then the two versions of the de Finetti theorem are related by the identity $P = \psi\mu$.

Remark 4 If $\mu \in \mathcal{M}_{\mathbb{N}}^{ex}$, then each finite-dimensional marginal distribution of μ on Ω_{I_n} is exchangeable, but not each $\mu \in \mathcal{M}_{I_n}^{ex}$ is a marginal distribution of some measure from $\mathcal{M}_{\mathbb{N}}^{ex}$, see [1].

Fix any $\mu = \int \nu^{\mathbb{N}}\, dP(\nu) \in \mathcal{M}_{\mathbb{N}}^{ex}$. Then, because of (3),

$$T_\epsilon \mu = \int T_\epsilon \nu^{\mathbb{N}}\, dP(\nu) = \int (\mathcal{T}_\epsilon(\nu))^{\mathbb{N}}\, dP(\nu) = \int \nu^{\mathbb{N}}\, d(\mathcal{T}_\epsilon P)(\nu) \ .$$

This shows that $T_\epsilon \mu$ is represented by the distribution $\mathcal{T}_\epsilon P$ on $\mathcal{P}$. In particular,

$$T_\epsilon \mu = \mu \quad \text{if and only if} \quad \mathcal{T}_\epsilon P = P \ .$$

2.6 Transition to the limit as $n \to \infty$

Consider Borel probabilities μ_n on Ω_{I_n} such that $T_\epsilon \mu_n = \mu_n$ for all n, and fix a subsequence μ_{n_i} such that the sequence $p\mu_{n_i}$ converges weakly to some Borel probability P on $\mathcal{P}$. Then

$$\mathcal{T}_\epsilon P = \lim_{i\to\infty} (\mathcal{T}_\epsilon \circ p)\mu_{n_i} = \lim_{i\to\infty} (p \circ T_\epsilon)\mu_{n_i} = \lim_{i\to\infty} p\mu_{n_i} = P \ . \tag{5}$$

Suppose now that the μ_n are exchangeable and that the subsequence μ_{n_i} converges weakly to some $\mu \in \mathcal{M}_{\mathbb{N}}$ in the sense that $\lim_{i\to\infty} \mu_{n_i}(\varphi) = \mu(\varphi)$ for each continuous $\varphi : \Omega_{\mathbb{N}} \to \mathbb{R}$ which depends only on a finite number of coordinates (so that $\mu_n(\varphi)$ is well defined for large enough n). Then $\mu \in \mathcal{M}_{\mathbb{N}}^{ex}$ as is easily seen, and our goal is to show that $\mu = \int \nu^{\mathbb{N}}\, dP(\nu)$ for the limiting distribution $P = \lim_{i\to\infty} p\mu_{n_i}$.

Denote by $C(\mathcal{P})$ the algebra of all continuous functions $F : \mathcal{P} \to \mathbb{R}$. Particularly simple elements from $C(\mathcal{P})$ are the functions $F_\varphi : \nu \mapsto \nu(\varphi)$ with $\varphi \in C(X)$. Denote $A_0(\mathcal{P}) := \{F_{\varphi_1} \cdots F_{\varphi_n} : n \in \mathbb{N}, \varphi_i \in C(X)\}$ and let $A(\mathcal{P})$ be the linear space generated by $A_0(\mathcal{P})$. Then $A(\mathcal{P})$ is a subalgebra of $C(\mathcal{P})$ which contains the constant functions and separates points in $\mathcal{P}$ so that, by the theorem of Stone and Weierstrass, $A(\mathcal{P})$ is dense in $C(\mathcal{P})$. Hence we have

Lemma 1 *A sequence $(P_k)_{k\in\mathbb{N}}$ of Borel probabilities on $\mathcal{P}$ converges weakly to P, if and only if*

$$\lim_{k\to\infty} \int \nu(\varphi_1) \cdots \nu(\varphi_n)\, dP_k(\nu) = \int \nu(\varphi_1) \cdots \nu(\varphi_n)\, dP(\nu)$$

for all $n \in \mathbb{N}$ and all $\varphi_1, \ldots, \varphi_n \in C(X)$.

Similarly, let $A_0(\Omega_{\mathbb{N}})$ be the family of all functions $\omega \mapsto \varphi_1(\omega_1) \cdots \varphi_n(\omega_n)$ from $\Omega_{\mathbb{N}}$ to $\mathbb{R}$ with $n \in \mathbb{N}$ and $\varphi_i \in C(X)$. Denote by $A(\Omega_{\mathbb{N}})$ the linear space generated by $A_0(\Omega_{\mathbb{N}})$. Then $A(\Omega_{\mathbb{N}})$ is a subalgebra of $C(\Omega_{\mathbb{N}})$ which contains the constant functions and separates points in $\Omega_{\mathbb{N}}$ so that, by the theorem of Stone and Weierstrass again, $A(\Omega_{\mathbb{N}})$ is dense in $C(\Omega_{\mathbb{N}})$. Hence we have

Lemma 2 *A sequence* $(\mu_{n_i})_{i\in\mathbb{N}}$ *of Borel probabilities on* Ω_{n_i} *converges weakly to* $\mu \in \mathcal{M}_{\mathbb{N}}$*, if and only if*

$$\lim_{i\to\infty} \int \varphi_1(\omega_1)\cdots\varphi_n(\omega_n)\, d\mu_{n_i} = \int \varphi_1(\omega_1)\cdots\varphi_n(\omega_n)\, d\mu$$

for all $n \in \mathbb{N}$ *and all* $\varphi_1, \dots, \varphi_n \in C(X)$*.*

Now we are ready to prove

Proposition 1 *Consider a sequence* $\mu_n \in \mathcal{M}^{ex}_{I_n}$*. A subsequence* $(\mu_{n_i})_{i\in\mathbb{N}}$ *converges weakly to a measure* $\mu \in \mathcal{M}^{ex}_{\mathbb{N}}$ *if and only if the sequence* $(p\mu_{n_i})_{i\in\mathbb{N}}$ *converges weakly to a Borel probability* P *on* $\mathcal{P}$*. In this case,* $\mu = \int \nu^{\mathbb{N}}\, dP(\nu)$*.*

Proof: Suppose first that $\mu_{n_i} \to \mu$. Then $\mu \in \mathcal{M}^{ex}_{\mathbb{N}}$ and hence $\mu = \int \nu^{\mathbb{N}}\, dP(\nu)$ for some Borel probability P on $\mathcal{P}$. Let $F_{\varphi_1}\cdots F_{\varphi_n} \in A_0(\mathcal{P})$. Then, as the μ_{n_i} are exchangeable,

$$\begin{aligned}
\int F_{\varphi_1}(\nu)\dots F_{\varphi_n}(\nu)\, d(p\mu_{n_i})(\nu) &= \int \nu(\varphi_1)\cdots\nu(\varphi_n)\, d(p\mu_{n_i})(\nu) &(6)\\
&= \int \prod_{k=1}^{n} \left(\frac{1}{n_i} \sum_{j=1}^{n_i} \varphi_k(\omega_j) \right) d\mu_{n_i}(\omega)\\
&= \frac{1}{n_i^n} \sum_{j_1=1}^{n_i} \cdots \sum_{j_n=1}^{n_i} \int \varphi_1(\omega_{j_1})\cdots\varphi_n(\omega_{j_n})\, d\mu_{n_i}(\omega)\\
&= \int \varphi_1(\omega_1)\cdots\varphi_n(\omega_n)\, d\mu_{n_i}(\omega) + O(n_i^{-1})\\
&\to \int \left[\int \varphi_1(\omega_1)\cdots\varphi_n(\omega_n)\, d\nu^{\mathbb{N}}(\omega) \right] dP(\nu)\\
&= \int F_{\varphi_1}(\nu)\dots F_{\varphi_n}(\nu)\, dP(\nu) &(7)
\end{aligned}$$

as $i \to \infty$. Hence $p\mu_{n_i} \to P$ weakly.

Suppose conversely that $p\mu_{n_i} \to P$ for some Borel probability P on $\mathcal{P}$. Let $\varphi_1, \dots, \varphi_n \in C(X)$. Then

$$\begin{aligned}
\int \varphi_1(\omega_1)\cdots\varphi_n(\omega_n)\, d\mu_{n_i}(\omega) &= \int F_{\varphi_1}(\nu)\cdots F_{\varphi_n}(\nu)\, d(p\mu_{n_i})(\nu) + O(n_i^{-1}) \quad \text{as in (6)}\\
&\to \int F_{\varphi_1}(\nu)\dots F_{\varphi_n}(\nu)\, dP(\nu)\\
&= \int \left[\int \varphi_1(\omega_1)\cdots\varphi_n(\omega_n)\, d\nu^{\mathbb{N}} \right] dP(\nu)\ .
\end{aligned}$$

Hence $\mu_{n_i} \to \int \nu^{\mathbb{N}}\, dP(\nu)$. □

Corollary 1 *Consider a sequence* $\mu_n \in \mathcal{M}^{ex}_{I_n}$ *and a probability measure* $\nu \in \mathcal{P}$*. For any subsequence* $(\mu_{n_i})_{i\in\mathbb{N}}$ *the following are equivalent:*

1) $(\mu_{n_i})_{i\in\mathbb{N}}$ *converges weakly to the product measure* $\nu^{\mathbb{N}}$*,*

2) $(p\mu_{n_i})_{i\in\mathbb{N}}$ *converges weakly to the point mass* δ_ν*,*

3) $\lim_{i\to\infty} \int \varphi(\omega_1)\psi(\omega_2)\, d\mu_{n_i}(\omega) = \nu(\varphi)\nu(\psi)$ *for all* $\varphi, \psi \in C(X, \mathbb{R})$*,* *Proof:* 1)$\Leftrightarrow$2) follows from the previous proposition, 1)$\Rightarrow$3) is trivial. We turn to 3)$\Rightarrow$2): Because of the weak compactness of $\mathcal{P}$ it suffices to show that if $\lim_{i\to\infty}\mu_{n_i}=\mu=\int\tilde{\nu}^{\mathbb{N}}\,dP(\tilde{\nu})$, then $P=\delta_\nu$. But, if $\lim_{i\to\infty}\mu_{n_i}=\mu$, then $\int\varphi(\omega_1)\psi(\omega_2)\,d\mu_{n_i}(\omega)\to\int\tilde{\nu}(\varphi)\tilde{\nu}(\psi)\,dP(\tilde{\nu})$ for all $\varphi,\psi\in C(X)$, and it follows from 3) that for each $\varphi\in C(X)$

$$\begin{aligned}\nu(\varphi)^2 &= \lim_{i\to\infty}\int\varphi(\omega_1)\varphi(\omega_2)\,d\mu_{n_i}(\omega)=\int\tilde{\nu}(\varphi)^2\,dP(\tilde{\nu})\\ &\geq \left(\int\tilde{\nu}(\varphi)\,dP(\tilde{\nu})\right)^2=\left(\lim_{i\to\infty}\int\varphi(\omega_1)\,d\mu_{n_i}(\omega)\right)^2=\nu(\varphi)^2\ .\end{aligned}$$

Hence $\tilde{\nu}(\varphi)=\nu(\varphi)$ for P-a.e. $\tilde{\nu}\in\mathcal{P}$, and as $C(X)$ has a countable dense subset, it follows that $\tilde{\nu}=\nu$ for P-a.e. $\tilde{\nu}\in\mathcal{P}$, i.e. $P=\delta_\nu$. □

As a consequence of Proposition 1, a suitable dynamical model for the infinite globally coupled map lattice is the topological dynamical system $(\mathcal{T}_\epsilon,\mathcal{P})$. Invariant measures P for $\mathcal{T}_\epsilon$ should be considered to be dynamically relevant (in the sense of an SRB-measure), if they are weak limits of sequences $(p\mu_{n_j})_j$ where the μ_n are SRB-measures for $T_\epsilon:\Omega_{I_n}\to\Omega_{I_n}$. This suggests the following precise definition:

Definition 1 *A globally coupled system* violates the 0-1-law*, if there exists a subsequence $(\mu_{n_i})_i$ of unique SRB-measures for $T_\epsilon:\Omega_{I_{n_i}}\to\Omega_{I_{n_i}}$ such that the $p\mu_{n_i}$ converge weakly to a probability distribution P on $\mathcal{P}$ that is not just a point mass in a fixed point of $\mathcal{T}_\epsilon$.*

3 Systems with piecewise differentiable units

In Section 4 we shall see that if the individual map τ is a piecewise C^2 and expanding mixing interval map (like the tent map), then for each n there is a unique SRB-measure μ_n, which is absolutely continuous and has a density of bounded variation. Beforehand we show in this section that weak limits P of such measures μ_n are concentrated on a relatively small subset $\mathcal{D}$ of $\mathcal{P}$, namely on the set of those measures ν on X that have densities (with respect to Lebesgue measure) of bounded variation.

3.1 Functions of bounded variation

A Lebesgue-integrable function $f:\mathbb{R}^n\to\mathbb{R}$ has bounded variation, if its derivative in the distributional sense is a finite $\mathbb{R}^n$-valued measure. A rather complete treatment of the basic properties of these functions is given in Chapter 5 of [6]. As for our purposes a slight variation of the notations introduced there is preferable, we start with some basic definitions.

For $f\in L^1(\mathbb{R}^n)$ let

$$\mathrm{var}_i(f)=\sup\left\{\int f\,\frac{\partial\varphi}{\partial x_i}\,dx:\varphi\in C_c^1(\mathbb{R}^n;\mathbb{R}),|\varphi|\leq 1\right\}$$

and

$$\mathrm{var}(f)=\sum_{i=1}^n\mathrm{var}_i(f)\ .$$

If $\Omega\subset\mathbb{R}^n$ is open and bounded and if $f:\Omega\to\mathbb{R}$ is integrable, then we define $\mathrm{var}_i(f):=\mathrm{var}_i(f\,1_\Omega)$. Therefore, $\mathrm{var}(f)$ is the sum of the variation of f inside Ω plus the contributions caused by the jumps of f to zero on the boundary of Ω. Denote by $BV=BV(\Omega)$ the space of functions of bounded variation on Ω [2]. It is a linear space which, equipped with the norm

[2] More precisely, BV is the space of all equivalence classes (modulo Lebesgue measure zero) of such functions

$\|f\|_{BV} := \mathrm{var}(f) + \|f\|_1$, becomes a Banach space. This fact, as well as the compact embedding of $BV(\Omega)$ into $L^1(\Omega)$ and many further results follow directly from the corresponding statements in [6, Chapter 5] if one observes that $\frac{1}{n}\mathrm{var}(f) \leq \|Df\|(\mathbb{R}^n) \leq \mathrm{var}(f)$ where $\|Df\|$ denotes the total variation measure of the distributional derivative of f which can be interpreted as a vector-valued Radon measure. In particular, $\|.\|_{BV}$ and the variation norm defined in [6, p.170] are equivalent norms on $BV(\Omega)$. [3]

If μ is a finite signed measure on $\mathbb{R}^n$, we define similarly

$$\mathrm{var}_i(d\mu) = \sup\left\{\int \frac{\partial\varphi}{\partial x_i}\, d\mu(x) : \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}), |\varphi| \leq 1\right\}$$

and

$$\mathrm{var}(d\mu) = \sum_{i=1}^{n} \mathrm{var}_i(d\mu)\ .$$

Then $\mathrm{var}_i(d\mu) = \mathrm{var}_i(h)$, if μ has density h with respect to Lebesgue measure on $\mathbb{R}^n$. [4]

Lemma 3 *If μ is a finite signed measure on $\mathbb{R}^n$ and if $\mathrm{var}(d\mu) < \infty$, then μ has a density h with respect to Lebesgue measure and $\mathrm{var}_i(d\mu) = \mathrm{var}_i(h)$ for $i = 1, \dots, n$.*

Proof: In view of the preceding remark, only the absolute continuity of μ remains to be proved. Consider $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R})$ and let $\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon})$ $(\epsilon > 0)$ be a standard mollifier (see [6, 4.2.1]). As $\frac{\partial\varphi}{\partial x_i}$ is continuous with compact support, $\lim_{\epsilon\to 0} \frac{\partial\varphi}{\partial x_i} * \eta_\epsilon = \frac{\partial\varphi}{\partial x_i}$ uniformly on $\mathbb{R}^n$.

The measures $\mu * \eta_\epsilon$ have densities, say h_ϵ, w.r.t. Lebesgue measure on $\mathbb{R}^n$. Therefore, if $|\varphi| \leq 1$, then

$$\int \frac{\partial\varphi}{\partial x_i} h_\epsilon\, dx = \int \frac{\partial\varphi}{\partial x_i} d(\mu * \eta_\epsilon) = \int \left(\frac{\partial\varphi}{\partial x_i} * \eta_\epsilon\right) d\mu = \int \frac{\partial}{\partial x_i}(\varphi * \eta_\epsilon)\, d\mu \leq \mathrm{var}_i(d\mu)$$

so that $\mathrm{var}_i(h_\epsilon) \leq \mathrm{var}_i(d\mu)$ for all $\epsilon > 0$. Hence there exist a sequence $\epsilon_k \to 0$ and a probability density h such that $\lim_{k\to\infty} \|h_{\epsilon_k} - h\|_1 = 0$. Therefore, for each $\psi \in C_c(\mathbb{R}^n;\mathbb{R})$,

$$\int \psi\, d\mu = \lim_{k\to\infty} \int \psi * \eta_{\epsilon_k}\, d\mu = \lim_{k\to\infty} \int \psi h_{\epsilon_k}\, dx = \int \psi h\, dx$$

so that $d\mu = h\, dx$. □

The most important property of functions of bounded variation we make use of in this section is the following compactness property: If $\Omega \subset \mathbb{R}^n$ is bounded and if $f_j : \mathbb{R}^n \to \mathbb{R}$ are of bounded variation $(j \in \mathbb{N})$ with $\mathrm{var}_i(f) \leq C_i < \infty$ for all $i = 1, \dots, n$ and all $j \in \mathbb{N}$, then there is a subsequence f_{j_k} converging in L^1 to some integrable function f on Ω and $\mathrm{var}_i(f) \leq C_i$ for all $i = 1, \dots, n$.

3.2 Measures with densities of bounded variation

In this section we make no dynamical assumptions. Suppose that $X = [0,1]$ (or any other interval or the unit circle line with normalized Lebesgue measure m). Let $\Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \dots \subset \mathbb{N}$ be a filtration of $\mathbb{N}$ by finite sets. We use the following notations:

$\pi_J : \Omega_{\mathbb{N}} \to \Omega_J$ for $J \subseteq \mathbb{N}$ is the canonical projection, $\pi_n := \pi_{\Lambda_n}$
$\mathcal{F}_J$ for $J \subseteq \mathbb{N}$ is the Borel σ-algebra of Ω_J, $\hat{\mathcal{F}}_J := \pi_J^{-1}\mathcal{F}_J$,
$\hat{A} := \pi_{\Lambda_J}^{-1}(A)$ for $J \subseteq \mathbb{N}$ and $A \in \mathcal{F}_J$.

[3] As Ω is open and bounded, the Sobolev inequality [6, Theorem 5.6.1-1] implies that there is a constant $C > 0$ such that $\|f\|_1 \leq C\,\mathrm{var}(f)$ for all $f \in BV(\Omega)$. Hence $\|f\|_{BV} \leq (1 + C)\,\mathrm{var}(f) \leq \|f\|_{BV}$, so $\mathrm{var}(.)$ is itself a norm equivalent to $\|.\|_{BV}$.

[4] So $\mathrm{var}(d\mu)$ is not related to the total variation (= total absolute mass) of μ.

Fix a probability measure μ on $\Omega_{\mathbb{N}}$. We define conditional probability distributions K_n and K_∞ on $\mathcal{F}_{\Lambda_0}$ given $\hat{\mathcal{F}}_{\Lambda_n\setminus\Lambda_0}$ (resp. $\hat{\mathcal{F}}_{\mathbb{N}\setminus\Lambda_0}$) by

$$K_n(\omega, A) := E_\mu[\hat{A} \mid \hat{\mathcal{F}}_{\Lambda_n\setminus\Lambda_0}](\omega)\ , \quad K_\infty(\omega, A) := E_\mu[\hat{A} \mid \hat{\mathcal{F}}_{\mathbb{N}\setminus\Lambda_0}](\omega)\ .$$

As $\omega \mapsto K_n(\omega, A)$ is $\hat{\mathcal{F}}_{\Lambda_n\setminus\Lambda_0}$-measurable, we may write as well $K_n((\omega_i)_{i\in\Lambda_n\setminus\Lambda_0}, A)$.

Lemma 4 *Suppose that all $\pi_n\mu$ are absolutely continuous with respect to Lebesgue measure on Ω_{Λ_n} with densities h_n and that these densities are of bounded variation. In particular, assume that there are constants C_i ($i \in \Lambda_0$) such that $\sup_n \mathrm{var}_i(h_n) \le C_i$ for all $i \in \Lambda_0$.*

Then, for μ-a.e. ω, $K_\infty(\omega, .)$ is absolutely continuous with respect to Lebesgue measure on Ω_{Λ_0} with density $k_\infty(\omega, x)$ of bounded variation and $\int_{\Omega_{\mathbb{N}}} \mathrm{var}_i(x \mapsto k_\infty(\omega, x))\, d\mu(\omega) \le C_i$ for all $i \in \Lambda_0$.

Proof: For $\varphi \in C_c(\mathbb{R}^{\Lambda_0}; \mathbb{R})$ we have $E_\mu[\varphi \circ \pi_0 \mid \hat{\mathcal{F}}_{\Lambda_n\setminus\Lambda_0}](\omega) = \int_{\Omega_{\Lambda_0}} \varphi(x)\, K_n(\omega, dx)$, and the same holds for $n = \infty$. Hence, by the martingale convergence theorem, $\int_{\Omega_{\Lambda_0}} \varphi(x)\, K_\infty(\omega, dx) = \lim_{n\to\infty} \int_{\Omega_{\Lambda_0}} \varphi(x)\, K_n(\omega, dx)$ for μ-a.e. ω and fixed φ. As $C_c(\mathbb{R}^{\Lambda_0}; \mathbb{R})$ is separable, $K_n(\omega, .)$ converges weakly to $K_\infty(\omega, .)$ for μ-a.e. ω. Let $A_0 \in \mathcal{F}_{\Lambda_0}$ and $A \in \mathcal{F}_{\Lambda_n\setminus\Lambda_0}$. Then

$$\begin{aligned}
\int_{A_0\times A} h_n(x,y)\, d(x,y) &= (\pi_n\mu)(A_0 \times A) \\
&= E_\mu\left[1_{\hat{A}} \cdot E_\mu[1_{\hat{A}_0} \mid \hat{\mathcal{F}}_{\Lambda_n\setminus\Lambda_0}]\right] \\
&= E_\mu\left[1_{\hat{A}} \cdot K_n(\,.\,, A_0)\right] \\
&= \int_{\Omega_{\Lambda_n\setminus\Lambda_0}} \int_{\Omega_{\Lambda_0}} 1_A(y)\, K_n(y, A_0)\, h_n(x,y)\, dx\, dy \\
&= \int_A K_n(y, A_0)\, \bar{h}_n(y)\, dy
\end{aligned}$$

where $\bar{h}_n(y) := \int_{\Omega_{\Lambda_0}} h_n(x,y)\, dx$. Hence $K_n(y, A_0) \cdot \bar{h}_n(y) = \int_{A_0} h_n(x,y)\, dx$ for $\pi_{\Lambda_n\setminus\Lambda_0}\mu$-a.e. y so that for such y the measure $K_n(y, .)$ is absolutely continuous with respect to Lebesgue measure.

We estimate the variation of the densities $x \mapsto k_n(y,x)$ of the measures $K_n(y, .)$. Observe first that $k_n(y,x) = \bar{h}_n(y)^{-1}\, h_n(x,y)$. Hence

$$\begin{aligned}
&\int_{\Omega_{\Lambda_n\setminus\Lambda_0}} \mathrm{var}_i(x \mapsto k_n(y,x))\, d(\pi_{\Lambda_n\setminus\Lambda_0}\mu)(y) \\
=\ &\int_{\Omega_{\Lambda_n\setminus\Lambda_0}} \mathrm{var}_i(x \mapsto h_n(x,y))\, \bar{h}_n(y)^{-1}\, \bar{h}_n(y)\, dy \\
=\ &\mathrm{var}_i((x,y) \mapsto h_n(x,y)) \le C_i
\end{aligned} \tag{8}$$

The last equality is trivial for differentiable h_n, and it caries over to general h_n by approximation with smoothened functions.

Define $k_n(\omega, x) := k_n((\omega_i)_{i\in\Lambda_n\setminus\Lambda_0}, x)$. As $K_n(\omega, .) \to K_\infty(\omega, .)$ weakly for μ-a.e. ω, it follows that for μ-a.e. ω and each $\varphi \in C_c^1(\mathbb{R}^{\Lambda_0}, \mathbb{R})$ with $|\varphi| \le 1$

$$\begin{aligned}
\int \frac{\partial\varphi}{\partial x_i}\, K_\infty(\omega, dx) = \lim_{n\to\infty} \int \frac{\partial\varphi}{\partial x_i}\, K_n(\omega, dx) &= \lim_{n\to\infty} \int \frac{\partial\varphi}{\partial x_i} k_n(\omega, x)\, dx \\
&\le \liminf_{n\to\infty} \mathrm{var}_i(x \mapsto k_n(\omega, x))
\end{aligned}$$

for each $i \in \Lambda_0$. Therefore $\mathrm{var}_i(dK_\infty(\omega, .)) \le \liminf_{n\to\infty} \mathrm{var}_i(x \mapsto k_n(\omega, x))$. for μ-a.e. ω so that

$$\begin{aligned}
\int \mathrm{var}_i(dK_\infty(\omega, .))\, d\mu(\omega) &\le \liminf_{n\to\infty} \int \mathrm{var}_i(x \mapsto k_n(\omega, x))\, d\mu(\omega) \\
&= \liminf_{n\to\infty} \int_{\Omega_{\Lambda_n\setminus\Lambda_0}} \mathrm{var}_i(x \mapsto k_n(y,x))\, d(\pi_{\Lambda_n\setminus\Lambda_0}\mu)(y) \\
&\le C_i\ .
\end{aligned}$$

In particular, $\mathrm{var}_i(dK_\infty(\omega,.)) < \infty$ for μ-a.e. ω and each i such that $K_\infty(\omega,.)$ has density $k_\infty(\omega,x)$ with $\mathrm{var}_i(x\mapsto k_\infty(\omega,x)) = \mathrm{var}_i(dK_\infty(\omega,\,.))$ by Lemma 3. It follows that $\int_{\Omega_\mathbb{N}} \mathrm{var}_i(x\mapsto k_\infty(\omega,x))\,d\mu(\omega) \le C_i$ for all $i\in\Lambda_0$. □

Next we define conditional probability distributions K^n and K^∞ on $\mathcal{F}_{\Lambda_0}$ given $\hat{\mathcal{F}}_{\mathbb{N}\setminus\Lambda_n}$ (resp. $\mathcal{F}^{tail}$) by

$$K^n(\omega,A) := E_\mu[\hat{A}\mid \hat{\mathcal{F}}_{\mathbb{N}\setminus\Lambda_n}](\omega)\ ,\quad K^\infty(\omega,A) := E_\mu[\hat{A}\mid \mathcal{F}^{tail}](\omega)\ .$$

Then $(K^n(\omega,A),\hat{\mathcal{F}}_{\mathbb{N}\setminus\Lambda_n})_n$ is a uniformly bounded reverse martingale so that $\lim_{n\to\infty} K^n(\omega,A) = K^\infty(\omega,A)$ μ-a.s. and in L^1_μ and $E_\mu[K^0(\,.\,,A)\mid\mathcal{F}^{tail}] = K^\infty(\,.\,,A)$. As in the proof of the previous lemma it follows that $\lim_{n\to\infty} K^n(\omega,\,.\,) = K^\infty(\omega,\,.\,)$ weakly for μ-a.e. ω.

Lemma 5 *Under the assumptions of Lemma 4, let $\mu = \int \mu_\omega\,d\mu(\omega)$ be the $\mathcal{F}^{tail}$-decomposition of μ, i.e. $(\omega,A)\mapsto\mu_\omega(A)$ is a conditional probability distribution on Ω given $\mathcal{F}^{tail}$.*

Then, for μ-a.e. ω, $\pi_{\Lambda_0}\mu_\omega = K^\infty(\omega,\,.\,)$ and $K^\infty(\omega,\,.\,)$ is absolutely continuous with respect to Lebesgue measure on Ω_{Λ_0} with density $k^\infty(\omega,x) = \int_{\Omega_\mathbb{N}} k_\infty(\tilde\omega,x)\,d\mu_\omega(\tilde\omega)$ of bounded variation and

$$\int_{\Omega_\mathbb{N}} \mathrm{var}_i(x\mapsto k^\infty(\omega,\,.\,))\,d\mu(\omega) \le C_i \tag{9}$$

for all $i\in\Lambda_0$.

Proof: $\pi_{\Lambda_0}\mu_\omega = K^\infty(\omega,\,.\,)$ for μ-a.e. ω is just the definition of conditional distributions. As $K^0(\omega,A) = K_\infty(\omega,A)$, we have

$$\begin{aligned}
K^\infty(\omega,A) &= E_\mu[K^0(\,.\,,A)\mid\mathcal{F}^{tail}](\omega)\\
&= E_\mu[K_\infty(\,.\,,A)\mid\mathcal{F}^{tail}](\omega)\\
&= E_\mu\left[\int_A k_\infty(\,.\,,x)\,dx\mid\mathcal{F}^{tail}\right](\omega)\\
&= \int_{\Omega_\mathbb{N}}\int_A k_\infty(\tilde\omega,x)\,dx\,d\mu_\omega(\tilde\omega)\\
&= \int_A\left[\int_{\Omega_\mathbb{N}} k_\infty(\tilde\omega,x)\,d\mu_\omega(\tilde\omega)\right]dx
\end{aligned}$$

so that $K^\infty(\omega,\,.\,)$ has density $x\mapsto \int_{\Omega_\mathbb{N}} k_\infty(\tilde\omega,x)\,d\mu_\omega(\tilde\omega)$. The average variation bound for this density follows at once from that for $k_\infty(\omega,x)$. □

3.3 Exchangeable densities of bounded variation

In the special case of exchangeable distributions μ on $\Omega_\mathbb{N}$ the previous lemma yields at once:

Proposition 2 *Let μ be an exchangeable distribution on $\Omega_\mathbb{N}$ all of whose finite-dimensional marginals are absolutely continuous with respect to Lebesgue measure and have densities of bounded variation, namely $\mathrm{var}_1(h_n)\le C$. Let $\mu = \int \nu_\omega^\mathbb{N}\,d\mu(\omega)$ be the de Finetti-decomposition of μ (see Theorem 2). Then μ-a.a. measures ν_ω of this decomposition are absolutely continuous with densities $k^\infty(\omega,x)$ of bounded variation and $\int \mathrm{var}_1(x\mapsto k^\infty(\omega,x))\,d\mu(\omega)\le C$.*

Proof: Apply Lemma 5 to $\mu\in\mathcal{M}_\mathbb{N}^{ex}$ with $\Lambda_0=\{1\}$. As the de Finetti-decomposition coincides with the $\mathcal{F}^{tail}$-decomposition from that lemma, $\mu_\omega = \nu_\omega^\mathbb{N}$ so that $\pi_{\Lambda_0}\mu_\omega = \nu_\omega$. □

In the dynamical setting discussed in Section 2 this proposition leads to the following theorem:

Theorem 3 *Suppose that for some fixed ϵ each $T_\epsilon : \Omega_{I_n} \to \Omega_{I_n}$ $(n > 0)$ has an absolutely continuous invariant probability measure μ_n with density h_n w.r.t. Lebesgue measure and that these densities satisfy the following variation bound:*

$$\exists C > 0 \ \forall i \in \mathbb{N} \ \forall n \geq i : \ \mathrm{var}_i(h_n) \leq C \ .$$

Let P be a weak accumulation point of the measures $p\mu_n$ on $\mathcal{P}$ (these are the Borel probability measures on $X = [0,1]$). Denote by $\mathcal{D} := \{\nu \in \mathcal{P} : d\nu(x) = h_\nu(x)\,dx$ and $\mathrm{var}_1(h_\nu) < \infty\}$. Then P is $\mathcal{T}_\epsilon$-invariant (see (5)),

$$P(\mathcal{P} \setminus \mathcal{D}) = 0 \quad \text{and} \quad \int \mathrm{var}_1(h_\nu)\,dP(\nu) \leq C \ . \tag{10}$$

Proof: Suppose that $P = \lim_{j\to\infty} p\mu_{n_j}$. In view of Proposition 1 and Remark 3,

$$\lim_{j\to\infty} \mu_{n_j} = \mu := \int \nu^{\mathbb{N}}\,dP(\nu) = \int \nu_\omega^{\mathbb{N}}\,d\mu(\omega) \ . \tag{11}$$

Fix $k \in \mathbb{N}$ and consider $\pi_k\mu_{n_j}$ for $n_j \geq k$. By assumption, $\pi_k\mu_{n_j}$ has density

$$h_{n_j}^{(k)}(x) := \int_{\Omega_{\{k+1,\dots,n_j\}}} h_{n_j}(x,y)\,dy \quad \text{with} \quad \mathrm{var}_i(h_{n_j}^{(k)}) \leq C \quad (i = 1,\dots,k) \ .$$

As $\lim_{j\to\infty} \mu_{n_j} = \mu$ in the weak sense, it follows from the relative compactness of the family of densities $(h_{n_j}^{(k)})_j$ that $\pi_k\mu$ is absolutely continuous w.r.t. Lebesgue measure with some density $h^{(k)}$ and that $\lim_{j\to\infty} \|h^{(k)} - h_{n_j}^{(k)}\|_1 = 0$. In particular, $\mathrm{var}_i(h^{(k)}) \leq C$ for all $i = 1\dots,k$. Now Proposition 2 implies that μ-a.e. ν_ω is absolutely continuous with density $k^\infty(\omega,x)$ of bounded variation and $\int \mathrm{var}_1(x \mapsto k^\infty(\omega,x))\,d\mu(\omega) \leq C$. In view of (11) this is equivalent to (10). □

In the next section we show that for piecewise expanding interval maps τ the assumptions of Theorem 3 are satisfied and that, even more, there is some $K > 0$ such that $P(\mathcal{P} \setminus \mathcal{D}_K) = 0$ where $\mathcal{D}_K := \{\nu \in \mathcal{D} : \mathrm{var}_1(h_\nu) \leq K\}$.

4 Systems with piecewise expanding units

In this section we specialize our setting. Now X is either a compact interval (for definitiveness we let $X = [0,1]$ in this case), or X is the unit circle line. In both cases we denote the normalized Lebesgue measure on X by m. The map $\tau : X \to X$ is continuous, piecewise C^2 and piecewise expanding in the following sense: There are points $0 = a_0 < a_1 < \dots < a_N = 1$ such that

1. $\tau_{|(a_{i-1},a_i)}$ is monotone and C^2 for all $i = 1,\dots,N$,
2. there are $\kappa > 2$ and $k \geq 1$ such that
   (a) $\tau^j a_i \notin \{a_1,\dots,a_{N-1}\}$ for $j = 0,\dots,k-1$ and $i = 0,\dots,N$ and
   (b) $|(\tau^k)'(x)| \geq \kappa$ for all $x \in X$ for which the derivative is well defined,
3. $\sup_x |\tau''(x)/\tau'(x)^2| < \infty$ where the supremum extends over all $x \notin \{a_0,\dots,a_N\}$.

These are traditional assumptions guaranteeing that τ has an absolutely continuous invariant measure with density of bounded variation. (There are weaker assumptions under which the same conclusion holds.) For a detailed review of this type of results see [3].

The two main results of this section show that expanding circle maps with a coupling as in (2) do not violate the 0-1-law (Theorem 4) and that the same holds for tent maps τ with $|\tau'| > 1.65$ and such that the critical trajectory of τ ends in a fixed point (Theorem 5 and Remark 5).

4.1 A specific class of couplings

In order to be able to use some results from the literature we consider only couplings of the following type:

$$S_\epsilon(x,\nu) = x + \epsilon\cdot\gamma(x,\bar{\nu}) \quad \text{where} \quad \bar{\nu} = \int_X F\circ\tau\,d\nu \ (\in \mathbb{R}^s) \tag{12}$$

for some fixed $F \in C^2(X,\mathbb{R}^s)$ and $\gamma \in C^2(X\times\mathbb{R}^s,\mathbb{R})$. Denoting, as in Examples 1 and 2 (coupled tent maps and circle maps),

$$\Phi_\epsilon : \Omega \to \Omega, \quad (\Phi_\epsilon\omega)_i = \omega_i + \epsilon\cdot\gamma(\omega_i, \int_X F\,dp_\omega)$$

we can write $T_\epsilon = \Phi_\epsilon \circ T$. Therefore the first model fits directly into the framework of [15] and of the thesis [16], the results of which are also summarized in [12]. In the second model, the structure of the coupling map is not exactly the one studied in these references, but a glance at the proofs given there shows that indeed any coupling as described in (12) allows the same conclusions. All one has to check is that there are constants $A_0, A_1, A_2 > 0$, which are independent of the lattice size n, such that for all $\omega \in \Omega$

$$|\Phi_\epsilon(\omega) - \omega|_\infty \le A_0\epsilon, \quad \|D\Phi_\epsilon(\omega) - E\|_\infty \le A_1\epsilon, \quad \|D_i D\Phi_\epsilon(\omega)\|_\infty \le A_2\epsilon \text{ for all } i\ ,$$

where D denotes the total derivative, D_i the partial derivative with respect to ω_i, E the identity matrix, $|\,.\,|_\infty$ the maximum norm for vectors, and $\|\,.\,\|_\infty$ the corresponding operator norm for matrices (maximal row sum). [5] It is not hard to check that these inequalities are satisfied for twice continuously differentiable F and γ. Therefore the dynamics of T_ϵ can be studied as in [15, 16, 12] by means of its associated Perron-Frobenius operator $\mathcal{L}_\epsilon : L^1_\Omega \to L^1_\Omega$. This operator is uniquely defined by the property

$$\int_\Omega \varphi(T_\epsilon\omega)\psi(\omega)\,dm(\omega) = \int_\Omega \varphi(\omega)(\mathcal{L}_\epsilon\psi)(\omega)\,dm(\omega) \quad \text{for all } \varphi \in L^\infty_\Omega, \psi \in L^1_\Omega.$$

It is proved in these references that there are constants $C_1, C_2 > 0$ and $\epsilon_0 > 0$ depending only on τ and A_0, A_1, A_2 and not on the dimension n of Ω_{I_n} such that for all $\epsilon \in [0,\epsilon_0]$ and all $f \in L^1_\Omega$ of bounded variation holds:

$$\mathrm{var}(\mathcal{L}^k_\epsilon f) \le \left(\frac{2}{\kappa} + C_1\epsilon\right)\cdot \mathrm{var}(f) + C_2\,n\cdot\|f\|_1 \tag{13}$$

where $k \in \mathbb{N}$ and $\kappa > 2$ is the lower expansion bound of τ^k as introduced above. Hence $\mathcal{L}_\epsilon$ leaves the space $BV(\Omega)$ of functions of bounded variation on Ω invariant and acts on it as a quasicompact operator such that $T_\epsilon : \Omega_{I_n} \to \Omega_{I_n}$ has an absolutely continuous (w.r.t. Lebesgue measure) invariant probability measure with density $h_{\epsilon,n}$ of bounded variation. Let $\alpha \in (\frac{2}{\kappa}, 1)$. Then ϵ_0 can be chosen such that $(\frac{2}{\kappa} + C_1\epsilon) < \alpha$ for all $\epsilon \in [0,\epsilon_0]$. Therefore $\mathrm{var}(h_{\epsilon,n}) \le \frac{C_2}{1-\alpha}n$. If T_ϵ is ergodic, then it has a unique invariant density. In any case we can assume that the density is exchangeable, because the invariance of $h_{\epsilon,n}$ implies immediately that also $\bar{h}_{\epsilon,n} := |\Sigma(I_n)|^{-1}\sum_{\sigma\in\Sigma(I_n)} h_{\epsilon,n}\circ\sigma^*$ is invariant (see Section 2.1). This implies that $\mathrm{var}_i(h_{\epsilon,n}) = \frac{1}{n}\mathrm{var}(h_{\epsilon,n}) \le \frac{C_2}{1-\alpha}$ for all $i \in I_n$.

For coupled tent maps with slope larger than $\sqrt{2}$ and for a coupling as in (1), it was proved in [13] that there is indeed $\epsilon_2 \in (0,\epsilon_1]$ independent of n such that T_ϵ is even mixing for $\epsilon \in [0,\epsilon_2]$. This result carries over to arbitrary mixing τ and also to more general couplings. If τ is a C^r-endomorphism of the circle ($r \in \{2,3,\dots,\infty\}\cup\{\omega\}$) and if the coupling is the one from (2), then it is well known that T_ϵ is mixing and that the unique density h_ϵ is of class C^{r-1}, see [17].

In any case we are now in a position to apply Theorem 3 which tells us that all weak accumulation points of the sequence $(p\mu_{\epsilon,n})_n$ (where $d\mu_{\epsilon,n}(\omega) = h_{\epsilon,n}(\omega)d\omega$) are measures on $\mathcal{P}$

[5] These are the essential properties of the more special couplings studied in [16]. They are listed in Proposition 2.15 of that thesis, and all further proofs use only these properties.

concentrated on the set of probability measures on X with densities of bounded variation. Even more is true. Recall the notations $\mathcal{D} = \{\nu \in \mathcal{P} : d\nu(x) = h_\nu(x)\,dx \text{ and } \mathrm{var}_1(h_\nu) < \infty\}$ and $\mathcal{D}_K = \{\nu \in \mathcal{D} : \mathrm{var}_1(h_\nu) \leq K\}$. Denote also by $\mathcal{L}$ the Perron Frobenius operator acting on $L^1_m(X)$ which is associated with τ and by $\mathcal{S}_{\epsilon,\nu}$ the one associated with $S_{\epsilon,\nu} : X \to X$.

Proposition 3 *For maps τ and couplings $S_{\epsilon,\nu}$ as introduced above let $K_0 := \frac{C_2}{1-\alpha}$. Then each weak accumulation point P of the sequence $(p\mu_{\epsilon,n})_n$ satisfies $P(\mathcal{P} \setminus \mathcal{D}_{K_0}) = 0$.*

Proof: We know already that $P(\mathcal{P} \setminus \mathcal{D}) = 0$. As P is invariant under $\mathcal{T}_\epsilon$, it suffices to show that for each $\nu \in \mathcal{D}$ all weak accumulation points of the sequence $(\mathcal{T}_\epsilon^i(\nu))_i$ belong to $\mathcal{D}_{K_0}$. If $\nu \in \mathcal{D}$ has density h, then $\mathcal{T}_\epsilon(\nu) = (S_{\epsilon,\nu} \circ \tau)\nu$ has density $\mathcal{S}_{\epsilon,\nu}(\mathcal{L}h)$. A simple calculation, similar to the one leading to (13) in the case $n = 1$, shows that for all $h \in BV(\Omega)$

$$\mathrm{var}_1((\mathcal{S}_{\epsilon,\nu}\mathcal{L})^k h) \leq \left(\frac{2}{\kappa} + C_1\epsilon\right) \cdot \mathrm{var}_1(h) + C_2 \tag{14}$$

with $k \in \mathbb{N}$ and $\kappa > 2$ as introduced above. Hence, iterating this estimate shows that the variation per variable of the densities of the $\mathcal{T}_\epsilon^k(\nu)$ tend to be at most $\frac{C_2}{1-\alpha}$ as $k \to \infty$, thus proving the claim. □

Denote by $\tilde{\mathcal{L}}_\epsilon$ the operator which associates with a density h (corresponding to the measure $\nu = hm$) the density $\mathcal{S}_{\epsilon,\nu}\mathcal{L}h$. This operator is often called the *self consistent Perron Frobenius operator*. The proof shows that the nonlinear operator $\tilde{\mathcal{L}}_\epsilon : L^1_m(X) \to L^1_m(X)$ and the single site Perron-Frobenius operator $\mathcal{L}$ are closely related. In general, however, $\tilde{\mathcal{L}}_\epsilon$ is not a small perturbation of $\mathcal{L}$ neither in the sense of the L^1-operator norm nor of the BV-operator norm. We shall see that this fact is responsible for the *violation of the 0-1-law* mentioned in the introduction. On the other hand, there are cases, where $\tilde{\mathcal{L}}_\epsilon$ is a small perturbation of $\mathcal{L}$, namely, when τ and $S_{\epsilon,\nu}$ satisfy suitable analyticity assumptions (see [7, 8]). We shall have a closer look at systems not violating the 0-1-law in Section 4.3.

4.2 Fixed points of $\mathcal{T}_\epsilon$

Proposition 3 leaves us with the task to study $\mathcal{T}_\epsilon$-invariant measures on $\mathcal{D}_{K_0}$. This is particularly simple, if $\tilde{\mathcal{L}}_\epsilon$ is a $\|.\|_{BV}$-contraction on $\mathcal{D}_{K_0}$ with a unique fixed point h_ϵ, because in this case $P = \delta_{\nu_\epsilon}$ with $\nu_\epsilon = h_\epsilon m$ is the only $\mathcal{T}_\epsilon$-invariant measure. So we start with a simple characterization of fixed points for $\mathcal{T}_\epsilon$.

With the notation and assumptions of the previous section denote $\tau_{\epsilon,\nu} := S_{\epsilon,\nu} \circ \tau$. Suppose that $\tau_{0,\nu} = \tau$ is ergodic and therefore has a unique absolutely continuous invariant measure. Then, for ϵ sufficiently close to 0, also $\tau_{\epsilon,\nu}$ has a unique invariant measure $\mu_{\epsilon,\nu} \in \mathcal{D}_{K_0}$ with density $h_{\epsilon,\nu}$ such that $\mathcal{S}_{\epsilon,\nu}\mathcal{L}h_{\epsilon,\nu} = h_{\epsilon,\nu}$. Furthermore, $G_\epsilon : \nu \mapsto h_{\epsilon,\nu}$ is continuous as a function from $\mathcal{P}$ to $L^1_m(X)$, see [11, Corollary 14]. In particular, the map $G_{\epsilon|\mathcal{D}_{K_0}} : \mathcal{D}_{K_0} \to \mathcal{D}_{K_0}$, $fm \mapsto h_{\epsilon,fm}$, is continuous w.r.t. the $\|.\|_1$-norm. Since $\mathcal{D}_{K_0}$ is compact convex, we have

Lemma 6 *$fm \mapsto G_\epsilon(f)m$ has at least one fixed point in $\mathcal{D}_{K_0}$.*

The following observation is well known [5, 10], its proof is straightforward:

Lemma 7 *$\nu \in \mathcal{P}$ is a fixed point of $\mathcal{T}_\epsilon$ if and only if ν has density $f = G_\epsilon(fm)$ for some f. In this case $f = h_{\epsilon,fm}$.*

Therefore, if $G_{\epsilon|G_\epsilon(\mathcal{P})}$ is a contraction, then $\mathcal{T}_\epsilon$ has a unique fixed point. Now $G_{\epsilon|G_\epsilon(\mathcal{P})}$ is certainly a contraction for small ϵ if it is Lipschitz continuous with a Lipschitz constant l_ϵ that can be made arbitrarily small as $\epsilon \to 0$. Unfortunately this is not true in general. The best known general result seems to be

$$\|G_\epsilon(fm) - G_\epsilon(\tilde{f}m)\|_1 = O\left(\epsilon\|f - \tilde{f}\|_1 \cdot |\log(\epsilon\|f - \tilde{f}\|_1)|\right) \quad \text{as } \epsilon \to 0. \tag{15}$$

for a coupling satisfying (12), see [11]. Under more restrictive assumptions, however, such a unique fixed point may exist (see Example 3 below), and it may even be an attractive fixed point for the map $\mathcal{T}_\epsilon$, as we shall see in the next section.

Example 3 [Tent maps] A particularly simple case is that of a tent map $\tau : [0,1] \to [0,1], x \mapsto a(1-|2x-1|)$ with $\frac{1}{2} < a < 1$. Choosing a simple coupling as in (1) we find that $\tau_{\epsilon,\nu}(x) = \tau_{\epsilon,0}(x) + \epsilon\bar{\nu}$ where $\bar{\nu} = \int \tau(y)\, d\nu(y)$. So $\tau_{\epsilon,\nu}$ is a tent map on the interval $[-\frac{\epsilon\bar{\nu}}{2a(1-\epsilon)-1}, 1+\frac{\epsilon\bar{\nu}}{2a(1-\epsilon)-1}]$ with the same slope $a(1-\epsilon)$ as $\tau_{\epsilon,0}$ and it is therefore linearly conjugated to $\tau_{\epsilon,0}$. Recall that $h_{\epsilon,\nu}$ denotes the invariant density of $\tau_{\epsilon,\nu}$. An elementary computation shows that

$$h_{\epsilon,\nu}(x) = \lambda_{\epsilon,\bar{\nu}}^{-1} \cdot h_{\epsilon,0}\left(\frac{1}{2} + \lambda_{\epsilon,\bar{\nu}}^{-1}(x - \frac{1}{2})\right) \quad \text{where} \quad \lambda_{\epsilon,t} := 1 + \frac{2\epsilon t}{2a(1-\epsilon)-1} .$$

Therefore, if $\nu_1, \nu_2 \in \mathcal{P}$, then $h_{\epsilon,\nu_1} = h_{\epsilon,\nu_2}$ if and only if $\bar{\nu}_1 = \bar{\nu}_2$. So we may write $h_{\epsilon,\bar{\nu}}$ instead and study $\bar{G} : \mathbb{R} \to \mathbb{R},\ t \mapsto \int \tau(x) h_{\epsilon,t}(x)\, dx = \epsilon A_{\tau,\epsilon} t + B_{\tau,\epsilon}$ for suitable constants $A_{\tau,\epsilon}, B_{\tau,\epsilon}$ that are bounded uniformly for ϵ ranging in a neighbourhood of 0. As $G_\epsilon(\nu) = \nu$ if and only if $\bar{G}_\epsilon(\bar{\nu}) = \bar{\nu}$, it follows that G_ϵ has a unique fixed point ν_ϵ which is also the unique fixed point of $\mathcal{T}_\epsilon$ (see Lemma 7).

The reader should be aware that this does not imply that tent maps do not violate the 0-1-law, since the $\mathcal{T}_\epsilon$-invariant measure δ_{ν_ϵ} need not be a weak accumulation point of unique SRB-measures μ_n for $T_\epsilon : \Omega_{I_n} \to \Omega_{I_n}$, see Definition 1. Indeed, there is both, numerical and theoretical evidence (not to be confused with a rigorous mathematical proof, however), that for many parameters tent maps violate the 0-1-law [5, 18, 4], while e.g. for tent maps with eventually fixed critical trajectory there are convincing arguments (close to a mathematical proof) that the 0-1-law is not broken [4]. Below we prove a result that covers also this latter case.

4.3 Globally coupled C^3 circle maps

In this section we prove that globally coupled expanding circle maps where both the maps and the coupling are of class C^3 do not violate the 0-1-law. For the case where maps and coupling are analytic this was proved in [7].

Theorem 4 *Let $\tau : \mathbb{T}^1 \to \mathbb{T}^1$ be C^3 and strictly expanding, and suppose that the coupling is of the type described in (12) and satisfies also $\sup_{(x,\bar{\nu})} \left|\frac{\partial}{\partial\bar{\nu}}\frac{\partial^2\gamma}{\partial x^2}\right| < \infty$. Then there is $\epsilon_0 > 0$ such that for all $\epsilon \in [0,\epsilon_0]$ holds: Let h_ϵ be a fixed point of G_ϵ (uniqueness is not assumed* a priori*), $\nu_\epsilon := h_\epsilon m$. Then*

$$\mathcal{T}_\epsilon \nu_\epsilon = \nu_\epsilon \quad \textit{and} \quad \lim_{i\to\infty} \mathcal{T}_\epsilon^i \nu = \nu_\epsilon \quad \textit{exponentially for all } \nu \in \mathcal{D}$$

(in the sense that there are $C > 0$ and $\gamma \in (0,1)$ independent of ν and such that for the density u of ν we have $\|\tilde{\mathcal{L}}_\epsilon^i u - h_\epsilon\|_1 \le C\gamma^i \|u\|_{BV}$).

We give two different proofs for this theorem. The first one makes use of the following perturbation property for invariant densities of C^3 circle maps, see [2, Theorem 1] or, for a slightly weaker assertion [17, Theorem 1]:

$$\text{There are } l_\epsilon > 0 \text{ with } \lim_{\epsilon\to 0} l_\epsilon = 0 \text{ such that } \|h_{\epsilon,\nu} - h_{\epsilon,\nu'}\|_{BV} \le l_\epsilon\, |\bar{\nu} - \bar{\nu}'| \text{ for all } \nu,\nu' \in \mathcal{P}. \quad (16)$$

The second one uses simply the C^2-smoothness of invariant densities of C^3 circle maps, see e.g. [17, Theorem 1]. For both proofs we need the following lemma:

Lemma 8 *Under the assumptions of Theorem 4 there are $\epsilon_1 > 0$, $C > 0$ and $\beta \in (0,1)$ such that for all $\epsilon \in [0,\epsilon_1]$ and for all $\nu_1, \dots, \nu_n \in \mathcal{P}$ holds*

$$\|\mathcal{S}_{\epsilon,\nu_n}\mathcal{L}\cdots\mathcal{S}_{\epsilon,\nu_1}\mathcal{L}(u)\|_{BV} \le C\,\beta^n \|u\|_{BV} \quad \textit{for all } u \in BV \textit{ with } \int u(x)\,dx = 0.$$

Proof: As the map $\tau : \mathbb{T}^1 \to \mathbb{T}^1$ is strictly expanding and of class C^2, it is mixing and satisfies the Lasota-Yorke type inequality (13) with $n = 1$, $\epsilon = 0$ and with a factor $\frac{1}{\kappa}$ instead of $\frac{2}{\kappa}$. Fix $\alpha \in (\frac{1}{\kappa}, 1)$. By continuity reasons there are $\epsilon_0 > 0$ and $C_3 > 0$ such that each $\tau_{\epsilon,\nu}$ with $\epsilon \in [0, \epsilon_0]$ and $\nu \in \mathcal{P}$ has a minimal expansion rate α and satisfies the estimate $|\tau''_{\epsilon,\nu}|/|\tau'_{\epsilon,\nu}|^2 \leq C_3$. Therefore $\tau_{\epsilon,\nu}$ is mixing and

$$\|\mathcal{S}_{\epsilon,\nu}\mathcal{L}u\|_{BV} \leq \alpha\|u\|_{BV} + (C_3 + 1)\,\|u\|_1 \tag{17}$$

for all $\epsilon \in [0, \epsilon_0]$, $\nu \in \mathcal{P}$ and $u \in BV$. Let $C_4 := \frac{C_3+1}{1-\alpha}$ and consider any sequence $\nu_1, \nu_2, \ldots \in \mathcal{P}$. Denote $\mathcal{Q}_{\epsilon,n} := \mathcal{S}_{\epsilon,\nu_n}\mathcal{L} \cdots \mathcal{S}_{\epsilon,\nu_1}\mathcal{L}$. Repeated application of (17) leads to

$$\|\mathcal{Q}_{\epsilon,n}u\|_{BV} \leq \alpha^n\|u\|_{BV} + C_4\,\|u\|_1$$

for all $\epsilon \in [0, \epsilon_0]$, $n \in \mathbb{N}$, $\nu \in \mathcal{P}$ and $u \in BV$. As $\|f\|_1 \leq \|f\|_{BV}$ by definition, it follows also that $\|\mathcal{Q}_{n,\epsilon}\|_{BV} \leq C_4 + 1$.

A well-known first consequence of estimate (17) is that the spectrum of $\mathcal{S}_{\epsilon,\nu_\epsilon}\mathcal{L}$ consists of 1 and a part strictly contained in the unit disc. It follows that there is $N \in \mathbb{N}$ such that

$$\|(\mathcal{S}_{\epsilon,\nu_\epsilon}\mathcal{L})^N(u)\|_1 \leq \frac{1}{8C_4}\|u\|_{BV} \quad \text{for all } u \in BV_0$$

where $BV_0 := \{u \in BV : \int u(x)\,dx = 0\}$. Choosing N even larger if necessary we can also make sure that

$$\alpha^N(C_4 + 1) < \frac{1}{4}\;.$$

Since $\|(\mathcal{S}_{\epsilon,\nu_j} - \mathcal{S}_{\epsilon,\nu_\epsilon})(u)\|_1 \leq \mathrm{const}\cdot\epsilon\cdot|\bar{\nu}_j - \bar{\nu}_\epsilon|\cdot\|u\|_{BV}$ for each $u \in BV$ (see [11, Lemma 13]), we have

$$\begin{aligned}
&\big|\|\mathcal{Q}_{\epsilon,N}(u)\|_1 - \|(\mathcal{S}_{\epsilon,\nu_\epsilon}\mathcal{L})^N(u)\|_1\big| \\
\leq\;& \sum_{j=1}^{N}\|\mathcal{S}_{\epsilon,\nu_N}\mathcal{L}\cdots\mathcal{S}_{\epsilon,\nu_{j+1}}\mathcal{L}\,(\mathcal{S}_{\epsilon,\nu_j} - \mathcal{S}_{\epsilon,\nu_\epsilon})\mathcal{L}\,(\mathcal{S}_{\epsilon,\nu_\epsilon}\mathcal{L})^{j-1}(u)\|_1 \\
\leq\;& \mathrm{const}_N\cdot\epsilon\cdot\|u\|_{BV}\cdot\|F\|_\infty
\end{aligned}$$

for all $\epsilon \in [0, \epsilon_0]$, $\nu_1, \ldots, \nu_N \in \mathcal{P}$ and $u \in BV_0$ with F from (12). Together with the previous estimates it follows that there is some $\epsilon_1 = \epsilon_1(N) \in (0, \epsilon_0]$ such that for all $\epsilon \in [0, \epsilon_1]$, $\nu_1, \ldots, \nu_N \in \mathcal{P}$ and $u \in BV_0$

$$\begin{aligned}
\|\mathcal{Q}_{\epsilon,2N}(u)\|_{BV} &\leq \alpha^N\|\mathcal{Q}_{\epsilon,N}(u)\|_{BV} + C_4\,\|\mathcal{Q}_{\epsilon,N}(u)\|_1 \\
&\leq \alpha^N(C_4 + 1)\|u\|_{BV} + C_4\left(\frac{1}{8C_4} + \mathrm{const}_N\,\epsilon\,\|F\|_\infty\right)\|u\|_{BV} \\
&\leq \frac{1}{2}\|u\|_{BV}
\end{aligned}$$

Now the lemma follows if one observes that $\mathcal{Q}_{\epsilon,n}(BV_0) \subseteq BV_0$ and $\|\mathcal{Q}_{\epsilon,n}\|_{BV} \leq C_4 + 1$ for all n. $\square$

Proof of Theorem 4: In most quantities occurring in this proof we skip the subscript ϵ. For $\nu \in \mathcal{D}$ let $\nu_i := \mathcal{T}^i(\nu)$. Denote by u_i the density of ν_i with respect to Lebesgue measure. Then $u_i = \tilde{\mathcal{L}}^i u_0$ and in particular $u_{i+1} = \tilde{\mathcal{L}}u_i = \mathcal{S}_{\epsilon,\bar{\nu}_i}\mathcal{L}u_i$.

First proof (using (16)): Observe first that G has only one fixed point so that also $\mathcal{T}$ has a unique fixed point: If $G(fm) = f$ and $G(\tilde{f}m) = \tilde{f}$, then $f = h_{\epsilon,fm}$, $\tilde{f} = h_{\epsilon,\tilde{f}m}$, and

$$\|f - \tilde{f}\|_1 = \|h_{fm} - h_{\tilde{f}m}\|_1 \leq l_\epsilon\cdot\|F\|_\infty\,\|f - \tilde{f}\|_1$$

by assumption (16), which proves the uniqueness for sufficiently small $\epsilon > 0$.

As $h_{\nu_i} = S_{\nu_i} \mathcal{L} h_{\nu_i}$,

$$\begin{aligned} u_{i+1} - h_{\nu_{i+1}} &= \mathcal{S}_{\nu_i}\mathcal{L}(u_i - h_{\nu_i}) + (h_{\nu_i} - h_{\nu_{i+1}}) \\ &\vdots \\ &= \mathcal{S}_{\nu_i}\mathcal{L}\ldots\mathcal{S}_{\nu_0}\mathcal{L}(u_0 - h_{\nu_0}) + \sum_{k=0}^{i} \mathcal{S}_{\nu_i}\mathcal{L}\ldots\mathcal{S}_{\nu_{k+1}}\mathcal{L}(h_{\nu_k} - h_{\nu_{k+1}})\ . \end{aligned}$$

Hence,

$$\begin{aligned} & u_{i+1} - u_i \\ = \ & \mathcal{S}_{\nu_i}\mathcal{L}(u_i - h_{\nu_i}) + (h_{\nu_i} - u_i) \\ = \ & (\mathcal{S}_{\nu_i}\mathcal{L} - \mathrm{Id}) \left(\mathcal{S}_{\nu_{i-1}}\mathcal{L}\ldots\mathcal{S}_{\nu_0}\mathcal{L}(u_0 - h_{\nu_0}) + \sum_{k=0}^{i-1} \mathcal{S}_{\nu_{i-1}}\mathcal{L}\ldots\mathcal{S}_{\nu_{k+1}}\mathcal{L}(h_{\nu_k} - h_{\nu_{k+1}}) \right)\ , \end{aligned}$$

and it follows from Lemma 8 that

$$\|u_{i+1} - u_i\|_{BV} \ \leq \ (C_4 + 2)\, C \left(\beta^i \|u_0 - h_{\nu_0}\|_{BV} + \sum_{k=0}^{i-1} \beta^{i-k-1} \|h_{\nu_k} - h_{\nu_{k+1}}\|_{BV} \right)\ .$$

As

$$\begin{aligned} \|h_{\nu_k} - h_{\nu_{k+1}}\|_{BV} &\leq l_\epsilon \cdot |\bar{\nu}_k - \bar{\nu}_{k+1}| = l_\epsilon \cdot |\nu_k(F \circ \tau) - \nu_{k+1}(F \circ \tau)| \\ &\leq l_\epsilon \cdot \|F\|_\infty \, \|u_k - u_{k+1}\|_1\ , \end{aligned}$$

we can conclude that

$$\|u_{i+1} - u_i\|_{BV} \ \leq \ (C_4 + 2)\, C \left(\beta^i \|u_0 - h_{\nu_0}\|_{BV} + \sum_{k=0}^{i-1} \beta^{i-k-1}\, l_\epsilon \cdot \|F\|_\infty \, \|u_{k+1} - u_k\|_1 \right)\ .$$

Now a simple induction argument shows that for $\gamma \in (\beta, 1)$ and sufficiently small $\epsilon > 0$ there is $\Gamma > 0$ such that $\|u_{i+1} - u_i\|_{BV} \leq \Gamma\, \gamma^i$ for all $i \in \mathbb{N}$. It follows that the sequence (u_i) converges to some $\tilde{\mathcal{L}}_\epsilon$-invariant density u_∞, and as $\mathcal{T}_\epsilon$ has only one fixed point, this density must coincide with $h_{\epsilon,\nu_\epsilon}$.

Second proof (using that the invariant density is twice differentiable): Denote $h_* := h_{\epsilon,\nu_\epsilon}$, $\nu_* := \nu_\epsilon$. Then

$$\begin{aligned} u_{i+1} - h_* &= \mathcal{S}_{\nu_i}\mathcal{L}(u_i - h_*) + (\mathcal{S}_{\nu_i} - \mathcal{S}_{\nu_*})\mathcal{L}h_* \\ &\vdots \\ &= \mathcal{S}_{\nu_i}\mathcal{L}\ldots\mathcal{S}_{\nu_0}\mathcal{L}(u_0 - h_*) + \sum_{k=0}^{i} \mathcal{S}_{\nu_i}\mathcal{L}\ldots\mathcal{S}_{\nu_{k+1}}\mathcal{L}(\mathcal{S}_{\nu_k} - \mathcal{S}_{\nu_*})\mathcal{L}h_* \end{aligned} \tag{18}$$

since $h_* = \mathcal{S}_{\nu_*}\mathcal{L}h_*$. It follows from Lemma 8 that

$$\|u_{i+1} - h_*\|_{BV} \ \leq \ C\, \beta^{i+1} \|u_0 - h_*\|_{BV} + C \sum_{k=0}^{i} \beta^{i-k} \|(\mathcal{S}_{\nu_k} - \mathcal{S}_{\nu_*})\mathcal{L}h_*\|_{BV}\ .$$

A straightforward calculation based on (12) and on the assumption $\sup_{(x,\bar{\nu})} \left| \frac{\partial}{\partial \bar{\nu}} \frac{\partial^2 \gamma}{\partial x^2} \right| < \infty$ shows that there is a constant $K > 0$ depending only on the coupling and on $\mathcal{L}$ such that

$$\sup_{x \in X} \left| ((\mathcal{S}_{\nu_k} - \mathcal{S}_{\nu_*})\mathcal{L}h_*)'\,(x) \right| \leq K\, \epsilon\, (\sup |h_*| + \sup |h_*'| + \sup |h_*''|)\, |\bar{\nu}_{k+1} - \bar{\nu}_*|\ .$$

In a similar way as in part 1 it follows that for $\gamma \in (\beta, 1)$ and sufficiently small $\epsilon > 0$ there is $C > 0$ such that $\|u_{i+1} - h_*\|_{BV} \leq C\,\gamma^i$ for all $i \in \mathbb{N}$. This proves at the same time the uniqueness of the invariant density $h_* = h_{\epsilon,\nu_\epsilon}$ of $\tilde{\mathcal{L}}_\epsilon$. □

4.4 Globally coupled piecewise linear maps

In this section we consider piecewise linear, expanding $\tau : [0,1] \to [0,1]$ and very simple couplings described by the function $\gamma(x, \bar{\nu}) = \bar{\nu}$, see (12).

There are two references which study such systems by mathematical methods. In [5] the authors argue that if $|\tau'| > 2$, then each fixed point of $\tilde{\mathcal{L}}_\epsilon$ is unstable so that one might expect a violation of the 0-1-law to happen. In [4] the authors mostly concentrate on post-critically finite tent maps and prove among others that the fixed point of $\tilde{\mathcal{L}}_\epsilon$ is stable and hence no violation of the 0-1-law happens if some iterate of the critical point is a fixed point of τ. Both papers are mathematically not really rigorous, because they rely on linear response theory.

Below we prove a result along the lines of reasoning of [4] which confirms the observation of [4] and shows at the same time that the assumption $|\tau'| > 2$ made in [5] is not sufficient to guarantee instability of the fixed point(s) of $\tilde{\mathcal{L}}_\epsilon$. Nevertheless I believe that "generically" these fixed points are unstable.

Theorem 5 *Let $\tau : [0,1] \to [0,1]$ be piecewise linear and expanding and suppose that the coupling is of the type described in (12) with a function $\gamma(x, \bar{\nu}) = \gamma(\bar{\nu})$. Assume that the conclusions of Lemma 8 are satisfied. Then there is $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0]$ holds:*

Let h_ϵ be a fixed point of G_ϵ (uniqueness is not assumed a priori*), $\nu_\epsilon := h_\epsilon m$. Assume that $\tau_{\epsilon,\nu_\epsilon}$ has a fixed point z, different from all its turning points, and that there exists an integer N such that $\tau^N_{\epsilon,\nu_\epsilon}(c) = z$ for all turning points c of $\tau_{\epsilon,\nu_\epsilon}$. Then*

$$\mathcal{T}_\epsilon \nu_\epsilon = \nu_\epsilon \quad \textit{and} \quad \lim_{i\to\infty} \mathcal{T}^i_\epsilon \nu = \nu_\epsilon \quad \textit{for all } \nu \in \mathcal{D}$$

(in the sense that there are $C > 0$ and $\gamma \in (0,1)$ independent of ν and such that for the density u of ν we have $\|\tilde{\mathcal{L}}^i_\epsilon u - h_\epsilon\|_1 \leq C\gamma^{\sqrt{i}}\,\|u\|_{BV}$).

Remark 5 For a broad class of piecewise linear maps it was proved in [14] that the conclusions of Lemma 8 are satisfied. This is in particular true for tent maps τ with slope $|\tau'| > 1.65$ and for each τ which is an iterate of such a tent map.

In the pure tent map case all maps $\tau_{\epsilon,\nu}$ are conjugated to the basic map τ, see Example 3. Therefore the extra assumption on the turning point of $\tau_{\epsilon,\nu_\epsilon}$ is satisfied if and only if it is satisfied for τ itself.

Proof of Theorem 5: As in the proof of Theorem 4 consider $\nu \in \mathcal{D}$ and let $\nu_i = \mathcal{T}^i\nu$ with density u_i so that $u_{i+1} = \tilde{\mathcal{L}}u_i = \mathcal{S}_{\nu_i}\mathcal{L}u_i$. (Again we mostly skip the subscript ϵ.) Fix $\nu_* \in \mathcal{D}$ with density h_* such that $\mathcal{T}\nu_* = \nu_*$, i.e. $\tilde{\mathcal{L}}h_* = h_*$. Recall that $\bar{\nu}_i = \int F(\tau x)u_i(x)\,dx$ and let

$$\delta_i := |\bar{\nu}_i - \bar{\nu}_*| \; .$$

Denote $s := |\tau'_{\epsilon,\nu_*}(z)| > 1$ where z is the fixed point of τ_{ϵ,ν_*} with $\tau^N_{\epsilon,\nu_*}(C) = \{z\}$, C being the set of turning points of τ_{ϵ,ν_*}. We are going to prove that there is $Q > 0$ such that $\delta_i \leq Q \cdot s^{-\sqrt{i}} \cdot \|u_0\|_{BV}$ for all $i \in \mathbb{N}_0$. The proof proceeds by induction, and Q will be chosen in the course of the proof. Note that all restrictions on Q will be satisfied if Q is sufficiently large and that Q depends neither on ϵ nor on the initial density u_0. Other constants that neither depend on ϵ and u_0 nor on Q are denoted by "const".

Recall from (18) that

$$u_{i+1} - h_* = \mathcal{S}_{\nu_i}\mathcal{L}\ldots\mathcal{S}_{\nu_0}\mathcal{L}(u_0 - h_*) + \sum_{k=0}^{i} \mathcal{S}_{\nu_i}\mathcal{L}\ldots\mathcal{S}_{\nu_{k+1}}\mathcal{L}(\mathcal{S}_{\nu_k}\mathcal{S}^{-1}_{\nu_*} - \mathrm{Id})h_*$$

where we used that $(\mathcal{S}_{\nu_k} - \mathcal{S}_{\nu_*})\mathcal{L}h_* = (\mathcal{S}_{\nu_k}\mathcal{S}_{\nu_*}^{-1} - \mathrm{Id})h_*$. Denote

$$H_{k,i} := \mathcal{S}_{\nu_i}\mathcal{L}\dots\mathcal{S}_{\nu_{k+1}}\mathcal{L}(\mathcal{S}_{\nu_k}\mathcal{S}_{\nu_*}^{-1} - \mathrm{Id})h_*$$

and let $p = p(i) := [\sqrt{i}]$. Then

$$\begin{aligned}\delta_{i+1} &= \left|\int (u_{i+1} - h_*)(x)F(\tau x)\,dx\right| \qquad (19)\\ &\leq \|F\|_\infty \cdot \left(\|\mathcal{S}_{\nu_i}\mathcal{L}\dots\mathcal{S}_{\nu_0}\mathcal{L}(u_0 - h_*)\|_1 + \sum_{k=0}^{i-p}\|H_{k,i}\|_1\right) + \sum_{k=i-p+1}^{i}\left|\int H_{k,i}(x)F(\tau x)\,dx\right|\end{aligned}$$

Since the conclusions of Lemma 8 are assumed to hold true and since $\|\mathcal{S}_{\nu_k}\mathcal{S}_{\nu_*}^{-1}h_*\|_{BV} = \|h_*\|_{BV}$, the first term of the r.h.s. of this inequality can be estimated by

$$\mathrm{const}\cdot\left(\beta^{i+1}\|u_0 - h_*\|_{BV} + \sum_{k=0}^{i-[Ap]}\beta^{i-k}2\|h_*\|_{BV} + \sum_{k=i-[Ap]+1}^{i-p}\beta^{i-p-k}\|H_{k,k+p}\|_{BV}\right) \qquad (20)$$

where $A := \frac{\log s}{-\log\beta}$.

In order to further estimate this term and also the second one on the r.h.s. of (19), we have a closer look at $(\mathcal{S}_{\nu_k}\mathcal{S}_{\nu_*}^{-1} - \mathrm{Id})h_*$. Observe first that h_*, as an invariant density of a post-critically finite piecewise linear map, is piecewise constant with all discontinuities in the finite set C. Since $S_\nu(x) = x + \epsilon\gamma(\bar\nu)$, we have

$$\left|S_{\nu_k}S_{\nu_*}^{-1}(x) - x\right| = |\epsilon\left(\gamma(\bar\nu_k) - \gamma(\bar\nu_*)\right)| \leq \epsilon\cdot\|\gamma'\|_\infty\cdot\delta_k\ .$$

The function $(\mathcal{S}_{\nu_k}\mathcal{S}_{\nu_*}^{-1} - \mathrm{Id})h_*$ is therefore supported in a neighbourhood of the set C with radius $\epsilon\cdot\|\gamma'\|_\infty\cdot\delta_k$. As $\tau_{\epsilon,\nu_*}^N(C) = \{z\}$, the function $H_{k,k+N}$ is thus supported in a neighbourhood of z with radius $\epsilon\cdot\mathrm{const}\cdot\sum_{j=0}^N\delta_{k+j}$. Since, by assumption, z is not a turning point of τ_{ϵ,ν_*}, there is $\rho > 0$ such that $\tau'_{\epsilon,\nu_*}(x) = \tau'_{\epsilon,\nu_*}(z) = \pm s$ for all $x \in (z-\rho, z+\rho)$. It follows that for $\ell = k+N,\dots,i$ the function $H_{k,\ell}$ is supported in a neighbourhood of z with radius

$$\begin{aligned}r_{k,\ell} &:= \epsilon\cdot\mathrm{const}\cdot\left(\dots\left(\left(\sum_{j=0}^N\delta_{k+j}\cdot s + \delta_{k+N+1}\right)\cdot s + \delta_{k+N+2}\right)\cdot s + \dots\right)\cdot s + \delta_\ell\Bigg)\\ &= \epsilon\cdot\mathrm{const}\cdot\left(s^{\ell-k-N}\sum_{j=0}^N\delta_{k+j} + \sum_{j=1}^{\ell-k-N}s^{\ell-k-N-j}\delta_{k+N+j}\right)\ .\end{aligned}$$

Now we proceed to the inductive estimate. Choose $Q > 0$ such that $\delta_j \leq Q\cdot s^{-\sqrt{j}}$ for $j = 0,\dots,N^2$. Let $i \geq N^2$ and suppose by induction that $\delta_j \leq Q\cdot s^{-\sqrt{j}}\cdot\|u_0\|_{BV}$ for $j = 0,\dots,i$. (Since $\|u_0\|_{BV} \geq \|u_0\|_1 = 1$, this assumption is satisfied for $i = N^2$.) Then

$$\begin{aligned}r_{k,\ell} &\leq \epsilon\cdot\mathrm{const}\cdot Q\cdot\|u_0\|_{BV}\left(s^{\ell-k-N}s^{-\sqrt{k}} + s^{-\sqrt{k+N+1}}\cdot\frac{s^{\ell-k-N}-1}{s-1}\right)\\ &\leq \epsilon Q\cdot\|u_0\|_{BV}\cdot\mathrm{const}\cdot s^{\ell-k-\sqrt{k}}\end{aligned} \qquad (21)$$

for $\ell = k+N,\dots,i$.

If $\max\{i-[Ap], \ell-p\} \leq k \leq \ell \leq i$, this yields in view of $p = [\sqrt{i}]$:

$$r_{k,\ell} \leq \epsilon Q\cdot\|u_0\|_{BV}\cdot\mathrm{const}\cdot s^{p-\sqrt{i-[Ap]}} \leq \epsilon Q\cdot\|u_0\|_{BV}\cdot\mathrm{const}\ .$$

Therefore, if $\epsilon < \frac{\rho}{Q\cdot\mathrm{const}}$, then $r_{k,\ell} < \rho$ for these k and ℓ so that the function $H_{k,\ell}$ is supported in the interval $(z-\rho, z+\rho)$.

Since all S_ν and also τ are affine on $(z-\rho, z+\rho)$, it follows at once that

$$\operatorname{var}(H_{k,\ell}) = s^{-(\ell-k-N)} \cdot \operatorname{var}(H_{k,k+N})$$

for $\max\{i-[Ap], \ell-p\} \le k \le \min\{\ell, i-N\}$, and as $\int H_{k,\ell}(x)\,dx = 0$, it follows that

$$\|H_{k,\ell}\|_{BV} \le \text{const} \cdot s^{-(\ell-k)} \cdot \|H_{k,k+N}\|_{BV} \le \text{const} \cdot s^{-(\ell-k)}$$

where we used the Lasota-Yorke type inequality (14) in the last step and included $\|h_*\|_{BV}$ in the constant. We apply this estimate to $k = i-[Ap], \dots, i-p$ and $\ell = k+p$ (observe that $p = [\sqrt{i}] \ge N$):

$$\|H_{k,k+p}\|_{BV} \le \text{const} \cdot s^{-p} = \text{const} \cdot \beta^{[Ap]} \ .$$

Combined with (19) and (20) this yields

$$\delta_{i+1} \le \text{const} \cdot \left(\beta^{i+1}\|u_0 - h_*\|_{BV} + \beta^{[Ap]} + \beta^{[Ap]}\right) + \sum_{k=i-p+1}^{i} \left| \int H_{k,i}(x) F(\tau x)\,dx \right| \tag{22}$$

Observe next that $\int H_{k,i}(x)\,dx = 0$ for all k. Therefore we can apply (21) with $k = i-p, \dots, i$ and $\ell = i$ and obtain

$$\begin{aligned} \left| \int H_{k,i}(x) F(\tau x)\,dx \right| &\le \operatorname{Lip}(F \circ \tau) \cdot r_{k,i} \cdot \|H_{k,i}\|_1 \\ &\le \epsilon Q \cdot \text{const} \cdot s^{i-k-\sqrt{k}} \cdot \left\| (\mathcal{S}_{\nu_k} \mathcal{S}_{\nu_*}^{-1} - \text{Id}) h_* \right\|_1 \ . \end{aligned}$$

As $|S_{\nu_k} S_{\nu_*}^{-1}(x) - x| \le \epsilon \cdot \|\gamma'\|_\infty \cdot \delta_k$, it follows from [11, Lemma 13] that

$$\left\| (\mathcal{S}_{\nu_k} \mathcal{S}_{\nu_*}^{-1} - \text{Id}) h_* \right\|_1 \le \epsilon \cdot \text{const} \cdot \delta_k \tag{23}$$

so that

$$\left| \int H_{k,i}(x) F(\tau x)\,dx \right| \le \epsilon^2 Q^2 \cdot \text{const} \cdot s^{i-k-2\sqrt{k}}$$

for $k = i-p, \dots, i$. Inserting this estimate into (22) and observing that $1 = \|u_0\|_1 \le \|u_0\|_{BV}$ we arrive at

$$\delta_{i+1} \le \text{const} \cdot \|u_0\|_{BV} \cdot \left(\beta^{i+1} + \beta^{[Ap]} + \epsilon^2 Q^2 \sum_{k=i-p+1}^{i} s^{i-k-2\sqrt{k}} \right)$$

We estimate the sum in this expression: As $\sqrt{i} - \sqrt{i-j} \le 1$ for $j = 0, \dots, p = [\sqrt{i}]$, we have

$$s^{\sqrt{i}} \cdot \sum_{k=i-p+1}^{i} s^{i-k-2\sqrt{k}} = \sum_{j=0}^{p-1} s^{j-\sqrt{i}+2(\sqrt{i}-\sqrt{i-j})} \le \sum_{j=0}^{p-1} s^{j-p+2} \le \frac{s^2}{s-1} \ .$$

Therefore there is a constant $K > 0$ independent of Q, ϵ and u_0 such that

$$\frac{\delta_{i+1}}{Q \cdot s^{-\sqrt{i}} \cdot \|u_0\|_{BV}} \le K \cdot \left(\frac{1}{Q} \beta^{i+1} s^{\sqrt{i+1}} + \frac{1}{Q} + \epsilon^2 Q \right) \ .$$

Choosing Q large enough we may assume that $Q \ge 2K \cdot (1 + \max_{i \ge 0} \beta^{i+1} s^{\sqrt{i+1}})$. Then, if $\epsilon < \sqrt{\frac{1}{2KQ}}$ (and $\epsilon < \frac{\rho}{\text{const} \cdot Q}$ from above), we have

$$\delta_{i+1} \le Q \cdot s^{-\sqrt{i+1}} \cdot \|u_0\|_{BV}$$

which finishes the induction.

Finally we use these bounds to estimate $\|u_{i+1} - h_*\|_1$: With a decomposition as in (19) and using (23) and the conclusions of Lemma 8 we have

$$\begin{aligned}
\|u_{i+1} - h_*\|_1 &\leq \text{const} \cdot \left(\beta^{i+1} + \sum_{k=0}^{i-[Ap]} \beta^{i-k} \right) \|u_0\|_{BV} + \sum_{k=1-[Ap]+1}^{i} \|H_{k,i}\|_1 \\
&\leq \text{const} \cdot \left(\beta^{[Ap]} \|u_0\|_{BV} + \sum_{k=i-[Ap]+1}^{i} \delta_k \right) \leq \text{const} \cdot \sqrt{i} \cdot s^{-\sqrt{i}} \|u_0\|_{BV}
\end{aligned}$$

which gives the assertion of the theorem. □

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