Complete asymptotics for correlations of Laplace integrals in the semi-classical limit.

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Abstract. In this paper we study the exponential asymptotics of correlations at large distance associated to a measure of Laplace type. As in [S1], [BJS], we look at a semi-classical limit. While in those papers we got the exponential decay rates and the prefactor only up to some factor $(1+\mathcal{O}(h^{1/2}))$, where h denotes the small semi-classical parameter, we now get full asymptotic expansions. The main strategy is the same as in the quoted papers, namely to use an identity ([HS]) involving the Witten Laplacian of degree 1, and a Grushin (Feshbach) reduction for the bottom of the spectrum of this operator. The essential difference is however that we now have to use higher order Grushin problems (amounting to the study of a larger part of the bottom of th spectrum). In a perturbative case, the strategy of higher order Grushin problems was recently implemented by W.M.Wang [W] to get a few terms in the perturbative expansion of the decay rate.

Resumé. Dans cet article nous étudions l'asymptotique exponentielle des corrélations à grande distance associées à une mesure de type de Laplace. Comme dans [S1], [BJS], on se place dans la limite semi-classique. Dans ces travaux nous avons obtenu le taux de décroissance exponentielle et le préfacteur à un facteur $(1+\mathcal{O}(h^{1/2}))$ près, où h désigne le petit paramètre semi-classique. Nous obtenons maintenant des développements asymptotiques complèts. La stratégie est la même que dans [S1,BJS]: Utiliser une identité ([HS]), qui comporte le laplacien de Witten en degré 1, ainsi qu'une réduction de Grushin (Feshbach) pour le bas du spectre de cet opérateur. La nouveauté est que nous devons maintenant utiliser des problèmes de Grushin d'ordre supérieur, ce qui revient à étudier une plus large fraction du spectre. Dans un cas perturbatif W.M.Wang [W] a récemment utilisé des problèmes de Grushin d'ordre supérieur pour obtenir quelques termes dans le développement du taux de décroissance.

0. Introduction.

In recent years there has been an attempt by B. Helffer, the author and others ([BJS], [H1–4], [HS], [J], [S1–6], [SW], [W]) to apply direct methods to the study of integrals and operators in high dimension, of the type that may appear naturally in statistical mecanics and Euclidean field theory. In the first works, we applied asymptotic methods and noticed already that there is a very strong interplay between asymptotic expansions for integrals obtained by some variant of the stationary phase method and asymptotic solutions of certain Schrödinger type operators obtained by the WKB method ([S3–5]). In later works ([S6], [HS], [S4]) we noticed that a suitable version of the maximum principle could be used in the proof of certain asymptotic expansions and to obtain exponential decay of correlations. (In the the work [SW] this was even applied to integrals in the complex domain, and was applied to show exponential decay of the expectation of the Green function for discrete Schrödinger operators with random potentials.)

In the present work we are interested in correlations for Laplace integrals at large distance. In physics language we are interested in the correlations at large distance for

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continuous spin systems. Under assumptions that imply the exponential decay of these correlations, we want to know the precise rate of exponential decay and to determine the possible polynomial prefactor. The original inspiration came from a talk given by R.Minlos in St Peterburg in 1993 and a corresponding joint paper by him and E.Zhizhina [MZ], about the asymptotics of correlations for discrete spin models at high temperature. Even though we never quite understood the methods used in [MZ], it prompted us to further develop our own methods in the continous spin case and in [S1] we were able to get the leading exponential decay asymptotics for correlations associated to measures of the type $\frac{e^{-\phi(x)/h}dx}{\int e^{-\phi(x)/h}dx,}$ at large distance, in the semi-classical limit ($h\to 0$). Here $\phi\in C^\infty(\mathbf{R}^\Lambda;\mathbf{R})$, and Λ is a finite subset of $\mathbf{Z}^d$ or a discrete torus of dimension d, and we study the limit when Λ is large. Recall that the correlation of two functions u, v which do not grow too fast at infinity is given by

$$\mathrm{Cor}\,(u,v)=\langle (u-\langle u\rangle)(v-\langle v\rangle)\rangle, \tag{0.1}$$

where

$$\langle u\rangle=\frac{\int u(x)e^{-\phi(x)/h}dx}{\int e^{-\phi(x)/h}dx}$$

denotes the expectation. We also observed that certain associated Schrödinger operators, already found with B.Helffer in [HS], are Witten Laplacians in degree 0 or 1. We also managed to replace the use of the maximum principle at many places by L^2 methods and consequently we got rid of a certain rigidity in the conditions. The elimination of the maximum principle was not complete however, and the results were flawed by a certain number of unnatural assumptions, in particular that of global uniform strict convexity of the function ϕ. Another short-coming of [S1] was that we only determined the decay rate and the prefactor up to a factor $(1+\mathcal{O}(h^{1/2}))$. Moreover, we did not work out the thermodynamical limit ($\Lambda\to\mathbf{Z}^d$) so oscillations within a factor $1+\mathcal{O}(h^{1/2})$ could not be excluded, when Λ varies.

With V.Bach and T.Jecko [BJS] we elimintated completely the use of the maximum principle and were able to give simpler and more natural conditions. In particular we could allow the exponent ϕ to be strictly convex only near the point where ϕ is minimal. The new assumptions still imply that there is only one critical point however. Again we obtained the decay rate and the prefactor only up to a factor $(1+\mathcal{O}(h^{1/2}))$, and we did not treat the thermodynamical limit.

The aim of the present paper is to get full asymptotic expansions in powers of h of the decay rate and in powers of the inverse distance and in h of the prefactor, and we shall also treat the thermodynamical limit. To get such more precise results, one has to get higher in the spectrum of the associated Witten Laplacians (or rather something close to that), and we do so by using higher order Grushin problems that we explain more in detail later in this introduction. The main idea of this strategy was rather clear in the author's mind since the writing of [S1], and has become practically realizable with the improvements of [BJS]. W.M.Wang [W] has recently used similar ideas in order to study the rate of exponential decay of correlations when $h=1$ and ϕ is a small perturbation of a non-degenerate quadratic form. For the decay rate, she got several terms in an expansion in powers of the perturbation parameter. In principle the method should give full asymptotic expansions and also the prefactor in that case too. [W] also has an interesting application to the exponential decay rate of the Green function for discrete random Schrödinger operators.

We now start to formulate the main result of this paper, and after that we will outline some ideas of the proof. To understand the idea behind many of the assumptions, it may be helpful to have in mind the special case when $\phi = \phi_\Lambda$ is of the form

$$\phi(x) = \sum_{j\in\Lambda} f(x(j)) + \sum_{|j-k|_1=1} w(x(j), x(k)), \tag{0.2}$$

when

$$\Lambda = (\mathbf{Z}/L\mathbf{Z})^d, \tag{0.3}$$

is a discrete torus with $L \geq 2$ and ϕ is of the form

$$\phi(x) = \sum_{j\in\Lambda} f(x(j)) + \sum_{\substack{|j-k|_1=1 \\ j \text{ or } k\in\Lambda}} w(\widetilde{x}(j), \widetilde{x}(k)), \tag{0.4}$$

when Λ is a finite subset of $\mathbf{Z}^d$. Here $|\cdot|_p$ will always denote the ℓ^p norm, and when x is the discrete torus Λ in (0.3), we let $|j|_p$ denote $\inf_{\widetilde{j}\in\pi_\Lambda^{-1}(j)} |\widetilde{j}|_p$, where $\pi_\Lambda : \mathbf{Z}^d \to \Lambda$ is the natural projection. In the formula (0.4) we let $\widetilde{x}(j)$ be equal to $x(j)$ for $j \in \Lambda$ and $= 0$ otherwise. f and w are smooth real valued functions on $\mathbf{R}$ and $\mathbf{R}^2$ respectively, with

$$w(x, y) = w(y, x). \tag{0.5}$$

In the proofs the special forms (0.2), (0.4) will never be of any real advantage, so we shall formulate our main result with more general functions. Notice that if we let Λ grow, then the expressions (0.2), (0.4) will in general diverge, while the derivatives of all orders will converge. For that reason (and with the termodynamical limit in mind) we will start by fixing the limiting Hessian, and after that we will see how to get a family of functions $\phi(x) = \phi_\Lambda(x)$ both in the torus case and in the subset case.

We say that a function f on $\mathbf{R}^{\mathbf{Z}^d}$ is smooth if it is continuous for the ℓ^∞ topology, differentiable in each of the variables with continuous derivatives (for the ℓ^∞ topology) and the derivatives enjoy the same properties et c. Let $\Phi_{j,k}(x)$, $j, k \in \mathbf{Z}^d$ be smooth and real on $\mathbf{R}^{\mathbf{Z}^d}$ and satisfy

$$\Phi_{j,k}(x) = \Phi_{k,j}(x), \tag{A.1}$$

$$\partial_{x_\ell}\Phi_{j,k} = \partial_{x_j}\Phi_{\ell,k}, \tag{A.2}$$

$$\Phi = (\Phi_{j,k}) \text{ is 2 standard}, \tag{A.3}$$

$$\Phi(0) \geq \text{Const.} > 0. \tag{A.4}$$

Here we use the terminology of [S2] concerning k standard tensors. Let $a = (a_{\Lambda;j,k}(x))$, $x \in \mathbf{R}^\Lambda$, $j, k \in \Lambda$ be a family of matrices (i.e. 2 tensors) depending on some family of finite sets Λ. We say that $a = a_\Lambda$ is 2 standard if we have uniformly in $x \in \mathbf{R}^\Lambda$, Λ, the estimates

$$\langle \nabla^k a(x), t_1 \otimes \ldots \otimes t_{k+2}\rangle = \mathcal{O}_k(1)|t_1|_{p_1}..|t_k|_{p_{k+2}}, \tag{0.6}$$

for all $t_j \in \mathbf{C}^\Lambda$ and $p_j \in [1, +\infty]$ with $1 = \frac{1}{p_1} + .. + \frac{1}{p_{k+2}}$. Here $|\cdot|_p$ denotes the standard ℓ^p norm on $\mathbf{C}^\Lambda$. When x varies in $\mathbf{R}^{\mathbf{Z}^d}$ and $j, k \in \mathbf{Z}^d$, we require the $a_{j,k}(x)$ to be smooth in the sense mentioned earlier and say that a is 2-standard if (0.6) holds with $\Lambda = \mathbf{Z}^d$ and $t_j \in \mathbf{C}^\Lambda$, with $t_j(\lambda) \to 0$, $\Lambda \ni \lambda \to \infty$. Notice that a 2-standard matrix is $\mathcal{O}(1) : \ell^p \to \ell^p$, for $1 \le p \le \infty$.

If we think of the formal expression (0.2) with $\Lambda = \mathbf{Z}^d$ and assume that $\nabla^k f$ and $\nabla^k w$ are bounded for all $k \ge 2$, then the matrix $\Phi(x) = (\phi''_{j,k}(x))$ fulfills the assumptions (A.1–3). $\Phi(0)$ is the matrix of a convolution and (A.4) amounts to the assumption:

$$f''(0) + 4d\partial_x^2 w(0,0) > 4d|\partial_x \partial_y w(0,0)|. \tag{0.7}$$

If $\mathbf{Z}^d$ is replaced by a finite set Λ, then (A.1,2) becomes a necessary and sufficient condition for the existence of a realvalued function $\phi \in C^\infty(\mathbf{R}^\Lambda)$ with $\phi''_{j,k} = \Phi_{j,k}$. In the $\mathbf{Z}^d$ case we shall now see how to produce two different finite dimensional versions of such a function.

Let $U \subset \mathbf{Z}^d$ be finite. If $x \in \mathbf{R}^U$, let $\widetilde{x} \in \mathbf{R}^{\mathbf{Z}^d}$ be the zero extension of x, so that $\widetilde{x}(j) = x(j)$ for $j \in U$, $\widetilde{x}(j) = 0$, for $j \in \mathbf{Z}^d \setminus U$. Then

$$\Phi_{U;j,k}(x) := \Phi_{j,k}(\widetilde{x}),\ \ j, k \in U$$

is a smooth tensor on $\mathbf{R}^U$ which satisfies (A.1,2) with $j, k, \ell \in U$. Hence there exists a function $\phi_U(x) \in C^\infty(\mathbf{R}^U; \mathbf{R})$ with

$$\phi''_{U;j,k}(x) = \Phi_{U;j,k}(x),\ \ x \in \mathbf{R}^U,\ j, k \in U. \tag{0.8}$$

We make ϕ_U unique up to a constant, by requiring that

$$\phi'_U(0) = 0. \tag{0.9}$$

It is easy to check that ϕ''_U is 2-standard.

We next do the same with U replaced by a discrete torus $\Lambda = (\mathbf{Z}/L\mathbf{Z})^d$. If $\lambda \in \mathbf{Z}^d$, we define $\tau_\lambda x \in \mathbf{R}^{\mathbf{Z}^d}$, by $(\tau_\lambda x)(\nu) = x(\nu - \lambda)$. We will assume translation invariance for Φ:

$$\Phi_{j+\lambda,k+\lambda}(\tau_\lambda x) = \Phi_{j,k}(x),\ \ j, k, \lambda \in \mathbf{Z}^d. \tag{A.7}$$

(In section 10 we discuss a larger set of conditions and reproduce here only the most important ones with the same numbering as in section 10.) Notice that if $\Phi_{j,k}$ were the Hessian of a smooth function $\phi \in C^\infty(\mathbf{R}^{\mathbf{Z}^d})$ (and the discussion remains valid if we replace $\mathbf{Z}^d$ by a discrete torus Λ) then (A.7) would be a consequence of the simpler translation invariance property:

$$\phi(\tau_\lambda x) = \phi(x). \tag{0.10}$$

If $x \in \mathbf{R}^\Lambda$, let $\widetilde{x} = x \circ \pi_\Lambda \in \mathbf{R}^{\mathbf{Z}^d}$ be the corresponding $L\mathbf{Z}^d$ periodic lift, where $\pi_\Lambda : \mathbf{Z}^d \to \Lambda$ is the natural projection. Replacing x by $\widetilde{x}$ in (A.7), we get

$$\Phi_{j-\lambda,k-\lambda}(\widetilde{x}) = \Phi_{j,k}(\widetilde{x}),\ \ \lambda \in L\mathbf{Z}^d. \tag{0.11}$$

If we view Φ as a matrix, this is equivalent to

$$\tau_\lambda \circ \Phi(\widetilde{x}) = \Phi(\widetilde{x}) \circ \tau_\lambda,\ \ \lambda \in L\mathbf{Z}^d, \tag{0.12}$$

so $\Phi(\widetilde{x})$ maps $L\mathbf{Z}^d$ periodic vectors into the same kind of vectors. Hence there is a naturally defined $\Lambda \times \Lambda$ matrix $\Phi_\Lambda(x)$, defined by

$$\widetilde{\Phi_\Lambda(x)t} = \Phi(\widetilde{x})\widetilde{t}, \tag{0.13}$$

where again the tilde indicates that we take the periodic lift. On the matrix level, we get

$$\Phi_{\Lambda;j,k}(x) = \sum_{\widetilde{k}\in\pi_\Lambda^{-1}(k)} \Phi_{\widetilde{j},\widetilde{k}}(\widetilde{x}), \tag{0.14}$$

for any $\widetilde{j} \in \pi_\Lambda^{-1}(j)$. Alternatively, we have

$$\Phi_{\Lambda;j,k}(x) = \sum_{\widetilde{j}\in\pi_\Lambda^{-1}(j)} \Phi_{\widetilde{j},\widetilde{k}}(\widetilde{x}),\ \ \widetilde{k} \in \pi_\Lambda^{-1}(k), \tag{0.15}$$

and $\Phi_{\Lambda;j,k}$ is symmetric (cf. (A.1)).

In section 10, we shall verify that Φ_Λ satisfies (A.1–4), so there exists $\phi_\Lambda \in C^\infty(\mathbf{R}^\Lambda;\mathbf{R})$, unique up to a constant, such that

$$\Phi_{\Lambda;j,k}(x) = \partial_{x_j}\partial_{x_k}\phi_\Lambda(x),\ \ \phi_\Lambda'(0) = 0. \tag{0.16}$$

In the case when Φ is the Hessian of the formal expression (0.2) with Λ there replaced by $\mathbf{Z}^d$, and if we assume that 0 and (0,0) are critical points of f and w respectively, then it is easy to see that ϕ_Λ is given by the expression (0.2) when Λ is a discrete torus with $L \geq 3$ and by (0.4) when $\Lambda = U$ is a bounded subset of $\mathbf{Z}^d$.

We assume that $\Phi(0)$ is ferromagnetic in the sense that

$$\Phi_{j,k}(0) \leq 0,\ \ j \neq k. \tag{A.9}$$

We have

$$\Phi(0) = 1 - \widetilde{v}_0 *, \tag{0.17}$$

where $0 \leq \widetilde{v}_0 \in \ell^1(\mathbf{Z}^d)$ is even with $\widetilde{v}_0(0) = 0$ and the star indicates that $\widetilde{v}_0$ acts as a convolution. Actually, the constant 1 should be replaced by a more general constant $a > 0$, but we may always reduce ourselves to the case $a = 1$, by a dilation in h.

Assume that there exists a finite set $K \subset \mathbf{Z}^d$ such that

$$\widetilde{v}_0(j) > 0,\, j \in K,\ \ \mathrm{Gr}\,(K) = \mathbf{Z}^d, \tag{A.10}$$

where $\mathrm{Gr}\,(K)$ denotes the smallest subgroup of $\mathbf{Z}^d$ which contains K. We also make the following finite range assumption:

$$\exists C_0,\ \text{such that}\ \Phi_{j,k}(x) = 0\ \text{for}\ |j-k| > C_0. \tag{A.fr}$$

We introduce the 2 standard matrix

$$A(x)=\int_0^1 \Phi(tx)dt, \tag{0.18}$$

The following assumption is a weakened convexity assumption and will be used in section 10 together with a maximum principle (from [S4]) to obtain other more explicit conditions.

$$\begin{aligned}&\exists\epsilon_0>0 \text{ such that for every } x\in\mathbf{R}^{\mathbf{Z}^d},\ A(x) \text{ satisfies } (\mathrm{mp}\,\epsilon_0)\text{: If} \qquad \text{(A.mp)}\\ &t\in\ell^1(\mathbf{Z}^d),\ s\in\ell^\infty(\mathbf{Z}^d), \text{ and } \langle t,s\rangle=|t|_1|s|_\infty, \text{ then } \langle A(x)t,s\rangle\ge\epsilon_0|t|_1|s|_\infty.\end{aligned}$$

Notice that this assumption is fulfilled if $A(x)=1+B(x)$ with $\|B(x)\|_{\mathcal{L}(\ell^\infty,\ell^\infty)}\le 1-\epsilon_0$. Also notice that (A.4) is a consequence of (A.mp).

Let $U_j\in\mathbf{Z}^d$, $j=1,2,..$ be an increasing sequence of finite sets containing 0 and converging to $\mathbf{Z}^d$. Let $2\le L_j\nearrow\infty$ be a sequence of integers with

$$U_j\subset[-\frac{L_j}{4},\frac{L_j}{4}]^d, \tag{0.19}$$

and let $\Lambda=\Lambda_j=(\mathbf{Z}/L_j\mathbf{Z})^d$ be a corresponding sequence of discrete tori, so that we can view U_j as a subset of Λ_j in the natural way.

The following is the main result of our work:

Theorem 0.1. *Let $\Phi_{j,k}(x)\in C^\infty(\mathbf{R}^{\mathbf{Z}^d})$ satisfy (A.1–3, 7, 9, 10, fr, mp) and define $\phi_U(x)\in C^\infty(\mathbf{R}^U;\mathbf{R})$, $\phi_\Lambda\in C^\infty(\mathbf{R}^\Lambda;\mathbf{R})$ as above, when $U\subset\mathbf{Z}^d$ is finite and $\Lambda=(\mathbf{Z}/L\mathbf{Z})^d$ is a discrete torus. Let U_j, Λ_j be as above, and put $r_j:=\mathrm{dist}\,(0,\mathbf{Z}^d\setminus U_j)$, so that $r_j\nearrow+\infty$ when $j\to\infty$.*

Then there exist $C_0\ge1$, $j_0\in\mathbf{N}$, $\theta>0$, $h_0>0$, such that for $j\ge j_0$, $0<h\le h_0$, we have:

$$\begin{aligned}&\mathrm{Cor}_{\phi_{U_j}}(x_\nu,x_\mu),\ \mathrm{Cor}_{\phi_{\Lambda_j}}(x_\nu,x_\mu)\\ &=\mathcal{O}(h)e^{-\theta r_j/4}+he^{-p^\infty_{1,h}(\nu-\mu)}q^\infty(\nu-\mu;h),\ \text{for } |\nu|,|\mu|\le\frac{r_j}{C_0}.\end{aligned} \tag{0.20}$$

Here, for the statement about $\mathrm{Cor}_{\phi_{\Lambda_j}}$, we view U_j as a subset of Λ_j in the natural way. $p^\infty_{1,h}\in C^\infty(\mathbf{R}^d\setminus\{0\})$ is positively homogenous of degree 1 and has the h asymptotic expansion

$$p^\infty_{1,h}(\nu)\sim\sum_{\ell=0}^\infty p^\infty_{1,\ell}(\nu)h^\ell,\ h\to0, \tag{0.21}$$

in the space of such functions. Here $p^\infty_{1,0}$ is a norm, strictly convex transversally to the radial direction. Further,

$$q^\infty(\nu;h)=|\nu|^{-\frac{d-1}{2}}e^{-s^\infty(\nu;h)},\ \nu\in\mathbf{Z}^d,\ |\nu|\ge C_0, \tag{0.22}$$

where

$$s^\infty(\nu;h)\sim\sum_{-\infty}^{0}s_\alpha^\infty(\nu;h),\,|\nu|\to\infty, \tag{0.23}$$

uniformly with respect to h*, and* $s_\alpha^\infty(\cdot;h)\in C^\infty(\mathbf{R}^d\setminus\{0\})$ *is positively homogeneous of degree* α*. Here*

$$s_\alpha^\infty(\nu;h)\sim\sum_{0}^{\infty}s_{\alpha,\beta}^\infty(\nu)h^\beta,\ \ h\to 0, \tag{0.24}$$

in the space of smooth functions on $\mathbf{R}^d\setminus\{0\}$*, positively homogeneous of degree* α*.*

$p_{1,0}^\infty(\nu)$*,* $s_{0,0}^\infty(\nu)$ *were computed in [S1]. They appear in the asymptotics of* $(\Phi(0)^{-1})_{\nu,\mu}$*, when* $\nu-\mu\to\infty$

It should be remarked that our result only covers one of many possible situations, and new difficulties appear when f in (0.2) has more than one critical point. In that case phase transitions may appear and the correlations do not necessarily decay exponentially. Hopefully our methods can be useful in that case also if suitably combined with others. See [DZ], [S4], [Z1,2].

The proof follows the general strategy of [S1], [BJS] with the additional idea of looking at higher order Grushin problems, which in some sense amounts to analyzing a larger part of the bottom of the spectrum of the Witten Laplacian in degree 1. Let $\phi=\phi_\Lambda$, with Λ equal to Λ_j or U_j as in the theorem, and assume after adding an (h,j)-dependent constant, that

$$\int e^{-\phi/h}dx=1. \tag{0.25}$$

Then the correlation $\mathrm{Cor}\,(u,v)=\mathrm{Cor}_\phi(u,v)$, of two functions u and v is given by

$$\mathrm{Cor}\,(u,v)=\langle(u-\langle u\rangle)(v-\langle v\rangle)\rangle=((u-\langle u\rangle)e^{-\phi/2h}|(v-\langle v\rangle)e^{-\phi/2h})_{L^2}, \tag{0.26}$$

where

$$\langle u\rangle=\int u(x)e^{-\phi/h}dx \tag{0.27}$$

denotes the expectation value. Introduce the annihilation and creation operators

$$Z_\nu=h^{\frac12}\partial_{x_\nu}+h^{-\frac12}\partial_{x_\nu}\phi/2,\quad Z_\nu^*=-h^{\frac12}\partial_{x_\nu}+h^{-\frac12}\partial_{x_\nu}\phi/2, \tag{0.28}$$

for $\nu\in\Lambda$. The Witten exterior differentiation ([Wi]) is obtained as a conjugate of the de Rahm exterior differentiation d together with a normalizing factor:

$$d_\phi u=h^{\frac12}e^{-\phi/2h}\circ d\circ e^{\phi/2h}. \tag{0.29}$$

It takes differential ℓ forms into $(\ell+1)$ forms. In the scalar case ($\ell=0$) we have $d_\phi u=\sum Z_\nu(u)dx_\nu$. The corresponding Hodge Laplacian is called the Witten Laplacian:

$$\Delta_\phi=d_\phi^*d_\phi+d_\phi d_\phi^*. \tag{0.30}$$

It conserves the degree of differential forms, and we let $\Delta_\phi^{(\ell)}$ denote the restriction to ℓ forms. So far, it seems that only

$$\Delta_\phi^{(0)} = \sum Z_j^* Z_j \text{ and } \Delta_\phi^{(1)} = 1 \otimes \Delta_\phi^{(0)} + \phi''(x) \tag{0.31}$$

have been really useful in the the study of high-dimensional integrals. (In the last expression in (0.31) we identify the space of differential 1 forms with L^2 coefficients with the space $\ell^2(\Lambda) \otimes L^2(\mathbf{R}^\Lambda)$.) Philosophically speaking, this may be due to the fact that differential forms carry an antisymmetric structure (corresponding to fermions), while the method of higher order Grushin problems developed in the present paper (and in [W]) leads to some kind of bosonic quasi-particles.

In our case, $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ are self-adjoint non-negative operators with discrete spectrum, and even though we eventually avoid spectral theory and work in a pair of dual spaces, it may illustrate some ideas to speak about spectra. The lowest eigenvalue of $\Delta_\phi^{(0)}$ is 0, the corresponding eigenspace is of dimension 1 and is generated by the normalized vector $e^{-\phi/2h}$. The second eigenvalue μ_1 of $\Delta^{(0)}$ is positive and using the intertwining relation

$$\Delta_\phi^{(1)} d_\phi = d_\phi \Delta_\phi^{(0)}, \tag{0.32}$$

it can be shown that μ_1 is among the eigenvalues of $\Delta_\phi^{(1)}$. On the other hand, the assumptions above will imply that $\Delta_\phi^{(1)} \geq \text{const.} > 0$ uniformly in j. If we had assumed (as in [S1]) that $\phi''(x) \geq \text{const.} > 0$ uniformly in x, j, that would have been immediate from (0.31). As in [BJS] we only assume this at $x = 0$ however, and the idea (exploited in [BJS]) is then to make a limited Taylor expansion,

$$\phi''(x) = \phi''(0) + \sum_\nu A_\nu(x) \phi'_{x_\nu}(x),$$

to write $\phi'_{x_\nu}(x)$ as $h^{\frac{1}{2}}(Z_\nu + Z_\nu^*)$, and to use a priori estimates that give control over $\|Z_\nu u\|$.

In [HS], we established a general formula for the correlations and in [S1] we observed that it is related to Witten Laplacians. In this formalism and under the normalization condition (0.25) it reads:

$$\mathrm{Cor}\,(u, v) = ({\Delta_\phi^{(1)}}^{-1} d_\phi(e^{-\phi/2h}u) | d_\phi(e^{-\phi/2h}v)) = h({\Delta_\phi^{(1)}}^{-1}(e^{-\phi/2h}du)|(e^{-\phi/2h}dv)). \tag{0.33}$$

In [HS], we used such a formula to establish the exponential decay of the correlations $\mathrm{Cor}\,(x_\nu, x_\mu)$ when $\mathrm{dist}\,(\nu, \mu)$ is large. This is based on the simple idea that since we have a uniform bound on the norm of $(\Delta_\phi^{(1)})^{-1}$, then we should also have such a bound after a conjugation of this operator by an exponential weight $\rho(\nu) = e^{r(\nu)}$, $\nu \in \Lambda$, provided that r does not vary too fast.

In [S1] we obtained the leading behaviour of $\mathrm{Cor}\,(x_\nu, x_\mu)$ for large $\mathrm{dist}\,(\nu, \mu)$ by using a Feshbach (or Grushin) approach to $\Delta_\phi^{(1)}$ which in many ways amounts to study the bottom of the spectrum of this operator. We introduced the auxiliary operator $R_+ = R_+^{1,0} : L^2(\mathbf{R}^\Lambda) \to \ell^2(\Lambda)$ by

$$(R_+^{1,0}u)(j) = (u|e^{-\phi/2h}dx_j) = (u_j|e^{-\phi/2h}), \; j \in \Lambda, \tag{0.34}$$

where $u = \sum u_j dx_j \simeq (u_j)_{j\in\Lambda}$, so in each component, we project onto the kernel of $\Delta_\phi^{(0)}$. Let $R_-^{1,0} = (R_+^{1,0})^*$ be the adjoint.

Let

$$\mathcal{H}_1 = \{u \in L^2(\mathbf{R}^\Lambda);\, Z_\nu u \in L^2, \forall \nu \in \Lambda\}$$

with the corresponding norm

$$\|u\|_{\mathcal{H}_1}^2 = \|u\|^2 + \sum_\nu \|Z_\nu\|^2,$$

where $\|\cdot\|$ denotes the L^2 norm. Let $\mathcal{H}_{-1} = \mathcal{H}_1^*$ denote the dual space. Then as we shall prove below (and as was essentially proved in [S1] and in greater generality in [BJS]), the operator

$$\mathcal{P}^{0,1}(z) = \begin{pmatrix} \Delta_\phi^{(1)} - z & R_-^{0,1} \\ R_+^{0,1} & 0 \end{pmatrix} : (\ell^2(\Lambda)\otimes\mathcal{H}_1)\times(\ell^2(\Lambda)\otimes\mathbf{C}) \to \tag{0.35}$$
$$(\ell^2(\Lambda)\otimes\mathcal{H}_{-1})\times(\ell^2(\Lambda)\otimes\mathbf{C})$$

is bijective with a uniformly bounded inverse

$$\mathcal{E}^{0,1}(z) = \begin{pmatrix} E^{0,1}(z) & E_+^{0,1}(z) \\ E_-^{0,1}(z) & E_{-+}^{0,1}(z) \end{pmatrix} \tag{0.36}$$

for

$$-C \le z \le 2\lambda_{\min}(\phi''(0)) - \frac{1}{C}, \tag{0.37}$$

when h is small enough depending on C, and $C \ge 1$ may be arbitrary. *Here and in the following we follow the convention that all estimates and assumptions will be uniform w.r.t. Λ, if nothing else is specified.* By $\lambda_{\min}(\phi''(0))$, we denote the lowest eigenvalue of $\phi''(0)$. (As a matter of fact, we will need the invertibility only for $z = 0$ but keeping track of the spectral parameter will help the understanding. In the end classes of exponential weights will be the more appropriate objects.)

Further, as we shall see (and as was established in [S1], [BJS]), we have

$$\begin{aligned} &E_+^{0,1} = R_-^{0,1} + \mathcal{O}(h^{\frac12}), \quad E_-^{0,1} = R_+^{0,1} + \mathcal{O}(h^{\frac12}), \\ &E_{-+}^{0,1} = z - \phi''(0) + \mathcal{O}(h^{\frac12}), \end{aligned} \tag{0.38}$$

in the respective spaces of bounded operators. Notice that $E_{-+}^{0,1}(z)$ is invertible for $-C \le z \le \lambda_{\min}(\phi''(0)) - \frac{1}{C}$, i.e. in a smaller domain than (0.37). Actually, instead of varying the spectral parameter, we shall take $z = 0$ and conjugate $\mathcal{P}^{0,1}(0)$ by an exponential weight $\begin{pmatrix} \rho\otimes 1 & 0 \\ 0 & \rho\otimes 1 \end{pmatrix}$, with $\rho = e^r : \Lambda \to]0,\infty[$. We shall then see that the conjugated operator $\mathcal{P}^{0,1}$ is uniformly invertible for ρ in a large class of weights. Notice that the inverse is simply

$$\begin{pmatrix} \rho\otimes 1 & 0 \\ 0 & \rho\otimes 1 \end{pmatrix} \mathcal{E}^{0,1}(0) \begin{pmatrix} \rho^{-1}\otimes 1 & 0 \\ 0 & \rho^{-1}\otimes 1 \end{pmatrix}.$$

Moreover we shall see that (0.38) remains valid for the conjugated operators. $(E_{-+}^{0,1})^{-1}$ will cope with conjugation only with weights in a smaller class, and starting with the case when Λ is a discrete torus (implying that $E_{-+}^{0,1}$ is a convolution), we shall be able to analyze quite precisely the rate of decay of this inverse, and see that it corresponds to weights in the larger class of weights with which $\mathcal{E}^{1,0}$ accomodates conjugation. Since

$$(\Delta_\phi^{(1)})^{-1} = E^{0,1}(0) - E_+^{0,1}(0)(E_{-+}(0))^{-1}E_-^{0,1}(0),$$

we can apply (0.33) and get

$$\begin{aligned}\operatorname{Cor}(x_\nu, x_\mu) = h(E^{0,1}(0)(e^{-\phi/2h}dx_\nu)|(e^{-\phi/2h}dx_\mu)) - \\ h((E_{-+}(0))^{-1}E_-^{0,1}(0)(e^{-\phi/2h}dx_\nu)|E_-^{0,1}(0)(e^{-\phi/2h}dx_\mu)).\end{aligned} \tag{0.39}$$

Because $\mathcal{E}^{0,1}$ can cope with conjugation with stronger exponential weights than $(E_{-+}^{0,1})^{-1}$, we see that the first term of the RHS in (0.39) has faster decay, than the second, when $\operatorname{dist}(\nu,\mu) \to \infty$ and the more precise information evocated about the inverse of E_{-+} together with (0.38) leads to a result of the type (0.20), where a priori the $p_{1,h}$ and q will depend also on Λ through factors $1 + \mathcal{O}(h^{1/2})$. So far the ideas were already developed in [S1] and [BJS].

In order to get complete expansions as stated in the theorem, we will introduce higher order Grushin problems. Let $\mathbf{N}_j^\Lambda$ be the set of multiindices $\alpha : \Lambda \to \mathbf{N}$ of length j: $|\alpha| = |\alpha|_1 = j$. If J is a finite subset of $\mathbf{N}$, we put $\mathbf{N}_J = \cup_{j\in J}\mathbf{N}_j^\Lambda$. Since the Z_ν^* form a commutative family, the operator $\frac{1}{\alpha!}(Z^*)^\alpha$ is well-defined. For $u \in L^2(\mathbf{R}^\Lambda)$, put

$$(R_+^{N,0}u)(\alpha) = (u|\frac{1}{\alpha!}(Z^*)^\alpha(e^{-\phi/2h})),\ |\alpha| \le N,$$

so that $R_+^{N,0} : L^2(\mathbf{R}^\Lambda) \to \ell^2(\mathbf{N}_{[0,N]}^\Lambda)$. (We will see in section 4 that this operator is uniformly bounded.) Notice that $\frac{1}{\alpha!}(Z^*)^\alpha(e^{-\phi/2h})$ are Hermite functions when ϕ is a quadratic form. When $u \in \ell^2(\Lambda)\otimes L^2(\mathbf{R}^\Lambda)$, we put $(R_+^{N,1}u)(j,\alpha) = (R_+^{N,0}u_j)(\alpha)$, $(j,\alpha) \in \Lambda\times\mathbf{N}_{[0,N]}^\Lambda$. Let $R_-^{N,k} = (R_+^{N,k})^*$, $k = 0, 1$, and introduce the auxiliary (Grushin) operators for $k = 0, 1$:

$$\mathcal{P}^{N,k}(z) = \begin{pmatrix} \Delta_\phi^{(k)} & R_+^{N,k} \\ R_-^{N,k} & 0 \end{pmatrix} : \tag{0.40}$$

$$\begin{cases} \mathcal{H}_1 \times \ell^2(\mathbf{N}_{[0,N]}^\Lambda) \to \mathcal{H}_{-1}\times \ell^2(\mathbf{N}_{[0,N]}^\Lambda),\ k = 0, \\ (\ell^2(\Lambda)\otimes\mathcal{H}_1)\times(\ell^2(\Lambda)\otimes\ell^2(\mathbf{N}_{[0,N]}^\Lambda)) \to (\ell^2(\Lambda)\otimes\mathcal{H}_{-1})\times(\ell^2(\Lambda)\otimes\ell^2(\mathbf{N}_{[0,N]}^\Lambda)),\ k = 1.\end{cases}$$

We will see in section 5 that $\mathcal{P}^{N,0}(z)$ is uniformly invertible for $-C \le z \le (N+1)\lambda_{\min}(\phi''(0)) - \frac{1}{C}$ and that the same is true for $\mathcal{P}^{N,1}(z)$ in the range $-C \le z \le (N+2)\lambda_{\min}(\phi''(0)) - \frac{1}{C}$. This will be proved following the inductice scheme

$$(N,1) \to (N+1,0) \to (N+1,1),$$

starting with the case $(-1,1)$, where by definition $\mathcal{P}^{-1,1}(z) = \Delta_\phi^{(1)} - z$.

If

$$\mathcal{E}^{N,k}(z) = \begin{pmatrix} E^{N,k}(z) & E_+^{N,k}(z) \\ E_-^{N,k}(z) & E_{-+}^{N,k}(z) \end{pmatrix}$$

denotes the inverse of $\mathcal{P}^{N,k}(z)$ and $E_{-+;\nu,\mu}^{N,k}(z)$ denotes the operator matrix element of $E_{-+}^{N,0}$ corresponding to the decomposition $\ell^2(\mathbf{N}_{[0,N]}^\Lambda) = \oplus_{\nu=0}^N \ell^2(\mathbf{N}_\nu^\Lambda)$, we will further see that

$$E_{-+;\nu,\mu}^{N,0}(z) = h^{\frac{1}{2}|\nu-\mu|} B_{\nu,\mu}^N(z;h) + \mathcal{O}(h^{\frac{1}{2}(|N+1-\nu|+|N+1-\mu|)}) \text{ in } \mathcal{L}(\ell^2,\ell^2), \tag{0.41}$$

where $B_{\nu,\mu}^N$ has a complete asymptotic expansion in powers h^ℓ, $\ell \in \mathbf{N}$. Essentially the same result holds for $E_{-+}^{N,1}$ and similar results hold for $E_\pm^{N,k}$. The idea behind this result is to consider the matrix of $\Delta_\phi^{(0)}$ (and similarly for $\Delta_\phi^{(1)}$) with respect to the decomposition

$$L^2(\mathbf{R}^\Lambda) = \mathcal{L}_0 \oplus .. \oplus \mathcal{L}_N \oplus \mathcal{L}_{[0,N]}^\perp,$$

where $\mathcal{L}_j = R_-^{N,0}(\ell^2(\mathbf{N}_j^\Lambda))$ and $\mathcal{L}_{[0,N]}^\perp$ is the orthogonal of $\mathcal{L}_0 \oplus .. \oplus \mathcal{L}_N = \mathcal{L}_{[0,N]}$, for which the corresponding matrix elements of $(\Delta_\phi^{(0)})_{\nu,\mu}$ should behave as in (0.41).

Notice that (0.41) gives increasing precision in the asymptotics for a fixed (μ,ν), when N increases. It is possible to describe $\mathcal{E}^{M,k}$ in terms of $\mathcal{E}^{N,k}$, for $M \le N$, and using this with $M = 0$ and $N \to \infty$, we arrive at a complete asymptotic expansion of $E_{-+}^{0,1}(z;h)$ and at similar almost complete descriptions of $E_\pm^{0,1}(z;h)$. In other words, by using higher order Grushin problems it is possible to improve (0.38) and to get full asymptotics. This improvement also survives the conjugation by exponential weights in a sufficiently large class, and leads to a complete asymptotic description of $(E_{-+}^{0,1}(z))^{-1}$, including the decay rate at large distances. Finally we use this improved information in (0.39) to get complete asymptotics of the correlations. The handling of the thermodynamical limit requires some additional arguments that we do not discuss here.

A major motivation for this paper was the hope (yet to be fulfilled) that the use of higher order Grushin problems may be useful in the study of correlations in cases when ϕ is only weakly convex at its critical point. In such cases we do not always expect the correlations to decay exponentially any more and one may expect phenomena like phonons in crystals. Though we are still far from such a result, we may point out that the parameter N can be interpreted as a maximum number of particles under consideration, and that the k particle space $\ell^2(\mathbf{N}_k^\Lambda)$ can be identified with the k fold symmetric tensor product of $\ell^2(\Lambda)$ with itself. In other words, our particles are bosons.

In sections 1–9, we do all the essential work, adding successively the assumptions that we need. At the end of section 9, we arrive at the main result. In section 10, we consider a slightly less general framework and extract a main result which is more easily formulated.

In section 1 we review some standard facts about Witten Laplacians.
In section 2 we introduce some special Sobolev spaces, which are the natural ones for our variational point of view.
In section 3 we discuss how to reshuffle creation and annihilation operators. The reason for doing so will appear very naturally, and we are aware of the fact that such reorderings also appear in quantum field theory.
In section 4 we apply the result of the preceding section to study certain scalar producs.
Section 5 is devoted to the well-posedness of higher order Grushin problems.
In section 6 we get asymptotics for the solutions of these problems and in section 7, we show that these asymptotics for $\mathcal{P}^{N,1}$ remain after introducing certain exponential weights on the $\ell^2(\Lambda)$ component of $\mathcal{P}^{N,1}$.
In section 8, we study the effect of parameter dependence in order to treat the thermodynamical limit.
In section 9 we arrive at the main result on the asymptotics of the correlations also in the thermodynamical limit.
In section 10 we extract the main result as it is formulated in Theorem 0.1 above.
The two appendices can be read when referred to in the main text.

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1. Assumptions on ϕ.

Let $\phi \in C^\infty(\mathbf{R}^\Lambda;\mathbf{R})$, where Λ is a finite set. We shall let Λ and consequently ϕ vary with some parameter, but all assumptions are uniform w.r.t. Λ, if nothing else is specified. Our first assumption is

$$\begin{gathered}\phi^{(2)} = \phi'' \text{ is 2 standard in the sense that for every } k \geq 2, \text{ we have uniformly} \\ \text{in } \Lambda \text{ and in } x \in \mathbf{R}^\Lambda : \langle \phi^{(k)}(x), t_1 \otimes \ldots \otimes t_k\rangle = \mathcal{O}(1)|t_1|_{p_1}\ldots|t_k|_{p_k},\ t_j \in \mathbf{C}^\Lambda, \\ \text{whenever } 1 \leq p_j \leq \infty, \text{ and } 1 = \frac{1}{p_1} + \ldots + \frac{1}{p_k}.\end{gathered} \tag{H1}$$

Here $\phi^{(k)} = \nabla^k\phi$ is the symmetric tensor of kth order derivatives. See [S2] for definitions and basic properties concerning standard tensors. Recall that by complex interpolation it suffices to have the estimate in (H1) in the extreme cases

$$p_\nu = \begin{cases} 1, & \nu = j, \\ \infty, & \nu \neq j, \end{cases}$$

for $j = 1,..,k$. Notice that (H1) implies that $\phi''(x) : \ell^p \to \ell^p$ is uniformly bounded for $x \in \mathbf{R}^\Lambda$, $1 \leq p \leq \infty$.

The next three assumptions imply that $x = 0$ is a non-degenerate critical point of ϕ and the only critical point:

$$\phi'(0) = 0, \tag{H2}$$

$$\phi''(0) \geq \text{const.} > 0, \tag{H3}$$

$$\phi'(x) = A(x)x \text{ where } A(x) \text{ is 2 standard and has an inverse } B(x) \text{ which is } \mathcal{O}(1) : \ell^p \to \ell^p,\ 1 \le p \le \infty. \tag{H4}$$

We observe that B will also be 2 standard. Also notice that (H1), (H2) imply that

$$|\phi'(x)|_p \le \mathcal{O}(1)|x|_p,\ 1 \le p \le \infty, \tag{1.1}$$

while (H4) implies the reverse estimate

$$|x|_p \le \mathcal{O}(1)|\phi'(x)|_p,\ 1 \le p \le \infty. \tag{1.2}$$

It would be of interest to know if conversely (1.2) and (H1–3) imply (H4). Also notice that (H4) (or (1.2)) implies that $\phi''(0)^{-1}$ exists and is $\mathcal{O}(1) : \ell^p \to \ell^p$. When checking (H4), a natural candidate for $A(x)$ is $\int_0^1 \phi''(tx)dt$, which is 2 standard by (H1).

We end this section by introducing Witten Laplacians and related objects (cf. [S]). For that purpose we shall work on $\mathbf{R}^\Lambda$, where Λ is some finite set. Let $d = \sum_{\ell\in\Lambda} dx_\ell^\wedge \otimes \partial_{x_\ell}$ denote the DeRahm exterior differentiation which takes differential k forms on $\mathbf{R}^\Lambda$ to differential $k+1$ forms. Here $dx_\ell^\wedge$ denotes the operator of left exterior multiplication by dx_ℓ and we let $dx_\ell^\rfloor$ denote the adjoint operator of contraction, which is well defined if we view $\mathbf{R}^\Lambda$ as a Riemannian manifold with the standard metric. Recall that d is a complex in the sense that $d \circ d = 0$. Using the standard scalar product on the space of smooth k forms, we can define the adjoint $d^* = \sum_{\ell\in\Lambda} dx_\ell^\rfloor \otimes (-\partial_{x_\ell})$, taking $k+1$ forms into k forms. The corresponding Hodge Laplacian is then $d^*d + dd^*$. It conserves k forms and commutes with d and d^*.

The Witten exterior differentiation is obtained from d by conjugation by $e^{\phi/h}$ and multiplication by a cosmetic factor:

$$d_\phi := h^{1/2}e^{-\phi/2h} \circ d \circ e^{\phi/2h} = \sum_{\ell\in\Lambda} dx_\ell^\wedge \otimes Z_\ell, \tag{1.3}$$

where

$$Z_\ell = e^{-\phi/2h} \circ h^{1/2}\partial_{x_\ell} \circ e^{\phi/2h} = h^{1/2}\partial_{x_\ell} + h^{-1/2}\partial_{x_\ell}\phi/2. \tag{1.4}$$

We view Z_ℓ as annihilation operators. The corresponding creation operators are

$$Z_\ell^* = -h^{1/2}\partial_{x_\ell} + h^{-1/2}\partial_{x_\ell}\phi/2. \tag{1.5}$$

We have the commutation relations:

$$[Z_j, Z_k] = 0,\ [Z_j, Z_k^*] = \phi''_{j,k}(x),\ j, k \in \Lambda. \tag{1.6}$$

d_ϕ is a complex and the corresponding Hodge Laplacian is called the Witten Laplacian and is given by:

$$\Delta_\phi = d_\phi^* d_\phi + d_\phi d_\phi^*. \tag{1.7}$$

It conserves the degree of forms and we denote by $\Delta_\phi^{(k)}$ the restriction to k forms. Only the cases $k = 0, 1$ will be of importance to us and maybe the explanation of this fact is that by working with differential forms, we impose a fermionic structure, while the problems in this paper have a bosonic structure with the degree k viewed as the number of particles. It would be interesting to know if there are some other operators better adapted to the bosonic structure. A general formula for Δ_ϕ is

$$\Delta_\phi = I \otimes \Delta_\phi^0 + \sum_{j,k} \phi''_{j,k}(x) dx_j^\wedge \, dx_k^\rfloor, \tag{1.8}$$

where we from now adopt the convention of letting the form component be the first factors and the function components to be the last factors when we represent differential forms and corresponding operators as tensor products. $\Delta_\phi^{(0)}$ acts on scalar functions and is given by:

$$\Delta_\phi^{(0)} = \sum_j Z_j^* Z_j. \tag{1.9}$$

When $k = 1$, the formula (1.8) simplifies to

$$\Delta_\phi^{(1)} = I \otimes \Delta_\phi^{(0)} + \phi''(x), \tag{1.10}$$

provided that we view 1 forms as functions with values in $\mathbf{C}^\Lambda$. Again d_ϕ and d_ϕ^* commute with Δ_ϕ and in particular,

$$d_\phi \Delta_\phi^{(0)} = \Delta_\phi^{(1)} d_\phi, \; d_\phi^* \Delta_\phi^{(1)} = \Delta_\phi^{(0)} d_\phi^*. \tag{1.11}$$

Under the assumptions (H1-4) we know (see for instance [BJS] or [Jo]) that $\Delta_\phi^{(k)}$ can be realized as a selfadjoint operator by means of the Friedrichs extension. We will use the same symbol to denote this selfadjoint operator. Moreover, the spectrum is discrete and contained in $[0, +\infty[$. When $k = 0$ the lowest eigenvalue is simple and equal to 0. The corresponding eigenspace is spanned by $e^{-\phi/2h}$. When $k = 1$, the lowest eigenvalue is > 0 (see for instance [S]).

2. The spaces $\mathcal{H}_{\pm 1}$.

There will be two versions of these spaces, one for scalar ($\mathbf{C}$ valued) functions and one for functions with values in $\mathbf{C}^\Lambda$. We start with the scalar case. We assume (H1–4) throughout this section.

If $u \in C_0^\infty(\mathbf{R}^\Lambda)$, we put

$$\|u\|_{\mathcal{H}_1}^2 = \|u\|_1^2 = \|u\|^2 + \sum_{\ell \in \Lambda} \|Z_\ell u\|^2, \tag{2.1}$$

and let $\mathcal{H}_1$ be the closure of C_0^∞ for this norm. $\mathcal{H}_1$ is the form domain of $\Delta_\phi^{(0)}$, and by a standard regularization argument we know that

$$\mathcal{H}_1 = \{u \in L^2(\mathbf{R}^\Lambda); \, Z_\ell u \in L^2, \, \forall \ell \in \Lambda\}. \tag{2.2}$$

Let $\mathcal{H}_{-1}$ be the dual space. Using the standard L^2 inner product, we view $\mathcal{H}_{-1}$ as a space of temperate distributions and have the natural inclusions:

$$\mathcal{S}(\mathbf{R}^\Lambda) \subset \mathcal{H}_1 \subset \mathcal{H}_0 \subset \mathcal{H}_{-1} \subset \mathcal{S}'(\mathbf{R}^\Lambda). \tag{2.3}$$

Here the two inclusions in the middle correspond to inclusion operators of norm ≤ 1 and $\mathcal{H}_0$ denotes $L^2(\mathbf{R}^\Lambda)$. $\mathcal{H}_1$ is a Hilbert space with scalar product

$$[u|v]_1 = (u|v) + \sum_{\ell\in\Lambda}(Z_\ell u|Z_\ell v) = ((1+\Delta_\phi^{(0)})u|v), \tag{2.4}$$

where $(\cdot|\cdot\cdot)$ is the usual innerproduct in L^2. From this it follows that $1+\Delta_\phi^{(0)}$ is unitary from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$. We also remark that $\mathcal{H}_{-1}$ is the space of all

$$u = u^0 + \sum Z_\ell^* u_\ell, \tag{2.5}$$

with $u^0, u_\ell \in L^2$. Moreover $\|u\|_{-1}^2$ is the infimum of $\|u^0\|^2 + \sum \|u_\ell\|^2$ over all decompositions as in (2.5).

We now pass to spaces of 1 forms, whenever there is a possibility of confusion we indicate the degree of the forms by a superscript (k), so that the spaces just defined are $\mathcal{H}_{\pm 1}^{(0)}$. Put

$$\mathcal{H}_{\pm1}^{(1)} = \ell^2(\Lambda)\otimes\mathcal{H}_{\pm1}^{(0)}. \tag{2.6}$$

The corresponding scalar product of two 1 forms $u = \sum u_j dx_j$, $v = \sum v_j dx_j$ is then

$$[u|v]_1 = \sum [u_j|v_j]_1 = \sum_j (u_j|v_j) + \sum_{j,k}(Z_j u_k|Z_j v_k) = (u|v) + \sum_{j,k}(Z_j u_k|Z_j v_k). \tag{2.7}$$

It can also be written $((1+1\otimes\Delta_\phi^{(0)})u|v)$. Again $\mathcal{H}_{-1}^{(1)}$ is the dual space of $\mathcal{H}_1^{(1)}$, and

$$(1+1\otimes\Delta_\phi^{(0)}) : \mathcal{H}_1^{(1)} \to \mathcal{H}_{-1}^{(1)} \text{ is unitary.} \tag{2.8}$$

Later we will need to approximate $\phi''(x)$ by $\phi''(0)$ in these spaces, and for that we shall use the following lemma.

Lemma 2.1. *The operator $u(x)\mapsto(\phi''(x)-\phi''(0))u(x)$ is bounded $\mathcal{H}_1^{(1)}\to\mathcal{H}_{-1}^{(1)}$ and of norm $\mathcal{O}(h^{1/2})$.*

Proof. Using Proposition A.1, we see that

$$\phi''_{j,k}(x) - \phi''_{j,k}(0) = h\phi_{j,k}^{(1)}(x) + h^{1/2}\sum_\ell Z_\ell^*\circ\phi_{j,k,\ell}^{(0)}(x) + h^{1/2}\sum_\ell \phi_{j,k,\ell}^{(0)}(x)\circ Z_\ell, \tag{2.9}$$

where $\phi^{(\nu)}$ are standard tensors. Let $u, v\in C_0^\infty(\mathbf{R}^\Lambda)$ and use (2.9) to get

$$\begin{aligned}&((\phi''(x)-\phi''(0))u|v) \\ &\quad = h^{1/2}\sum_{j,k,\ell}(\phi_{j,k,\ell}^{(0)}u_k|Z_\ell v_j) + h^{1/2}\sum_{j,k,\ell}(\phi_{j,k,\ell}^{(0)}Z_\ell u_k|v_j) + h\sum_{j,k}(\phi_{j,k}^{(1)}u_k|v_j).\end{aligned} \tag{2.10}$$

Since $\phi^{(1)}$ is 2 standard, the last sum is $\mathcal{O}(h)\|u\|\|v\|$. For the two other sums, we use Lemma B.2 and get for the first sum:

$$|\sum_{\ell,j,k}\phi^{(0)}_{j,k,\ell}u_k\overline{Z_\ell v_j}|\leq\mathcal{O}(1)(\sum_k|u_k(x)|^2)^{1/2}(\sum_{j,\ell}|Z_\ell v_j|^2)^{1/2}. \tag{2.11}$$

This implies that the first sum in (2.10) is $\mathcal{O}(1)\|u\|\|v\|_1$. Similarly the second sum is $\mathcal{O}(1)\|u\|_1\|v\|$. We then get

$$((\phi''(x)-\phi''(0))u|v)=\mathcal{O}(h^{1/2})\|u\|_1\|v\|_1, \tag{2.12}$$

which implies the lemma. #

3. Reshuffling of Z and Z^*.

Let $J=\{1,..,N\}$, $K=\{1,..,M\}$, $f\in C^\infty(\mathbf{R}^\Lambda)$. Then for $j\in\Lambda^J$, $k\in\Lambda^K$, we want to rewrite

$$(\prod_{\nu\in J}Z_{j(\nu)})\circ f\circ(\prod_{\mu\in K}Z^*_{k(\mu)}) \tag{3.1}$$

as a sum of similar terms with the Z^* to the left and the Z to the right. We first move each factor $Z_{j(\nu)}$ as far as possible to the right, taking into account the appearance of commutator terms, due to the relations

$$[Z_j,f]=h^{\frac{1}{2}}\partial_{x_j}f(x)=[f,Z^*_j],\ [Z_j,Z^*_k]=\phi''_{j,k}(x). \tag{3.2}$$

After that, we move the surviving factors Z^*_k as a far as possible to the left, generating new commutator terms. The expression (3.1) becomes

$$\sum_{P\geq 0}\frac{1}{P!}\sum_{\substack{J=J_0\cup..\cup J_{P+1}\\ K=K_0\cup..\cup K_{P+1}\\ \text{partitions with}\\ J_p\neq\emptyset\neq K_p,\ \text{for } 1\leq p\leq P}}(\prod_{\mu\in K_0}Z^*_{k(\mu)})\circ h^{\frac{1}{2}(\#J_{P+1}+\#K_{P+1})}((\prod_{\substack{\mu\in K_{P+1},\\ \nu\in J_{P+1}}}\partial_{x_{k(\mu)}}\partial_{x_{j(\nu)}})f)\times \tag{3.3}$$
$$\prod_{p=1}^{P}(h^{-1+\frac{1}{2}(\#J_p+\#K_p)}(\prod_{\substack{\mu\in K_p\\ \nu\in J_p}}(\partial_{x_{k(\mu)}}\partial_{x_{j(\nu)}}))\phi)\prod_{\nu\in J_0}Z_{j(\nu)}.$$

Here and in the following we use the term partition for a union of pairwise disjoint sets. The factor $\frac{1}{P!}$ can be eliminated if we let the second summation be over all similtaneous partitions of J and K which are non-ordered in the indices $1\leq p\leq P$.

Define a map $m:\Lambda^N\to\mathbf{N}^\Lambda$, by

$$m(j)(\lambda)=\#\{k;\,j(k)=\lambda\},\ \lambda\in\Lambda. \tag{3.4}$$

If $\alpha\in\mathbf{N}^\Lambda$, we put $|\alpha|=|\alpha|_1=\sum_{\lambda\in\Lambda}\alpha(\lambda)$. Then $|m(j)|=N$. We write

$$\mathbf{N}^\Lambda_N=\{\alpha\in\mathbf{N}^\Lambda;\,|\alpha|=N\}, \tag{3.5}$$

and more generally

$$\mathbf{N}^\Lambda_A = \{\alpha \in \mathbf{N}^\Lambda;\ |\alpha| \in A\}, \tag{3.6}$$

if $A \subset \mathbf{N}$.

If $j \in \Lambda^J$, $k \in \Lambda^K$ as above, then

$$\prod_{\nu \in J} Z_{j(\nu)} = Z^\alpha,\ \prod_{\mu \in K} Z^*_{k(\mu)} = (Z^*)^\beta, \tag{3.7}$$

where $\alpha = m(j)$, $\beta = m(k)$, and where we use standard multiindex notation, $Z^\alpha = \prod_{\lambda\in\Lambda} Z_\lambda^{\alpha(\lambda)}$. Conversely for a given $\alpha \in \mathbf{N}^\Lambda_N$, the number of $j \in \Lambda^J$ with $m(j) = \alpha$ is equal to $\frac{N!}{\alpha!} = \frac{|\alpha|!}{\alpha!}$. For a typical term in (3.3), write

$$\prod_{\mu \in K_p} \partial_{x_{k(\mu)}} = \partial_x^{\beta_p},\ \prod_{\nu \in J_p} \partial_{x_{j(\nu)}} = \partial_x^{\alpha_p}, \tag{3.8}$$

and similarly with ∂_x replaced by Z or Z^*. Then

$$\alpha = \alpha_0 + .. + \alpha_{P+1},\ \beta = \beta_0 + .. + \beta_{P+1},\ \text{with } \alpha_p \neq 0 \neq \beta_p \text{ when } 1 \leq p \leq P. \tag{3.9}$$

Conversely, for such a decomposition of $\alpha = m(j)$, we consider the decomposition $\alpha(\lambda) = \alpha_0(\lambda) + .. + \alpha_{P+1}(\lambda)$ for every $\lambda \in \Lambda$, and see that there are $\dfrac{\alpha!}{\alpha_0!..\alpha_{P+1}!}$ corresponding partitions of J into $J_0 \cup .. \cup J_{P+1}$. The equality of the expressions in (3.1) and in (3.3) becomes

$$\begin{aligned} &\frac{Z^\alpha}{\alpha!} \circ f \circ \frac{(Z^*)^\beta}{\beta!} = \\ &\sum_{P\geq 0} \frac{1}{P!} \sum_{\substack{\alpha=\alpha_0+..+\alpha_{P+1},\\ \beta=\beta_0+..+\beta_{P+1},\\ \alpha_j,\beta_j\neq 0 \text{ for } 1\leq j\leq P.}} \frac{(Z^*)^{\alpha_0}}{\alpha_0!} h^{\frac{1}{2}(|\alpha_{P+1}|+|\beta_{P+1}|)} \frac{\partial_x^{\alpha_{P+1}+\beta_{P+1}} f}{\alpha_{P+1}!\beta_{P+1}!} \times \\ &\qquad \prod_{p=1}^{P} (h^{-1+\frac{1}{2}(|\alpha_p|+|\beta_p|)} \frac{\partial_x^{\alpha_p+\beta_p}\phi}{\alpha_p!\beta_p!}) \frac{Z^{\beta_0}}{\beta_0!}. \end{aligned} \tag{3.10}$$

We shall transform our expression further by using non-commutative expansions of the tensors appearing in (3.3), (3.10). For this, it seems easier to work with (3.3), and we assume that f is 0 standard, or possibly M standard, depending on M additional indices. For simplicity, we write

$$\prod_{\mu\in K_0} Z^*_{k(\mu)} = Z^*_{k|K_0},\quad (\prod_{\mu\in K_p} \partial_{x_{k(\mu)}} \prod_{\nu\in J_p} \partial_{x_{j(\nu)}})\phi = \phi_{k|K_p,j|J_p}.$$

Now apply Proposition A.1 to one of the tensors:

$$\begin{aligned} \phi_{k|K_p,j|J_p}(x) = \phi_{k|K_p,j|J_p}(0) + h^{\frac{1}{2}} \sum_{\ell\in\Lambda} Z^*_\ell \circ \phi^{(0)}_{k|K_p,j|J_p,\ell}(x) + \\ h^{\frac{1}{2}} \sum_{\ell\in\Lambda} \phi^{(0)}_{k|K_p,j|J_p,\ell}(x) \circ Z_\ell + h\phi^{(1)}_{k|K_p,j|J_p}(x). \end{aligned} \tag{3.11}$$

When substituting this into (3.3), the effect of the first term of the RHS will be to freeze the corresponding factor to $x = 0$. For the contributions of the second term of the RHS of (3.11) in (3.3), we move the Z_ℓ^* to the left, until either it joins the factors $Z^*_{k|K_0}$ or until it forms a commutator with a $\phi_{k|K_q,j|J_q}$ or with $f_{k|K_{P+1},j|J_{P+1}}$, that we denote by $\phi_{k|K_q,j|J_q,\ell}$ (also for $q = P+1$). In the second case the ℓ summation amounts to the contraction of two standard tensors, which produces a standard tensor and an additional power of h. For the contribution of the last sum in (3.11), we move the factors Z_ℓ as far as possible to the right and repeat the same discussion. The contribution from the last term in (3.11) in (3.3) is simply to introduce an extra power of h. The procedure can be iterated a finite number of times, and we see that the general term term in (3.3) becomes a finite sum of terms of the type

$$\sum_{\substack{\ell\in\Lambda^{L_1\cup..\cup L_{Q+1}}\\ r\in\Lambda^{R_1\cup..\cup R_{Q+1}}}} h^X Z^*_{k|K_0} Z^*_{\ell|L_1\cup..\cup L_{Q+1}} \circ \Phi^{(1)}_{k|K_1,j|J_1,\ell|L_1,r|R_1}(x).. \tag{3.12}$$
$$\Phi^{(Q+1)}_{k|K_{Q+1},j|J_{Q+1},\ell|L_{Q+1},r|R_{Q+1}}(x) \circ Z_{j|J_0} Z_{r|R_1\cup..\cup R_Q}.$$

Here $K = K_0\cup..\cup K_{Q+1}$, $J = J_0\cup..\cup J_{Q+1}$ are partitions and $K_q \neq \emptyset \neq J_q$ for $1 \le q \le Q$. $\Phi^{(q)}$ are standard, L_q, R_q are finite disjoint sets, possibly empty, and

$$X = \frac{1}{2}\#(L_1\cup..\cup L_Q)+\frac{1}{2}\#(R_1\cup..\cup R_Q)+N+\sum_1^Q(\frac{1}{2}(\#K_q+\#J_q)-1)+\frac{1}{2}(\#K_{Q+1}+\#J_{Q+1}),$$

where $N \in \mathbf{N}$, and we have arranged that $\Phi^{(Q+1)}$ is the factor which contains the contribution from f under the contraction procedure.

The point with the further Taylor expansions of the term (0.3) is to arrive at terms with constant factors $\Phi^{(\nu)}$. More precisely, we can introduce a stopping rule, so that we only get terms of the form (3.12) with

$$\#(K_0 \cup L_1 \cup .. \cup L_{Q+1}) \le A, \tag{3.13}$$

$$\#(J_0 \cup R_1 \cup .. \cup R_{Q+1}) \le B, \tag{3.14}$$

$$N \le N_0, \tag{3.15}$$

where A, B, N_0 are given integers ≥ 0 with $A \ge \#K$, $B \ge \#J$, and so that the factors $\Phi^{(\nu)}$ are constant for all terms for which we have strict inequality in all the three equations (3.13–15).

We will also need a slight variation of the arguments above. Assume for simplicity that $f = 1$. Then from (3.3), we see that (3.1) takes the the form

$$\sum_{P\ge 0}\frac{1}{P!} \sum_{\substack{J=J_0\cup..\cup J_P\\ K=K_0\cup..\cup K_P\\ \text{partitions with}\\ J_p\neq\emptyset\neq K_p \text{ for } 1\le p\le P}} Z^*_{k|K_0} \prod_{p=1}^P (h^{-1+\frac{1}{2}(\#J_p+\#K_p)} \partial_{x_{k|K_p}} \partial_{x_{j|J_p}} \phi(x)) Z_{j|J_0}. \tag{3.16}$$

We now want the coefficients to the left, so we move all the factors $\partial_{x_{k|K_p}}\partial_{x_{j|J_p}}\phi(x)$ to the left, taking into account the commutator terms. Then the expression (3.1) becomes:

$$\sum_{P\ge 0}\sum_{\substack{J=J_0\cup..\cup J_P\\ K=K_0\cup..\cup K_P\\ \text{partitions with}\\ J_p\neq\emptyset\neq K_p \text{ for } 1\le p\le P}} C^{J_0,..,J_P}_{K_0,..,K_P}\prod_{p=1}^{P}(h^{-1+\frac{1}{2}(\#J_p+\#K_p)}\partial_{x_{k|K_p}}\partial_{x_{j|J_p}}\phi(x))Z^*_{k|K_0}Z_{j|J_0}. \tag{3.17}$$

Here the combinatorial coefficients $C^{...}_{...}$ are independent of Λ.

4. Study of $(\frac{1}{\alpha!}(Z^*)^\alpha(e^{-\phi/h})|\frac{1}{\beta!}(Z^*)^\beta(e^{-\phi/h}))$.

After adding a h-dependent constant to ϕ, we assume that

$$\int e^{-\phi(x)/h}dx=1. \tag{4.1}$$

We want to study the matrix formed by the scalarproducts in the title of this section, for $|\alpha|,|\beta|\le N_0$, for some $N_0\in\mathbf{N}$. Equivalently, we want to study,

$$(Z^*_{k|K}(e^{-\phi/2h})|Z^*_{j|J}(e^{-\phi/2h})), \tag{4.2}$$

for $J=\{1,..,N\}$, $K=\{1,..,M\}$, $0\le N,M\le N_0$, $k\in\Lambda^K$, $j\in\Lambda^J$. Here we use the notation of section 3. We write this as

$$(Z_{j|J}Z^*_{k|K}(e^{-\phi/2h})|e^{-\phi/2h}), \tag{4.3}$$

and apply (3.12), with $f=1$, in which case the factor $\Phi^{(Q+1)}$ drops out. Since $Z(e^{-\phi/2h})=0$, we can further restrict our attention to the terms with K_0,L_q,J_0,R_q all empty, and it follows that (4.2) is a finite sum of terms of the type

$$h^X(\Phi^{(1)}_{k|K_1,j|J_1}..\Phi^{(Q)}_{k|K_Q,j|J_Q}e^{-\phi/2h}|e^{-\phi/2h}). \tag{4.4}$$

Here $K=K_1\cup..\cup K_Q$, $J=J_1\cup..\cup J_Q$ are partitions with $K_q\neq\emptyset\neq J_q$ for all q. Further,

$$X=N+\sum_1^Q(\frac{1}{2}(\#K_q+\#J_q)-1),\ N\in[0,N_1]\cap\mathbf{N}, \tag{4.5}$$

where N_1 is any fixed sufficiently large integer ≥ 0, and as we saw in section 3, we may arrange that $\Phi^{(\nu)}$ are constant for the terms with $N<N_1$. Because of the Hilbert-Schmidt property of standard tensors (Lemma B.2), we see that the term (4.4) defines a matrix which is $\mathcal{O}(h^X):\ell^2(\Lambda^K)\to\ell^2(\Lambda^J)$, and when $N<N_1$ it is also equal to h^X times the (constant) matrix $\Phi^{(1)}_{k|K_1,j|J_1}..\Phi^{(Q)}_{k|K_Q,j|J_Q}$.

Rewrite (4.5) as

$$X = N + \frac{1}{2}(\#K + \#J) - Q, \tag{4.6}$$

and notice that $1 \leq Q \leq \min(\#K, \#J)$. Then we can write

$$X = \frac{1}{2}|\#K - \#J| + \widetilde{N},\ \ \widetilde{N} \in \mathbf{N}. \tag{4.7}$$

We have proved most of the following result:

Proposition 4.1. *The matrix (4.2) has an asymptotic expansion*

$$\sim \sum_{\nu=0}^{\infty} h^{\frac{1}{2}|\#K-\#J|+\nu} m_{\nu,J,K}(j|J, k|K), \tag{4.8}$$

where the matrix $m_{\nu,J,K}$ is $\mathcal{O}(1) : \ell^2(\Lambda^K) \to \ell^2(\Lambda^J)$. Moreover,

$$m_{0,J,J}(j|J, k|J) = \sum_{\pi\in \mathrm{Perm}\,(J)} \prod_{p\in J} \phi''_{j(p),k(\pi(p))}(0), \tag{4.9}$$

where $\mathrm{Perm}\,(J)$ *is the group of permutations of J.*

It only remains to verify (4.9). It suffices to review the computations which lead to (4.4), (4.5) with a minimal X, i.e. with $N = 0$, $Q = N$, and we omit the details.

5. Higher order Grushin problems.

In this section we introduce a sequence of auxiliary, so-called Grushin problems for $\Delta_\phi^{(0)} - z$, $\Delta_\phi^{(1)} - z$ and show their well-posedness.

For $\alpha \in \mathbf{N}^\Lambda$, put

$$e_\alpha = \frac{1}{\sqrt{\alpha!}}(Z^*)^\alpha(e^{-\phi/2h}). \tag{5.1}$$

Then $e_\alpha \in L^2(\mathbf{R}^\Lambda)$. For $N \in \mathbf{N}$, define

$$R_+^{N,0}u(\alpha) = (u|e_\alpha),\ \ u \in L^2(\mathbf{R}^\Lambda),\ \ |\alpha| \leq N, \tag{5.2}$$

$$R_-^{N,0} = (R_+^{N,0})^*, \tag{5.3}$$

so that

$$R_+^{N,0} : L^2(\mathbf{R}^\Lambda) \to \ell^2(\mathbf{N}^\Lambda_{[0,N]}),$$
$$R_-^{N,0} : L^2(\mathbf{R}^\Lambda) \leftarrow \ell^2(\mathbf{N}^\Lambda_{[0,N]}),$$

where $\ell^2(\mathbf{N}^\Lambda_{[0,N]})$ is equipped with the standard scalar product.

These operators can also be described in an equivalent and sometimes more convenient way. Put

$$\widetilde{e}_0 = e_0,\ \ \widetilde{e}_j = \frac{1}{\sqrt{M!}} Z^*_{j(1)}..Z^*_{j(M)}(e^{-\phi/2h}),\ \ j \in \Lambda^M, \tag{5.4}$$

$$\widetilde{R}_+^{N,0}u(j) = (u|\widetilde{e}_j),\ j \in \Lambda^0 \cup \Lambda^1 \cup .. \cup \Lambda^N. \tag{5.5}$$

Here $\Lambda^0 = \{0\}$ by definition. Then $\widetilde{v}_+ := \widetilde{R}_+^{N,0}u$ is an element of $\ell^2(\Lambda^0 \cup \Lambda^1 \cup .. \cup \Lambda^N)$, which is invariant under the permutations: $\widetilde{v}_+(j) = \widetilde{v}_+(j \circ \pi)$, $j \in \Lambda^M$, $\pi \in \mathrm{Perm}\,(\{1, .., M\})$. We say that $\widetilde{v}_+$ belongs to the bosonic space $\ell_b^2(\Lambda^0 \cup \Lambda^1 \cup .. \cup \Lambda^N)$ of ℓ^2 functions that are invariant under the permutations above. The latter space can also be viewed as a direct (orthogonal) sum of symmetric tensor products; $\oplus_{M=0}^N(\odot_1^M \ell^2(\Lambda))$. The identification of $\widetilde{R}_+^{N,0}u$ and $R_+^{N,0}u$ is given by

$$\widetilde{R}_+^{N,0}u(j) = \frac{\sqrt{\alpha!}}{\sqrt{|\alpha|!}} R_+^{N,0}u(\alpha),\ \text{if } m(j) = \alpha,\ |a| = M. \tag{5.6}$$

This identification also respects the ℓ^2 structures:

$$\begin{aligned} |\widetilde{R}_+^{N,0}u|_2^2 &= \sum_{j \in \Lambda^0 \cup .. \cup \Lambda^N} |\widetilde{R}_+^{N,0}u(j)|^2 = \sum_{|\alpha| \le N} \frac{\alpha!}{|\alpha|!} |R_+^{N,0}u(\alpha)|^2 \sum_{j \in m^{-1}(\alpha)} 1 \\ &= \sum_{|\alpha| \le N} |R_+^{N,0}u(\alpha)|^2 = |R_+^{N,0}u|_2^2. \end{aligned} \tag{5.7}$$

Put

$$R_+^{N,1} = 1 \otimes R_+^{N,0} : \ell^2(\Lambda) \otimes L^2(\mathbf{R}^\Lambda) \to \ell^2(\Lambda) \otimes \ell^2(\mathbf{N}_{[0,N]}^\Lambda), \tag{5.8}$$

$$R_-^{N,1} = 1 \otimes R_-^{N,0} : \ell^2(\Lambda) \otimes L^2(\mathbf{R}^\Lambda) \leftarrow \ell^2(\Lambda) \otimes \ell^2(\mathbf{N}_{[0,N]}^\Lambda). \tag{5.9}$$

We consider the following Grushin problems for $\nu = 0, 1$:

$$\begin{cases} (\Delta_\phi^{(\nu)} - z)u + R_-^{N,\nu}u_- = v, \\ R_+^{N,\nu}u = v_+, \end{cases} \tag{Gr(N,ν)}$$

where $v \in \mathcal{H}_{-1}^{(\nu)}$, $u \in \mathcal{H}_1^{(\nu)}$,

$$u_-, v_+ \in \begin{cases} \ell^2(\mathbf{N}_{[0,N]}^\Lambda),\ \nu = 0 \\ \ell^2(\Lambda) \otimes \ell^2(\mathbf{N}_{[0,N]}^\Lambda),\ \nu = 1, \end{cases}$$

and z belongs to a suitable bounded interval, that will be specified. In the problem above, v and v_+ are the given quantities and u, u_- are the unknown. The main goal of this section is to prove

Proposition 5.1. *For every $N \in \mathbf{N}$, $C \ge 1$, there is a constant $\widetilde{C} > 0$ such that the following holds for $h > 0$ small enough:*

(A) If $-C \le z \le (N+1)\lambda_{\min}(\phi''(0)) - 1/C$, then (Gr(N,0)) has a unique solution $(u, u_-) \in \mathcal{H}_1 \times \ell^2$ for every $(v, v_+) \in \mathcal{H}_{-1} \times \ell^2$, and

$$\|u\|_{\mathcal{H}_1} + |u_-|_2 \le \widetilde{C}(\|v\|_{\mathcal{H}_{-1}} + |v_+|_2). \tag{5.10}$$

(B) If $-C \leq z \leq (N+2)\lambda_{\min}(\phi''(0)) - 1/C$, then (Gr(N,1)) has a unique solution $(u, u_-) \in \mathcal{H}_1 \times \ell^2$ for every $(v, v_+) \in \mathcal{H}_{-1} \times \ell^2$ and (5.10) holds.

Here $\lambda_{\min}(\phi''(0)) > 0$ denotes the smallest eigenvalue of $\phi''(0)$ and $\mathcal{H}_{\pm 1} = \mathcal{H}^{(\nu)}_{\pm 1}$ in case ν, $\ell^2 = \ell^2(\mathbf{N}^{\Lambda}_{[0,N]})$ in the case $\nu = 0$, $\ell^2 = \ell^2(\Lambda) \otimes \ell^2(\mathbf{N}^{\Lambda}_{[0,N]})$, when $\nu = 1$.

We shall prove the proposition following the inductive scheme

$$\mathrm{Gr}\,(k,1) \to \mathrm{Gr}\,(k+1,0) \to \mathrm{Gr}\,(k+1,1),$$

where we start by considering $\mathrm{Gr}(-1,1)$, which by definition is the problem

$$(\Delta^{(1)}_{\phi} - z)u = v, \;\; u \in \mathcal{H}_1, \; v \in \mathcal{H}_{-1}. \tag{5.11}$$

Lemma 5.2. *For all $C \geq 1$, $-C \leq z \leq \lambda_{\min}(\phi''(0)) - 1/C$, and h sufficiently small, depending on C, the problem (5.11) has a unique solution $u \in \mathcal{H}_1$, for every $v \in \mathcal{H}_{-1}$. Moreover,*

$$\|u\|_{\mathcal{H}_1} \leq \widetilde{C} \|v\|_{\mathcal{H}_{-1}}, \tag{5.12}$$

where $\widetilde{C} > 0$ depends on C but not on z, h, Λ.

Proof. Recall that $\Delta^{(1)}_{\phi} = 1 \otimes \Delta^{(0)}_{\phi} + \phi''(x)$ and consider first the simplified problem,

$$(1 \otimes \Delta^{(0)}_{\phi} + \phi''(0) - z)u = v, \;\; u \in \mathcal{H}_1, \; v \in \mathcal{H}_{-1}. \tag{5.13}$$

If u solves (5.13), take the scalar product of this equation with u and get

$$\begin{aligned} \|v\|_{\mathcal{H}_{-1}} \|u\|_{\mathcal{H}_1} &\geq (v|u) = ((1 \otimes \Delta^{(0)}_{\phi} + \phi''(0) - z)u|u) \\ &\geq \epsilon((1 \otimes \Delta^{(0)}_{\phi} + 1)u|u) + ((\phi''(0) - z - \epsilon)u|u) \geq \epsilon \|u\|^2_{\mathcal{H}_1}, \end{aligned}$$

for $\epsilon > 0$ small enough, so

$$\|u\|_{\mathcal{H}_1} \leq C \|v\|_{\mathcal{H}_{-1}}. \tag{5.14}$$

This gives injectivity and the analogue of (5.12) for the problem (5.13). Since $(1 \otimes \Delta^{(0)}_{\phi} + \phi''(0) - z)$ is a bounded selfadjoint operator $\mathcal{H}_1 \to \mathcal{H}_{-1}$, it is also surjective, so (5.13) is uniquely solvable and satisfies (5.14). To get the lemma it suffices to use that $\phi''(x) - \phi''(0) = \mathcal{O}(h^{1/2}) : \mathcal{H}_1 \to \mathcal{H}_{-1}$. #

The preceding lemma gives well-posedness for $\mathrm{Gr}\,(-1,1)$ in the appropriate range. Let us now perform the step $\mathrm{Gr}\,(-1,1) \to \mathrm{Gr}\,(0,0)$, so consider

$$\begin{cases} (\Delta^{(0)}_{\phi} - z)u + R^{0,0}_{-} u_- = v \\ R^{0,0}_{+} u = v_+, \end{cases} \tag{5.15}$$

with $v \in \mathcal{H}_{-1}$, $u \in \mathcal{H}_1$, $u_-, v_+ \in \mathbf{C}$, $R_+^{0,0}u = (u|e^{-\phi/2h})$. We let z be in the range of the lemma above, and we first prove uniqueness in (5.15). Let $v = 0$, $v_+ = 0$ in (5.15). Since $d_\phi R_-^{0,0} = 0$, $d_\phi \Delta_\phi^{(0)} = \Delta_\phi^{(1)} d_\phi$, we get by applying d_ϕ to the first equation in (5.15):

$$(\Delta_\phi^{(1)} - z)d_\phi u = 0. \tag{5.16}$$

Here we only know apriori that $d_\phi u \in L^2$, so we cannot apply Lemma 5.2 directly. However, it is easy and standard to show that every L^2 solution w of $(\Delta_\phi^{(1)} - z)w = 0$, has to belong to $\mathcal{S}$ and in particular to $\mathcal{H}_1$. Consequently, we can apply Lemma 5.2 and conclude that $d_\phi u = 0$. Since $d_\phi = h^{1/2}e^{-\phi/2h} \circ d \circ e^{\phi/2h}$, it follows that $u = \lambda e^{-\phi/2h}$ for some constant $\lambda \in \mathbf{R}$. Using also that $v_+ = 0$ in (5.15), we see that $\lambda = 0$, so $u = 0$ and then $u_- = 0$, and we have proved uniqueness for solutions of (5.15). Define $z_0 \in [0, \infty[$ by

$$z_0 = \inf_{\substack{u \in \mathcal{H}_1 \cap (e^{-\phi/2h})^\perp \\ \|u\|=1}} (\Delta_\phi^{(0)} u|u). \tag{5.17}$$

Since the inclusion map $\mathcal{H}_1 \to L^2$ is compact, there exists $u_0 \in \mathcal{H}_1 \cap (e^{-\phi/2h})^\perp$ with $\|u_0\| = 1$, such that $z_0 = (\Delta_\phi^{(0)} u_0|u_0)$, i.e. $((\Delta_\phi^{(0)} - z_0)u_0|u_0) = 0$, while $((\Delta_\phi^{0)} - z_0)u|u) \geq 0$ for general $u \in \mathcal{H}_1 \cap (e^{-\phi/2h})^\perp$. It follows that $(\Delta_\phi^{(0)} - z_0)u_0 = \mu e^{-\phi/2h}$ for some $\mu \in \mathbf{R}$ and since $u_0 \perp e^{-\phi/2h}$ and $e^{-\phi/2h} \in \operatorname{Ker} \Delta_\phi^{(0)}$, we see that $\mu = 0$. Hence $(\Delta_\phi^{(0)} - z_0)u_0 = 0$, so $u = u_0$, $u_- = 0$ is a solution of (5.15) with $v = 0$, $v_+ = 0$ and $z = z_0$. Since we know that (5.15) is injective for $0 \leq z \leq \lambda_{\min}(\phi''(0)) - 1/2C$, for h small enough depending on C, we conclude that

$$z_0 \geq \lambda_{\min}(\phi''(0)) - \frac{1}{2C}. \tag{5.18}$$

Let us now restrict the attention to $-C \leq z \leq \lambda_{\min}(\phi''(0)) - 1/C$ and derive an apriori estimate for solutions to (5.15). Let first $v_+ = 0$ in (5.15), and take the scalar product of the first equation there with u, and use that $(R_-^{0,0}u_-|u) = (u_-|R_+^{0,0}u) = 0$. We get

$$((\Delta_\phi^{(0)} - z)u|u) = (v|u). \tag{5.19}$$

With $\delta > 0$ small enough, write

$$\Delta_\phi^{(0)} - z = \delta \Delta_\phi^{(0)} + (1-\delta)(\Delta_\phi^{(0)} - z_0) + (1-\delta)(z_0 - \frac{z}{1-\delta}),$$

and get

$$((\Delta_\phi^{(0)} - z)u|u) \geq \delta(\Delta_\phi^{(0)} u|u) + (1-\delta)(z_0 - \frac{z}{1-\delta})\|u\|^2 \geq \delta\|u\|_{\mathcal{H}_1}^2.$$

Hence from (5.19), we get

$$\|u\|_{\mathcal{H}_1}^2 \leq \widetilde{C}\|v\|_{\mathcal{H}_{-1}}\|u\|_{\mathcal{H}_1},$$

$$\|u\|_{\mathcal{H}_1} \leq \widetilde{C}\|v\|_{\mathcal{H}_{-1}}, \tag{5.20}$$

for solutions of (5.15) with $v_+ = 0$.

Now take the scalar product of the first equation in (5.15) with $R_-^{0,0}u_-$ and get

$$-z(u|R_-^{0,0}u_-) + |u_-|^2 = \overline{u}_-(v|e^{-\phi/h}).$$

With (5.20), this gives $|u_-|^2 \leq \widehat{C}\|v\|_{\mathcal{H}_{-1}}|u_-|$ and hence

$$|u_-| \leq \widetilde{C}\|v\|_{\mathcal{H}_{-1}}, \tag{5.21}$$

where we let $\widetilde{C}$ denote a new constant in every new formula.

If $v_+ \neq 0$, consider $\widetilde{u} := u - v_+e^{-\phi/2h}$, which solves

$$\begin{cases} (\Delta_\phi^{(0)} - z)\widetilde{u} + R_-^{0,0}u_- = v + zv_+e^{-\phi/2h} \\ R_+^{0,0}\widetilde{u} = 0. \end{cases} \tag{5.22}$$

Applying (5.20), (5.21) to this system, we get

$$\|\widetilde{u}\|_{\mathcal{H}_1} + |u_-| \leq \widetilde{C}(\|v\|_{\mathcal{H}_{-1}} + |v_+|),$$

leading to

$$\|u\|_{\mathcal{H}_1} + |u_-| \leq \widetilde{C}(\|v\|_{\mathcal{H}_{-1}} + |v_+|), \tag{5.23}$$

for solutions of (5.15). Since this problem is selfadjoint, we also have existence and we have proved the proposition for (Gr(0,0)).

Les us now prove that if for some $N \in \mathbf{N}$ the proposition is valid for (Gr(N,0)) then it is valid for (Gr(N,1)). So we assume for a fixed N that (A) holds for all C with $h > 0$ small enough depending on C, and we want to prove (B) with the same N. Using again that $\phi''(x) - \phi''(0) = \mathcal{O}(h^{1/2}) : \mathcal{H}_1 \to \mathcal{H}_{-1}$, we see that it suffices to treat the simplified problem

$$\begin{cases} (1 \otimes \Delta_\phi^{(0)} + \phi''(0) - z)u + R_-^{N,1}u_- = v \\ R_+^{N,1}u = v_+. \end{cases} \tag{5.24}$$

Since we have (A), for the chosen value of N, we know from the preceding discussion that

$$\inf_{\substack{w\in\mathcal{H}_1,\, \|w\|=1 \\ R_+^{N,0}w=0}} (\Delta_\phi^{(0)}w|w) \geq (N+1)\lambda_{\min}(\phi''(0)) - \frac{1}{2C}, \tag{5.25}$$

for $h > 0$ small enough depending on C. Consider (5.24) in the case $v_+ = 0$. Then $R_+^{N,0}u_j = 0$ for each component u_j of u and consequently

$$((1 \otimes \Delta_\phi^{(0)})u|u) \geq ((N+1)\lambda_{\min} - \frac{1}{2C})\|u\|^2.$$

Since $(\phi'(0)u|u) \geq \lambda_{\min}\|u\|^2$, we get

$$((1 \otimes \Delta_\phi^{(0)} + \phi''(0) - z)u|u) \geq \frac{1}{2C}\|u\|^2, \tag{5.26}$$

for z in the range of values of (B). As before, this leads to

$$((1\otimes\Delta_\phi^{(0)}+\phi''(0)-z)u|u)\geq\delta\|u\|^2_{\mathcal{H}_1}, \tag{5.27}$$

for some $\delta>0$. Take the scalar product of the first equation in (5.24) with u, and use that $(R_-^{N,1}u_-|u)=(u_-|R_+^{N,1}u)=0$. Then

$$\delta\|u\|^2_{\mathcal{H}_1}\leq\|v\|_{\mathcal{H}_{-1}}\|u\|_{\mathcal{H}_1},$$

which gives,

$$\|u\|_{\mathcal{H}_1}\leq\widetilde{C}\|v\|_{\mathcal{H}_{-1}} \tag{5.28}$$

for solutions of (5.24) with $v_+=0$, when z is in the range of (B).

We next want to take the scalarproduct with $R_-^{N,1}u_-$, and as a preparation we need to establish two results about $R_\pm$.

Lemma 5.3. *$R_+^{N,0}R_-^{N,0}$ is $\mathcal{O}(1):\ell^2(\mathbf{N}^\Lambda_{[0,N]})\to\ell^2(\mathbf{N}^\Lambda_{[0,N]})$ and has a uniformly bounded inverse. Moreover, if $r_P:\ell^2(\mathbf{N}^\Lambda_{[0,N]})\to\ell^2(\mathbf{N}^\Lambda_P)$ is the natural restriction operator, then for $0\leq P,Q\leq N$.*

$$r_PR_+^{N,0}R_-^{N,0}r_Q^*\sim\sum_{\nu=0}^\infty h^{\nu+\frac{1}{2}|P-Q|}M_{\nu;P,Q}, \tag{5.29}$$

in $\mathcal{L}(\ell^2(\mathbf{N}^\Lambda_Q),\ell^2(\mathbf{N}^\Lambda_P))$ uniformly with repect to Λ. Here

$$M^{(N)}_{0;P,P}=\phi''(0)\odot..\odot\phi''(0). \tag{5.30}$$

Proof. For simplicity, we work with the equivalent operators $\widetilde{R}_\pm^{N,0}$ between $L^2(\mathbf{R}^\Lambda)$ and $\ell^2_b(\Lambda^0\cup\Lambda^1\cup..\cup\Lambda^N)$, where the subscript b indicates that we take the "Bosonic" subspace of permutation invariant elements of ℓ^2. Then the matrix of $r_P\widetilde{R}_+^{N,0}\widetilde{R}_-^{N,0}r_Q^*$ is given by

$$\frac{1}{\sqrt{P!Q!}}(Z^*_{p|\mathcal{P}}(e^{-\phi/2h})|Z^*_{q|\mathcal{Q}}(e^{-\phi/2h})), \tag{5.31}$$

with $\mathcal{P}=\{1,..,P\}$, $\mathcal{Q}=\{1,..,Q\}$, $p\in\Lambda^\mathcal{Q}$, $p\in\Lambda^\mathcal{P}$. The uniform asymptotic expansion (5.29) then follows from Proposition 4.1. Moreover, the matrix $\widetilde{M}^{(N)}_{0,P,P}$ (corresponding to $M^{(N)}_{0,P,P}$) has the elements

$$\frac{1}{P!}\sum_{\pi\in\mathrm{Perm}\,(K)}\prod_{\nu\in\mathcal{P}}\phi''_{p(\nu),q(\pi(\nu))}(0), \tag{5.32}$$

which has the same action on $\ell^2_b(\Lambda^P)$ as the matrix

$$\prod_{\nu\in\mathcal{P}}\phi''_{p(\nu),q(\nu)}(0), \tag{5.33}$$

which is simply the matrix $\phi''(0)\otimes .. \otimes \phi''(0)$. #

By tensoring all the spaces with $\ell^2(\Lambda)$, we get the obvious analogue of Lemma 5.3 for $R_+^{N,1}R_-^{N,1}$. It also follows from Lemma 5.3, that $R_-^{N,0}$ is uniformly $\mathcal{O}(1):\ell^2\to L^2$. Consequently $R_+^{N,0}=\mathcal{O}(1):L^2\to\ell^2$, and we have the corresponding facts for $R_\pm^{N,1}$. This can be strengthened:

Lemma 5.4. *$R_-^{N,0}$ is uniformly bounded: $\ell^2(\mathbf{N}_{[0,N]}^\Lambda)\to\mathcal{H}_1$. Consequently $R_+^{N,0}$ is uniformly bounded $\mathcal{H}_{-1}\to\ell^2(\mathbf{N}_{[0,N]}^\Lambda)$.*

Proof. Again we think it is more convenient to work with the equivalent operator $\widetilde{R}_-^{N,0}$. Let $1\le M\le N$, $u\in\ell_b^2(\Lambda^M)$ and consider

$$Z_j\widetilde{R}_-^{N,0}u=\sum_{m\in\Lambda^{\mathcal{M}}}\frac{1}{\sqrt{M!}}Z_jZ^*_{m|\mathcal{M}}(e^{-\phi/2h})u(m), \tag{5.34}$$

where $\mathcal{M}=\{1,..,M\}$. We apply the expression (3.17) to obtain

$$Z_j\widetilde{R}_-^{N,0}u=\sum_{m\in\Lambda^{\mathcal{M}}}\sum_{\substack{\mathcal{M}=M_0\cup M_1\\ \text{partition with}\\ M_1\neq\emptyset}}C_{M_0,M_1}h^{-\frac{1}{2}+\frac{1}{2}\#M_1}\partial_{x_j}\partial_{x_{m|M_1}}\phi(x)Z^*_{m|M_0}(e^{-\phi/2h})u(m), \tag{5.35}$$

where C_{M_0,M_1} is independent of Λ, and equal to $1/\sqrt{M!}$ when $\#M_1=1$. For $u\in\ell_b^2(\Lambda^M)$, $v\in\ell_b^2(\Lambda^P)$, $1\le P,M\le N$, we get

$$\sum_j(Z_j\widetilde{R}_-^{N,0}u|Z_j\widetilde{R}_-^{N,0}v)=(B^{P,M}u|v), \tag{5.36}$$

where $B^{P,M}$ is given by a matrix $B^{P,M}_{p,m}$, $p\in\Lambda^P$, $m\in\Lambda^M$, which is a finite linear combination of terms

$$h^{-1+\frac{1}{2}\#M_1+\frac{1}{2}\#P_1}(Z_{p|P_0}\sum_j(\partial_{x_{p|P_1}}\partial_{x_j}\phi)(\partial_{x_j}\partial_{x_{m|M_1}}\phi)Z^*_{m|M_0}(e^{-\phi/2h})|e^{-\phi/2h}), \tag{5.37}$$

where $\mathcal{M}=M_0\cup M_1$, $\mathcal{P}=P_0\cup P_1$ are partitions with $M_1\neq\emptyset\neq P_1$. Here

$$\Phi_{p|P_1,m|M_1}:=\sum_j(\partial_{x_{p|P_1}}\partial_{x_j}\phi)(\partial_{x_j}\partial_{x_{m|M_1}}\phi)$$

is a standard tensor, being the contraction of two standard tensors of size $1+\#P_1$ and $1+\#M_1$, with at least one of the sizes ≥ 2 (cf. Lemma 8.2).

As in sections 3,4, in particular the discussion leading to (3.12-15), we see that (5.37) is a finite sum of terms

$$h^X(\Phi^{(1)}_{p|\widetilde{P}_1,m|\widetilde{M}_1}..\Phi^{(Q)}_{p|\widetilde{P}_Q,m|\widetilde{M}_Q}e^{-\phi/2h}|e^{-\phi/2h}), \tag{5.38}$$

where $\mathcal{P} = \widetilde{P}_1 \cup .. \cup \widetilde{P}_Q$, $\mathcal{M} = \widetilde{M}_1 \cup .. \cup \widetilde{M}_Q$ are partitions with $\widetilde{P}_q, \widetilde{M}_q \neq \emptyset$, $\widetilde{P}_1 \supset P_1$, $\widetilde{M}_1 \supset M_1$. $\Phi^{(q)}$ are standard tensors and

$$X = \widetilde{N} + \sum_1^Q (\frac{1}{2}(\#\widetilde{M}_q + \#\widetilde{P}_q) - 1),\ \ \widetilde{N} \in [0, N_1] \cap \mathbf{N}.$$

Here we can fix any $N_1 \in \mathbf{N}$, and the $\Phi^{(q)}$ are independent of x, when $\widetilde{N} < N_1$.

As in section 3, we conclude that $B^{P,M}$ has an asymptotic expansion

$$B^{P,M} \sim \sum_0^\infty h^{\nu + \frac{1}{2}|P-M|} B_\nu^{P,M} \text{ in } \mathcal{L}(\ell_b^2(\Lambda^M), \ell_b^2(\Lambda^P)), \tag{5.39}$$

uniformly w.r.t. Λ. From this and (5.36) it follows that

$$\sum_j \|Z_j \widetilde{R}_-^{N,0} u\|^2 \leq \mathcal{O}(1)|u|_2^2,\ \ u \in \ell_b^2(\Lambda^0 \cup \Lambda^1 \cup .. \cup \Lambda^N), \tag{5.40}$$

where $|u| = |u|_2$ denotes the ℓ^2 norm of u. Hence

$$\|\widetilde{R}_-^{N,0} u\|_{\mathcal{H}_1}^2 = \|\widetilde{R}_-^{N,0} u\|^2 + \sum_j \|Z_j \widetilde{R}_-^{N,0} u\|^2 \leq \mathcal{O}(1)|u|_2^2, \tag{5.41}$$

and the lemma follows. #

We can now return to the simplified problem (5.24) with z in the range of (B) in Proposition 5.1. As in (5.28), we assume that $v_+ = 0$ and take the scalar product of the first equation with $R_-^{N,1} u_-$. Lemma 5.4 shows that

$$\begin{aligned} ((1 \otimes \Delta_\phi^{(0)} + \phi''(0) - z)u | R_-^{N,1} u_-) &= \mathcal{O}(1)\|u\|_{\mathcal{H}_1} |u_-|_2 \\ (v | R_-^{N,1} u_-) &= \mathcal{O}(1)\|v\|_{\mathcal{H}_{-1}} |u_-|_2, \end{aligned}$$

while Lemma 5.3 implies that

$$\|R_-^{N,1} u_-\|^2 \sim |u_-|_2^2.$$

Using also (5.28), we get

$$|u_-|_2 \leq \widetilde{C}\|v\|_{\mathcal{H}_{-1}},$$

so with (5.28), we get

$$(\|u\|_{\mathcal{H}_1} + |u_-|_2) \leq \widetilde{C}\|v\|_{\mathcal{H}_{-1}}. \tag{5.42}$$

Now let $v_+ \neq 0$ in (5.24). Then

$$\widetilde{u} = u - R_-^{N,1}(R_+^{N,1} R_-^{N,1})^{-1} v_+$$

satisfies

$$\begin{cases} (1\otimes \Delta_\phi^{(0)}+\phi''(0)-z)\widetilde{u}+R_-^{N,1}u_-=\widetilde{v} \\ R_+^{N,1}\widetilde{u}=0, \end{cases} \tag{5.43}$$

where

$$\widetilde{v}-v=-(1\otimes\Delta_\phi^{(0)}+\phi''(0)-z)R_-^{N,1}(R_+^{N,1}R_-^{N,1})^{-1}v_+=\mathcal{O}(1)|v_+|_2,$$

in $\mathcal{H}_{-1}$, by Lemmas 5.3, 5.4. We can apply (5.42) to (5.43) and get

$$(\|\widetilde{u}\|_{\mathcal{H}_1}+|u_-|_2)\le \widetilde{C}(\|v\|_{\mathcal{H}_{-1}}+|v_+|_2). \tag{5.44}$$

Since $\|u\|_{\mathcal{H}_1}\le\|\widetilde{u}\|_{\mathcal{H}_1}+\mathcal{O}(1)|v_+|_2$, by Lemmas 5.3, 5.4, we get

$$(\|u\|_{\mathcal{H}_1}+|u_-|_2)\le \widetilde{C}(\|v\|_{\mathcal{H}_{-1}}+|v_+|_2), \tag{5.45}$$

for solutions of (5.24) with $u\in\mathcal{H}_1$. Since (5.24) is a selfadjoint problem, we also get existence of solutions for all $v\in\mathcal{H}_{-1}$, $v_+\in\ell^2$. Using finally that $\phi''(x)-\phi''(0)=\mathcal{O}(h^{1/2}):\mathcal{H}_1\to\mathcal{H}_{-1}$, we get (B) of the proposition for the given value of N.

As a preparation for the step $\mathrm{Gr}\,(N,1)\to\mathrm{Gr}\,(N+1,0)$, we have to study $d_\phi\widetilde{R}_-^{N+1,0}u_-$ for $u_-\in\ell_b^2(\Lambda^0\cup\Lambda^1\cup..\cup\Lambda^{N+1})$. Let u_-^M be the component of u_- in $\ell_b^2(\Lambda^M)$, so that

$$\widetilde{R}_-^{N+1,0}u_-^M=\frac{1}{\sqrt{M!}}\sum_{m\in\Lambda^{\mathcal{M}}}Z^*_{m|\mathcal{M}}(e^{-\phi/2h})u_-^M(m),\ \mathcal{M}=\{1,..,M\}.$$

The jth component of $d_\phi\widetilde{R}_-^{N+1,0}u_-^M$ is then

$$Z_j\widetilde{R}_-^{N+1,0}u_-^M=\frac{1}{\sqrt{M!}}\sum_{m\in\Lambda^{\mathcal{M}}}Z_jZ^*_{m|\mathcal{M}}(e^{-\phi/2h})u_-^M(m).$$

If $M=0$, we get 0. If $M\ge 1$, we are in the situation of the proof of Lemma 5.4, which shows that

$$\begin{aligned} &Z_j\widetilde{R}_-^{N+1,0}u_-^M=\frac{1}{\sqrt{M!}}\sum_{p=1}^{M}\sum_{m\in\Lambda^{\mathcal{M}}}Z^*_{m|\mathcal{M}\setminus\{p\}}(e^{-\phi/2h})\phi''_{j,m(p)}(0)u_-^M(m)+h^{\frac12}\Psi_j^{M,N}, \\ &\sum_{j\in\Lambda}\|\Psi_j^{M,N}\|^2\le\mathcal{O}(1)|u_-^M|_2^2. \end{aligned} \tag{5.46}$$

Since u_-^M is invariant under permutations of $\mathcal{M}$, the first term of the RHS in the first equation of (5.46) can be written

$$\begin{aligned} &\frac{M}{\sqrt{M!}}\sum_{m\in\Lambda^{M-1}}Z^*_m(e^{-\phi/2h})\sum_{n\in\Lambda}\phi''_{j,n}(0)u_-^M((n,m)) \\ &=M^{\frac12}\widetilde{R}_-^{N,0}(((\phi''(0)\otimes 1\otimes..\otimes 1)u_-^M)(j,\cdot)) \\ &=M^{\frac12}\widetilde{R}_-^{N,1}((\phi''(0)\otimes 1\otimes..\otimes 1)u_-^M), \end{aligned} \tag{5.47}$$

where in the last equation, we consider that $(\phi''(0)\otimes 1\otimes ..\otimes 1)u_-^M\in \ell^2(\Lambda)\otimes \ell_b^2(\Lambda^{M-1})$.

Summing up, we have

$$d_\phi \widetilde{R}_-^{N+1,0}u_- = \widetilde{R}_-^{N,1}(\widetilde{\Phi}u_-)+h^{\frac{1}{2}}\widetilde{\Psi}u_-, \tag{5.48}$$

where $\widetilde{\Psi}=\mathcal{O}(1)=\ell_b^2(\Lambda^0\cup ..\cup\Lambda^{N+1})\to \ell^2(\Lambda)\otimes L^2(\mathbf{R}^\Lambda)$, $\widetilde{\Psi}u_-$ and $\widetilde{\Phi}u_-$ vanish when $u_-\in \ell_b^2(\Lambda^0)$, and for general u_-, $\widetilde{\Phi}u_-$ has the $\ell^2(\Lambda)\otimes \ell_b^2(\Lambda^{M-1})$ component equal to $M^{1/2}(\phi''(0)\otimes 1\otimes ..\otimes 1)u_-^M$, for $1\le M\le N+1$. Notice that

$$|\widetilde{\Phi}u_-|_2^2\sim \sum_{M=1}^{N}|u_-^M|_2^2, \tag{5.49}$$

so that even though $\widetilde{\Phi}$ kills the $M=0$ component of u_-, it acts injectively on the remaining part of u_-. Clearly we have an analogue of (5.48), (5.49) without the tildes.

Now fix some $N\in\mathbf{N}$ and assume that Proposition 5.1(B) holds for this value of N (and for all C). We shall prove that (A) holds with N replaced by $N+1$. Let z vary in the range for Gr(N+1,0) which is the same as for Gr(N,1):

$$-C\le z\le (N+2)\lambda_{\min}(\phi''(0))-\frac{1}{C}. \tag{5.50}$$

Let us first show injectivity in Gr(N+1,0), so consider the homogeneous system

$$\begin{cases}(\Delta_\phi^{(0)}-z)u+R_-^{N+1,0}u_-=0\\ R_+^{N+1,0}u=0.\end{cases} \tag{5.51}$$

By standard arguments, we see that $u\in\mathcal{S}(\mathbf{R}^\Lambda)$. Here we apply d_ϕ everywhere and use (5.48,49) to get

$$\begin{cases}(\Delta_\phi^{(1)}-z)d_\phi u+R_-^{N,1}\Phi u_-=-h^{\frac{1}{2}}\Psi u_-\\ R_+^{N,1}d_\phi u=0.\end{cases} \tag{5.52}$$

Here $\|\Psi u_-\|\le\mathcal{O}(1)|\Phi u_-|_2$ so by part (B) of the proposition, we get

$$\|d_\phi u\|_{\mathcal{H}_1}+|\Phi u_-|_2\le \mathcal{O}(1)h^{\frac{1}{2}}|\Phi u_-|_2.$$

It follows that $d_\phi u=0$, $\Phi u_-=0$, when h is small enough. Consequently, $u=\lambda e^{-\phi/2h}$ for some $\lambda\in\mathbf{C}$, and using that $R_+^{N+1,0}u=0$, we get $u=0$. Then (5.51) and the injectivity of $R_-^{N+1,0}$ imply that $u_-=0$, and we have proved injectivity for Gr(N+1,0) when z varies in the range (5.50) (with $C>0$ as large as we like).

As before we conclude that

$$\inf_{\substack{u\in\mathcal{H}_1,\,\|u\|=1\\ R_+^{N+1,0}u=0}}(\Delta_\phi^{(0)}u|u)\ge (N+2)\lambda_{\min}(\phi''(0))-\frac{1}{2C},$$

for every $C > 0$ when $h > 0$ is small enough depending on C. By repeating earlier arguments, we obtain the apriori estimate (5.10) for solutions to Gr(N+1,0), as well as existence of such solutions for arbitrary $v \in \mathcal{H}_{-1}$ and $v_+ \in \ell^2$. In other words, we get part (A) of the proposition with N replaced by $N+1$ and this completes the inductive proof of Proposition 5.1.

Remark 5.5. Let us compute $(\Delta_\phi^{(0)}\widetilde{R}_-^{N,0}u|\widetilde{R}_-^{N,0}v)$ to leading order for $u, v \in \ell_b^2(\Lambda^0 \cup .. \cup \Lambda^N)$, i.e. modulo $\mathcal{O}(1)h^{1/2}|u|_2|v|_2$. The proof of Lemma 5.4 shows that the searched expression involves a block diagonal matrix, so we may assume that $u, v \in \ell_b^2(\Lambda^{\mathcal{P}})$, $\mathcal{P} = \{1, .., P\}$, for $1 \le P \le N$. (The case $P = 0$ will give 0.) Then if $\equiv$ indicates equality modulo $\mathcal{O}(1)h^{1/2}|u|_2|v|_2$, we get

$$(\Delta_\phi^{(0)}\widetilde{R}_-^{N,0}u|\widetilde{R}_-^{N,0}v) = \sum_{j\in\Lambda}(Z_j\widetilde{R}_-^{N,0}u|Z_j\widetilde{R}_-^{N,0}v) \equiv$$

$$\frac{1}{P!}\sum_{j\in\Lambda}\sum_{\widehat{p},\widehat{q}=1}^{P}\sum_{p,q\in\Lambda^{\mathcal{P}}}\phi''_{q(\widehat{q}),j}(0)\phi''_{j,p(\widehat{p})}(0)(Z^*_{p|\mathcal{P}\setminus\{\widehat{p}\}}(e^{-\phi/2h})|Z^*_{q|\mathcal{P}\setminus\{\widehat{q}\}}(e^{-\phi/2h}))u(p)\overline{v(q)}.$$

Using that $u, v \in \ell_b^2$, we can reduce the sum to the case $\widehat{p} = \widehat{q} = 1$, and get

$$\begin{aligned}
&\frac{P^2}{P!}\sum_{j\in\Lambda}\sum_{p,q\in\Lambda^{\mathcal{P}}}\phi''_{q(1),j}(0)\phi''_{j,p(1)}(0)(Z^*_{p|\mathcal{P}\setminus\{1\}}(e^{-\phi/2h})|Z^*_{q|\mathcal{P}\setminus\{1\}}(e^{-\phi/2h}))u(p)\overline{v(q)}\\
&= \frac{P^2}{P!}\sum_{p,q\in\Lambda^{\mathcal{P}}}(\phi''(0)^2)_{q(1),p(1)}(Z^*_{p|\mathcal{P}\setminus\{1\}}(e^{-\phi/2h})|Z^*_{q|\mathcal{P}\setminus\{1\}}(e^{-\phi/2h}))u(p)\overline{v(q)}\\
&\equiv \frac{P^2}{P!}\sum_{p,q\in\Lambda^{\mathcal{P}}}(\phi''(0)^2)_{q(1),p(1)}\sum_{\pi\in\mathrm{Perm}(\{2,..,P\})}(\prod_{\nu=2}^{P}\phi''_{p(\nu),q(\pi(\nu))}(0))u(p)\overline{v(q)} =\\
&P\sum_{p,q\in\Lambda^{\mathcal{P}}}(\phi''(0)^2)_{q(1),p(1)}\prod_{\nu=2}^{P}(\phi''_{p(\nu),q(\nu)}(0))u(p)\overline{v(q)} = (P\phi''(0)^2\otimes\phi''(0)\otimes..\otimes\phi''(0)u|v)_{\ell^2}\\
&\qquad = (P(\phi''(0)\otimes 1\otimes..\otimes 1)(\phi''(0)^{\frac{1}{2}}\otimes..\otimes\phi''(0)^{\frac{1}{2}})u|(\phi''(0)^{\frac{1}{2}}\otimes..\otimes\phi''(0)^{\frac{1}{2}})v),
\end{aligned}$$

where we again used that $u, v \in \ell_b^2$. Using this property once more, we can replace $P(\phi''(0)\otimes 1\otimes..\otimes 1)$ by the more suggestive expression

$$\Phi_P := \phi''(0)\otimes 1\otimes..\otimes 1 + 1\otimes\phi''(0)\otimes 1..\otimes 1 + .. + 1\otimes..\otimes 1\otimes\phi''(0). \tag{5.53}$$

If $\lambda_1, .., \lambda_{\#\Lambda}$ denote the eigenvalues of $\phi''(0)$, then the eigenvalues of (5.53) are of the form

$$\sum_{\nu=1}^{P}\lambda_{p(\nu)},\ \ p \in \{1, .., \#\Lambda\}^{\mathcal{P}}. \tag{5.54}$$

Summing up the discussion, for $u \in \ell_b^2(\Lambda^{\mathcal{P}})$, $v \in \ell_b^{\mathcal{Q}}$, $\mathcal{P} = \{1,..,P\}$, $\mathcal{Q} = \{1,..,Q\}$, we have

$$\begin{aligned}&(\Delta_\phi^{(0)} \widetilde{R}_-^{N,0} u | \widetilde{R}_-^{N,0} v) &(5.55)\\ &= \mathcal{O}(h^{\frac{1}{2}})|u|_2|v|_2 + \begin{cases} 0, \text{ if } P \neq Q \\ (\Phi_P(\phi''(0)^{\frac{1}{2}} \otimes .. \otimes \phi''(0)^{\frac{1}{2}})u|(\phi''(0)^{\frac{1}{2}} \otimes .. \otimes \phi''(0)^{\frac{1}{2}})v), \ P = Q. \end{cases}\end{aligned}$$

This should be compared with the following consequence of section 4:

$$\begin{aligned}&(\widetilde{R}_-^{N,0} u | \widetilde{R}_-^{N,0} v) &(5.56)\\ &= \mathcal{O}(h^{\frac{1}{2}})|u|_2|v|_2 + \begin{cases} 0, \text{ if } P \neq Q \\ ((\phi''(0)^{\frac{1}{2}} \otimes .. \otimes \phi''(0)^{\frac{1}{2}})u|(\phi''(0)^{\frac{1}{2}} \otimes .. \otimes \phi''(0)^{\frac{1}{2}})v), \ P = Q. \end{cases}\end{aligned}$$

6. Asymptotics of the solutions of the Grushin problems.

We first work with the scalar case and denote by $\mathcal{L}_j$ the span of all $\frac{(Z^*)^\alpha}{\alpha!}(e^{-\phi/2h})$, $|\alpha| = j$. Equivalently, $\mathcal{L}_j$ is equal to $R_-^{N,0}(\ell^2(\mathbf{N}_j^\Lambda))$, if $j \leq N$. If A is a finite subset of $\mathbf{N}$, we write $\mathcal{L}_A = \oplus_{j\in A}\mathcal{L}_j \subset L^2$. Notice that the orthogonal projection onto $\mathcal{L}_{[0,N]}$ is given by

$$R_-^{N,0}(R_+^{N,0}R_-^{N,0})^{-1}R_+^{N,0}. \tag{6.1}$$

By section 4 we know that

$$\|R_-^{N,0}v_+\| \sim |v_+|, \ u \in \ell^2(\mathbf{N}_{[0,N]}^\Lambda). \tag{6.2}$$

We can identify $\mathcal{L}_j$ with $\ell^2(\mathbf{N}_j^\Lambda)$ by means of $r_jR_+^{N,0}$, where $r_j : \ell^2(\mathbf{N}_{[0,N]}^\Lambda) \to \ell^2(\mathbf{N}_j^\Lambda)$ is the natural restriction map, and again by section 4 we know that

$$|R_+^{N,0}u|_2 \sim \|u\|, \ u \in \mathcal{L}_{[0,N]}. \tag{6.3}$$

We have the decomposition

$$L^2(\mathbf{R}^\Lambda) = \mathcal{L}_0 \oplus .. \oplus \mathcal{L}_N \oplus \mathcal{L}_{[0,N]}^\perp, \ u = u_0 + .. + u_N + u_{N+1} \in L^2, \tag{6.4}$$

and correspondingly

$$\|u\|^2 \sim \sum_0^{N+1} \|u_j\|^2. \tag{6.5}$$

For $j \leq N$, the projection onto $\mathcal{L}_j$ is given by

$$\Pi_j = R_-^{N,0} r_j^* r_j (R_+^{N,0}R_-^{N,0})^{-1} R_+^{N,0}.$$

Lemma 5.4 and (6.2) imply that

$$\|u\|_{\mathcal{H}_1} \leq \mathcal{O}(1)\|u\|, \ u \in \mathcal{L}_{[0,N]}, \tag{6.6}$$

and the same lemma with (6.3) implies that

$$\|u\| \leq \mathcal{O}(1)\|u\|_{\mathcal{H}_{-1}}, \ u \in \mathcal{L}_{[0,N]}. \tag{6.7}$$

In other words, the norms of $\mathcal{H}_1$, $\mathcal{H}_{-1}$ and L^2 are (uniformly) equivalent on $\mathcal{L}_{[0,N]}$, and we also know that the projections (6.1) and Π_j are bounded in these spaces.

We are interested in the block matrix of $\Delta_\phi^{(0)}$, viewed as an operator

$$\begin{aligned}\Delta_\phi^{(0)} :&\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus .. \oplus \mathcal{L}_N \oplus (\mathcal{H}_1 \cap \mathcal{L}_{[0,N]}^\perp) \to \\ &\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus .. \oplus \mathcal{L}_N \oplus (\mathcal{H}_{-1} \cap \mathcal{L}_{[0,N]}^\perp).\end{aligned} \tag{6.8}$$

(3.12) shows that for $j \in \Lambda$, $\mathcal{M} = \{1, .., M\}$, $0 \leq M \leq N$, $m \in \Lambda^{\mathcal{M}}$:

$$\begin{aligned}&Z_j Z^*_{m|\mathcal{M}}(e^{-\phi/2h}) = \text{ a finite sum of terms of the type} \\ &\sum_{\ell \in \Lambda^L} h^X Z^*_{m|M_0} Z^*_{\ell|L} \circ \Phi_{j,\ell|L,m|M_1}(x)(e^{-\phi/2h}), \ \#M_0 + \#L \leq N,\end{aligned} \tag{6.9}$$

where $\mathcal{M} = M_0 \cup M_1$ is a partition with $M_1 \neq \emptyset$, L is finite, and Φ is standard. Moreover,

$$X = \frac{1}{2}\#L + \widetilde{N} + \frac{1}{2}(\#M_1 - 1), \ 0 \leq \widetilde{N} \leq N_1 \in \mathbf{N}. \tag{6.10}$$

Here N_1 is any sufficiently large integer and $\Phi_{j,\ell|L,m|M_1}$ is independent of x, when $\#M_0 + \#L < N$ and $\widetilde{N} < N_1$. Using the representation $\Delta_\phi^{(0)} = \sum_{j\in\Lambda} Z_j^* Z_j$, it follows that with standard tensors Φ:

$$\begin{aligned}&\Delta_\phi^{(0)}(Z^*_{m|\mathcal{M}} e^{-\phi/2h}) = \text{ a finite sum of terms of the type} \\ &\sum_{\ell \in \Lambda^L} h^Y Z^*_{m|M_0} Z^*_{\ell|L} \circ \Phi_{\ell|L,m|M_1}(x)(e^{-\phi/2h}), \ \#M_0 + \#L \leq N+1,\end{aligned} \tag{6.11}$$

where $\mathcal{M} = M_0 \cup M_1$ is a partition with $M_1 \neq \emptyset$, $\emptyset \neq L \subset \mathbf{N}$ is finite. Moreover,

$$Y = \frac{1}{2}(\#L - 1) + \frac{1}{2}(\#M_1 - 1) + \widetilde{N}, \ 0 \leq \widetilde{N} \leq N_1 \in \mathbf{N}, \tag{6.12}$$

and $\Phi_{\ell|L,m|M_1}$ is independent of x, when $\widetilde{N} < N_1$, $\#M_0 + \#L < N + 1$.

If we put $P = \#M_0 + \#L$ in (6.11), then since $\#M_0 + \#M_1 = M$, we get $P - M = (\#L - 1) - (\#M_1 - 1)$ and it follows that

$$Y = \frac{1}{2}|P - M| + \widetilde{N}, \ \widetilde{N} \leq \widehat{N} \in \mathbf{N}. \tag{6.13}$$

We conclude as in sections 3, 5, that for every $N_1 \in \mathbf{N}$, we have for $u \in \ell_b^2(\Lambda^M)$:

$$\Delta_\phi^{(0)} \widetilde{R}_-^{N,0} u = \sum_{P=0}^{N+1} h^{\frac{1}{2}|P-M|} D_{P,M} u + h^{N_1} R u, \tag{6.14}$$

with for $0 \leq P \leq N$

$$D_{P,M} = \mathcal{O}(1) : \ell^2 \to \mathcal{L}_P, \tag{6.15}$$

$$D_{P,M} \sim \sum_{\nu=0}^{\infty} h^\nu E_{P,M}^{(\nu)} \text{ in } \mathcal{L}(\ell^2, \mathcal{L}_P), \tag{6.16}$$

where the sum is finite and we identify $\mathcal{L}_P$ and $\ell^2(\mathbf{N}_P^\Lambda)$ by means of $R_-^{N,0}$. Moreover $\|Ru\| \leq \mathcal{O}(1)\|u\|$ and Ru is of the form (6.17) below, with M_0, M_1, L as in (6.11). Further, $D_{N+1,M}u$ is a finite sum of terms

$$\sum_{\ell\in\Lambda^L,\, m\in\Lambda^M} h^Z Z_{m|M_0}^* Z_{\ell|L}^* \circ \Phi_{\ell|L,m|M_1}(x)(e^{-\phi(x)/2h})u(m), \tag{6.17}$$

with M_0, M_1, L as in (6.11) and with $\#M_0 + \#L = N+1$, $Z \in \mathbf{N}$, and

$$D_{N+1,M} = \mathcal{O}(1) : \ell^2 \to L^2. \tag{6.18}$$

We whall next decompose $D_{N+1,M}u$ into $\mathcal{L}_{[0,N]} \oplus \mathcal{L}_{[0,N]}^\perp$, and consider first $r_P \widetilde{R}_+^{N,0} D_{N+1,M}u$, for $0 \leq P \leq N$, $u \in \ell_b^2(\Lambda^M)$. The matrix element of $r_P \widetilde{R}_+^{N,0} D_{N+1,M}$ at p, m, with $p \in \Lambda^{\mathcal{P}}$, $m \in \Lambda^{\mathcal{M}}$, $\mathcal{P} = \{1, .., P\}$, $\mathcal{M} = \{1, .., M\}$, is a finite sum of terms of the type

$$h^\nu \sum_{\ell\in\Lambda^L} (Z_{p|\mathcal{P}} Z_{m|M_0}^* Z_{\ell|L}^* \Phi_{\ell|L,m|M_1}(x) e^{-\phi/2h} | e^{-\phi/2h}), \tag{6.19}$$

with $M = M_0 \cup M_1$ being a partition with $M_1 \neq \emptyset$, $L \subset \mathbf{N}$ finite with $\#M_0 + \#L = N+1$, $\nu \in \mathbf{N}$. As before, we get an asymptotic expansion in $\mathcal{L}(\ell_b^2, \ell_b^2)$:

$$r_P \widetilde{R}_+^{N,0} D_{N+1,M} \sim \sum_{\nu=0}^{\infty} h^{\frac{1}{2}(|N+1-P|)+\nu} F_{P,M;N+1}^{(\nu)}. \tag{6.20}$$

From this and Proposition 4.1 it follows that

$$r_P (\widetilde{R}_+^{N,0} \widetilde{R}_-^{N,0})^{-1} \widetilde{R}_+^{N,0} D_{N+1,M} \sim \sum_{\nu=0}^{\infty} h^{\frac{1}{2}|N+1-P|+\nu} G_{P,M;N+1}^{(\nu)} \text{ in } \mathcal{L}(\ell^2, \ell^2), \tag{6.21}$$

and the promised decomposition of $D_{N+1,M}u$ is given by

$$D_{N+1,M}u = \sum_{P=0}^{N+1} D_{P,M;N+1}u, \tag{6.22}$$

with

$$D_{P,M;N+1}u = \widetilde{R}_-^{N,0} r_P (\widetilde{R}_+^{N,0} \widetilde{R}_-^{N,0})^{-1} \widetilde{R}_+^{N,0} D_{N+1,M}u \in \mathcal{L}_P, \tag{6.23}$$

for $0\le P\le N$, and with $D_{N+1,M:N+1}u$ being the remainder. Notice that $D_{N+1,M;N+1}u\in \mathcal{L}_{[0,N]}^\perp$ and that

$$\|D_{P,M;N+1}u\|\le \mathcal{O}(1)h^{\frac{1}{2}|N+1-P|}|u|_2,\ 0\le P\le N+1. \tag{6.24}$$

Since we can use $R_\pm^{N,0}$ to parametrize the spaces $\mathcal{L}_j$, $0\le j\le P$, we obtain the following result.

Proposition 6.1. *Fix $N\in\mathbf{N}$ and let $\Delta_{i,j}^{(0)}$, $0\le i,j\le N+1$, be the block matrix decomposition of $\Delta_\phi^{(0)}$ corresponding to (6.8). Then*

$$\|\Delta_{i,j}^{(0)}\|_{\mathcal{L}(L^2,L^2)}=\mathcal{O}(1)h^{\frac{1}{2}|i-j|},\ (i,j)\in\{0,1,..,N+1\}^2\setminus\{(N+1,N+1)\}. \tag{6.25}$$

Moreover, for $0\le i,j\le N$, we have

$$\Delta_{i,j}^{(0)}\sim\sum_{\nu=0}^{\infty}h^{\frac{1}{2}|i-j|+\nu}A_{i,j}^{(\nu)},\ \textit{in } \mathcal{L}(\mathcal{L}_j,\mathcal{L}_i), \tag{6.26}$$

where, in the RHS, we identify $\mathcal{L}_k$ with $\ell^2(\mathbf{N}_k^\Lambda)$ by means of $R_-^{N,0}$.

Here we have already verified (6.25) for $j\le N$, and the cases with $j=N+1$ follow if we write with $R_\pm=R_\pm^{N,0}$

$$\Delta_{i,N+1}=\Pi_i\Delta_\phi^{(0)}(1-\Pi_{\mathcal{L}_{[0,N]}})=\sum_{P=0}^{N}R_-r_i^*r_i(R_+R_-)^{-1}r_P^*r_PR_+\Delta_\phi^{(0)}(1-\Pi_{\mathcal{L}_{[0,N]}}),$$

and observe that $r_PR_+\Delta_\phi^{(0)}(1-\Pi_{\mathcal{L}_{[0,N]}})$ is the adjoint of an operator with $L^2\to L^2$ norm $\mathcal{O}(h^{\frac{1}{2}|P-(N+1)|})$. Notice also that $\Delta_{N+1,N+1}^{(0)}=\mathcal{O}(1):\mathcal{H}_1\to\mathcal{H}_{-1}$.

Let $d_0,..,d_{N+1}>0$ be constants with $d_{N+1}=1$, such that

$$d_{j+1}/d_j\in[h^{\frac{1}{2}},h^{-\frac{1}{2}}],\ 0\le j\le N, \tag{6.27}$$

or satifying the sharper assumption

$$d_{j+1}/d_j\in[\delta,1/\delta],\ 0\le j\le N, \tag{6.28}$$

for some $h^{\frac{1}{2}}\le\delta\le 1$. Let $\widetilde{d}:\ell^2(\mathbf{N}_{[0,N]}^\Lambda)\to\ell^2(\mathbf{N}_{[0,N]}^\Lambda)$ be given by the block diagonal matrix $\operatorname{diag}(d_j)_{0\le j\le N}$, with respect to the orthogonal decomposition $\ell^2(\mathbf{N}_{[0,N]}^\Lambda)=\oplus_{j=1}^N\ell^2(\mathbf{N}_j^\Lambda)$. Put

$$K_+=(R_+R_-)^{-\frac{1}{2}}R_+,\ K_-=R_-(R_+R_-)^{-\frac{1}{2}}$$

so that

$$K_+^*=K_-,\ K_+K_-=1,\ K_-K_+=\Pi=\Pi_{[0,N]}.$$

Put $d = K_-\widetilde{d}K_+ + (1-\Pi)$, where we notice that the first term commutes with Π; $\Pi K_-\widetilde{d}K_+ = K_-\widetilde{d}K_+\Pi = K_-\widetilde{d}K_+$. We observe that d and $\widetilde{d}$ are selfadjoint and that d^{-1} corresponds to $\widetilde{d}^{-1}$: $d^{-1} = K_-\widetilde{d}^{-1}K_+ + (1-\Pi)$.

Consider $\widetilde{d}^{-1}R_+d = \widetilde{d}^{-1}(R_+R_-)^{\frac{1}{2}}\widetilde{d}(R_+R_-)^{-\frac{1}{2}}R_+$. Here we know from Proposition 4.1 that the block matrix elements $((R_+R_-)^{\frac{1}{2}})_{j,k}$ are $\mathcal{O}(h^{|j-k|/2})$ and it follows that

$$\widetilde{d}^{-1}(R_+R_-)^{\frac{1}{2}}\widetilde{d} = \mathcal{O}(1) : \ell^2 \to \ell^2$$

under the assumption (6.27) and that

$$\widetilde{d}^{-1}(R_+R_-)^{\frac{1}{2}}\widetilde{d} - (R_+R_-)^{\frac{1}{2}} = \mathcal{O}(1)\frac{h^{\frac{1}{2}}}{\delta} : \ell^2 \to \ell^2$$

under the assumption (6.28). We conclude that under the latter assumption

$$\begin{aligned} \widetilde{d}^{-1}R_+d - R_+ &= \mathcal{O}(1)\frac{h^{\frac{1}{2}}}{\delta} : \mathcal{H}_{-1} \to \ell^2 \\ d^{-1}R_-\widetilde{d} - R_- &= \mathcal{O}(1)\frac{h^{\frac{1}{2}}}{\delta} : \ell^2 \to \mathcal{H}_1. \end{aligned} \tag{6.29}$$

Here the second relation follows from the first by duality and in both relations, we are allowed to replace $(\widetilde{d}, d)$ be $(\widetilde{d}^{-1}, d^{-1})$.

Now recall that the $\mathcal{H}_{\pm 1}$ norms and the L^2 norm are all equivalent on $\mathcal{L}_{[0,N]}$, and consider

$$\begin{aligned} &d^{-1}\Delta_\phi^{(0)}d - \Delta_\phi^{(0)} = \\ &K_-(\widetilde{d}^{-1}(R_+R_-)^{-\frac{1}{2}}R_+\Delta_\phi^{(0)}R_-(R_+R_-)^{-\frac{1}{2}}\widetilde{d} - (R_+R_-)^{-\frac{1}{2}}R_+\Delta_\phi^{(0)}R_-(R_+R_-)^{-\frac{1}{2}})K_+ \\ &+ K_-(\widetilde{d}^{-1}-1)(R_+R_-)^{-\frac{1}{2}}R_+\Delta_\phi^{(0)}(1-\Pi) + (1-\Pi)\Delta_\phi^{(0)}R_-(R_+R_-)^{-\frac{1}{2}}(\widetilde{d}-1)K_+. \end{aligned}$$

Here the block matrix element of $(R_+R_-)^{-\frac{1}{2}}R_+\Delta_\phi^{(0)}R_-(R_+R_-)^{-\frac{1}{2}}$ at (j,k) is $\mathcal{O}(h^{\frac{1}{2}|j-k|})$, so

$$\begin{aligned} \widetilde{d}^{-1}(R_+R_-)^{-\frac{1}{2}}R_+\Delta_\phi^{(0)}R_-(R_+R_-)^{-\frac{1}{2}}\widetilde{d} - (R_+R_-)^{-\frac{1}{2}}R_+\Delta_\phi^{(0)}R_-(R_+R_-)^{-\frac{1}{2}} \\ = \mathcal{O}(1)\frac{h^{\frac{1}{2}}}{\delta} : \ell^2 \to \ell^2. \end{aligned}$$

Similarly

$$\begin{aligned} (\widetilde{d}^{-1}-1)(R_+R_-)^{-\frac{1}{2}}R_+\Delta_\phi^{(0)}(1-\Pi) &= \mathcal{O}(1)\frac{h^{\frac{1}{2}}}{\delta} : L^2 \to \ell^2 \\ (1-\Pi)\Delta_\phi^{(0)}R_-(R_+R_-)^{-\frac{1}{2}}(\widetilde{d}-1) &= \mathcal{O}(1)\frac{h^{\frac{1}{2}}}{\delta} : \ell^2 \to L^2, \end{aligned}$$

and we conclude that

$$d^{-1}\Delta_\phi^{(0)}d - \Delta_\phi^{(0)} = \mathcal{O}(1)\frac{h^{\frac{1}{2}}}{\delta} : \mathcal{H}_1 \to \mathcal{H}_{-1}. \tag{6.30}$$

Define

$$D = \begin{pmatrix} d & 0 \\ 0 & \widetilde{d} \end{pmatrix} = \mathcal{H}_{\pm 1} \times \ell^2(\mathbf{N}^{\Lambda}_{[0,N]}) \to \mathcal{H}_{\pm 1} \times \ell^2(\mathbf{N}^{\Lambda}_{[0,N]}). \tag{6.31}$$

If

$$\mathcal{P}^{N,0} = \begin{pmatrix} \Delta^{(0)}_{\phi} - z & R^{N,0}_{-} \\ R^{N,0}_{+} & 0 \end{pmatrix}, \tag{6.32}$$

then under the assumption (6.27)

$$D^{-1}\mathcal{P}^{N,0}D = \mathcal{O}(1) : \mathcal{H}_1 \times \ell^2 \to \mathcal{H}_{-1} \times \ell^2, \tag{6.33}$$

and if (6.28) holds, then

$$D^{-1}\mathcal{P}^{N,0}D - \mathcal{P}^{N,0} = \mathcal{O}(1)h^{\frac{1}{2}}/\delta : \mathcal{H}_1 \times \ell^2 \to \mathcal{H}_{-1} \times \ell^2. \tag{6.34}$$

Under the assumptions of Proposition 5.1(A), we introduce

$$\mathcal{E}^{N,0} = (\mathcal{P}^{N,0})^{-1} : \mathcal{H}_{-1} \times \ell^2 \to \mathcal{H}_1 \times \ell^2. \tag{6.35}$$

Under the assumption (6.27), we have

$$D^{-1}\mathcal{E}^{N,0}D = \mathcal{O}(1) : \mathcal{H}_{-1} \times \ell^2 \to \mathcal{H}_1 \times \ell^2, \tag{6.36}$$

(noticing that we have (6.28) with $\delta = Ch^{1/2}$ and C large enough) and if we assume (6.30), then

$$D^{-1}\mathcal{E}^{N,0}D - \mathcal{E}^{N,0} = \mathcal{O}(1)h^{\frac{1}{2}}/\delta. \tag{6.37}$$

Write

$$\mathcal{E}^{N,0} = \begin{pmatrix} E^{N,0} & E^{N,0}_{+} \\ E^{N,0}_{-} & E^{N,0}_{-+} \end{pmatrix}. \tag{6.38}$$

We shall derive approximations of $E_{\pm}$, E_{-+}, where we sometimes drop the superscript $N,0$ and for that we look for an approximate solution of the system

$$\begin{cases} (\Delta^{(0)}_{\phi} - z)u + R_{-}u_{-} = 0 \\ R_{+}u = v_{+}. \end{cases} \tag{6.39}$$

Try

$$u_0 = R_{-}(R_{+}R_{-})^{-1}v_{+} =: E^0_{+}v_{+}, \tag{6.40}$$

so that $R_{+}u_0 = v_{+}$. We will choose $u_{-} = u^0_{-}$ in order to satisfy the $\mathcal{L}_{[0,N]}$ component of the first equation of (6.39). Since the orthogonal projection onto that component is given by $R_{-}(R_{+}R_{-})^{-1}R_{+}$, this means that we look for $u^0_{-} \in \ell^2$, such that

$$R_{+}(\Delta^{(0)}_{\phi} - z)u_0 + R_{+}R_{-}u^0_{-} = 0,$$

i.e. we take

$$u_-^0 = (R_+R_-)^{-1}R_+(z-\Delta_\phi^{(0)})R_-(R_+R_-)^{-1}v_+ =: E_{-+}^0 v_+. \tag{6.41}$$

If $v_+ \in \ell^2(\mathbf{N}_M^\Lambda)$, $0 \le M \le N$, then

$$\begin{cases} (\Delta_\phi^{(0)} - z)E_+^0 v_+ + R_- E_{-+}^0 v_+ = \sum_{0\le \widetilde{M}\le N} h^{\frac{1}{2}|N+1-\widetilde{M}|} D_{N+1,\widetilde{M};N+1} r_{\widetilde{M}}^* r_{\widetilde{M}} (R_+R_-)^{-1}v_+ \\ R_+E_+^0 v_+ = v_+, \end{cases} \tag{6.42}$$

where $D_{P,M;N+1}$ is defined as in (6.22), with $\widetilde{R}_-^{N,0}$ replaced by the equivalent operator $R_-^{N,0}$. Then (dropping the superscripts in (6.38)) we get

$$\begin{cases} E_+v_+ = E_+^0 v_+ - \sum_{0\le \widetilde{M}\le N} h^{\frac{1}{2}|N+1-\widetilde{M}|} ED_{N+1,\widetilde{M};N+1} r_{\widetilde{M}}^* r_{\widetilde{M}} (R_+R_-)^{-1}v_+ \\ E_{-+}v_+ = E_{-+}^0 v_+ - \sum_{0\le \widetilde{M}\le N} h^{\frac{1}{2}|N+1-\widetilde{M}|} E_-D_{N+1,\widetilde{M};N+1} r_{\widetilde{M}}^* r_{\widetilde{M}} (R_+R_-)^{-1}v_+. \end{cases} \tag{6.43}$$

Recall that Π_j, $j=0,..,N+1$ are the projections associated to the decomposition (6.4) and that $\Pi_j = R_-r_j^*r_j(R_+R_-)^{-1}R_+$, $0\le j\le N$, $\Pi_{N+1} = 1-\Pi_{[0,N]}$. Let $A:\mathcal{H}_{-1}\to\mathcal{H}_1$. We claim that the following two statements are equivalent:

(a) $d^{-1}Ad = \mathcal{O}(1) : \mathcal{H}_{-1}\to\mathcal{H}_1$ for all (d_j) satisfying (6.27).

(b) $\Pi_jA\Pi_k = \mathcal{O}(h^{\frac{1}{2}|j-k|}) : \mathcal{H}_{-1}\to\mathcal{H}_1$ for all $j,k\in\{0,..,N+1\}$.

To see that, we introduce the orthogonal projections $\widehat{\Pi}_j$, $0\le j\le N+1$, with $\widehat{\Pi}_j\widehat{\Pi}_k=0$ for $j\ne k$, $1=\widehat{\Pi}_0+..+\widehat{\Pi}_{N+1}$, by

$$\widehat{\Pi}_j = R_-(R_+R_-)^{-\frac{1}{2}}r_j^*r_j(R_+R_-)^{-\frac{1}{2}}R_+ \text{ for } 0\le j\le N,\ \widehat{\Pi}_{N+1}=\Pi_{N+1}.$$

Then $d=\sum_0^{N+1} d_j\widehat{\Pi}_j$ and

$$d^{-1}Ad = \sum_{j,k=0}^{N+1} \frac{d_k}{d_j}\widehat{\Pi}_jA\widehat{\Pi}_k. \tag{$*$}$$

We also notice that $\widehat{\Pi}_j = \mathcal{O}(1) = \mathcal{H}_{\pm1}\to\mathcal{H}_{\pm1}$. We shall show that (a) and (b) are both equivalent to the statement

(c) $\widehat{\Pi}_jA\widehat{\Pi}_k = \mathcal{O}(h^{\frac{1}{2}|j-k|}) : \mathcal{H}_{-1}\to\mathcal{H}_1$, for all $0\le j,k\le N+1$.

That (c) implies (a) is obvious if we use $(*)$, and to get from (a) to (c), it suffices to write

$$\mathcal{O}(1) = \widehat{\Pi}_jd^{-1}Ad\widehat{\Pi}_k = \frac{d_k}{d_j}\widehat{\Pi}_jA\widehat{\Pi}_k,$$

and choose d_ν satisfying (6.27) such that $d_k/d_j = h^{-\frac{1}{2}|j-k|}$.

The equivalence between (b) and (c) is an easy consequence of the following estimates

$$\Pi_j\widehat{\Pi}_k,\ \widehat{\Pi}_j\Pi_k = \mathcal{O}(h^{\frac{1}{2}|j-k|}) : \mathcal{H}_{\pm1}\to\mathcal{H}_{\pm1},$$

that we shall verify:

When $j = k = N+1$, we have $\Pi_{N+1}\widehat{\Pi}_{N+1} = \widehat{\Pi}_{N+1}\Pi_{N+1} = \Pi_{N+1} = \widehat{\Pi}_{N+1}$.

When $j \neq k$ and $N+1 \in \{j,k\}$, then $\widehat{\Pi}_j\Pi_k = \Pi_j\widehat{\Pi}_k = 0$.

For $0 \le j,k \le N$, we get

$$\Pi_j\widehat{\Pi}_k = R_- r_j^* r_j (R_+R_-)^{-\frac{1}{2}} r_k^* r_k (R_+R_-)^{-\frac{1}{2}} R_+$$

and the block matrix element $r_j(R_+R_-)^{-\frac{1}{2}}r_k^*$ is $\mathcal{O}(h^{\frac{1}{2}|j-k|}) : \ell^2 \to \ell^2$. Consequently $\Pi_j\widehat{\Pi}_k = \mathcal{O}(h^{\frac{1}{2}|j-k|}) : \mathcal{H}_{-1} \to \mathcal{H}_1$. Similarly,

$$\widehat{\Pi}_j\Pi_k = R_-(R_+R_-)^{-\frac{1}{2}} r_j^* r_j (R_+R_-)^{\frac{1}{2}} r_k^* r_k (R_+R_-)^{-1} R_+ = \mathcal{O}(h^{\frac{1}{2}|j-k|}) : \mathcal{H}_{-1} \to \mathcal{H}_1.$$

Combining (6.24), (6.36) and (6.43), we get:

Proposition 6.2. *With E_+^0, E_{-+}^0 given by (6.40), (6.41) and under the assumptions of Proposition 5.1(A), we have*

$$\begin{cases} \Pi_P(E_+^{N,0} - E_+^0)r_Q^* = \mathcal{O}(1)h^{\frac{1}{2}(|N+1-Q|+|N+1-P|)} : \ell^2 \to \mathcal{H}_1 \\ r_Q(E_-^{N,0} - E_-^0)\Pi_P = \mathcal{O}(1)h^{\frac{1}{2}(|N+1-Q|+|N+1-P|)} : \mathcal{H}_{-1} \to \ell^2 \\ r_{\widetilde{P}}(E_{-+}^{N,0} - E_{-+}^0)r_Q^* = \mathcal{O}(1)h^{\frac{1}{2}(|N+1-Q|+|N+1-\widetilde{P}|)} : \ell^2 \to \ell^2, \end{cases} \tag{6.44}$$

for $0 \le P \le N+1$, $0 \le \widetilde{P}, Q \le N$, where $E_-^0 := (E_+^0)^*$.

Here the second equation in (6.44) is obtained by duality, using that Π_{N+1} is the orthogonal projection (6.1) and that Π_P is given after (6.5).

Now let $M > N$ and let us compare

$$\mathcal{P}^{N,0}, \mathcal{P}^{M,0}$$

and their inverses for z in the domain of wellposedness of the "smaller" problem $\mathcal{P}^{N,0}$. Let $r = r_{[0,N]}$ denote the natural restriction operator: $\ell^2(\mathbf{N}^\Lambda_{[0,M]}) \to \ell^2(\mathbf{N}^\Lambda_{[0,N]})$, and notice that

$$R_+^{N,0} = r_{[0,N]}R_+^{M,0},\ R_-^{N,0} = R_-^{M,0}r_{[0,N]}^*, \tag{6.45}$$

where $r_{[0,N]}^*$ is the adjoint: $\ell^2(\mathbf{N}^\Lambda_{[0,N]}) \to \ell^2(\mathbf{N}^\Lambda_{[0,N]})$. To shorten the notations, we write $\mathcal{P} = \mathcal{P}^{N,0}$, $\widetilde{\mathcal{P}} = \mathcal{P}^{M,0}$, and similarly for the associated quantities. In order to solve Gr(N,0):

$$\begin{cases} (\Delta_\phi^{(0)} - z)u + R_-u_- = v \\ R_+u = v_+, \end{cases} \tag{6.46}$$

we consider the bigger problem Gr(M,0)

$$\begin{cases} (\Delta_\phi^{(0)} - z)u + \widetilde{R}_-\widetilde{u}_- = v \\ \widetilde{R}_+u = \widetilde{v}_+, \end{cases} \tag{6.47}$$

and write the solution as

$$\begin{cases} u = \widetilde{E}v + \widetilde{E}_+\widetilde{v}_+ \\ \widetilde{u}_- = \widetilde{E}_-v + \widetilde{E}_{-+}\widetilde{v}_+. \end{cases} \tag{6.48}$$

We want (6.46) to be fulfilled, so we get the condition

$$R_-u_- = \widetilde{R}_-\widetilde{u}_-. \tag{6.49}$$

The necessary and sufficient condition on $\widetilde{u}_-$ for (6.49) to have a solution u_- is

$$r_{[N+1,M]}\widetilde{u}_- = 0, \tag{6.50}$$

where $r_{[N+1,M]} : \ell^2(\mathbf{N}^\Lambda_{[0,M]}) \to \ell^2(\mathbf{N}^\Lambda_{[N+1,M]})$ is the restriction operator, and the corresponding u_- is then

$$u_- = r_{[0,N]}\widetilde{u}_-. \tag{6.51}$$

We then get a solution of (6.46) iff

$$r_{[N+1,M]}\widetilde{E}_-v + r_{[N+1,M]}\widetilde{E}_{-+}\widetilde{v}_+ = 0, \tag{6.52}$$

$$r_{[0,N]}\widetilde{v}_+ = v_+. \tag{6.53}$$

(6.52) is equivalent to

$$\widetilde{E}_{-+}\widetilde{v}_+ + r^*_{[0,N]}w_- = -\widetilde{E}_-v, \text{ for some } w_- \in \ell^2(\mathbf{N}^\Lambda_{[0,N]}). \tag{6.54}$$

(6.54), (6.53) lead to a new Grushin problem, namely to invert the matrix

$$\widetilde{\mathcal{E}}_{-+} := \begin{pmatrix} \widetilde{E}_{-+} & r^*_{[0,N]} \\ r_{[0,N]} & 0 \end{pmatrix}, \tag{6.55}$$

which is wellposed in the range of wellposedness of $\mathcal{P}$ given in Proposition 5.1(A). This follows from Remark 5.5 and the fact that

$$\widetilde{E}_{-+} = (\widetilde{R}_+\widetilde{R}_-)^{-1}\widetilde{R}_+(z - \Delta^{(0)}_\phi)\widetilde{R}_-(\widetilde{R}_+\widetilde{R}_-)^{-1} + \mathcal{O}(h^{\frac{1}{2}}),$$

by Proposition 6.2. Modulo $\mathcal{O}(h^{\frac{1}{2}})$ we obtain a block diagonal matrix and the diagonal block at (j,j) with $0 \le j \le M$ is given by

$$(\phi''(0)^{-\frac{1}{2}}\otimes..\otimes\phi''(0)^{-\frac{1}{2}})(z-(\phi''(0)\otimes 1\otimes..\otimes 1+...1\otimes..\otimes 1\otimes\phi''(0)))(\phi''(0)^{-\frac{1}{2}}\otimes..\otimes\phi''(0)^{-\frac{1}{2}}).$$

Here the tensor products are of length j and for $j = 0$ the expression above should be replaced by z.

Let

$$\begin{pmatrix} F & F_+ \\ F_- & F_{-+} \end{pmatrix} \tag{6.56}$$

be the inverse of (6.55), so that

$$\begin{pmatrix} \widetilde{v}_+ \\ w_- \end{pmatrix} = \begin{pmatrix} F & F_+ \\ F_- & F_{-+} \end{pmatrix} \begin{pmatrix} -\widetilde{E}_- v \\ v_+ \end{pmatrix}, \tag{6.57}$$

i.e.

$$\widetilde{v}_+ = -F\widetilde{E}_- v + F_+ v_+, \ w_- = -F_-\widetilde{E}_- v + F_{-+} v_+. \tag{6.58}$$

The solution of (6.46) is then given from (6.48), (6.51), (6.58):

$$\begin{aligned} u &= \widetilde{E}v + \widetilde{E}_+(-F\widetilde{E}_- v + F_+ v_+) \\ u_- &= r_{[0,N]}(\widetilde{E}_- v + \widetilde{E}_{-+}(-F\widetilde{E}_- v + F_+ v_+)), \end{aligned}$$

i.e.

$$\begin{cases} u = (\widetilde{E} - \widetilde{E}_+ F\widetilde{E}_-)v + \widetilde{E}_+ F_+ v_+ \\ u_- = r_{[0,N]}(\widetilde{E}_- - \widetilde{E}_{-+}F\widetilde{E}_-)v + r_{[0,N]}\widetilde{E}_{-+}F_+ v_+. \end{cases} \tag{6.59}$$

This can be further simplified, if we use the identity $\widetilde{E}_{-+}F + r^*F_- = 1$, with $r = r_{[0,N]}$ for short. Then

$$(\widetilde{E}_- - \widetilde{E}_{-+}F\widetilde{E}_-) = (1 - \widetilde{E}_{-+}F)\widetilde{E}_- = r^*F_-\widetilde{E}_-,$$

and since $rr^* = 1$, we get

$$\begin{cases} u = (\widetilde{E} - \widetilde{E}_+ F\widetilde{E}_-)v + \widetilde{E}_+ F_+ v_+ \\ u_- = F_-\widetilde{E}_- v + r_{[0,N]}\widetilde{E}_{-+}F_+ v_+. \end{cases} \tag{6.60}$$

We next use the relation $\widetilde{E}_{-+}F_+ + r^*F_{-+} = 0$, to rewrite the last equation as

$$u_- = F_-\widetilde{E}_- v - rr^*F_{-+}v_+,$$

and using again that $rr^* = 1$, we get the solution of the small problem $\mathcal{P} = \mathcal{P}^{N,0}$ as

$$\begin{cases} u = Ev + E_+ v_+ \\ u_- = E_- v + E_{-+} v_+ \end{cases} \text{with } \begin{cases} E = \widetilde{E} - \widetilde{E}_+ F\widetilde{E}_-, \ E_+ = \widetilde{E}_+ F_+ \\ E_- = F_-\widetilde{E}_-, \ E_{-+} = -F_{-+} \end{cases}. \tag{6.61}$$

The next goal is to get asymptotics for F_{-+}, F_- similar to (6.44) with N replaced by M. Define $\widetilde{E}^0_+$ and $\widetilde{E}^0_{-+}$ as in (6.40) (6.41):

$$\begin{cases} \widetilde{E}^0_+ = \widetilde{R}_-(\widetilde{R}_+\widetilde{R}_-)^{-1} \\ \widetilde{E}^0_{-+} = (\widetilde{R}_+\widetilde{R}_-)^{-1}\widetilde{R}_+(z - \Delta^{(0)}_\phi)\widetilde{R}_-(\widetilde{R}_+\widetilde{R}_-)^{-1}, \end{cases} \tag{6.62}$$

so that analogously to (6.44)

$$\begin{cases} \Pi_P(\widetilde{E}_+ - \widetilde{E}^0_+)r^*_Q = \mathcal{O}(1)h^{\frac{1}{2}(|M+1-Q|+|M+1-P|)} : \ell^2 \to \mathcal{H}_1 \\ r_Q(\widetilde{E}_- - \widetilde{E}^0_-)\Pi_P = \mathcal{O}(1)h^{\frac{1}{2}(|M+1-Q|+|M+1-P|)} : \mathcal{H}_{-1} \to \ell^2 \\ r_{\widetilde{P}}(\widetilde{E}_{-+} - \widetilde{E}^0_{-+})r^*_Q = \mathcal{O}(1)h^{\frac{1}{2}(|M+1-Q|+|M+1-\widetilde{P}|)} : \ell^2 \to \ell^2, \end{cases} \tag{6.63}$$

for $0 \le P \le M+1$, $0 \le Q, \widetilde{P} \le M$, where $\widetilde{E}_-^0$ is defined to be the adjoint of $\widetilde{E}_+^0$.

Let

$$\begin{pmatrix} F^0 & F_+^0 \\ F_-^0 & F_{-+}^0 \end{pmatrix} = \begin{pmatrix} \widetilde{E}_{-+}^0 & r_{[0,N]}^* \\ r_{[0,N]} & 0 \end{pmatrix}^{-1} =: (\widetilde{\mathcal{E}}_{-+}^0)^{-1}. \tag{6.64}$$

Let $\lambda_j > 0$, $0 \le j \le M+1$ and let $\lambda_{[0,M]} = \mathrm{diag}\,(\lambda_j)_{0\le j\le M}$, $\lambda_{[0,N]} = \mathrm{diag}\,(\lambda_j)_{0\le j\le N}$. Let

$$\Lambda = \begin{pmatrix} \lambda_{[0,M]} & 0 \\ 0 & \lambda_{[0,N]} \end{pmatrix}, \tag{6.65}$$

viewed as an operator on

$$(\oplus_0^M \ell^2(\mathbf{N}_j^\Lambda)) \times (\oplus_0^N \ell^2(\mathbf{N}_j^\Lambda)).$$

As before we define the action of λ on $\mathcal{H}_{\pm 1}$ and on $L^2(\mathbf{R}^\Lambda)$. We also need a second system of weights μ_j and define $\mu_{[0,N]}$, $\mu_{[0,M]}$ and $\mathcal{M} = \begin{pmatrix} \mu_{[0,M]} & 0 \\ 0 & \mu_{[0,N]} \end{pmatrix}$ analogously. (6.63) can be reformulated as

$$\begin{cases} \lambda(\widetilde{E}_+ - \widetilde{E}_+^0)\mu_{[0,M]}^{-1} = \mathcal{O}(1) : \ell^2 \to \mathcal{H}_1, \\ \mu_{[0,M]}(\widetilde{E}_- - \widetilde{E}_-^0)\lambda^{-1} = \mathcal{O}(1) : \mathcal{H}_{-1} \to \ell^2, \\ \lambda_{[0,M]}(\widetilde{E}_{-+} - \widetilde{E}_{-+}^0)\mu_{[0,M]}^{-1} = \mathcal{O}(1) : \ell^2 \to \ell^2, \end{cases} \tag{6.66}$$

for all λ, μ as above with

$$\frac{\lambda_{j+1}}{\lambda_j}, \frac{\mu_{j+1}}{\mu_j} \in [h^{\frac{1}{2}}, h^{-\frac{1}{2}}],\ 0 \le j \le M, \tag{6.67}$$

$$\lambda_{M+1} = \mu_{M+1}. \tag{6.68}$$

It follows from (6.66) that

$$\Lambda(\widetilde{\mathcal{E}}_{-+} - \widetilde{\mathcal{E}}_{-+}^0)\mathcal{M}^{-1} = \mathcal{O}(1), \tag{6.69}$$

for all μ, λ which satisfy (6.67,68). We also have

$$\Lambda\widetilde{\mathcal{E}}_{-+}\Lambda^{-1} = \mathcal{O}(1) \tag{6.70}$$

and similarly with Λ replaced by one of Λ^{-1}, $\mathcal{M}^{\pm 1}$ and/or $\widetilde{\mathcal{E}}_{-+}$ replaced by $(\widetilde{\mathcal{E}}_{-+}^0)^{\pm 1}$, $(\widetilde{\mathcal{E}}_{-+})^{-1}$. Then

$$\Lambda(\widetilde{\mathcal{E}}_{-+}^{-1} - (\widetilde{\mathcal{E}}_{-+}^0)^{-1})\mathcal{M}^{-1} = \Lambda\widetilde{\mathcal{E}}_{-+}^{-1}\Lambda^{-1}\Lambda(\widetilde{\mathcal{E}}_{-+}^0 - \widetilde{\mathcal{E}}_{-+})\mathcal{M}^{-1}\mathcal{M}(\widetilde{\mathcal{E}}_{-+}^0)^{-1}\mathcal{M}^{-1} = \mathcal{O}(1), \tag{6.71}$$

for all μ, λ satisfying (6.67), (6.68). Equivalently, if we introduce the block matrix notation $A_{j,k} = r_j A r_k^*$, then

$$\begin{aligned} (F - F^0)_{j,k} &= \mathcal{O}(h^{\frac{1}{2}(|M+1-j|+|M+1-k|)}),\ 0 \le j,k \le M, \\ (F_+ - F_+^0)_{j,k} &= \mathcal{O}(h^{\frac{1}{2}(|M+1-j|+|M+1-k|)}),\ 0 \le j \le M,\ 0 \le k \le N, \\ (F_- - F_-^0)_{j,k} &= \mathcal{O}(h^{\frac{1}{2}(|M+1-j|+|M+1-k|)}),\ 0 \le j \le N,\ 0 \le k \le M, \\ (F_{-+} - F_{-+}^0)_{j,k} &= \mathcal{O}(h^{\frac{1}{2}(|M+1-j|+|M+1-k|)}),\ 0 \le j,k \le N. \end{aligned} \tag{6.72}$$

For $\widetilde{E}^0_{-+}$ we have a complete asymptotic expansion

$$\widetilde{E}^0_{-+;j,k} \sim h^{\frac{1}{2}|j-k|} \sum_{\nu=0}^{\infty} h^{\nu} A^{\nu}_{j,k} \text{ in } \mathcal{L}(\ell^2,\ell^2), \tag{6.73}$$

and it follows that the inverse (cf. (6.64)) of the corresponding Grushin problem $\widetilde{\mathcal{E}}^0_{-+}$ has the same structure.

$$\begin{aligned} F^0_{j,k} &\sim h^{\frac{1}{2}|j-k|} \sum_{\nu=0}^{\infty} h^{\nu} B^{0,\nu}_{j,k},\ 0 \le j,k \le M, \\ F^0_{+;j,k} &\sim h^{\frac{1}{2}|j-k|} \sum_{\nu=0}^{\infty} h^{\nu} B^{0,\nu}_{+;j,k},\ 0 \le j \le M,\ 0 \le k \le N, \\ F^0_{-;j,k} &\sim h^{\frac{1}{2}|j-k|} \sum_{\nu=0}^{\infty} h^{\nu} B^{0,\nu}_{-;j,k},\ 0 \le j \le N,\ 0 \le k \le M, \\ F^0_{-+;j,k} &\sim h^{\frac{1}{2}|j-k|} \sum_{\nu=0}^{\infty} h^{\nu} B^{0,\nu}_{-+;j,k},\ 0 \le j,k \le N. \end{aligned} \tag{6.74}$$

Combining this with (6.72) and letting $M \to \infty$ we obtain a complete asymptotic expansion for $F_{-+} = -E^{N,0}_{-+}$:

$$-F_{-+;j,k} \sim h^{\frac{1}{2}|j-k|} \sum_{\nu=0}^{\infty} h^{\nu} B^{\nu}_{-+;j,k},\ 0 \le j,k \le N. \tag{6.75}$$

For $F_{\pm}$ we only have a partial asymptotics with the limitations of (6.72) but again it will be advantageous to let $M \to \infty$.

We now look at E_+. The first equation in (6.63) tells us that

$$(\widetilde{E}_+ - \widetilde{E}^0_+) r^*_Q = \mathcal{O}(1) h^{\frac{1}{2}|M+1-Q|} : \ell^2 \to \mathcal{H}_1.$$

Write

$$E_+ = \widetilde{E}_+ F_+ = (\widetilde{E}_+ - \widetilde{E}^0_+) F_+ + \widetilde{E}^0_+ (F_+ - F^0_+) + \widetilde{E}^0_+ F^0_+.$$

Here

$$(\widetilde{E}_+ - \widetilde{E}^0_+) F_+ = \mathcal{O}(1) \sum_{Q=0}^{M} h^{\frac{1}{2}(|M+1-Q| + (Q-N)_+)} : \ell^2 \to \mathcal{H}_1,$$

so that $(\widetilde{E}_+ - \widetilde{E}^0_+) F_+ = \mathcal{O}(1) h^{\frac{1}{2}|M+1-N|} : \ell^2 \to \mathcal{H}_1$. We also have

$$\widetilde{E}^0_+ (F_+ - F^0_+) = \mathcal{O}(1) \sum_{Q=0}^{M} h^{\frac{1}{2}(|M+1-Q| + |M+1-N|)} = \mathcal{O}(1) h^{\frac{1}{2}|M+1-N|} : \ell^2 \to \mathcal{H}_1.$$

Consequently,
$$E_+^{N,0} = \widetilde{E}_+^0 F_+^0 + \mathcal{O}(1) h^{\frac{1}{2}(M+1-N)} : \ell^2 \to \mathcal{H}_1.$$

Here F_+^0 has a complete asymptotic expansion given by (6.74) and $\widetilde{E}_+^0$ is given by (6.62):

$$E_+^{N,0} = R_-^{M,0}(R_+^{M,0} R_-^{M,0})^{-1} F_+^0 + \mathcal{O}(1) h^{\frac{1}{2}(M+1-N)} : \ell^2 \to \mathcal{H}_1,$$

where $(R_+^{M,0} R_-^{M,0})^{-1}$ has a complete asymptotic expansion of the same type as $R_+^{M,0} R_-^{M,0}$ (c.f. (5.29)), so we conclude that for every $M \geq N$:

$$E_+^{N,0} = R_-^{M,0} C^M + \mathcal{O}(1) h^{\frac{1}{2}(M+1-N)} : \ell^2 \to \mathcal{H}_1, \tag{6.76}$$

where
$$C_{j,k}^M \sim \sum_{\nu=0}^{\infty} h^{\frac{1}{2}|j-k|+\nu} D_{j,k}^{M,\nu}, \text{ in } \mathcal{L}(\ell^2, \ell^2), \tag{6.77}$$

for $0 \leq j \leq M$, $0 \leq k \leq N$. Summing up, we have

Proposition 6.3. *$E_{-+}^{N,0}$ has a complete asymptotic expansion in $\mathcal{L}(\ell^2, \ell^2)$, that can be written at the level of block matrix elements:*

$$E_{-+;j,k}^{N,0} \sim h^{\frac{1}{2}|j-k|} \sum_{\nu=0}^{\infty} h^{\nu} B_{-+;j,k}^{\nu}, \ 0 \leq j, k \leq N. \tag{6.78}$$

For every $M \geq N$, we have (6.76,77) for $E_+^{N,0}$.

Using Proposition 6.2, (6.40), (6.41), we also get the leading terms in the asymptotic expansions (6.78), (6.77):

$$\begin{aligned} B_{-+;j,j}^0 = {} & (\phi''(0) \otimes .. \otimes \phi''(0))^{-\frac{1}{2}} \\ & (z - (\phi''(0) \otimes 1 \otimes .. \otimes 1 + .. + 1 \otimes .. \otimes 1 \otimes \phi''(0)))(\phi''(0) \otimes .. \otimes \phi''(0))^{-\frac{1}{2}}, \end{aligned} \tag{6.79}$$

$$D_{j,j}^{M,0} = (\phi''(0) \otimes .. \otimes \phi''(0))^{-1}, \ 0 \leq j \leq M. \tag{6.80}$$

We now want to do the same job with Gr(N,1) as we did with Gr(N,0), and the only slightly new thing is to analyze the block matrix of $\phi''(x)$, viewed as an operator

$$\begin{aligned} & \ell^2(\Lambda) \otimes (\mathcal{L}_0 \oplus .. \oplus \mathcal{L}_N \oplus (\mathcal{H}_1 \cap \mathcal{L}_{[0,N]}^{\perp})) \to \\ & \ell^2(\Lambda) \otimes (\mathcal{L}_0 \oplus .. \oplus \mathcal{L}_N \oplus (\mathcal{H}_{-1} \cap \mathcal{L}_{[0,N]}^{\perp})). \end{aligned} \tag{6.81}$$

According to (3.12) (or by reviewing more directly the arguments leading to that result), if $\mathcal{M} = \{1, .., M\}$, for $1 \leq M \leq N$, or $\mathcal{M} = \emptyset$, then for $\nu, \mu \in \Lambda$, $m \in \Lambda^{\mathcal{M}}$, $\phi''_{\nu,\mu}(x) Z_{m|\mathcal{M}}^*(e^{-\phi/2h})$ is a finite sum of terms of the type

$$\sum_{\ell \in \Lambda^L} h^X Z_{m|M_0}^* Z_{\ell|L}^* (\Phi_{\nu,\mu,m|M_1,\ell|L}(x) e^{-\phi/2h}), \tag{6.82}$$

where $\mathcal{M} = M_0 \cup M_1$ is a partition and L is a finite set, possibly with M_1 or L empty. Φ is a standard tensor and

$$X = \frac{1}{2}(\# M_1 + \# L) + \widetilde{N},\ \mathbf{N} \ni \widetilde{N} \leq N_1 \in \mathbf{N}, \tag{6.83}$$

where $N_1 \in \mathbf{N}$ is any fixed number. Moreover Φ is independent of x, when

$$\begin{cases} \# M_0 + \# L < N + 1 \\ \widetilde{N} < N_1. \end{cases}$$

We conclude as before, that for every $N_1 \in \mathbf{N}$ we have for all $u \in \ell^2(\Lambda) \otimes \ell_b^2(\Lambda^M)$:

$$\phi''(x) \widetilde{R}_-^{N,1} u = \sum_{P=0}^{N+1} h^{\frac{1}{2}|P-M|} D_{P,M} u + h^{N_1} R u, \tag{6.84}$$

where

$$\begin{cases} D_{P,M} = \mathcal{O}(1) : \ell \otimes \ell_b^2 \to \ell^2 \otimes \mathcal{L}_P, \text{ for } 0 \leq P \leq N, \\ D_{P,M} \sim \sum_{\nu=0}^{\infty} h^\nu E_{P,M}^{(\nu)}, \text{ in } \mathcal{L}(\ell^2 \otimes \ell_b^2, \ell^2 \otimes \mathcal{L}_P), \end{cases} \tag{6.85}$$

where we identify $\mathcal{L}_P$ with ℓ_b^2 by means of $\widetilde{R}_-^{N,0}$, and where the sum in (6.85) is finite. Further $D_{N+1,M}$ is a finite sum of terms

$$\sum_{\substack{\ell \in \Lambda^L \\ m \in \Lambda^{\mathcal{M}} \\ \mu \in \Lambda}} h^Z Z_{m|M_0}^* Z_{\ell|L}^* (\Phi_{\nu,\mu,\ell|L,m|M_1}(x) e^{-\phi(x)/2h}) u(\mu, m), \tag{6.86}$$

with M_0, M_1, L as in (6.82) and with $\# M_0 + \# L = N + 1$, $Z \in \mathbf{N}$, and

$$D_{N+1,M} = \mathcal{O}(1) : \ell^2 \otimes \ell_b^2 \to \ell^2 \otimes L^2. \tag{6.87}$$

Finally Ru is a finite sum of terms of the type (6.86) and we have $\|Ru\| \leq \mathcal{O}(1)\|u\|$.

We shall next decompose $D_{N+1,M} u$ into $(\ell^2 \otimes \mathcal{L}_{[0,N]}) \oplus (\ell^2 \otimes \mathcal{L}_{[0,N]}^\perp)$ and we consider first $(1 \otimes r_P) \widetilde{R}_+^{N,1} D_{N+1,M} u$, for $0 \leq P \leq N$, $u \in \ell^2(\Lambda) \otimes \ell_b^2(\Lambda^M)$. The matrix element of $(1 \otimes r_P) \widetilde{R}_+^{N,1} D_{N+1,M}$ at $(\nu, p), (\mu, m)$, with $\nu, \mu \in \Lambda$, $p \in \Lambda^{\mathcal{P}}$, $m \in \Lambda^{\mathcal{M}}$, $\mathcal{P} = \{1, .., P\}$ ($= \emptyset$ for $P = 0$), is a finite sum of terms of the type

$$h^{\widetilde{\nu}} \sum_{\ell \in \Lambda^L} (Z_{p|\mathcal{P}} Z_{m|M_0}^* Z_{\ell|L}^* \Phi_{\nu,\mu,\ell|L,m|M_1}(x) e^{-\phi(x)/2h} | e^{-\phi/2h}), \tag{6.88}$$

with $\mathcal{M} = M_0 \cup M_1$ being a partition, L a finite set, $\# M_0 + \# L = N + 1$, $\widetilde{\nu} \in \mathbf{N}$. As before, we get an asymptotic expansion in $\mathcal{L}(\ell^2 \otimes \ell_b^2, \ell^2 \otimes \ell_b^2)$:

$$(1 \otimes r_P) \widetilde{R}_+^{N,1} D_{N+1,M} \sim \sum_{\widetilde{\nu}=0}^{\infty} h^{\frac{1}{2}|N+1-P| + \widetilde{\nu}} F_{P,M;N+1}^{(\widetilde{\nu})}. \tag{6.89}$$

From this and Proposition 4.1 it follows that

$$(1\otimes r_P)(\widetilde{R}_+^{N,1}\widetilde{R}_-^{N,1})^{-1}\widetilde{R}_+^{N,1}D_{N+1,M}\sim\sum_{\widetilde{\nu}=0}^{N+1}h^{\frac{1}{2}|N+1-P|}G_{P,M;N+1}^{(\widetilde{\nu})},\ \text{in }\mathcal{L}(\ell^2\otimes\ell_b^2,\ell^2\otimes\ell_b^2),$$

and the desired decomposition of $D_{N+1,M}u$ is given by

$$D_{N+1,M}u=\sum_{P=0}^{N+1}D_{P,M;N+1}u, \tag{6.90}$$

with

$$D_{P,M;N+1}u=\widetilde{R}_-^{N,1}(1\otimes r_P^*r_P)(\widetilde{R}_+^{N,1}\widetilde{R}_-^{N,1})^{-1}\widetilde{R}_+^{N,1}D_{N+1,M}u\in\ell^2\otimes\mathcal{L}_P, \tag{6.90}$$

for $0\le P\le N$, and with $D_{N+1,M;N+1}$ being the remainder. We notice that $D_{N+1,M;N+1}u$ belongs to $\ell^2\otimes\mathcal{L}_{[0,N]}^{\perp}$, and that

$$\|D_{P,M;N+1}u\|\le\mathcal{O}(1)h^{\frac{1}{2}|N+1-P|}|u|_2,\ 0\le P\le N+1. \tag{6.92}$$

Since we can use $R_{\pm}^{N,1}$ to parametrize the spaces $\ell^2(\Lambda)\otimes\mathcal{L}_j$, $0\le j\le P$, we obtain the following analogue of Proposition 6.1:

Proposition 6.4. *Fix $N\in\mathbf{N}$ and let $\Phi_{i,j}$, $0\le i,j\le N+1$ be the block matrix elements of $\phi''(x)$ corresponding to (6.81). Then*

$$\|\Phi_{i,j}\|_{\mathcal{L}(\ell^2\otimes L^2,\ell^2\otimes L^2)}=\mathcal{O}(1)h^{\frac{1}{2}|i-j|},\ (i,j)\in\{0,1,..,N+1\}^2\setminus\{(N+1,N+1)\}. \tag{6.93}$$

Moreover, for $0\le i,j\le N$, we have

$$\Phi_{i,j}\sim\sum_{\nu=0}^{\infty}h^{\frac{1}{2}|i-j|+\nu}\Psi_{i,j}^{\nu},\ \textit{in }\mathcal{L}(\ell^2\otimes\mathcal{L}_j,\ell^2\otimes\mathcal{L}_i), \tag{6.94}$$

where in the RHS we identify $\mathcal{L}_j$ and $\ell^2(\mathbf{N}_j^{\Lambda})$ by means of $R_-^{N,0}$.

Since $\Delta_\phi^{(1)}=1\otimes\Delta_\phi^{(0)}+\phi''(x)$, we can combine Propositions 6.1, 6.4 to arrive at a complete analogue of Proposition 6.1 for $\Delta_\phi^{(1)}$: Let $\Delta_{i,j}^{(1)}$, $0\le i,j\le N+1$ be the block matrix decomposition of $\Delta_\phi^{(1)}$ corresponding to (6.81). Then (6.93,94) remain valid with $\Phi_{i,j}$ replaced by $\Delta_{i,j}^{(1)}$ and with $\Psi_{i,j}^{\nu}$ replaced by some new matrices independent of h. We can repeat the discussion following Proposition 6.1. The only change is to replace all spaces by their tensor products with $\ell^2(\Lambda)$, we now write

$$\mathcal{E}^{N,1}=\begin{pmatrix}E^{N,1} & E_+^{N,1}\\ E_-^{N,1} & E_{-+}^{N,1}\end{pmatrix}, \tag{6.95}$$

and we look for approximations of $E_+^{N,1}$, $E_{-+}^{N,1}$, by trying to find approximate solutions of (6.39), now with $R_\pm = R_\pm^{N,1}$ and with $\Delta_\phi^{(0)}$ replaced by $\Delta_\phi^{(1)}$. With these changes we still try $u = u_0 = E_+^0 v_+$ as in (6.40), now with ($R_\pm = R_\pm^{N,1}$) and with $u_- = u_-^0 = E_{-+}^0 v_+$ as in (6.41). We then arrive at the obvious analogue of Proposition 6.2. (Replace Π_P, r_Q, r_Q^*, by $1 \otimes \Pi_P$, $1 \otimes r_Q$, $1 \otimes r_Q^*$ and $\mathcal{H}_{\pm 1}$ by $\ell^2 \otimes \mathcal{H}_{\pm 1}$.) The discussion after Proposition 6.2 also goes through with the obvious changes. For the invertibility of the new matrix in (6.55), we need the analogue of Remark 5.5 for $\Delta_\phi^{(1)}$. Instead of (5.55) we now have for $u \in \ell^2 \otimes \ell_b^2(\Lambda^{\mathcal{Q}})$, $v \in \ell^2 \otimes \ell_b^2(\Lambda^{\mathcal{P}})$:

$$
\begin{aligned}
(\Delta_\phi^{(1)} R_-^{N,1} u | R_-^{N,1} v) &= \mathcal{O}(h^{\frac{1}{2}})|u|_2 |v|_2 + \\
&\begin{cases} 0, \text{ if } P \neq Q, \\ ((\phi''(0) \otimes 1 \otimes .. \otimes 1 + .. + 1 \otimes .. \otimes 1 \otimes \phi''(0))(1 \otimes \phi''(0) \otimes .. \otimes \phi''(0))^{\frac{1}{2}} u \\ \qquad | (1 \otimes \phi''(0) \otimes .. \otimes \phi''(0))^{\frac{1}{2}} v), \text{ if } P = Q, \end{cases}
\end{aligned}
\tag{6.97}
$$

with $P+1$ factors in the tensor products, and instead of (5.56):

$$
\begin{aligned}
(R_-^{N,1} u | R_-^{N,1} v) &= \mathcal{O}(h^{\frac{1}{2}})|u|_2 |v|_2 + \\
&\begin{cases} 0, \text{ if } P \neq Q \\ ((1 \otimes \phi''(0) \otimes .. \otimes \phi''(0))^{\frac{1}{2}} u | (1 \otimes \phi''(0) \otimes .. \otimes \phi''(0))^{\frac{1}{2}} v), \text{ if } P = Q. \end{cases}
\end{aligned}
\tag{6.98}
$$

We arrive at the following analogue of Proposition 6.3.

Proposition 6.5. *$E_{-+}^{N,1}$ has a complete asymptotic expansion in $\mathcal{L}(\ell^2 \otimes \ell^2, \ell^2 \otimes \ell^2)$ that can be written for the block matrix elements:*

$$
E_{-+;j,k}^{N,1} \sim h^{\frac{1}{2}|j-k|} \sum_{\nu=0}^{\infty} h^\nu B_{-+;j,k}^\nu, \ \ 0 \leq j, k \leq N. \tag{6.99}
$$

For every $M \geq N$, we have

$$
E_+^{N,1} = R_-^{M,1} C^M + \mathcal{O}(1) h^{\frac{1}{2}(M+1-N)} : \ell^2 \otimes \ell^2 \to \ell^2 \otimes \mathcal{H}_1, \tag{6.100}
$$

where C^M has a complete asymptotic expansion, which we can write in block matrix form:

$$
C_{j,k}^M \sim \sum_{\nu=0}^{\infty} h^{\frac{1}{2}|j-k|+\nu} D_{j,k}^{M,\nu}, \ \ 0 \leq j \leq M, \ \ 0 \leq k \leq N. \tag{6.101}
$$

As for Gr(N,0), we also have the leading terms in (6.99), (6.101):

$$
\begin{aligned}
B_{-+;j,j}^0 &= (1 \otimes \phi''(0) \otimes .. \otimes \phi''(0))^{-\frac{1}{2}} \\
&\quad (z - (\phi''(0) \otimes 1 \otimes .. \otimes 1 + ... + 1 \otimes .. \otimes 1 \otimes \phi''(0))) \\
&\qquad (1 \otimes \phi''(0) \otimes .. \otimes \phi''(0))^{-\frac{1}{2}},
\end{aligned}
\tag{6.102}
$$

$$D^{M,0}_{j,j} = (1 \otimes \phi''(0) \otimes .. \otimes \phi''(0))^{-1},\ 0 \le j \le N. \tag{6.103}$$

7. Exponential weights.

Let $W = W_\Lambda$ be a set of positive weight functions $\rho : \Lambda \to]0,\infty[$, with $1 \in W$ and such that $\rho \in W \Rightarrow 1/\rho \in W$. We sharpen the assumption (H1) to

$$\rho^{-1}\phi''(x)\rho \text{ is 2 standard, uniformly w.r.t. } \rho \in W. \tag{$\widetilde{\mathrm{H1}}$}$$

Lemma 2.1 extends and becomes:

$$\rho^{-1}(\phi''(x) - \phi''(0))\rho \text{ is of norm } \mathcal{O}(h^{\frac{1}{2}}) : \mathcal{H}^{(1)}_1 \to \mathcal{H}^{(1)}_{-1}, \text{ uniformly w.r.t. } \rho \in W. \tag{7.1}$$

In fact, the tensors $\phi^{(1)}_{j,k}(x)$ and $\phi^{(0)}_{j,k,\ell}$ in the non-commutative Taylor expansion (2.9) have the property that $\rho(j)^{-1}\phi^{(1)}_{j,k}\rho(k)$, $\rho(j)^{-1}\phi^{(0)}_{j,k,\ell}\rho(k)$ are standard, uniformly for $\rho \in W$. We can then conjugate all the tensors in (2.10) by ρ and the remainder of the proof of the lemma leads to (7.1).

Let $a \in]-\infty, \lambda_{\min}(\phi''(0))]$ and let $W_a \subset W$ contain 1, satisfy $\rho \in W_a \Rightarrow 1/\rho \in W_a$ as well as:

$$(\rho^{-1}\phi''(0)\rho t|t) \ge a|t|_2^2,\ \forall t \in \mathbf{R}^\Lambda, \rho \in W_a. \tag{7.2}$$

Let

$$\begin{aligned} R_\rho = \begin{pmatrix} \rho\otimes 1 & 0 \\ 0 & \rho \otimes 1 \end{pmatrix} :&(\ell^2(\Lambda)\otimes\mathcal{H}_{\pm 1}) \times (\ell^2(\Lambda)\otimes \ell^2(\mathbf{N}^\Lambda_{[0,N]})) \\ &\to (\ell^2(\Lambda)\otimes\mathcal{H}_{\pm 1}) \times (\ell^2(\Lambda)\otimes \ell^2(\mathbf{N}^\Lambda_{[0,N]})). \end{aligned}$$

We have the following extension of Proposition 5.1.

Proposition 7.1. *For every $C \ge 1$ and $N \in \mathbf{N}$, the weighted Grushin operator*

$$R_\rho^{-1}\mathcal{P}^{N,1}R_\rho : (\ell^2\otimes\mathcal{H}_1)\times(\ell^2\otimes\ell^2) \to (\ell^2\otimes\mathcal{H}_{-1})\times(\ell^2\otimes\ell^2) \tag{7.3}$$

is bijective with a uniformly bounded inverse $R_\rho^{-1}\mathcal{E}^{N,1}R_\rho$, when $\rho \in W_a$ and

$$-C \le z \le (N+1)\lambda_{\min}(\phi''(0)) + a - \frac{1}{C}. \tag{7.4}$$

Proof. Since the range (7.4) is contained in the range of wellposedness of $\mathcal{P}^{N,1}$ in Proposition 5.1, we already know that that the operator (7.3) is bijective and consequently, we only need to establish the a priori estimate

$$\|u\|_1 + |u_-|_2 \le \widetilde{C}(\|v\|_{-1} + |v_+|_2), \tag{7.5}$$

for solutions $u \in \ell^2\otimes\mathcal{H}_1$, $u_- \in \ell^2\otimes\ell^2$ of

$$R_\rho^{-1}\mathcal{P}^{N,1}R_\rho \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}. \tag{7.6}$$

In view of (7.1), it suffices to establish (7.5) for solutions of the simplified problem

$$\begin{pmatrix} 1\otimes \Delta_\phi^{(0)}+\rho^{-1}\phi''(0)\rho - z & R_-^{N,1} \\ R_+^{N,1} & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}. \tag{7.7}$$

In the case $N=-1$ (which could be included in the statement of the proposition), (7.7) reduces to

$$(1\otimes \Delta_\phi^{(0)}+\rho^{-1}\phi''(0)\rho - z)u = v, \tag{7.8}$$

and we obtain the analogue of (7.5)

$$\|u\|_1 \le \widetilde{C}\|v\|_{-1}, \tag{7.9}$$

by taking the scalar product with u and repeating some of the estimates of the proof of Proposition 5.1. To obtain (7.5) from (7.7) is a straight forward adaptation of the arguments of the step Gr(N,0)→Gr(N,1) in the proof of Proposition 5.1, and we do not repeat the details. #

The next goal is to show that the aymptotic expansions in Proposition 6.5 are compatible with exponential weights. For a given $N\in \mathbf{N}$ and $C\ge 1$, we continue to let z vary in the interval (7.4). Let (d_j) satisfy (6.27). Define D by (6.31), so that analogously to (6.36), we have

$$(1\otimes D^{-1})\mathcal{E}^{N,1}(1\otimes D) = \mathcal{O}(1) : \ell^2\otimes(\mathcal{H}_{-1}\times \ell^2)\to \ell^2\otimes(H_1\times\ell^2). \tag{7.10}$$

Recall that the argument leading to (7.10) is based on the fact that $(1\otimes d^{-1})\Delta_\phi^{(1)}(1\otimes d) - \Delta_\phi^{(1)} : \ell^1\otimes\mathcal{H}_1\to \ell^2\otimes\mathcal{H}_{-1}$ is of arbitrarily small norm if $d_{j+1}/d_j\in[Ch^{1/2},C^{-1}h^{-1/2}]$, with $C\ge 1$ sufficiently large. It is then clear that we can can combine the weight d with a weight $\rho\in W_a$ and conclude that if

$$R_{\rho,d} = \begin{pmatrix} \rho\otimes d & 0 \\ 0 & \rho\otimes\widetilde{d} \end{pmatrix}, \tag{7.11}$$

then

$$R_{\rho,d}^{-1}\mathcal{E}^{N,1}R_{\rho,d} = \mathcal{O}(1) : \ell^2\otimes(\mathcal{H}_{-1}\times\ell^2)\to\ell^2\otimes(\mathcal{H}_1\times\ell^2), \tag{7.12}$$

for $\rho\in W_a$, d_j satisfying (6.28), and z in the range (7.4).

Using the assumption $(\widetilde{\mathrm{H1}})$ again, we see that the arguments for the analogue of Proposition 6.1 give

Proposition 7.2. *Let $\Delta_{i,j}^{(1)}$, $0\le i,j\le N+1$ be the block matrix decomposition corresponding to (6.81). Then uniformly w.r.t. $\rho\in W$, we have*

$$\|(\rho\otimes 1)^{-1}\Delta_{i,j}^{(1)}(\rho\otimes 1)\|_{\mathcal{L}(\ell^2\otimes L^2,\ell^2\otimes L^2)} = \mathcal{O}(1)h^{\frac{1}{2}|i-j|}, \tag{7.13}$$
$$(i,j)\in\{0,1,..,N+1\}^2\setminus\{(N+1,N+1)\}.$$

Moreover, for $0\le i,j\le N$, we have uniformly w.r.t. $\rho\in W_a$:

$$\begin{aligned}&(\rho^{-1}\otimes 1)\Delta_{i,j}^{(1)}(\rho\otimes 1)\\ &\qquad\qquad\sim\sum_{\nu=0}^{\infty}h^{\frac{1}{2}|i-j|+\nu}(\rho\otimes 1)^{-1}A_{i,j}^{(\nu)}(\rho\otimes 1),\ \ \textit{in}\ \mathcal{L}(\ell^2\otimes\mathcal{L}_i,\ell^2\otimes\mathcal{L}_j),\end{aligned}\tag{7.14}$$

with $A_{i,j}^{(\nu)}$ independent of ρ. Here we identify $\mathcal{L}_i$ with ℓ^2 by means of $R_-^{N,0}$.

As before, we define E_+^0, E_{-+}^0 by (6.40,41), now with $R_\pm=R_\pm^{N,1}$, and analogously to (6.42), we have

$$\begin{cases}(\Delta_\phi^{(1)}-z)E_+^0v_++R_-E_{-+}^0v_+=\sum_{0\le\widetilde{M}\le N}h^{\frac{1}{2}|N+1-\widetilde{M}|}D_{N+1,\widetilde{M},N+1}r_{\widetilde{M}}^*r_{\widetilde{M}}(R_+R_-)^{-1}v_+\\ R_+E_+^0v_+=v_+\in\ell^2(\mathbf{N}_M^\Lambda),\ 0\le M\le N,\end{cases}\tag{7.15}$$

where $D_{N+1,\widetilde{M};N+1}u\in\ell^2\otimes\mathcal{L}_{[0,N]}^\perp$, and

$$\begin{cases}(\rho\otimes 1)^{-1}D_{N+1,\widetilde{M};N+1}(\rho\otimes 1)=\mathcal{O}(1):\ell^2\otimes\ell^2\to\ell^2\otimes L^2,\\ (\rho\otimes d)^{-1}E_+^0(\rho\otimes\widetilde{d})=\mathcal{O}(1):\ell^2\otimes\ell^2\to\ell^2\otimes\mathcal{H}_1,\\ (\rho\otimes\widetilde{d})^{-1}E_{-+}^0(\rho\otimes\widetilde{d})=\mathcal{O}(1):\ell^2\otimes\ell^2\to\ell^2\otimes\ell^2,\end{cases}\tag{7.16}$$

for $\rho\in W$ and d_j satisfying (6.27).

Using $\mathcal{E}^{N,1}$, we can correct for the non-vanishing RHS in (7.15) and the correction can be estimated thanks to (7.10). We get the following analogue of Proposition 6.2 for z as in (7.4):

Proposition 7.3. *We have*

$$\begin{cases}(\rho^{-1}\otimes\Pi_P)(E_+^{N,1}-E_+^0)(\rho\otimes r_Q^*)=\mathcal{O}(1)h^{\frac{1}{2}(|N+1-Q|+|N+1-P|)}:\ell^2\otimes\ell^2\to\ell^2\otimes\mathcal{H}_1\\ (\rho^{-1}\otimes r_Q)(E_-^{N,1}-E_-^0)(\rho\otimes\Pi_P)=\mathcal{O}(1)h^{\frac{1}{2}(|N+1-Q|+|N+1-P|)}:\ell^2\otimes\mathcal{H}_{-1}\to\ell^2\otimes\ell^2,\\ (\rho^{-1}\otimes r_{\widetilde{P}})(E_{-+}^{N,1}-E_{-+}^0)(\rho\otimes r_Q^*)=\mathcal{O}(1)h^{\frac{1}{2}(|N+1-Q|+|N+1-\widetilde{P}|)}:\ell^2\otimes\ell^2\to\ell^2\otimes\ell^2,\end{cases}\tag{7.17}$$

for $\rho\in W_a$, $0\le P\le N+1$, $0\le\widetilde{P},Q\le N$, where $E_-^0=(E_+^0)^*$.

We can continue to follow the discussion after Proposition 6.2, that we have already adapted to $\mathcal{P}^{N,1}$ after Proposition 6.4. Only the inversion of

$$\widetilde{\mathcal{E}}_{-+}:=\begin{pmatrix}E_{-+}^{M,1} & 1\otimes r_{[0,N]}^*\\ 1\otimes r_{[0,N]} & 0\end{pmatrix}\tag{7.18}$$

for $M\ge N$, now with exponential weights, requires some special attention. We already know that if $d_0,..,d_M$ satisfy (6.27) for $0\le j\le M-1$ and if $\rho\in W_a$, then

$$(\rho^{-1}\otimes(d^{-1}\oplus d_{[0,N]}^{-1}))\widetilde{\mathcal{E}}_{-+}(\rho\otimes(d\oplus d_{[0,N]}))=\mathcal{O}(1),\tag{7.19}$$

and we need the same estimate for $\widetilde{\mathcal{E}}_{-+}^{-1}$. Here d and $d_{[0,N]}$ denote the natural block diagonal matrices. In addition to (7.19), we know that the description of the leading part of $E_{-+}^{M,1}$ survives the addition of exponential weights:

$$(\rho^{-1}\otimes 1)(E_{-+}^{M,1}-\operatorname{diag}(B_{-+;j,j}^{0}))(\rho\otimes 1)=\mathcal{O}(h^{\frac{1}{2}}):\ell^2\to\ell^2, \tag{7.20}$$

where $B_{-+,j,j}^{0}$ is given by (6.102). For $j\geq N+1$, we see that

$$(\rho^{-1}\otimes 1)(B_{-+;j,j}^{0})^{-1}(\rho\otimes 1)=\mathcal{O}(1), \tag{7.21}$$

for $\rho\in W_a$ and z in the interval (7.4), and it follows that

$$(\rho^{-1}\otimes(d^{-1}\oplus d_{[0,N]}^{-1}))(\widetilde{\mathcal{E}}_{-+})^{-1}(\rho\otimes(d\oplus d_{[0,N]}))=\mathcal{O}(1), \tag{7.22}$$

for the same ρ and z.

It is now straight forward to repeat the arguments leading to Proposition 6.5, now with additional exponential weights, and we get

Proposition 7.4. *We fix $N\in\mathbf{N}$, $C\geq 1$, $a\in\mathbf{R}$ and let z vary in the interval (7.4). Then for $h>0$ sufficiently small and uniformly for all $\rho\in W_a$, we have the asymptotic expansion in $\mathcal{L}(\ell^2\otimes\ell^2,\ell^2\otimes\ell^2)$, that can be written for the block diagonal elements:*

$$(\rho\otimes 1)^{-1}E_{-+;j,k}^{N,1}(\rho\otimes 1)\sim h^{\frac{1}{2}|j-k|}\sum_{\nu=0}^{\infty}h^{\nu}(\rho\otimes 1)^{-1}B_{-+;j,k}^{\nu}(\rho\otimes 1),\ \ 0\leq j,k\leq N. \tag{7.23}$$

Here $B_{-+;j,k}^{\nu}$ are the same as in (6.99).

For $M\geq N$, we have:

$$(\rho\otimes 1)^{-1}E_{+}^{N,1}(\rho\otimes 1)=R_{-}^{M,1}(\rho\otimes 1)^{-1}C^{M}(\rho\otimes 1)+\mathcal{O}(1)h^{\frac{1}{2}(M+1-N)}:\ell^2\otimes\ell^2\to\ell^2\otimes\mathcal{H}_1. \tag{7.24}$$

Here C^M is the same as in Proposition 6.5, and we have the asymptotic expansion for the block matrix elements, valid uniformly with respect to $\rho\in W_a$:

$$(\rho\otimes 1)^{-1}C_{j,k}^{M}(\rho\otimes 1)\sim\sum_{\nu=0}^{\infty}h^{\frac{1}{2}|j-k|+\nu}(\rho\otimes 1)^{-1}D_{j,k}^{M,\nu}(\rho\otimes 1), \tag{7.25}$$

for $0\leq j\leq M$, $0\leq k\leq N$.

8. Parameter dependent exponents.

In this section we carry out an essential preparation for controling the thermodynamical limit of the correlations. For that, we need to estimate the variation of the correlations, when the exponent $\phi=\phi_t(x)$ depends on a parameter $t\in[0,1]$:

$$\phi_t(x)=\phi_{t,0}(x)+C(t;h), \tag{8.1H}$$

with $C(t;h)$ independent of x and with $\phi_{t,0}$ independent of h. We assume that $\phi_t(x)$ is of class C^1 in t and smooth in $x \in \mathbf{R}^\Lambda$. Further assumptions will be given later on. We assume that $C(t;h)$ is chosen so that

$$\int e^{-\phi_t(x)/h} dx = 1. \tag{8.2H}$$

We start the section by making some formal computations. After that we will introduce some precise assumptions on ϕ_t that justify the formal computations. Finally we will estimate the various terms that we get. The estimates will be summed up in Proposition 8.4.

We are interested in

$$\mathrm{Cor}_t(u,v) = \int e^{-\phi_t(x)/h}(u - \langle u\rangle_t)(v - \langle v\rangle_t)dx, \tag{8.3}$$

where $\langle u\rangle_t$ denotes the expectation of u with respect to $e^{-\phi_t/h}dx$, and where u and v are supposed to be independent of t. Since $u - \langle u\rangle_t$ and $v - \langle v_t\rangle$ have expectation 0, we get

$$-\partial_t \mathrm{Cor}_t(u,v) = \int e^{-\phi_t(x)/h}\frac{\partial_t\phi_t(x)}{h}(u - \langle u\rangle_t)(v - \langle v\rangle_t)dx. \tag{8.4}$$

Assume that

$$\partial_t\phi_{t,0}(0) = 0,\ \ \partial_x\partial_t\phi_{t,0}(0) = 0. \tag{8.5H}$$

Then

$$\partial_t\phi_{t,0}(x) = \sum_{j,k}\widetilde{\Phi}_{j,k}(x)x_jx_k = \langle\widetilde{\Phi}(x)x, x\rangle, \tag{8.6}$$

with $\widetilde{\Phi} = \widetilde{\Phi}_t$ given by

$$\widetilde{\Phi}_{j,k}(x) = \int_0^1 (1-s)\partial_{x_j}\partial_{x_k}\partial_t\phi_{t,0}(sx)ds. \tag{8.7}$$

Now assume that

$$\partial_x\phi_t(x) = A_t(x)x, \tag{8.8H}$$

where the matrix $A_t(x)$ is C^1 in t, C^∞ in x and invertible. Combining this with (8.6), we get

$$\partial_t\phi_{t,0}(x) = \sum_{j,k}\Phi_{j,k}(x)\partial_{x_j}\phi\partial_{x_k}\phi = \langle\Phi(x)\partial_x\phi(x), \partial_x\phi(x)\rangle, \tag{8.9}$$

with

$$\Phi(x) = {}^tA(x)^{-1}\widetilde{\Phi}(x)A(x)^{-1}. \tag{8.10}$$

Here and in the following, we often drop the subscript t. Later on it will be useful to keep in mind that $\Phi_{j,k}(x)$ is symmetric.

With $\phi = \phi_t$, we define Z_j, Z_j^* as in section 1. A straight forward computation shows that

$$e^{-\phi/2h}\frac{\partial_t \phi}{h} = \sum_{j,k} Z_j^* Z_k^*(e^{-\phi/2h}\Phi_{j,k}) + h^{\frac{1}{2}}\sum_j Z_j^*(e^{-\phi/2h}\Psi_j) + D(x;h)e^{-\phi/2h}, \tag{8.11}$$

where

$$\Psi_j = 2\sum_k \partial_{x_k}\Phi_{j,k}, \tag{8.12}$$

$$D = \sum_{j,k}(\partial_{x_j}\partial_{x_k}\phi)\Phi_{j,k} + h\sum_{j,k}\partial_{x_j}\partial_{x_k}\Phi_{j,k} + \frac{\partial_t C(t;h)}{h}. \tag{8.13}$$

Using this in (8.4), we get

$$-\partial_t \mathrm{Cor}_t(u,v) = \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{8.14}$$

where

$$\begin{cases} \mathrm{I} = \int \sum_{j,k} Z_j^* Z_k^*(e^{-\phi/2h}\Phi_{j,k})e^{-\phi/2h}(u-\langle u\rangle)(v-\langle v\rangle)dx, \\ \mathrm{II} = \int h^{\frac{1}{2}}\sum_j Z_j^*(e^{-\phi/2h}\Psi_j)e^{-\phi/2h}(u-\langle u\rangle)(v-\langle v\rangle)dx, \\ \mathrm{III} = \int e^{-\phi/2h}De^{-\phi/2h}(u-\langle u\rangle)(v-\langle v\rangle)dx. \end{cases} \tag{8.15}$$

Since $Z_j \circ e^{-\phi/2h} = e^{-\phi/2h}h^{1/2}\partial_{x_j}$, we get after an integration by parts,

$$\begin{aligned} \mathrm{I} &= \int \sum_{j,k} e^{-\phi/2h}\Phi_{j,k}Z_jZ_k(e^{-\phi/2h}(u-\langle u\rangle)(v-\langle v\rangle))dx \\ &= h\int \sum_{j,k} e^{-\phi/2h}\Phi_{j,k}e^{-\phi/2h}\partial_{x_j}\partial_{x_k}((u-\langle u\rangle)(v-\langle v\rangle))dx \\ &= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3, \end{aligned} \tag{8.16}$$

where

$$\mathrm{I}_1 = 2h\int \sum_{j,k}\Phi_{j,k}(e^{-\phi/2h}\partial_{x_j}u)(e^{-\phi/2h}\partial_{x_k}v)dx, \tag{8.17}$$

and

$$\begin{cases} \mathrm{I}_2 = h\int \sum_{j,k} e^{-\phi/2h}\Phi_{j,k}(\partial_{x_j}\partial_{x_k}u)e^{-\phi/2h}(v-\langle v\rangle)dx, \\ \mathrm{I}_3 = h\int \sum_{j,k} e^{-\phi/2h}\Phi_{j,k}(\partial_{x_j}\partial_{x_k}v)e^{-\phi/2h}(u-\langle u\rangle)dx. \end{cases}$$

Here we used the symmetry of $(\Phi_{j,k})$ to get I_1. We need to transform $\mathrm{I}_2, \mathrm{I}_3$ further.

Later in this section we shall solve

$$\begin{cases} e^{-\phi/2h}(u-\langle u\rangle) = \sum_\nu Z_\nu^*(\widetilde{u}_\nu) \ (= d_\phi^*(\sum \widetilde{u}_\nu dx_\nu)), \\ e^{-\phi/2h}(v-\langle v\rangle) = \sum_\nu Z_\nu^*(\widetilde{v}_\nu), \\ e^{-\phi/2h}D = \sum Z_\nu^*(\widetilde{D}_\nu). \end{cases} \tag{8.18}$$

As for the last equation, we see formally that

$$\langle D\rangle = 0. \tag{8.19}$$

This follows from (8.11), since

$$\langle \frac{\partial_t \phi}{h} \rangle = -\partial_t \int e^{-\phi_t/h} dx = -\partial_t(1) = 0,$$

and

$$\int e^{-\phi/2h} Z_j^* w dx = \int Z_j(e^{-\phi/2h}) w dx = 0,$$

unders suitable assumptions on w which will be verified.

Substitution of the second equation of (8.18) into the expression for I_2 and integration by parts gives

$$\mathrm{I}_2 = h^{\frac{3}{2}} \int \sum_{j,k,\nu} e^{-\phi/2h} \partial_{x_\nu}(\Phi_{j,k} \partial_{x_j} \partial_{x_k} u) \widetilde{v}_\nu dx = \mathrm{I}_{2,1} + \mathrm{I}_{2,2}, \tag{8.20}$$

where

$$\begin{cases} \mathrm{I}_{2,1} = h^{\frac{3}{2}} \int \sum_{j,k,\nu} e^{-\phi/2h} (\partial_{x_\nu} \Phi_{j,k})(\partial_{x_j} \partial_{x_k} u) \widetilde{v}_\nu dx, \\ \mathrm{I}_{2,2} = h^{\frac{3}{2}} \int \sum_{j,k,\nu} e^{-\phi/2h} \Phi_{j,k} (\partial_{x_\nu} \partial_{x_j} \partial_{x_k} u) \widetilde{v}_\nu dx. \end{cases} \tag{8.21}$$

Since I_3 is obtained from I_2 by exchanging u and v, we get $\mathrm{I}_3 = \mathrm{I}_{3,1} + \mathrm{I}_{3,2}$, with $\mathrm{I}_{3,1}$, $\mathrm{I}_{3,2}$ as in (8.21), with u replaced by v and $\widetilde{v}_\nu$ by $\widetilde{u}_\nu$.

After an integration by parts and application of (8.18), we get

$$\mathrm{II} = \mathrm{II}_1 + \mathrm{II}_2, \tag{8.22}$$

$$\begin{cases} \mathrm{II}_1 = h \int \sum_{j,\nu} \Psi_j (e^{-\phi/2h} \partial_{x_j} u) Z_\nu^* \widetilde{v}_\nu dx, \\ \mathrm{II}_2 = h \int \sum_{j,\nu} \Psi_j (e^{-\phi/2h} \partial_{x_j} v) Z_\nu^* \widetilde{u}_\nu dx. \end{cases} \tag{8.23}$$

We observe that II_1 and II_2 differ only by a permutation of u and v and their related quantities. By integration by parts, we get

$$\mathrm{II}_1 = \mathrm{II}_{1,1} + \mathrm{II}_{1,2}, \tag{8.24}$$

$$\begin{cases} \mathrm{II}_{1,1} = h^{\frac{3}{2}} \int \sum_{j,\nu} (\partial_{x_\nu} \Psi_j)(e^{-\phi/2h} \partial_{x_j} u) \widetilde{v}_\nu dx, \\ \mathrm{II}_{1,2} = h^{\frac{3}{2}} \int \sum_{j,\nu} \Psi_j (e^{-\phi/2h} \partial_{x_\nu} \partial_{x_j} u) \widetilde{v}_\nu dx. \end{cases} \tag{8.25}$$

Similarly, we have $\mathrm{II}_2 = \mathrm{II}_{2,1} + \mathrm{II}_{2,2}$, where $\mathrm{II}_{2,i}$ is obtained from $\mathrm{II}_{1,i}$, by replacing u by v and $\widetilde{v}_\nu$ by $\widetilde{u}_\nu$.

Next consider III in (8.15). Using (8.18) and an integration by parts, we get

$$\mathrm{III} = \mathrm{III}_1 + \mathrm{III}_2, \tag{8.26}$$

$$\begin{cases} \mathrm{III}_1 = h^{\frac{1}{2}} \int \sum_\nu \widetilde{D}_\nu (\partial_{x_\nu} u) e^{-\phi/2h} (v - \langle v \rangle) dx, \\ \mathrm{III}_2 = h^{\frac{1}{2}} \int \sum_\nu \widetilde{D}_\nu (\partial_{x_\nu} v) e^{-\phi/2h} (u - \langle u \rangle) dx. \end{cases} \tag{8.27}$$

Again the two terms are analogous. Applying (8.18) and integrating by parts, we get

$$\mathrm{III}_1 = \mathrm{III}_{1,1} + \mathrm{III}_{1,2}, \tag{8.28}$$

$$\begin{cases} \mathrm{III}_{1,1} = h^{\frac{1}{2}} \int \sum_{\nu,\mu} \widetilde{v}_\mu (Z_\mu \widetilde{D}_\nu)(\partial_{x_\nu} u) dx, \\ \mathrm{III}_{1,2} = h \int \sum_{\nu,\mu} \widetilde{v}_\mu \widetilde{D}_\nu (\partial_{x_\mu} \partial_{x_\nu} u) dx. \end{cases} \tag{8.29}$$

Clearly $\mathrm{III}_2 = \mathrm{III}_{2,1} + \mathrm{III}_{2,2}$, where $\mathrm{III}_{2,i}$ is obtained from $\mathrm{III}_{1,i}$ by replacing u by v and $\widetilde{v}_\mu$ by $\widetilde{u}_\mu$. This completes our formal calculations:

$$\begin{aligned} -\partial_t \mathrm{Cor}\,(u,v) = \mathrm{I}_1 + \mathrm{I}_{2,1} + \mathrm{I}_{2,2} + \mathrm{I}_{3,1} + \mathrm{I}_{3,2} + \\ \mathrm{II}_{1,1} + \mathrm{II}_{1,2} + \mathrm{II}_{2,1} + \mathrm{II}_{2,2} + \\ \mathrm{III}_{1,1} + \mathrm{III}_{1,2} + \mathrm{III}_{2,1} + \mathrm{III}_{2,2}. \end{aligned} \tag{8.30}$$

Let $W = W_\Lambda$ be a set of positive weight functions $\rho : \Lambda \to]0,\infty[$, with $1 \in W$ and such that $\rho \in W \Rightarrow 1/\rho \in W$. First of all we assume that $\phi = \phi_t$ satisfies the assumptions of the earlier sections uniformly w.r.t. t. (Actually in the next sections we shall see that the sets of weights we use in this section are smaller than the corresponding sets of weights in sections 7.) More precisely we assume that $(\widetilde{\mathrm{H1}})$ (section 7) holds uniformly in t. We assume (H2) and we assume that (H3) holds uniformly in t. We strengthen (H4) to:

> $\phi_t'(x) = A_t(x)x$, where $\rho^{-1}A_t(x)\rho$ is 2 standard and $\rho(\ell)(\partial_{x_\ell} A_t) \circ \rho^{-1}$, $\rho^{-1} \circ \rho(\ell)\partial_{x_\ell} A_t$ are 3 standard uniformly for $\rho \in W,\ t \in [0,1]$. Moreover, $A_t(x)$ has an inverse $B_t(x)$ such that $\rho^{-1}B_t(x)\rho = \mathcal{O}(1) : \ell^p \to \ell^p,\ 1 \le p \le \infty$, uniformly for $0 \le t \le 1,\ \rho \in W$. $(\widetilde{\mathrm{H4}})$

It follows that $\rho^{-1}A_t^{-1}(x)\rho$ is 2 standard and that $\rho(\ell)(\partial_{x_\ell} A_t^{-1}) \circ \rho^{-1}$, $\rho^{-1} \circ \rho(\ell)\partial_{x_\ell} A_t^{-1}$ are 3 standard, uniformly for $0 \le t \le 1,\ \rho \in W$. The most natural choice of A_t seems to be $A_t = \int_0^1 \phi_t''(sx)ds$. With that choice we only have to check the statement about the inverse of A_t, since the other properties follow from the previous assumptions on ϕ. Let $W_a \subset W$ be a set of weights as with $\rho \in W_a \Rightarrow 1/\rho \in W_a$ such that (7.2) holds, with $\phi = \phi_t$. We let a be fixed with

$$0 < a < \lambda_{\min}(\phi_t''(0)). \tag{8.32}$$

Let $1 \le \rho_0 = \rho_{0,\Lambda} \in W_a$ be a weight and assume

$$\rho_0(j)\rho_0(k)(\partial_t \phi_t)''_{j,k} \text{ is 2 standard,} \tag{H5}$$

uniformly in t.

Lemma 8.1. *$\rho_0(j)\Psi_j$ and $\rho_0(\ell)\partial_{x_\ell} D$ are 1 standard.*

Proof. recall that $\Phi(x)$ is given by (8.10), with $\widetilde{\Phi}$ given by (8.7). It is clear from (H5) that $\rho_0(j)\rho_0(k)\widetilde{\Phi}_{j,k}(x)$ is 2 standard, and from $(\widetilde{\mathrm{H4}})$ that $\rho^{-1}A(x)\rho$ and its inverse are 2 standard for every $\rho \in W$. Consequently, $\rho_0(j)\rho_0(k)\Phi_{j,k}(x)$ is 2 standard, and it follows that $\rho_0(j)\Phi_{j,k}(x)$ is 2 standard.

We conclude that $\rho_0(j)\partial_{x_\ell}\Phi_{j,k}(x)$ is 3 standard, and using the trace lemma, we see that $\rho_0(j)\Psi_j = \sum_k \rho_0(j)\partial_{x_k}\Phi_{j,k}(x)$ is 1 standard.

Next consider D, given by (8.13). Let $D^{(1)}$, $D^{(2)}$ be the first and the second terms of the RHS of (8.13), so that

$$\partial_{x_\ell} D = \partial_{x_\ell} D^{(1)} + \partial_{x_\ell} D^{(2)}.$$

For $D^{(2)}$, we use that $\rho_0(\ell)\partial_{x_\ell}\Phi_{j,k}$ is 3 standard, so that $\rho_0(\ell)\partial_{x_\ell}\partial_{x_j}\partial_{x_k}\Phi_{\widetilde{j},\widetilde{k}}(x)$ is 5 standard. Using the trace lemma twice, we conclude that $h^{-1}\rho_0(\ell)\partial_{x_\ell}D^{(2)}$ is 1 standard, uniformly in h.

Next look at

$$\rho_0(\ell)\partial_{x_\ell}D^{(1)} = \sum_{j,k}(\rho_0(\ell)\partial_{x_\ell}\partial_{x_j}\partial_{x_k}\phi)\Phi_{j,k} + \sum_{j,k}(\partial_{x_j}\partial_{x_k}\phi)(\rho_0(\ell)\partial_{x_\ell}\Phi_{j,k}). \tag{8.33}$$

The first term to the right can be written

$$\sum_{j,k}(\rho_0(\ell)\rho_0(j)^{-1}\partial_{x_\ell}\partial_{x_j}\partial_{x_k}\phi)(\rho_0(j)\Phi_{j,k}). \tag{8.34}$$

Here $\rho_0(\ell)\rho_0(j)^{-1}\partial_{x_\ell}\partial_{x_j}\partial_{x_k}\phi$ is 3 standard by $(\widetilde{\mathrm{H1}})$ since $\rho_0 \in W$, and $\rho_0(j)\Phi_{j,k}$ is 2 standard. Now we need

Lemma 8.2. *Let $a_{j_1,..,j_p}$ be p standard and let $b_{k_1,..,k_q}$ be q standard, where $p, q \geq 1$. Let $1 \leq r \leq \min(p,q)$, $r < \max(p,q)$. Then*

$$c_{j_{r+1},..,j_p,k_{r+1},..,k_q} := \sum_{j_1,..,j_r} a_{j_1,..,j_p} b_{j_1,..,j_r,k_{r+1},..,k_q}$$

is $(p-r)+(q-r)$ standard.

Proof. Let us first consider the case $r = 1$. Then we have a contraction via one summation index and

$$\langle c, t_2 \otimes .. \otimes t_p \otimes s_2 \otimes .. \otimes s_q \rangle = \langle \langle a, t_2 \otimes .. \otimes t_p \rangle, \langle b, s_2 \otimes .. \otimes s_q \rangle \rangle. \tag{8.35}$$

Here

$$|\langle a, t_2 \otimes .. \otimes t_p \rangle|_{\widetilde{p}} = \mathcal{O}(1)|t_2|_{p_2}..|t_p|_{p_p},$$

if

$$\frac{1}{\widetilde{p}} = \frac{1}{p_2} + .. + \frac{1}{p_p}.$$

If $p = 1$ the right hand side of the last equation is 0 by definition and $\widetilde{p} = \infty$. Similarly

$$|\langle b, s_2 \otimes .. \otimes s_q \rangle|_{\widetilde{q}} = \mathcal{O}(1)|s_2|_{q_2}..|s_q|_{q_q},$$

if

$$\frac{1}{\widetilde{q}}=\frac{1}{q_2}+..+\frac{1}{q_q},$$

and we conclude that the expression (8.35) is $\mathcal{O}(1)|t_2|_{p_2}..|t_{p_p}||s_2|_{q_2}..|s_q|_{q_q}$, if

$$1=\frac{1}{p_2}+..+\frac{1}{p_p}+\frac{1}{q_2}+..+\frac{1}{q_q}.$$

The gradients are treated the same way and we have verified the lemma in the case when $r=1$. Notice that the argument breaks down when $p=q=1$.

If $r\ge 2$, we first sum in $j_1=k_1$ as above and treat the remaining summations with the trace lemma. Again the argument breaks down in the last step, when $r=p=q$. #

Applying Lemma 8.2, we see that (8.34) is 1 standard. The 1 standardness of the second term to the right in (8.33) follows more directly from Lemma 8.2 and the fact that $\phi''_{j,k}$ and $\rho_0(\ell)\partial_{x_\ell}\Phi_{j,k}$ are 2 and 3 standard respectively. This completes the proof of Lemma 8.1. #

We next consider the equations (8.18). The assumptions on u, v will require a generalization of the notion of standardness.

Definition 8.3. Let $k\in\mathbf{N}$, $1\le p\le+\infty$. A k tensor $a(x)=a_{j_1,..,j_k}(x)$ is said to be (k,p) standard if

$$\begin{aligned}&\langle\nabla^m a,t_1\otimes..\otimes t_{k+m}\rangle=\mathcal{O}(1)|t_1|_{p_1}..|t_{k+m}|_{p_{k+m}},\ \text{uniformly}\\&\text{for } 1\le p_j\le\infty,\ 1=\frac{1}{p}+\frac{1}{p_1}+..+\frac{1}{p_{k+m}},\ t_j\in\ell^{p_j}.\end{aligned}\tag{8.36}$$

We observe that "k standard" and "(k,∞) standard" are the same thing.

We assume for some $1\le\rho_u,\rho_v\in W_a$:

$$\rho_u(\ell)\partial_{x_\ell}u,\ \rho_v(\ell)\partial_{x_\ell}v \text{ are (1,2) standard.}\tag{H6}$$

This implies that

$$\|\rho_u(\ell)e^{-\phi/2h}\partial_{x_\ell}u\|_{\ell^2\otimes L^2}=\mathcal{O}(1),\tag{8.37}$$

and similarly for v. We now solve the first equation in (8.18) for $\widetilde{u}=\sum\widetilde{u}_\nu dx_\nu$ in the spirit of [HS], [S1]. The wellposedness of Gr(0,0) for $z=0$ means that we can solve

$$\Delta_\phi^{(0)}f=e^{-\phi/2h}(u-\langle u\rangle)$$

with $f\in\mathcal{H}_1$ unique up to a multiple of $e^{-\phi/2h}$. Since the RHS is in $\mathcal{S}$, we have $f\in\mathcal{S}$. Since $\Delta_\phi^{(0)}=d_\phi^*d_\phi$, it suffices to take $\widetilde{u}=d_\phi f$. Using that $d_\phi\Delta_\phi^{(0)}=\Delta_\phi^{(1)}d_\phi$, we get

$$d_\phi(e^{-\phi/2h}u)=\Delta_\phi^{(1)}\widetilde{u},\tag{8.38}$$

and since $\rho_u \in W_a$, we know from the proof of Proposition 7.1 (valid also in the case $N = -1$) that the solution $\widetilde{u}$ satisfies

$$\sum_\nu \|\rho_u(\nu)\widetilde{u}_\nu\|_{L^2}^2 + \sum_\nu \sum_\mu \|\rho_u(\nu) Z_\mu \widetilde{u}_\nu\|_{L^2}^2 \le \mathcal{O}(1)\|(\rho_u \otimes 1) d_\phi(e^{-\phi/2h}u)\|_{\ell^2 \otimes L^2}^2 \le \mathcal{O}(1)h, \tag{8.39}$$

where we used (8.37) in the last estimate.

Similarly, with $\widetilde{v} = \sum \widetilde{v}_\nu dx_\nu = \Delta_\phi^{(1)^{-1}} d_\phi(e^{-\phi/2h}v)$, we have

$$\sum_\nu \|\rho_v(\nu)\widetilde{v}_\nu\|_{L^2}^2 + \sum_\nu \sum_\mu \|\rho_v(\nu) Z_\mu \widetilde{v}_\nu\|_{L^2}^2 \le \mathcal{O}(1)h. \tag{8.40}$$

As for the last equation in (8.18), we recall that $\rho_0(\ell)\partial_{x_\ell} D$ is $(1,\infty)$ standard and we have to accept some loss since we need an ℓ^2 estimate. We choose to take the loss in the exponential weight. Let $\rho_0 \ge \rho_1 \in W_a$ and assume that $\rho_1(\ell)\partial_{x_\ell} D$ is (1,2) standard, or the weaker assumption

$$\sup_x |\rho_1(\ell)\partial_{x_\ell} D(x)|_2 = \mathcal{O}(1). \tag{H7}$$

In the application later on this will be achieved by taking $\rho_1(\nu) = \rho_0(\nu)e^{-\epsilon \mathrm{dist}(\nu_0,\nu)}$ for some $\epsilon > 0$. Then for the solution in $\ell^2 \otimes \mathcal{H}_1$ of the last equation in (8.18), we have

$$\sum_\nu \|\rho_1(\nu)\widetilde{D}_\nu\|^2 + \sum_\nu \sum_\mu \|\rho_1(\nu) Z_\mu \widetilde{D}_\nu\|^2 \le \mathcal{O}(1)h. \tag{8.41}$$

The justification of our derivation of (8.30) is now immediate, and next we shall estimate the various terms that appear in that equation.

Estimate of I_1 *(see (8.17)).* We rewrite the integrand in (8.17) as

$$e^{-\phi/h} \sum_{j,k} (\rho_0(j)\rho_0(k)\Phi_{j,k}(x)) \frac{1}{\rho_u(j)\rho_0(j)} (\rho_u(j)\partial_{x_j} u) \frac{1}{\rho_v(k)\rho_0(k)} (\rho_v(k)\partial_{x_k} v). \tag{8.42}$$

In the proof of Lemma 8.1 we have seen that $\rho_0(j)\rho_0(k)\Phi_{j,k}(x)$ is 2 standard, so the double sum in (8.42) is

$$\mathcal{O}(1)\frac{1}{\inf \rho_u\rho_0}\frac{1}{\inf \rho_v\rho_0}|\rho_u(j)\partial_{x_j}u|_2 |\rho_v(k)\partial_{x_k}v|_2.$$

Combining this with (8.37) and the analogous estimate for v, we get

$$\mathrm{I}_1 = \mathcal{O}(1)h\frac{1}{\inf \rho_u\rho_0}\frac{1}{\inf \rho_v\rho_0}. \tag{8.43}$$

Estimate of $I_{2,1}$ *(see (8.21)).* We write the integrand in (8.21) as

$$\sum_{j,k,\nu} (\rho_0(j)\rho_0(\nu)\partial_{x_\nu}\Phi_{j,k}) \frac{1}{\rho_u(j)\rho_0(j)} (e^{-\phi/2h}\rho_u(j)\partial_{x_j}\partial_{x_k} u) \frac{1}{\rho_v(\nu)\rho_0(\nu)} (\rho_v(\nu)\widetilde{v}_\nu). \tag{8.44}$$

Here we observe that $\rho_0(j)\rho_0(\nu)\partial_{x_\nu}\Phi_{j,k}$ is (3,∞) standard. If a is (2,2) standard, then $|a|_2 = \mathcal{O}(1)$ by Lemma B.1. By Lemma B.2 in the same section, we know that if b is (3,∞) standard, a a 2-tensor and c a 1 tensor, then

$$\langle b, a \otimes c\rangle = \mathcal{O}(1)|a|_2|c|_2.$$

Hence the expression (8.44) is

$$\mathcal{O}(1)\frac{1}{\inf(\rho_u\rho_0)}e^{-\phi/2h}\mathcal{O}(1)\frac{1}{\inf(\rho_v\rho_0)}|\rho_v(\nu)\widetilde{v}_\nu(x)|_2. \tag{8.45}$$

We use this and (8.40) in (8.21), and get

$$\mathrm{I}_{2,1} = \mathcal{O}(1)\frac{h^2}{\inf(\rho_u\rho_0)\inf(\rho_v\rho_0)}. \tag{8.46}$$

Estimate of $\mathrm{I}_{2,2}$ *(see (8.21)).* Rewrite the integrand in (8.21) as

$$\sum_{j,k,\nu}\rho_0(j)\Phi_{j,k}e^{-\phi/2h}\frac{1}{\rho_0(j)\rho_u(j)}(\rho_u(j)\partial_{x_\nu}\partial_{x_j}\partial_{x_k}u)\widetilde{v}_\nu = e^{-\phi/2h}\sum_{j,k}\rho_0(j)\Phi_{j,k}B_{j,k}, \tag{8.47}$$

with

$$B_{j,k} = \frac{1}{\rho_0(j)\rho_u(j)}\sum_\nu(\rho_u(j)\partial_{x_\nu}\partial_{x_j}\partial_{x_k}u)\widetilde{v}_\nu. \tag{8.48}$$

Here we recall that $\rho_u(j)\partial_{x_\nu}\partial_{x_j}\partial_{x_k}u$ is (3,2) standard, so if we view $B = (B_{j,k})$ as a matrix,

$$\|B\|_{\mathcal{L}(\ell^\infty,\ell^1)} = \mathcal{O}(1)\frac{1}{\inf(\rho_0\rho_u)}|\widetilde{v}_\nu|_2. \tag{8.49}$$

On the other hand $\rho_0(j)\Phi_{j,k}$ is 2 standard and hence $\mathcal{O}(1) : \ell^1 \to \ell^1$. If we view the last sum in (8.47) as $\mathrm{tr}\left((\rho_0(j)\Phi_{j,k}) \circ {}^tB\right)$, we conclude by the trace lemma that it is $\mathcal{O}(1)\frac{1}{\inf(\rho_0\rho_u)}|\widetilde{v}_\nu|_2$. Using this in (8.47) and then in (8.21), we get

$$\mathrm{I}_{2,2} = \mathcal{O}(h^{\frac{3}{2}})\frac{1}{\inf(\rho_0\rho_u)}\|\widetilde{v}\|_{\ell^2\otimes L^2} = \mathcal{O}(h^2)\frac{1}{\inf(\rho_0\rho_u)}. \tag{8.50}$$

Here we used (8.40) in the last step, with ρ_v replaced by 1. By playing with ρ_v also we could reach the estimate

$$\mathrm{I}_{2,2} = \mathcal{O}(h^2)\frac{1}{\inf(\rho_0\rho_u)}\frac{1}{\inf(\rho_0\rho_v)},$$

provided that we add to (H6), the assumption that

$$\sum_{j,k,\nu}\rho_u(j)\frac{\rho_v(k)}{\rho_v(\nu)}(\partial_{x_\nu}\partial_{x_j}\partial_{x_k}u)t_js_kr_\nu = \mathcal{O}(1)|t|_\infty|s|_\infty|r|_2.$$

Estimate of $II_{1,1}$ (see (8.25)). Write the integrand in (8.25) as

$$\sum_{j,\nu} \frac{1}{\rho_0(j)\rho_u(j)} (\rho_0(j)\partial_{x_\nu}\Psi_j)(e^{-\phi/2h}\rho_u(j)\partial_{x_j}u)\widetilde{v}_\nu = \frac{\mathcal{O}(1)}{\inf(\rho_0\rho_u)} e^{-\phi/2h}|\widetilde{v}_\nu|_2.$$

Here we used that $\rho_0(j)\partial_{x_\nu}\Psi_j$ is 2 standard by Lemma 8.1, and that $\rho_u(j)\partial_{x_j}u$ is (1,2) standard by (H6). Inserting this into (8.25) and using (8.40) with ρ_v replaced by 1, we get

$$\mathrm{II}_{1,1} = \mathcal{O}(1)\frac{h^2}{\inf(\rho_0\rho_u)}. \tag{8.51}$$

Estimate of $II_{1,2}$ (see (8.25)). Write the integrand in (8.25) as

$$e^{-\phi/2h}\sum_{j,\nu}(\rho_0(j)\Psi_j)\frac{1}{\rho_0(j)\rho_u(j)}(\rho_u(j)\partial_{x_\nu}\partial_{x_j}u)\widetilde{v}_\nu = \mathcal{O}(1)e^{-\phi/2h}\frac{1}{\inf(\rho_0\rho_u)}|\widetilde{v}_\nu|_2, \tag{8.52}$$

where we used Lemma 8.1 and (H6). Using this with (8.40) in (8.25), we get

$$\mathrm{II}_{1,2} = \mathcal{O}(1)\frac{h^2}{\inf(\rho_0\rho_u)}. \tag{8.53}$$

Estimate of $III_{1,1}$ (see (8.29)). We write the integrand in (8.29) as

$$\sum_{\nu,\mu}\widetilde{v}_\mu(\rho_1(\nu)Z_\mu\widetilde{D}_\nu)\frac{1}{\rho_1(\nu)\rho_u(\nu)}(\rho_u(\nu)\partial_{x_\nu}u) = \frac{\mathcal{O}(1)}{\inf(\rho_1\rho_u)}|\widetilde{v}|_2|\rho_1(\nu)Z_\mu\widetilde{D}_\nu|_{\ell^2\otimes\ell^2}, \tag{8.54}$$

where we used (H6). Using this in the expression for $\mathrm{III}_{1,1}$ together with (8.40), (8.41), we get

$$\mathrm{III}_{1,1} = \mathcal{O}(1)\frac{h^{\frac{3}{2}}}{\inf(\rho_1\rho_u)}. \tag{8.55}$$

Estimate of $III_{1,2}$ (see (8.29)). Write the integrand as

$$\sum_{\nu,\mu}\widetilde{v}_\mu(\rho_1(\nu)\widetilde{D}_\nu)\frac{1}{\rho_1(\nu)\rho_u(\nu)}(\rho_u(\nu)\partial_{x_\mu}\partial_{x_\nu}u) = \mathcal{O}(1)\frac{1}{\inf(\rho_1\rho_u)}|\widetilde{v}_\mu|_2|\rho_1(\nu)\widetilde{D}_\nu|_2, \tag{8.56}$$

since $|\rho_u(\nu)\partial_{x_\mu}\partial_{x_\nu}u|_{\ell^2\otimes\ell^2} = \mathcal{O}(1)$ by (H6) and Lemma B.1. Using this in (8.29) with (8.40), (8.41), we get

$$\mathrm{III}_{1,2} = \mathcal{O}(1)\frac{h^2}{\inf(\rho_1\rho_u)}. \tag{8.57}$$

Recall that for $X = \mathrm{I}, \mathrm{II}, \mathrm{III}$ and $i = 1, 2$, we get $X_{2,i}$ from $X_{1,i}$ by exchanging u and v as well as their associated quantities. This means that we get the estimates for $X_{2,i}$ from those for $X_{1,i}$, by exchanging u and v to the right, and we therefore obtain estimates for all terms in (8.30). Summing up, we have

Proposition 8.4. *Let $\phi_t(x) = \phi_t(x;h)$, $0 \le t \le 1$, $x \in \mathbf{R}^\Lambda$ be C^1 in t and smooth in x, of the form (8.1H), satisfying (8.2H). Let W be a set of weights $\rho : \Lambda \to]0,\infty[$, with $\rho \in W \Rightarrow 1/\rho \in W$, $1 \in W$. Assume that $\phi = \phi_t$ satisfies ($\widetilde{H}1$) (section 7), (H2), (H3) (section 1), ($\widetilde{H}4$) and (H5) of this section uniformly with respect to $t \in [0,1]$. Here W_a is defined prior to (H5) with some fixed a as in (8.32). Let $u, v \in C^\infty(\mathbf{R}^\Lambda;\mathbf{R})$ be independent of t and satisfy (H6). Finally choose $\rho_0 \ge \rho_1 \in W_a$ such that (H7) holds. (Cf Lemma 8.1.) Then*

$$\partial_t \mathrm{Cor}_{\phi_t}(u,v) = \mathcal{O}(1)\Big(\frac{h}{\inf(\rho_u\rho_0)\inf(\rho_v\rho_0)} + \frac{h^{\frac{3}{2}}}{\inf(\rho_1\rho_u)} + \frac{h^{\frac{3}{2}}}{\inf(\rho_1\rho_v)}\Big). \tag{8.58}$$

The estimate (8.58) could certainly be improved to the price of some further assumptions. Also notice that the assumptions of standardness could be weakened, since we only use derivatives up to some fixed finite order.

9. Asymptotics of the correlations.

This section is divided into three parts. In part A we make only the assumptions of section 1 and consider the correlation of two functions with (1,2)-standard gradients, which are independent of h. We show that it has an asymptotic expansion in powers of h, and that this expansion is valid uniformly with respect to Λ. In part B we let $\Lambda = (\mathbf{Z}/L\mathbf{Z})^d$ with $L \in \{2,3,..\}$. Adding assumptions on $\phi''(0)$; an assumption of translation invariance, as well as the assumption ($\widetilde{\mathrm{H}}1$) of section 7 for a suitable family of weights, we study the asymptotics of $\mathrm{Cor}\,(x_\nu, x_\mu)$ for $\nu, \mu \in \Lambda$, when $1 \ll \mathrm{dist}\,(\nu,\mu) \ll L$ and obtain the product of an exponentially decaying factor and a factor with power behaviour, in the limit $\nu - \mu \to \infty$. The exponent in the exponential factor is positively homogeneous of degree 1 in $\nu - \mu$ and we show that it has an asymptotic expansion in powers of h. We obtain a similar result for the power factor. In this result all terms in the asymptotic expansions, may depend on Λ but they remain bounded and the asymptotic expansions are valid uniformly in Λ. In part C we make some additional assumptions that allow us to pass to the thermodynamical limit. This part contains the final result of the paper, and the results here remain valid also with Λ equal to a finite subset of $\mathbf{Z}^d$ which contains a large ball centered at 0. (In section 10 we derive simplified sets of assumptions in order to reach the formulation of the main result in the introduction.) Throughout the whole section we make the assumptions (H1–4) of section 1 and let the functions ϕ be normalized as in (8.2H).

<u>A.</u> In this part we only assume that u, v are functions on $\mathbf{R}^\Lambda$ independent of h, such that ∇u, ∇v are (1,2) standard (as defined in section 8). We are interested in the asymptotics of

$$\mathrm{Cor}\,(u,v) = (e^{-\phi/2h}(u - \langle u\rangle)|e^{-\phi/2h}(v - \langle v\rangle)) = h({\Delta_\phi^{(1)}}^{-1} e^{-\phi/2h}du|e^{-\phi/2h}dv), \tag{9.1}$$

as $h \to 0$. The second equality was established in this explicit form in [S1], but already effectively used in earlier work by Helffer and the author [HS]. Under the present assumptions the derivation is very simple: Let $\widetilde{u} \in \mathcal{S}$ solve (8.18). After an integration by parts, we get $\mathrm{Cor}\,(u,v) = (d_\phi \widetilde{u} | d_\phi(e^{-\phi/2h} v))$. In view of (8.38), we have $d_\phi \widetilde{u} = (\Delta_\phi^{(1)})^{-1} d_\phi u$, which gives the last expression in (9.1).

We apply Proposition A.1, which extends to $(P,2)$ standard tensors, and write

$$du\, e^{-\phi/2h} = \sum_{n=0}^{N+1} \sum_{\nu=0}^{M} \sum_{j\in\Lambda^n} h^{\frac{1}{2}n+\nu} Z_j^*(\widetilde{u}_{j,\nu}^n e^{-\phi/2h}), \tag{9.2}$$

where $\widetilde{u}_{j,\nu}^n$ is $(1+j,2)$ standard, and for $\nu < M$, $n \le N$ it is independent of x. Applying the procedure of section 4 and Lemma B.1, we see that the $\ell^2 \otimes L^2$ norm of $\sum_{j\in\Lambda^n} Z_j^*(\widetilde{u}_{j,\nu}^n e^{-\phi/2h})$ is $\mathcal{O}(1)$, and consequently we have for M large enough

$$du\, e^{-\phi/2h} = \sum_{n=0}^{N} \sum_{\nu=0}^{M-1} h^{\frac{1}{2}n+\nu} Z_j^*(\widetilde{u}_{j,\nu}^n e^{-\phi/2h}) + \mathcal{O}(h^{\frac{N+1}{2}}) \text{ in } \ell^2 \otimes L^2, \tag{9.3}$$

where now $\widetilde{u}_{j,\nu}^n$ are constant. We have of course the analogous expression for $dv\, e^{-\phi/2h}$ and we let $\widetilde{v}_{j,\nu}^n$ be the corresponding coefficients.

Next we shall apply Proposition 6.5 together with the formula

$$\Delta_\phi^{(1)^{-1}} = E^{N,1} - E_+^{N,1}(E_{-+}^{N,1})^{-1}E_-^{N,1}. \tag{9.4}$$

Since the terms in the sum in (9.3) belong to the range of $R_-^{N,1}$ and $E^{N,1}R_-^{N,1} = 0$, $R_+^{N,1}E^{N,1} = 0$, $E_-^{N,1}R_-^{N,1} = 1$, $R_+^{N,1}E_+^{N,1} = 1$, we obtain from (9.1), (9.3), (9.4):

$$\begin{aligned}
&\mathrm{Cor}\,(u,v) = \mathcal{O}(h^{\frac{N+3}{2}}) + \qquad\qquad (9.5)\\
&h \sum_{n,m=0}^{N} \sum_{\nu,\mu=0}^{M-1} h^{\frac{1}{2}(n+m)+\nu+\mu} \sum_{\substack{j\in\Lambda^n\\ k\in\Lambda^m}} ((E_{-+}^{N,1})^{-1}E_-^{N,1} Z_j^*(e^{-\phi/2h}) | E_- Z_k^*(e^{-\phi/2h})) \widetilde{u}_{j,\nu} \cdot \overline{\widetilde{v}}_{k,\mu} \\
&= \mathcal{O}(h^{\frac{N+3}{2}}) + h \sum_{n,m=0}^{N} \sum_{\nu,\mu=0}^{M-1} h^{\frac{1}{2}(n+m)+\nu+\mu} \sqrt{n!m!}((E_{-+}^{N,1})^{-1} \widetilde{u}_{\cdot,\nu}^n | \widetilde{v}_{\cdot,\mu}^m).
\end{aligned}$$

From Proposition 6.5 we know that $E_{-+}^{N,1}$ as well as its inverse have asymptotic expansions of the type (6.99). We obtain

Proposition 9.1. *Assume that ϕ satisfies the assumptions of section 1 and let u, v be independent of h and have (1,2) standard differentials. Then uniformly in Λ, we have*

$$\mathrm{Cor}\,(u,v) \sim h \sum_{k=0}^{\infty} h^{k/2} C_k(u,v). \tag{9.6}$$

Here $C_k(u,v)$ can be expressed in terms of derivatives up to finite order of u, v at 0, and

$$C_0(u,v) = (\phi''(0)^{-1} du(0) | dv(0))_{\ell^2}. \tag{9.7}$$

B. In this part we assume that

$$\Lambda = (\mathbf{Z}/L\mathbf{Z})^d,\ L \in \{2, 3, ..\}. \tag{9.8}$$

We assume that ϕ is translation invariant,

$$\phi(\tau_\ell x) = \phi(x),\ \forall \ell \in \Lambda,\ x \in \mathbf{R}^\Lambda, \tag{H8}$$

where $(\tau_\ell x)(j) = x(j - \ell)$. It follows (as in [S1], see also [BJS]) that $\phi''(0)$ is a convolution on $\ell^2(\Lambda)$ and after a dilation in h, we may assume that

$$\phi''(0) = I - v_0 *, \tag{9.9}$$

where v_0 is a real valued even function on Λ with $v_0(0) = 0$.

We shall assume that $\phi''(0)$ is of ferromagnetic type:

$$\phi''_{j,k}(0) \leq 0,\ \text{when } j \neq k. \tag{H9}$$

In other words, we assume that $v_0 \geq 0$. Using Fourier expansions, we see that the lowest eigenvalue of $\phi''(0)$ is

$$\lambda_{\min}(\phi''(0)) = 1 - \sum v_0(j) \geq 1/\mathcal{O}(1), \tag{9.10}$$

where the lower bound follows from (H3). Similarly (H1) implies that

$$\sum v_0(j) = \mathcal{O}(1). \tag{9.11}$$

Let $\pi_\Lambda : \mathbf{Z}^d \to \Lambda$ be the natural projection and let $\widetilde{v}_0 : \mathbf{Z}^d \to [0, \infty[$ be a function such that

$$v_0(\lambda) = \sum_{\mu \in \pi_\Lambda^{-1}(\lambda)} \widetilde{v}_0(\mu), \tag{9.12}$$

such that

$$\widetilde{v}_0 \text{ is even}. \tag{9.13}$$

From now we assume that $L \geq L_0$ is sufficiently large and assume that there exists a finite set $K \subset \mathbf{Z}^d$ independent of Λ such that

$$\widetilde{v}_0(j) \geq \text{const.} > 0 \text{ for } j \in K, \text{ and } \mathrm{Gr}(K) = \mathbf{Z}^d, \tag{H10}$$

where $\mathrm{Gr}(K)$ denotes the smallest subgroup of $\mathbf{Z}^d$ which contains K. Put

$$F_{\widetilde{v}_0}(\eta) = \sum_{k \in \mathbf{Z}^d} e^{k \cdot \eta} \widetilde{v}_0(k),\ \eta \in \mathbf{R}^d, \tag{9.14}$$

where we know ([S1]) that

$$\{\eta \in \mathbf{R}^d;\, F_{\widetilde{v}_0}(\eta) < \infty\} \tag{9.15}$$

is convex and that $F_{\widetilde{v}_0}$ is a convex function, which is smooth on the interior of the set (9.15). Assume

$$\begin{aligned} &\exists \text{ an open convex even set } \widetilde{\Omega} \subset \mathbf{R}^d, \text{ independent of } \Lambda, \text{ such that} \\ &F_{\widetilde{v}_0}(\eta) \text{ is uniformly bounded on every compact subset of } \widetilde{\Omega}. \end{aligned} \tag{H11}$$

Here we define even sets to be the ones which are symmetric around 0. Then from [S1] (based on the fact that K is contained in no hyperplane of $\mathbf{R}^d$) we know that $F_{\widetilde{v}_0}$ is strictly convex:

$$\nabla^2 F_{\widetilde{v}_0}(\eta) \geq \frac{1}{\mathcal{O}(1)},\ \eta \in \widetilde{\Omega}. \tag{9.16}$$

(We even have that $\log F_{\widetilde{v}_0}$ is strictly convex.)

Let $\Omega \subset\subset \widetilde{\Omega}$ be an open even convex set, which is independent of Λ and assume that

$$\liminf_{\Omega \ni \eta \to \partial\Omega} F_{\widetilde{v}_0}(\eta) \geq 1 + 3\epsilon_0, \tag{9.17H}$$

where $\epsilon_0 > 0$ is independent of Λ. $F_{\widetilde{v}_0}(\eta)$ is then uniformly bounded in Ω and its derivatives are uniformly bounded on every fixed relatively compact subset of Ω. Using also (9.16), it is clear that the sets

$$\Omega_b := \{\eta \in \Omega; F_{\widetilde{v}_0} < b\},\ 1 \leq b \leq 1 + 2\epsilon_0, \tag{9.18}$$

are relatively compact in Ω and uniformly strictly convex with (uniformly) smooth boundary. Moreover they are even.

Let

$$p_b(x) = \sup_{\eta \in \Omega_b} x \cdot \eta,\ x \in \mathbf{R}^d, \tag{9.19}$$

be the corresponding support function. Then p_b are norms on $\mathbf{R}^d$, of class $C^\infty(\mathbf{R}^d \setminus \{0\})$, strictly convex transversally to the radial direction. Since

$$p_{1+3\epsilon_0/2}(x) \leq \sup_{1 \leq j \leq N} x \cdot \eta_j,$$

with $\eta_j \in \Omega_{1+2\epsilon_0}$ and N uniformly bounded, we see that

$$\sum_{x \in \mathbf{Z}^d} e^{p_{1+3\epsilon_0/2}(x)} \widetilde{v}_0(x) = \mathcal{O}(1). \tag{9.20}$$

Let $\widetilde{r} \in C^{1,1}(\mathbf{R}^d; \mathbf{R})$ with $\widetilde{r}(0) = 0$, $\nabla\widetilde{r}(x) \in \Omega_{1+\epsilon_0/2}$, $|\nabla^2\widetilde{r}(x)| \leq \epsilon_1$, $\forall x \in \mathbf{R}^d$. Then $\widetilde{r}(x)$ is arbitrarily well approximated in L^∞ by $\nabla\widetilde{r}(0) \cdot x$ for $|x| \leq R$ with $R > 0$ arbitrarily large, provided that we choose ϵ_1 small enough. Combining this with (9.20), we see as in [BJS] (and [S1], [SW])

$$\sum_{x\in\mathbf{Z}^d} e^{\widetilde{r}(x)}\widetilde{v}_0(x) \le 1+\epsilon_0. \tag{9.21}$$

Let $r \in C^{1,1}((\mathbf{R}/L\mathbf{Z})^d)$ be real and assume that $\nabla r(x) \in \Omega_{1+\epsilon_0/2}$, $|\nabla^2 r| \le \epsilon_1$ everywhere, where ϵ_1 is small enough. Then

$$\sum_{x\in\Lambda} e^{r(x)} v_0(x) = \sum_{x\in\mathbf{Z}^d} e^{r\circ\pi_\Lambda(x)}\widetilde{v}_0(x) \le 1+\epsilon_0, \tag{9.22}$$

since $\widetilde{r} := r\circ\pi_\Lambda$ satisfies the earlier assumptions.

If $r'(x)$ is merely Lipschitz on $(\mathbf{R}/L\mathbf{Z})^d$ with $r'(0)=0$, and $\nabla r'(x)\in\Omega_{1+\epsilon_0/2}$ a.e., then by regularization,we can find r with the above properties, such that $r-r'=\mathcal{O}_{\epsilon_0,\epsilon_1}(1)$.

Let $a=-\epsilon_0$, and let $W=W_a$ consist of all weights $\rho(x)=e^{r(x)}$, $x\in(\mathbf{R}/L\mathbf{Z})^d$, for which $\nabla r\in\Omega_{1+\epsilon_0/2}$, and $|\nabla^2 r|\le\epsilon_1$, with $\epsilon_1>0$ sufficiently small. Using Shur's lemma and (9.22) (with r there replaced by $r(x)-r(0)$) we see that for all $\rho\in W$:

$$\|\rho(v_0*)\rho^{-1}\|_{\mathcal{L}(\ell^2,\ell^2)}, \|\rho^{-1}(v_0*)\rho\|_{\mathcal{L}(\ell^2,\ell^2)} \le 1+\epsilon_0, \tag{9.23}$$

so that

$$(\rho^{-1}\phi''(0)\rho u|u) \ge -\epsilon_0|u|^2,\ \ u\in\ell^2(\Lambda). \tag{9.24}$$

We fix ϵ_0 with

$$0<\epsilon_0<1-|v_0|_1-\frac{1}{\mathcal{O}(1)}, \tag{9.25}$$

so that

$$-\epsilon_0 > -\lambda_{\min}(\phi''(0))+\frac{1}{\mathcal{O}(1)} = -1+|v_0|_1+\frac{1}{\mathcal{O}(1)}. \tag{9.26}$$

As for the higher derivates of ϕ, we will assume $(\widetilde{\mathrm{H1}})$ (section 7), with W equal to the set of weights defined above.

We apply Proposition 7.1 with $N=0$, $z=0$, and we shall drop the superscript (0,1) for simplicity. We recall the formula

$$\Delta_\phi^{(1)^{-1}} = E - E_+E_{-+}^{-1}E_-, \tag{9.27}$$

that we shall use in (9.1), with $u=x_\mu$, $v=x_\nu$, $\mu,\nu\in\Lambda$.

Let us first consider the contribution from E. According to Proposition 7.1, we know that $(\rho^{-1}\otimes 1)E(\rho\otimes 1)=\mathcal{O}(1):\ell^2\otimes L^2\to\ell^2\otimes L^2$, for all $\rho\in W$, and in view of the observation after (9.22), we know that this remains true for $\rho=e^r$, with $r\in\mathrm{Lip}\,((\mathbf{R}/L\mathbf{Z})^d)$, $\nabla r\in\Omega_{1+\epsilon_0/2}$ a.e. Now recall that we have introduced the norm p_b on $\mathbf{R}^d$ in (9.19). Let $d_b=d_b^\Lambda$ be the corresponding distance on Λ, given by

$$d_b(\nu,\mu) = \inf_{\substack{\widetilde{\nu}\in\pi_\Lambda^{-1}(\nu)\\ \widetilde{\mu}\in\pi_\Lambda^{-1}(\mu)}} p_b(\widetilde{\nu}-\widetilde{\mu}).$$

Then $(\rho_\mu \otimes 1)e^{-\phi/2h}dx_\mu = \mathcal{O}(1)$ in $\ell^2 \otimes L^2$ with $\rho_\mu = e^{d_{1+\epsilon_0/2}(\mu,\cdot)}$. By the weighted boundedness result for E, that we have established above, we have $(\rho_\mu \otimes 1)E(e^{-\phi/2h}dx_\mu) = \mathcal{O}(1)$ in $\ell^2 \otimes L^2$. It follows that (cf. (9.1))

$$h(Ee^{-\phi/2h}dx_\mu|e^{-\phi/2h}dx_\nu) = \mathcal{O}(1)he^{-d_{1+\epsilon_0/2}(\nu,\mu)}. \tag{9.28}$$

As a matter of fact, since $e^{-\phi/2h}dx_\mu$, $e^{-\phi/2h}dx_\nu$ belong to the image of R_- and $ER_- = 0$, the expression (9.28) vanishes, However the weaker formulation in (9.28) may be of interest for more general correlations. The main contribution to (9.1) will come from the second term in (9.27) and is equal to

$$-h(E_{-+}^{-1}E_-(e^{-\phi/2h}dx_\mu)|E_-(e^{-\phi/2h}dx_\nu)) = -h(E_{-+}^{-1}\delta_\mu, \delta_\nu), \tag{9.29}$$

since $e^{-\phi/2h}dx_\mu = R_-\delta_\mu$ and $E_-R_- = 1$. Here we are in the presence of convolution matrices. Indeed, from (H8) we deduce (cf. [S1]) a certain translation invariance for $\mathcal{P}^{0,1}$ and its inverse, which implies that E_{-+} and its inverse are convolutions and

$$E_-(e^{-\phi/2h}dx_\mu) = \tau_\mu E_-(e^{-\phi/2h}dx_0). \tag{9.30}$$

Proposition 7.4 can be applied together with the remark after (9.22) to show that

$$E_{-+} = -(1 - v*),\ v \sim \sum_{\nu=0}^{\infty} h^\nu v_\nu \text{ in } \mathcal{L}(\rho\ell^2, \rho\ell^2), \tag{9.31}$$

uniformly when $\rho = e^r$, with r Lipschitz, such that $\nabla r \in \Omega_{1+\epsilon_0/2}$ a.e., with v_0 as before. Here we equip $\rho\ell^2$ with the natural norm $\|\rho u\|_{\rho\ell^2} = \|u\|_{\ell^2}$. This implies that

$$|v(\ell)| \le \mathcal{O}(1)e^{-d_{1+\epsilon_0/2}(\ell)},\ |v_\nu(\ell)| \le \mathcal{O}_\nu(1)e^{-d_{1+\epsilon_0/2}(\ell)},\ \ell \in \Lambda. \tag{9.32}$$

We have already assumed that v_0 has a lift $\widetilde{v}_0$ to $\mathbf{Z}^d$ with certain properties including (9.13) and we know that $\widetilde{v}_0(\ell) = \mathcal{O}(1)e^{-p_{1+\epsilon_0/2}(\ell)}$. For v_ν, $\nu \ge 1$ and $v - v_0$ we use the following lift: If $\ell \in \Lambda$, let $A(\ell) \subset \mathbf{Z}^d$ be the set of $\widetilde{\ell}$ in $\pi_\Lambda^{-1}(\ell)$ for which $p_{1+\epsilon_0/2}(\widetilde{\ell})$ is minimal ($= d_{1+\epsilon_0/2}(0,\ell)$). Let $\widetilde{\Lambda}$ be the union of all such $A(\ell)$, and define $\widetilde{v}_\nu(\ell)$ to be the unique function on $\mathbf{Z}^d$ with support in $\widetilde{\Lambda}$, such that

$$v_\nu(\ell) = \sum_{\widetilde{\ell}\in\pi_\Lambda^{-1}(\ell)} \widetilde{v}_\nu(\widetilde{\ell}), \tag{9.33}$$

and such that $\widetilde{v}_\nu$ is constant on each $A(\ell)$. Define $\widetilde{v} - \widetilde{v}_0$ (where $\widetilde{v}_0$ is already known) by the same construction. This means that we have defined $\widetilde{v}$. Then

$$v(\ell) = \sum_{\widetilde{\ell}\in\pi_\Lambda^{-1}(\ell)} \widetilde{v}(\ell), \tag{9.34}$$

$$\widetilde{v} \sim \sum_0^\infty h^\nu \widetilde{v}_\nu \tag{9.35}$$

in ℓ^∞ and even in $e^{-p_{1+\epsilon_0/2}}\ell^\infty$. Let

$$\widehat{\widetilde{v}}(\xi) = \sum_{\ell\in\mathbf{Z}^d} \widetilde{v}(\ell) e^{-i\ell\xi},\ \xi \in (\mathbf{R}/2\pi\mathbf{Z})^d =: \mathbf{T}^d, \tag{9.36}$$

denote the Fourier transform of $\widetilde{v}$. From the above asymptotic expansions, it follows that $\widehat{\widetilde{v}}$ extends to a holomorphic function in $\mathbf{T}^d + i\Omega_{1+\epsilon_0/2}$ which is uniformly bounded and has a uniform asymptotic expansion

$$\widehat{\widetilde{v}}(\zeta) \sim \sum_{\nu=0}^\infty h^\nu \widehat{\widetilde{v}}_\nu(\zeta), \tag{9.37}$$

in $\mathbf{T}^d + i\Omega_b$, for every fixed $b < 1 + \epsilon_0/2$.

As in [S1], we can study the asymptotic behaviour of $(1 - \widetilde{v}*)^{-1}$ by means of Fourier inversion. In that paper (as well as in [BJS]) we only knew that $\widehat{\widetilde{v}}(\zeta) = \widehat{\widetilde{v}}_0(\zeta) + \mathcal{O}(h^{1/2})$ in $\mathbf{T}^d + i\Omega_b$, while we now have the full asymptotic expansion (9.37), but the discussion in the above mentioned papers goes through without any essential changes and will give the full h asymptotics. We only recall some steps. If

$$\widetilde{F}* = (1 - \widetilde{v}*)^{-1}, \tag{9.38}$$

then

$$\widetilde{F}(k) = \frac{1}{(2\pi)^d} \int_{\mathbf{T}^d} \frac{e^{ik\cdot\xi}}{1 - \widehat{\widetilde{v}}(\xi)} d\xi. \tag{9.39}$$

In section 4 of [S1], we discussed the corresponding inverse F_0* of $(1 - \widetilde{v}_0*)$,

$$\widetilde{F}_0(k) = \frac{1}{(2\pi)^d} \int_{\mathbf{T}^d} \frac{e^{ik\cdot\xi}}{1 - \widehat{\widetilde{v}}_0} d\xi, \tag{9.40}$$

denoted by $E(k)$ there. We observed that $1 - \widehat{\widetilde{v}}_0(\zeta) \neq 0$ for $\zeta = \xi + i\eta$, with $\eta \in \Omega_1$, and that $1 - \widehat{\widetilde{v}}_0(\zeta)$ vanishes for $\eta \in \partial\Omega_1$ precisely for $\xi = 0$. (Here is where the full power of (H10) is used.) We do not repeat the proof here, but simply recall that $\widehat{\widetilde{v}}_0(i\eta) = F_{\widetilde{v}_0}(\eta)$. For $k \in \mathbf{R}^d \setminus \{0\}$, let $\eta_0(k) = \eta_0(k/|k|) \in \partial\Omega_1$ be the unique point where the exterior normal of Ω_1 is equal to a positive multiple of k. We can write

$$\eta_0(k) = \eta_0'(k) + p_1(\frac{k}{|k|})\frac{k}{|k|},\ \eta_0'(k) \in (k)^\perp \tag{9.41}$$

and because of the strict convexity, we can represent the boundary of $\partial\Omega_1$ in a neighborhood of $\eta_0(k)$ as

$$\partial\Omega_1 = \{\eta_0'(k) + \eta' + (p_1(\frac{k}{|k|}) - g_{\frac{k}{|k|},0}(\eta'))\frac{k}{|k|};\ \eta' \in (k)^\perp \cap \mathrm{neigh}\,(0)\}, \tag{9.42}$$

where g is a real and analytic function vanishing to second order at 0 and with

$$g''_{\frac{k}{|k|},0}(0) > 0. \tag{9.43}$$

Near $i\eta(k)$, we can view the complex hypersurface $1 = \widehat{\widetilde{v}}_0(\zeta)$ as the complexification of the real-analytic hypersurface $i\partial\Omega_1$, so we get

$$(1 - \widehat{\widetilde{v}}_0)^{-1}(0) = \{i\eta_0'(k) + \zeta' + i(p_1(\frac{k}{|k|}) - g_{\frac{k}{|k|},0}(\frac{\zeta'}{i}))\frac{k}{|k|};\ \zeta' \in \mathrm{neigh}\,(0),\ k\cdot\zeta' = 0\}. \tag{9.43}$$

By contour deformation and residues, we got in [S1]:

$$\widetilde{F}_0(k) = \tag{9.44}$$
$$\frac{ie^{-p_1(k)}}{(2\pi)^{d-1}}\int_{\xi'\in V\cap(k)^\perp} \frac{e^{|k|g_{k/|k|,0}(-i\xi')}}{-(\frac{k}{|k|}\cdot\partial)(\widehat{\widetilde{v}}_0)(\xi' + i\eta_0'(k) + i(p_1(\frac{k}{|k|}) - g_{\frac{k}{|k|},0}(\frac{\xi'}{i}))\frac{k}{|k|})} d\xi'$$
$$+\mathcal{O}(1)e^{-p_1(k)-\delta_0|k|},$$

where V is a small real neighborhood of 0 in $\mathbf{R}^d$ and $\delta_0 > 0$ some fixed constant.

When passing from $\widetilde{v}_0$ to $\widetilde{v}$ very little changes. Let $\Omega_1(h)$ be the set of $\eta \in \Omega_{1+\epsilon_0/2}$ such that $F_{\widetilde{v}}(\eta) \le 1$. This set is very close to Ω_1, and is strictly convex. Again, we have $\widehat{\widetilde{v}}(i\eta) = F_{\widetilde{v}}(\eta)$. For $k \in \mathbf{R}^d \setminus \{0\}$, let $\eta(k) \sim \sum \eta_\nu(\frac{k}{|k|})h^\nu$ be the unique point in $\partial\Omega_1(h)$, where the exterior unit normal of $\Omega_1(h)$ is equal to a positive multiple of k. Write

$$\eta(k) = \eta'(k) + p_{1,h}(\frac{k}{|k|})\frac{k}{|k|},\ \eta'(k) \in (k)^\perp, \tag{9.45}$$

where $p_{1,h}(k) = \sup_{\eta\in\Omega_1(h)} k\cdot\eta$ is the support function of $\Omega_1(h)$. Then

$$p_{1,h} \sim p_1 + \sum_{\nu=1}^{\infty} p_1^{(\nu)}(k), \tag{9.46}$$

with $p_1^{(\nu)}$ positively homogeneous of degree 1. In a neighborhood of $\eta(k)$, we can represent $\partial\Omega_1(h)$ as

$$\partial\Omega_1(h) = \{\eta'(k) + \eta' + (p_{1,h}(\frac{k}{|k|}) - g_{\frac{k}{|k|}}(\eta'))\frac{k}{|k|};\ \eta' \in (k)^\perp \cap \mathrm{neigh}\,(0)\}, \tag{9.47}$$

where g is real and analytic and has the uniform asymptotic expansion

$$g_{\frac{k}{|k|}}(\eta') \sim \sum_{\nu=0}^{\infty} g_{\frac{k}{|k|},\nu}(\eta')h^\nu, \tag{9.48}$$

for $\eta' \in (k)^\perp \cap \widetilde{V}$, where $\widetilde{V}$ is a fixed complex neighborhood of 0.

Again, as in [S1], we get

$$\widetilde{F}(k) = \tag{9.49}$$
$$\frac{ie^{-p_{1,h}(k)}}{(2\pi)^{d-1}}\int_{\xi'\in V\cap(k)^\perp}\frac{e^{|k|g_{k/|k|}(\frac{\xi'}{i};h)}}{-(\frac{k}{|k|}\cdot\partial)(\widehat{\widetilde{v}})(\xi'+i\eta'(k)+i(p_{1,h}(\frac{k}{|k|})-g_{\frac{k}{|k|}}(\frac{\xi'}{i}))\frac{k}{|k|})}d\xi' + \mathcal{O}(1)e^{-p_{1,h}(k)-\delta_0|k|},$$

with V and δ_0 as in (9.44). $g_{\frac{k}{|k|}}(\frac{\xi'}{i};h)$ vanishes to second order at $\xi' = 0$ and $\mathrm{Re\,Hess}\, g_{\frac{k}{|k|}}(0;h) \leq -\mathrm{Const.} < 0$. The method stationary phase gives the large k asymptotics of $\widetilde{F}(k)$ uniformly in Λ and in h (for $h \leq h_0 > 0$ sufficiently small), where all the involved functions have uniform asymptotic expansions in powers of h:

$$\widetilde{F}(k) = \mathcal{O}(1)e^{-(p_{1,h}(k)+\delta_0|k|)} + e^{-p_{1,h}(k)}q(k;h), \tag{9.50}$$

$$q(k;h) \sim \sum_{-\infty}^{0} q_{-\frac{d-1}{2}-\nu}(k;h),\ k\to\infty, \tag{9.51}$$

where $q_j(k;h)$ is smooth and positively homogeneous of degree j in k and has an asymptotic expansion,

$$q_j(k;h) \sim \sum_{\nu=0}^{\infty} h^\nu q_{j,\nu}(k),\ h\to 0, \tag{9.52}$$

where $q_{j,\nu}$ is also positively homogeneous of degree j. In [S1], the leading term was computed:

$$q_{-\frac{d-1}{2},0}(k) = \frac{1}{(2\pi|k|)^{\frac{d-1}{2}}}\frac{((\frac{k}{|k|}\cdot\partial_\eta)F_{\widetilde{v_0}}(\eta(k)))^{\frac{d-3}{2}}}{\sqrt{\det(\partial_\eta'^2 F_{\widetilde{v_0}})(\eta(k))}}, \tag{9.53}$$

where η' indicates some orthonormal coordinates in $(k)^\perp$.

The convolution operator $1 - v*$ on Λ has the inverse $F*$, where

$$F(k) = \sum_{\widetilde{k}\in\pi_\Lambda^{-1}(k)}\widetilde{F}(\widetilde{k}). \tag{9.54}$$

For $\delta > 0$, let Λ_δ, be the set of $k \in \Lambda$, such that

1° there is a unique $\widetilde{k}(k) \in \pi_\Lambda^{-1}(k)$, such that $d_1(k,0) = p_1(\widetilde{k}(k))$,

2° $p_1(\ell) \geq (1+\delta)d_1(k,0)$, whenever $\widetilde{k}(k) \neq \ell \in \pi_\Lambda^{-1}(k)$.

Fix $\delta > 0$. Then from (9.50), (9.54) we get the uniform asymptotics for $k \in \Lambda_\delta$:

$$F(k) = \mathcal{O}(1)e^{-(d_{1,h}(0,k)+\delta_0|\widetilde{k}(k)|)} + e^{-d_{1,h}(0,k)}q(\widetilde{k}(k);h), \tag{9.55}$$

where $d_{1,h}$ is the distance on Λ, induced by $p_{1,h}$ and $\delta_0 > 0$ a constant. We also have the bound

$$F(k) = \mathcal{O}(1)|k|^{-\frac{d-1}{2}} e^{-d_{1,h}(0,k)}, \ k \in \Lambda. \tag{9.56}$$

Since $E_{-+}^{-1} = -F*$, (9.55), (9.56) give us an asymptotic expansion for $E_{-+}^{-1}(\mu, \nu)$ for $\mu - \nu \in \Lambda_\delta$ and a precise upper bound for all $\mu, \nu \in \Lambda$.

Before using this, it may be instructive to study (cf. (9.30))

$$E_-(e^{-\phi/2h} dx_0), \tag{9.57}$$

having in mind also more general correlations. Since $E_- = E_+^*$, we can apply Proposition 7.4 and conclude that

$$\begin{aligned} E_-(e^{-\phi/2h} dx_0) &\equiv (C^M)^* R_+^{M,1}(e^{-\phi/2h} dx_0) \equiv (C^M)^*(\delta_0) \\ &\equiv \sum_{\nu=0}^{\widetilde{M}} h^\nu D_{0,0}^{M,\nu}(\delta_0 \otimes e_0) + \mathcal{O}(1) h^{M/2}, \end{aligned} \tag{9.58}$$

modulo $\mathcal{O}(h^{M/2})$ in $e^{-d_{1+\epsilon_0/2}}\ell^2$, where $e_0 \in \ell_b^2(\Lambda^0 \cup \Lambda^1 \cup .. \cup \Lambda^N)$ is the element given by $1 \in \mathbf{C} \simeq \ell^2(\Lambda^0)$. Here M can be chosen arbitrarily large, so we get a full asymptotic expansion

$$E_-(e^{-\phi/2h} dx_0) \sim \sum_{\nu=0}^{\infty} h^\nu f_\nu \text{ in } e^{-d_{1+\epsilon_0/2}}\ell^2, \tag{9.59}$$

where we also know that $f_0 = \delta_0$.

Now we combine this with (9.31), (9.27), (9.28), (9.29), (9.55), (9.56) and get

Proposition 9.2. *Assume (H1–4) of section 1, (H8–11) of this section, (9.17H), and ($\widetilde{\text{H1}}$) of section 7 for the set of weights $\rho = e^{r(x)}$ with $\nabla r(x) \in \Omega_{1+\epsilon_0/2}$ a.e. (discussed after (9.22)). Then*

$$\mathrm{Cor}\,(x_\nu, x_\mu) = \mathcal{O}(h) \mathrm{dist}\,(\nu, \mu)^{-\frac{d-1}{2}} e^{-d_{1,h}(\nu,\mu)}, \ \nu, \mu \in \Lambda, \tag{9.60}$$

$$\mathrm{Cor}\,(x_\nu, x_\mu) = \mathcal{O}(h) e^{-(d_{1,h}(\nu,\mu) + \delta_0 \mathrm{dist}\,(\nu,\mu))} + h e^{-d_{1,h}(\nu,\mu)} q(\widetilde{k}(\nu - \mu); h), \ \nu, \mu \in \Lambda_\delta, \tag{9.61}$$

where q, $\widetilde{k}$, $d_{1,h}$ have been defined above (cf. (9.50), (9.55)).

<u>C.</u> Let U_j, V_j be increasing sequences of bounded subsets of $\mathbf{Z}^d$ with $U_j \subset V_j$, $U_j \nearrow \mathbf{Z}^d$, $j \to \infty$. Let $\rho_0 = \rho_{0,j} : \mathbf{Z}^d \to]0, \infty[$ be the weight

$$\rho_{0,j}(\nu) = \exp(\theta \,\mathrm{dist}\,(\nu, \mathbf{Z}^d \setminus U_j)), \tag{9.62}$$

for some fixed (possibly small) $\theta > 0$, and where dist denotes the standard Euclidean distance on $\mathbf{Z}^d$. Let $\phi = \phi_j \in C^\infty(\mathbf{R}^{V_j}; \mathbf{R})$ satisfy the assumptions of section 1 (with $V_j = \Lambda$) and assume

$$\begin{aligned} &\text{If } k > j, \text{ then } (\rho_0 \otimes \rho_0)(\phi_j \oplus \psi_{k,j} - \phi_k)'' \text{ is 2 standard,} \\ &\text{if } \psi_{k,j} \text{ is defined on } \mathbf{R}^{V_k \setminus V_j} \text{ with } \psi_{k,j}'' \text{ 2 standard.} \end{aligned} \tag{H12}$$

Notice that this condition is independent of the choice $\psi_{k,j}$, and we could for instance just take zero.

We also assume that $\phi = \phi_j$ satisfies $(\widetilde{\mathrm{H1}})$ (section 7), $(\widetilde{\mathrm{H4}})$ (section 8) with

$$W = \{\rho = e^r;\ |r(\nu) - r(\mu)| \le \theta|\nu - \mu|\},\ |\cdot| = |\cdot|_{\ell^2}, \tag{9.63}$$

so that $\rho_0 \in W$. Let

$$\rho_1(\nu) = \rho_0(\nu)e^{-\theta|\nu|/4}, \tag{9.64}$$

$$S = S_j = \{\nu \in U_j;\ |\nu| \le \mathrm{dist}\,(\nu, \mathbf{Z}^d \setminus U_j)\},\ r_j = \mathrm{dist}\,(0, \mathbf{Z}^d \setminus U_j), \tag{9.65}$$

so that

$$\rho_1(\nu) \ge e^{\theta r_j/4},\ \nu \in S_j. \tag{9.66}$$

Put

$$\phi_{j,k,t} = t\phi_k + (1-t)(\phi_j \oplus \psi_{k,j}),\ 0 \le t \le 1,$$

with $\psi_{k,j}(x) = \sum_{\nu \in V_k \setminus V_j} x_\nu^2$, where we drop the normalization constant (cf. (8.1H)) for simplicity. Assume that $\phi_{j,k,t}$ satisfies $(\widetilde{\mathrm{H4}})$ with W given in (9.63). Then we can apply Proposition 8.4 and obtain for $\nu, \mu \in S_j$:

$$\mathrm{Cor}_{\phi_k}(x_\nu, x_\mu) - \mathrm{Cor}_{\phi_j}(x_\nu, x_\mu) = \mathcal{O}(1)he^{-\theta r_j/4}. \tag{9.67}$$

Here we also used that

$$\mathrm{Cor}_{\phi_j}(x_\nu, x_\mu) = \mathrm{Cor}_{\phi_j \oplus \psi_{k,j}}(x_\nu, x_\mu).$$

Let $\Lambda = \Lambda_j = (\mathbf{Z}/L_j\mathbf{Z})^d$ be a sequence of discrete tori with $L_j \nearrow \infty$ large enough so that there exists a natural embedding

$$V_j \subset \Lambda_j. \tag{9.68}$$

We can view $\rho_0 = \rho_{0,j}$ as a function on Λ_j, with $\rho_0 = 1$ outside V_j. Assume that $\widetilde{\phi}_j \in C^\infty(\mathbf{R}^{\Lambda_j}; \mathbf{R})$ is a family which satisfies the assumptions of subsection B with a new set of weights W that contains ρ_0, such that

$$\begin{aligned} &(\rho_0 \otimes \rho_0)(\phi_j \oplus \psi_j - \widetilde{\phi}_j)'' \text{ is 2 standard, if} \\ &\psi_j \in C^\infty(\mathbf{R}^{\Lambda_j \setminus V_j}; \mathbf{R}) \text{ and } \psi_j'' \text{ is 2 standard.} \end{aligned} \tag{9.69}$$

Here ρ_0 is defined as in (9.62) with $\mathbf{Z}^d$ replaced by Λ_j. We also assume that $t\widetilde{\phi}_j + (1-t)(\phi_j \oplus \psi_j)$, $0 \le t \le 1$, satisfies $(\widetilde{\mathrm{H4}})$ of section 8. Similarly to (9.67), we get

$$\mathrm{Cor}_{\widetilde{\phi}_j}(x_\nu, x_\mu) - \mathrm{Cor}_{\phi_j}(x_\nu, x_\mu) = \mathcal{O}(1)he^{-\theta r_j/4},\ \nu, \mu \in S_j. \tag{9.70}$$

On the other hand we can apply Proposition 9.2 to $\mathrm{Cor}_{\widetilde{\phi}_j}(x_\nu, x_\mu)$ and get

$$\mathrm{Cor}_{\phi_j}(x_\nu, x_\mu) = \mathcal{O}(h)e^{-\theta r_j/4} + he^{-p^j_{1,h}(\nu-\mu)}q^j(\nu - \mu; h),\ \nu, \mu \in S_j, \tag{9.71}$$

with $p_{1,h}^j$, q_j as in subsection B.

(9.67) gives a thermodynamical limit of the correlations, while (9.71) describes their asymptotic behaviour. We now combine the two results, in order to show that $p_{1,h}^j$ has a limit when $j \to \infty$ and that the same thing holds for the terms in the large ν asymptotic expansion of q^j, as well as for the terms in the h asymptotic expansions of these quantities. Let $k \geq j \gg 1$ and take $\mu = 0$. For every sufficiently large $C_0 \geq 1$, there is a $C_1 > 0$, such that

$$|e^{-p_{1,h}^j(\nu)} q^j(\nu;h) - e^{-p_{1,h}^k(\nu)} q^k(\nu;h)| \leq \mathcal{O}(1) e^{-(p_{1,h}^j(\nu)+r_j/C_1)},$$

for $r_j/C_0^2 \leq |\nu| \leq r_j/C_0$. This implies that with a new constant $C_1 > 0$:

$$|1 - e^{p_{1,h}^j(\nu)-p_{1,h}^k(\nu)} \frac{q^k(\nu;h)}{q^j(\nu;h)}| \leq \mathcal{O}(1) e^{-r_j/C_1}, \ \frac{r_j}{C_0^2} \leq |\nu| \leq \frac{r_j}{C_0}. \tag{9.72}$$

Here it will be convenient to write

$$q^j(\nu;h) = |\nu|^{-\frac{d-1}{2}} e^{-s^j(\nu;h)}, \ |\nu| \geq C_0, \tag{9.73}$$

where

$$s^j(\nu;h) \sim \sum_{-\infty}^{0} s_\alpha^j(\nu;h), \ |\nu| \to \infty, \tag{9.74}$$

uniformly with respect to h, j, with s_α^j positively homogeneous of degree $-\alpha$ in ν, and

$$s_\alpha^j \sim \sum_{\beta=0}^{\infty} s_{\alpha,\beta}^j(\nu) h^\beta, \ h \to 0, \tag{9.75}$$

with $s_{\alpha,\beta}^j$ also positively homogeneous of degree $-\alpha$ in ν. All these functions are smooth and uniformly bounded in the appropriate spaces when j varies. From (9.72), we deduce that

$$p_{1,h}^j(\nu) - p_{1,h}^k(\nu) + s^j(\nu;h) - s^k(\nu;h) = \mathcal{O}(1) e^{-r_j/C_1}, \ \frac{r_j}{C_0^2} \leq |\nu| \leq \frac{r_j}{C_0}.$$

Using (9.74), we get

$$p_{1,h}^j(\nu) - p_{1,h}^k(\nu) + \sum_{-N}^{0} (s_\alpha^j(\nu;h) - s_\alpha(\nu;h)) = \mathcal{O}(r_j^{-(N+1)}), \ \frac{r_j}{C_0^2} \leq |\nu| \leq \frac{r_j}{C_0}, \tag{9.76}$$

where we recall that C_0 can be chosen arbitrarily large. For $m \in \mathbf{R}$, $a : \mathbf{Z}^d \setminus \{0\} \to \mathbf{R}$, put

$$(D_m a)(\nu) = a(\nu) - 2^{-m} a(2\nu). \tag{9.77}$$

If a is the restriction of a function on $\mathbf{R}^d \setminus \{0\}$ which is positively homogeneous of degree $n \in \mathbf{R}$, then $D_m a = (1 - 2^{n-m})a$, and we observe that the prefactor $1 - 2^{n-m}$ vanishes precisely for $n = m$. If $-N \leq n \leq 1$, we apply

$$\prod_{m \in \{-N,..,1\} \setminus \{n\}} D_m$$

to (9.76) and conclude that

$$p_{1,h}^{j}(\nu) - p_{1,h}^{k}(\nu) = \mathcal{O}(r_j^{-(N+1)}) \ \ (n=1) \tag{9.78}$$

$$s_{\alpha}^{j}(\nu;h) - s_{\alpha}^{k}(\nu;h) = \mathcal{O}(r_j^{-(N+1)}), \ \ (n=\alpha) \tag{9.79}$$

for $-N \le \alpha \le 0$, $\frac{r_j}{2C_N} \le |\nu| \le \frac{r_j}{C_N}$, $N \in \mathbf{Z}^d$, for some $C_N \in [C_0, C_0^2/2]$.

Lemma 9.3. *Let $\widetilde{\Omega} \subset\subset \Omega \subset \mathbf{R}^d$ be open, $N_0 \in \{1,2,..\}$, $\mathcal{E} \subset]0,1]$, $0 \in \overline{\mathcal{E}}$. Let $u = u_\epsilon(x) \in C^\infty(\Omega)$, $\epsilon \in \mathcal{E}$, and assume that*

$$|\partial_x^\alpha u_\epsilon(x)| \le C_\alpha, \ \ x \in \Omega, \ \epsilon \in \mathcal{E}, \tag{9.80}$$

$$|u_\epsilon(x)| \le \epsilon^{N_0}, \ \ x \in \epsilon\mathbf{Z}^d \cap \Omega, \tag{9.81}$$

where C_α is independent of ϵ. Then,

$$|\partial_x^\alpha u_\epsilon(x)| \le \widetilde{C}_\alpha \epsilon^{N_0 - |\alpha|}, \ \ x \in \widetilde{\Omega}, \ |\alpha| \le N_0. \tag{9.82}$$

Proof. For $1 \le j \le d$, let e_j be the jth unit vector in $\mathbf{R}^d$ and put

$$D_{j,\epsilon} u(x) = \frac{u(x+\epsilon e_j) - u(x)}{\epsilon} = \int_0^1 (\partial_{x_j} u)(x + t\epsilon e_j) dt.$$

Then

$$\begin{aligned} D_{j_1,\epsilon}..D_{j_k,\epsilon} u(x) &= \int_0^1 .. \int_0^1 (\partial_{x_{j_1}}..\partial_{x_{j_k}} u)(x + \epsilon t_1 e_1 + .. + \epsilon t_k e_{j_k}) dt_1..dt_k \\ &= \partial_{x_{j_1}}..\partial_{x_{j_k}} u(x) + \mathcal{O}_k(\epsilon) \sup_{|y-x|_\infty \le k\epsilon} \max_{|\alpha|=k+1} |\partial^\alpha u(y)|. \end{aligned} \tag{9.83}$$

Let $\widetilde{\Omega} =: \Omega_{N_0} \subset\subset \Omega_{N_0-1} \subset\subset ... \subset\subset \Omega_1 \subset\subset \Omega$ and choose $\epsilon > 0$ small enough. We first see that

$$D_{j_1,\epsilon}..D_{j_k,\epsilon} u(x) = \mathcal{O}(\epsilon^{N_0-k}), \ \ x \in \Omega_1 \cap \epsilon\mathbf{Z}^d, \ \ k \le N_0 - 1, \tag{9.84}$$

then using also (9.83), that $\partial^\alpha u = \mathcal{O}(\epsilon)$, $|\alpha| \le N_0 - 1$. Using (9.83) again, we see that

$$\partial^\alpha u = \mathcal{O}(\epsilon^2), \ \ |\alpha| \le N_0 - 2, \ x \in \Omega_2.$$

Iterating this argument, we get (9.82). #

We apply the lemma to (9.78), (9.79) with $\epsilon = \frac{1}{r_j}$, after the change of variables $\nu = r_j \mu$. Using also the homogeneity, and that N can be chosen arbitrarily large, we conclude that

$$\partial^\beta (p_{1,h}^j - p_{1,h}^k) = \mathcal{O}_{N,\beta}(r_j^{-N}) |\nu|^{1-|\beta|}, \ \ \nu \in \mathbf{R}^d \setminus \{0\}, \tag{9.85}$$

$$\partial^\beta (s_\alpha^j - s_\alpha^k) = \mathcal{O}_{N,\beta,\alpha}(r_j^{-N}) |\nu|^{\alpha-|\beta|}, \ \ \nu \in \mathbf{R}^d \setminus \{0\}, \tag{9.86}$$

for all multiindices β, when $k \geq 1$.

From (9.85) we conclude that there exists $p^{\infty}_{1,h}(\nu) \in C^{\infty}(\mathbf{R}^d \setminus \{0\})$ positively homogeneous of degree 1 in ν, and uniformly bounded in C^{∞}, when h varies, such that

$$\partial^{\beta}(p^{\infty}_{1,h} - p^{j}_{1,h}) = \mathcal{O}_{N,\beta}(r_j^{-N})|\nu|^{1-|\beta|}, \ \nu \in \mathbf{R}^d, \tag{9.87}$$

for every multiindex β.

Consider the truncated asymptotic expansion of $p^j_{1,h}$:

$$p^j_{1,h}(\nu) = \sum_{\ell=0}^{M-1} p^j_{1,\ell}(\nu)h^{\ell} + R^j_M(\nu;h), \tag{9.88}$$

with

$$\partial^{\beta} R^j_M = \mathcal{O}_{M,\beta}(h^M)|\nu|^{1-|\beta|}, \tag{9.89}$$

uniformly in j. Since all functions will be homogeneous for a while, we restrict the attention to a spherical shell in ν, and drop the obvious powers of $|\nu|$. Using (9.85), we see that

$$p^j_{1,0} - p^k_{1,0} = \mathcal{O}(r_j^{-N} + h),$$

for all N, implying

$$p^j_{1,0} - p^k_{1,0} = \mathcal{O}(r_j^{-N}). \tag{9.90}$$

Now we can use (9.85,88) once more, to see that

$$p^j_{1,1} - p^k_{1,1} = \mathcal{O}(\frac{r_j^{-2N}}{h} + h), \tag{9.91}$$

for every N. Choose $h = r_j^{-N}$, to get

$$p^j_{1,1} - p^k_{1,1} = \mathcal{O}(r_j^{-N}). \tag{9.92}$$

Continuing this way, we get

$$p^j_{1,\ell} - p^k_{1,\ell} = \mathcal{O}_{\ell}(r_j^{-N}), \tag{9.93}$$

for every N, and the same estimates hold for $\partial^{\beta}(p^j_{1,\ell} - p^k_{1,\ell})$. Using this and (9.85) in (9.89), we get

$$R^j_M - R^k_M = \mathcal{O}(r_j^{-N}), \ \forall N. \tag{9.94}$$

On the other hand, $R^j_M - R^k_M = \mathcal{O}(h^M)$, by (9.89), so interpolation with (9.94) gives $R^j_M - R^k_M = \mathcal{O}(h^{M-1}r_j^{-N})$ for every $N \geq 0$. Use this estimate with M replaced by $M+1$, as well as (9.93) in the identity

$$R^j_M - R^k_M = (p^j_{1,M} - p^k_{1,M})h^M + (R^j_{M+1} - R^k_{M+1}). \tag{9.95}$$

We get

$$R^j_M - R^k_M = \mathcal{O}(h^M r_j^{-N})|\nu|, \ \nu \in \mathbf{R}^d \setminus \{0\}, \tag{9.96}$$

and the analogous estimate holds for the βth derivative, with $|\nu|$ replaced by $|\nu|^{1-|\beta|}$. From (9.93), we get the existence of $p_{1,\ell}^{\infty} \in C^{\infty}(\mathbf{R}^d \setminus \{0\})$, such that

$$|p_{1,\ell}^{j} - p_{1,\ell}^{\infty}| = \mathcal{O}_{\ell,N}(r_j^{-N})|\nu|, \tag{9.97}$$

and similarly for the derivatives. Similarly

$$|R_M^j - R_M^{\infty}| = \mathcal{O}_{M,N}(h^M r_j^{-N})|\nu|. \tag{9.98}$$

We conclude that

$$p_{1,h}^{\infty}(\nu) \sim \sum_{\ell=0}^{\infty} p_{1,\ell}^{\infty}(\nu) h^{\ell}. \tag{9.99}$$

The same arguments apply to

$$s_{\alpha}^{j} \sim \sum_{\beta=0}^{\infty} s_{\alpha,\beta}^{j}(\nu) h^{\beta}, \tag{9.100}$$

and we get

$$s_{\alpha}^{j} - s_{\alpha}^{\infty} = \mathcal{O}_{N,\alpha}(r_j^{-N})|\nu|^{-\alpha}, \tag{9.101}$$

$$s_{\alpha,\beta}^{j} - s_{\alpha,\beta}^{\infty} = \mathcal{O}_{N,\beta,\alpha}(r_j^{-N})|\nu|^{-\alpha}, \tag{9.102}$$

$$s_{\alpha}^{\infty} \sim \sum_{\beta=0}^{\infty} s_{\alpha,\beta}^{\infty}(\nu) h^{\beta}. \tag{9.103}$$

We combine (9.67), (9.71) to get for $j \leq k$:

$$|e^{-p_{1,h}^{j}(\nu)} q^j(\nu; h) - e^{-p_{1,h}^{k}(\nu)} q^k(\nu; h)| \leq \mathcal{O}(1) e^{-\theta r_j/4}, \ |\nu| \leq \frac{r_j}{C_0}. \tag{9.104}$$

We know that $p_{1,h}^{j}(\nu)$ is uniformly of the order of $|\nu|$, so after multiplying with $e^{p_{1,h}^{j}(\nu)}$, we get (possibly with a new C_0):

$$|q^j(\nu; h) - q^k(\nu; h) + (1 - e^{p_{1,h}^{j}(\nu) - p_{1,h}^{k}(\nu)}) q^k(\nu; h)| \leq \mathcal{O}(1) e^{-r_j/C_0}, |\nu| \leq \frac{r_j}{C_0}. \tag{9.105}$$

Here $q^k = \mathcal{O}((|\nu| + 1)^{-\frac{d-1}{2}})$ uniformly with respect to k and we have (9.85), so

$$(1 - e^{p_{1,h}^{j} - p_{1,h}^{k}}) q^k = \mathcal{O}(r_j^{-N}), \ |\nu| \leq \frac{r_j}{C_0}.$$

Using this in (9.105), we get

$$|q^j(\nu; h) - q^k(\nu; h)| \leq \mathcal{O}(1) r_j^{-N}, \ |\nu| \leq \frac{r_j}{C_0}, \tag{9.106}$$

for every N. We conclude (cf. (9.87)) that there exists a function $q^\infty(\nu;h)$, $\nu \in \mathbf{Z}^d$, $0 < h \le h_0$, such that

$$|q^j(\nu;h) - q^\infty(\nu;h)| \le \mathcal{O}_N(1) r_j^{-N},\ |\nu| \le \frac{r_j}{C_0}. \tag{9.107}$$

Using this and (9.87), we see that

$$e^{-p_{1,h}^j(\nu)} q^j(\nu;h) - e^{-p_{1,h}^\infty(\nu)} q^\infty(\nu;h) = \mathcal{O}_N(1) r_j^{-N},\ |\nu| \le \frac{r_j}{C_0},$$

and if we compare this with (9.104), we get the sharper estimate

$$|e^{-p_{1,h}^j(\nu)} q^j(\nu;h) - e^{-p_{1,h}^\infty(\nu)} q^\infty(\nu;h)| \le \mathcal{O}(1) e^{-\theta r_j/4},\ |\nu| \le \frac{r_j}{C_0}. \tag{9.108}$$

Now (9.71) gives

$$\mathrm{Cor}_{\phi_j}(x_\nu, x_\mu) = \mathcal{O}(h) e^{-\theta r_j/4} + h e^{-p_{1,h}^\infty(\nu-\mu)} q^\infty(\nu-\mu;h),\ \text{for } |\nu|, |\mu| \le \frac{r_j}{C_0}. \tag{9.109}$$

Here we recall the h asymptotic expansion (9.99), where $p_{1,\ell}^\infty \in C^\infty(\mathbf{R}^d \setminus \{0\})$ are positively homogeneous of degree 1 (as $p_{1,\theta}^\infty$). If we combine (9.73), (9.106), we see that

$$q^\infty(\nu;h) = |\nu|^{-\frac{d-1}{2}} e^{-s^\infty(\nu;h)},\ \nu \in \mathbf{Z}^d \setminus B(0, C_0), \tag{9.110}$$

$$|s^j(\nu;h) - s^\infty(\nu;h)| \le \mathcal{O}(1) r_j^{-N},\ C_0 \le |\nu| \le \frac{r_j}{C_0}. \tag{9.111}$$

Now use (9.74) which is uniform in j, and (9.101), to get for $C_0 \le |\nu| \le r_j/C_0$ and for every $N \in \mathbf{N}$, uniformly in j:

$$\begin{aligned} s^\infty(\nu;h) = \mathcal{O}(r_j^{-N}) + s^j(\nu;h) &= \mathcal{O}(r_j^{-N}) + \sum_{1-N}^{0} s_\alpha^j(\nu;h) + \mathcal{O}(|\nu|^{-N}) \\ &= \sum_{1-N}^{0} s_\alpha^\infty(\nu;h) + \mathcal{O}(|\nu|^{-N}). \end{aligned}$$

Hence

$$s^\infty(\nu;h) \sim \sum_{-\infty}^{0} s_\alpha^\infty(\nu;h),\ |\nu| \to \infty. \tag{9.112}$$

The main result of our paper is the asymptotics (9.109) together with (9.99), (9.110), (9.112) and the fact that $s_\alpha^\infty(\nu;h) \in C^\infty(\mathbf{R}^d \setminus \{0\})$ is homogeneous of degree α and has the h asymptotic expansion (9.103) in the space of smooth functions on $\mathbf{R}^d \setminus \{0\}$, homogeneous of degree α.

10. Extraction of a main result.

We say that a function f on $\mathbf{R}^{\mathbf{Z}^d}$ is smooth if it is continuous for the ℓ^∞ topology, differentiable in each of the variables with continuous derivatives and the derivatives enjoy the same properties et c. Let $\Phi_{j,k}(x)$, $j,k\in\mathbf{Z}^d$ be smooth and real on $\mathbf{R}^{\mathbf{Z}^d}$ and satisfy

$$\Phi_{j,k}(x)=\Phi_{k,j}(x), \tag{A.1}$$

$$\partial_{x_\ell}\Phi_{j,k}=\partial_{x_j}\Phi_{\ell,k}, \tag{A.2}$$

$$\Phi=(\Phi_{j,k}) \text{ is 2 standard}, \tag{A.3}$$

$$\Phi(0)\geq \text{Const.}>0. \tag{A.4}$$

If we had been working on a finite dimensional space, then (A.1,2) would have been a necessary and sufficient condition for the existence of a smooth realvalued function ϕ with $\phi''_{j,k}=\Phi_{j,k}$. Such a function does not in general exist in the infinite dimensional case, but we shall now see how to produce two different finite dimensional versions of such a function.

Let $U\subset\mathbf{Z}^d$ be finite. If $x\in\mathbf{R}^U$, let $\widetilde{x}\in\mathbf{R}^{\mathbf{Z}^d}$ be the zero extension of x, so that $\widetilde{x}(j)=x(j)$ for $j\in U$, $\widetilde{x}(j)=0$, for $j\in\mathbf{Z}^d\setminus U$. Then

$$\Phi_{U;j,k}(x):=\Phi_{j,k}(\widetilde{x}),\ j,k\in U$$

is a smooth tensor on $\mathbf{R}^U$ which satisfies (A.1,2) with $j,k,\ell\in U$. Hence there exists a function $\phi_U(x)\in C^\infty(\mathbf{R}^U;\mathbf{R})$ with

$$\phi''_{U;j,k}(x)=\Phi_{U;j,k}(x),\ x\in\mathbf{R}^U,\ j,k\in U. \tag{10.1}$$

We make ϕ_U unique up to a constant, by requiring that

$$\phi'_U(0)=0. \tag{10.2}$$

It is obvious that ϕ''_U is 2-standard, so we have (H1) (section 1) with Λ replaced by U, and (H2,3) hold. In order to have (H4) of section 1, we introduce the 2 standard matrix

$$A(x)=\int_0^1\Phi(tx)dt, \tag{10.3}$$

and assume

$$\begin{aligned}&A(x):\ell^p\to\ell^p \text{ has an inverse } B(x):\ell^p\to\ell^p,\\ &\text{which is uniformly bounded for } x\in\mathbf{R}^{\mathbf{Z}^d},\ 1\leq p\leq\infty.\end{aligned} \tag{A.5}$$

Since A is 2 standard, we see that $B(x)$ is 2 standard.

With U as above, we take $x \in \mathbf{R}^U$ and let as before $\widetilde{x}$ denote the 0 extension of x to $\mathbf{R}^{\mathbf{Z}^d}$. Then we can introduce the 2 standard matrix

$$A_U(x) = \int_0^1 \phi_U''(tx)dt = r_U A(\widetilde{x}) r_U^*, \tag{10.4}$$

where $r_U : \mathbf{R}^{\mathbf{Z}^d} \to \mathbf{R}^U$ is the restriction map. We assume in addition to (A.5), that

$$\begin{aligned} &A_U(x) \text{ has an inverse } B_U(x) \text{ which} \\ &\text{is uniformly bounded for } x \in \mathbf{R}^U,\ 1 \le p \le \infty, \end{aligned} \tag{A.6}$$

uniformly for all U in some class of finite U under consideration. With these assumptions we have obtained smooth functions $\phi_U \in C^\infty(\mathbf{R}^U)$ which satisfy (H.1–4) for U in some class of finite subsets of $\mathbf{Z}^d$.

We next do the same with U replaced by a discrete torus $\Lambda = (\mathbf{Z}/L\mathbf{Z})^d$. If $\lambda \in \mathbf{Z}^d$, we define $\tau_\lambda x \in \mathbf{R}^{\mathbf{Z}^d}$, by $(\tau_\lambda x)(\nu) = x(\nu - \lambda)$. Eventually, we will assume complete translation invariance for Φ:

$$\Phi_{j+\lambda,k+\lambda}(\tau_\lambda x) = \Phi_{j,k}(x),\ \ j,k,\lambda \in \mathbf{Z}^d. \tag{A.7}$$

Notice that if $\Phi_{j,k}$ were the Hessian of a smooth function on $\phi \in C^\infty(\mathbf{R}^{\mathbf{Z}^d})$ (and the discussion remains valid if we replace $\mathbf{Z}^d$ by a discrete torus Λ) then (A.7) would be a consequence of the simpler translation invariance property:

$$\phi(\tau_\lambda x) = \phi(x). \tag{10.4}$$

However, to begin with, we only assume the weaker assumption

$$\Phi_{j+\lambda,k+\lambda}(\tau_\lambda x) = \Phi_{j,k}(x),\ \ j,k \in \mathbf{Z}^d,\ \lambda \in L\mathbf{Z}^d, \tag*{(A.7)$_L$}$$

for some given $L \in \{1, 2, ..\}$.

If $x \in \mathbf{R}^\Lambda$, let $\widetilde{x} \in \mathbf{R}^{\mathbf{Z}^d}$ be the corresponding $L\mathbf{Z}^d$ periodic lift. Replacing x by $\widetilde{x}$ in (A.7)$_L$, we get

$$\Phi_{j-\lambda,k-\lambda}(\widetilde{x}) = \Phi_{j,k}(\widetilde{x}),\ \ \lambda \in L\mathbf{Z}^d. \tag{10.5}$$

If we view Φ as a matrix, this is equivalent to

$$\tau_\lambda \circ \Phi(\widetilde{x}) = \Phi(\widetilde{x}) \circ \tau_\lambda,\ \ \lambda \in L\mathbf{Z}^d, \tag{10.6}$$

so $\Phi(\widetilde{x})$ maps $L\mathbf{Z}^d$ periodic vectors into the same kind of vectors. Hence there is a $\Lambda \times \Lambda$ matrix $\Phi_\Lambda(x)$, defined by

$$\widetilde{\Phi_\Lambda(x)t} = \Phi(\widetilde{x})\widetilde{t}, \tag{10.7}$$

where again the tilde indicates that we take the periodic lift. On the matrix level, we get

$$\Phi_{\Lambda;j,k}(x) = \sum_{\widetilde{k} \in \pi_\Lambda^{-1}(k)} \Phi_{\widetilde{j},\widetilde{k}}(\widetilde{x}), \tag{10.8}$$

for any $\widetilde{j}\in\pi_\Lambda^{-1}(j)$, where $\pi_\Lambda:\mathbf{Z}^d\to\Lambda$ is the natural projection. Alternatively, we have

$$\Phi_{\Lambda;j,k}(x)=\sum_{\widetilde{j}\in\pi_\Lambda^{-1}(j)}\Phi_{\widetilde{j},\widetilde{k}}(\widetilde{x}),\ \ \widetilde{k}\in\pi_\Lambda^{-1}(k),\tag{10.9}$$

and $\Phi_{\Lambda;j,k}$ is symmetric (cf. (A.1)).

Let us verify the analogue of (A.2). For $j,k,\ell\in\Lambda$ we have

$$\partial_{x_\ell}\Phi_{\Lambda;j,k}(x)=\partial_{x_\ell}\sum_{\widetilde{k}\in\pi_\Lambda^{-1}(k)}\Phi_{\widetilde{j},\widetilde{k}}(\widetilde{x})=\sum_{\widetilde{k}\in\pi_\Lambda^{-1}(k)}\sum_{\widetilde{\ell}\in\pi_\Lambda^{-1}(\ell)}\partial_{x_{\widetilde{\ell}}}\Phi_{\widetilde{j},\widetilde{k}}(\widetilde{x}).\tag{10.10}$$

From (A.1,2) we know that $\partial_{x_{\widetilde{\ell}}}\Phi_{\widetilde{j},\widetilde{k}}(\widetilde{x})=\partial_{x_{\widetilde{k}}}\Phi_{\widetilde{j},\widetilde{\ell}}(\widetilde{x})$, so the last expression in (10.10) is symmetric in ℓ,k and we get

$$\partial_{x_\ell}\Phi_{\Lambda;j,k}(x)=\partial_{x_k}\Phi_{\Lambda;j,\ell}(x).\tag{10.11}$$

Using also the symmetry of $\Phi_{\Lambda;j,k}$, we get the analogue of (A.2). It is now clear that there exists $\phi_\Lambda\in C^\infty(\mathbf{R}^\Lambda;\mathbf{R})$, unique up to a constant, such that

$$\Phi_{\Lambda;j,k}(x)=\partial_{x_j}\partial_{x_k}\phi_\Lambda(x),\ \ \phi_\Lambda'(0)=0.\tag{10.12}$$

Let us verify that ϕ_Λ'' is 2 standard. If $k\ge 2$, $t_1,..,t_k\in\mathbf{C}^\Lambda$, $x\in\mathbf{R}^\Lambda$, we have

$$\langle\nabla^k\phi_\Lambda(x),t_1\otimes..\otimes t_k\rangle=\langle\nabla^{k-2}\Phi_\Lambda,t_1\otimes..\otimes t_k\rangle=\langle\nabla^{k_2}\Phi(\widetilde{x}),1_E\widetilde{t}_1\otimes\widetilde{t}_2\otimes..\otimes\widetilde{t}_k\rangle,\tag{10.13}$$

if $E\subset\mathbf{Z}^d$ is a fundamental domain for $L\mathbf{Z}^d$ and $\widetilde{x}$, $\widetilde{t}_j$ denote the periodic lifts. Using that Φ is 2 standard, we deduce that

$$\langle\nabla^k\phi_\Lambda(x),t_1\otimes..\otimes t_k\rangle=\mathcal{O}_k(1)|t_1|_1|t_2|_\infty..|t_k|_\infty.\tag{10.14}$$

Here the index 1, can be replaced by any other index in $\{1,..,k\}$, and the RHS in (10.14) can therefore be replaced by

$$\mathcal{O}_k(1)|t_j|_1\prod_{\substack{1\le\nu\le k,\\ \nu\ne j}}|t_\nu|_\infty.\tag{10.15}$$

By complex interpolation, we get the desired relation

$$\langle\nabla^k\phi_\Lambda(x),t_1\otimes..\otimes t_k\rangle=\mathcal{O}_k(1)|t_1|_{p_1}..|t_k|_{p_k},\tag{10.16}$$

uniformly in x, t_j and p_j, when $1\le p_j\le\infty$, $1=\frac{1}{p_1}+..+\frac{1}{p_k}$.

We next check that ϕ_Λ satisfies (H.3) (section 1), so we put $x=0$, $\widetilde{x}=0$ and omit these quantities in the formulas. Choose a fundamental domain E and let $(\Psi_{j,k})$ be the block matrix of Φ with respect to the decomposition

$$\ell^2(\mathbf{Z}^d)=\oplus_{k\in\mathbf{Z}^d}\ell^2(E_k),$$

where $E_k := kL + E$. Then $\Psi_{j,k} = \Psi_{j-k}$ by slight abuse of notation. Since $\Phi = \mathcal{O}(1) : \ell^p \to \ell^p$, $1 \le p \le \infty$, we know that $\Phi_{j,k}$ satisfies the (equivalent) Shur condition

$$\sup_j \sum_k |\Phi_{j,k}|,\ \sup_k \sum_j |\Phi_{j,k}| < \infty,$$

and this implies that $\sum_k \|\Psi_k\| < \infty$, where $\|\cdot\|$ denotes the norm in $\mathcal{L}(\ell^2(E), \ell^2(E))$. Now we can write

$$\langle \Phi_\Lambda t, t\rangle = \sum_k \langle \Psi_k t, t\rangle, \tag{10.17}$$

identifying $t \simeq 1_E \widetilde{t}$. We compare this with

$$\begin{aligned} &\frac{1}{\#B(0,R)} \sum_{|j|,|k|\le R} \langle \Psi_{j-k} t, t\rangle = \\ &\frac{1}{\#B(0,R)} \sum_{\substack{|j|\le(1-\epsilon)R\\ k\in\mathbf{Z}^d}} .. - \frac{1}{\#B(0,R)} \sum_{\substack{|j|\le(1-\epsilon)R\\ |k|>R}} .. + \frac{1}{\#B(0,R)} \sum_{\substack{(1-\epsilon)R<|j|\le R\\ |k|\le R}} .. \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}, \end{aligned} \tag{10.18}$$

where $B(0,R) := \{j \in \mathbf{Z}^d; |j| \le R\}$. Here

$$\mathrm{I} = \frac{\#B(0,(1-\epsilon)R)}{\#B(0,R)} \sum \langle \Psi_{-k} t, t\rangle = \frac{\#B(0,(1-\epsilon)R)}{\#B(0,R)} \langle \Phi_\Lambda t, t\rangle, \tag{10.19}$$

$$|\mathrm{II}| \le \sum_{|\ell|>\epsilon R} \|\Psi_\ell\| |t|^2 = o_{\epsilon,t}(1),\ R \to \infty, \tag{10.20}$$

$$|\mathrm{III}| \le \frac{\#(B(0,R)\setminus B(0,(1-\epsilon)R))}{\#B(0,R)} \sum_k \|\Psi_k\| |t|^2 = o_t(1),\ \epsilon \to 0. \tag{10.21}$$

It follows that

$$\langle \Phi_\Lambda t, t\rangle = \lim_{R\to\infty} \frac{1}{\#B(0,R)} \sum_{|j|,|k|\le R} \langle \Psi_{j-k} t, t\rangle = \lim_{R\to\infty} \langle \Phi \widetilde{t}_R, \widetilde{t}_R\rangle, \tag{10.22}$$

where

$$\widetilde{t}_R = \frac{1}{\sqrt{\#B(0,R)}} \sum_{|k|\le R} 1_{E_k} \widetilde{t}.$$

The sum is orthogonal in ℓ^2, so

$$|\widetilde{t}_R|^2 = |1_E \widetilde{t}|^2 = |t|^2. \tag{10.23}$$

On the other hand, by (A.4) we have $\langle \Phi \widetilde{t}_R, \widetilde{t}_R\rangle \ge \frac{1}{\mathcal{O}(1)} |\widetilde{t}_R|^2$, so from this and (10.23,22) we get (H.3) for Φ_Λ with the same constant as in (A.4).

Next we verify (H.4) for ϕ_Λ. For that we notice that we can define a gradient $\phi'(x)$ at $x\in\mathbf{R}^{\mathbf{Z}^d}$ if $|x|_\infty<\infty$, by

$$\phi'(x)=\int_0^1\Phi(tx)xdt=A(x)x, \tag{10.24}$$

or more explicitly by

$$\phi_j'(x)=\sum_k\int_0^1\Phi_{j,k}(tx)x_kdt, \tag{10.25}$$

and we verify that

$$\partial_{x_\ell}\phi_j'(x)=\Phi_{j,\ell}(x)\quad (=\partial_{x_j}\phi_\ell'(x)\,)$$

by a straight forward computation:

$$\partial_{x_\ell}\phi_j'(x)=\sum_k\int_0^1(\partial_{x_\ell}\Phi_{j,k})(tx)tx_kdt+\int_0^1\Phi_{j,\ell}(tx)dt=$$
$$\int_0^1\sum_k(\partial_{x_k}\Phi_{j,\ell})(tx)tx_kdt+\int\Phi_{j,\ell}(tx)dt=\int_0^1(t\partial_t+1)(\Phi_{j,\ell}(tx))dt=\Phi_{j,\ell}(x).$$

Put $A_\Lambda(x)=\int_0^1\Phi_\Lambda(tx)dt$ so that

$$\phi_\Lambda'(x)=A_\Lambda(x)x. \tag{10.26}$$

The relation between $A_\Lambda(x)$ and $A(\widetilde{x})$ is the same as between Φ_Λ and Φ:

$$\widetilde{A_\Lambda(x)t}=A(\widetilde{x})\widetilde{t}. \tag{10.27}$$

Since $A(\widetilde{x})$ is uniformly invertible in $\ell^\infty(\mathbf{Z}^d)$ it has the same property on the invariant subspace of $L\mathbf{Z}^d$ periodic vectors. This means that $A_\Lambda(x):\ell^\infty(\Lambda)\to\ell^\infty(\Lambda)$ has a uniformly bounded inverse. Since A_Λ is symmetric, we have the same property on $\ell^1(\Lambda)$ and by interpolation on $\ell^p(\Lambda)$.

Assume that for every $C_0>0$

$\rho^{-1}\Phi(x)\rho$ is 2 standard, uniformly for every $\rho:\mathbf{Z}^d\to]0,\infty[$ of the form $\rho(j)=e^{r(j)}$ with $r:\mathbf{R}^d\to\mathbf{R}$ of Lipschitz class with $|\nabla r|\le C_0$ a.e. (A.8)

Let us then verify $(\widetilde{\mathrm{H.1}})$ (section 7) for ϕ_U and ϕ_Λ with a suitable class of weights W. In the first case we let $W=W_U$ be the set of weights of the form $\rho_{|_U}$ with ρ as in (A.8) for an arbitrary but fixed $C_0>0$. Then $(\widetilde{\mathrm{H.1}})$ holds for ϕ_U. In the second case, we let W_Λ be the set of $\rho(j)=e^{r(j)}$ with $r:(\mathbf{R}/L\mathbf{Z})^d\to\mathbf{R}$ of Lipschitz class with $|\nabla r|\le C_0$ a.e., and again $C_0>0$ is arbitrary but fixed. Let $\widetilde{\rho}=e^{\widetilde{r}}=\mathbf{Z}^d\to]0,\infty[$ be the corresponding periodic lift. Then we have the analogue of (10.13):

$$\langle\nabla^{k-2}\rho^{-1}\Phi_\Lambda(x)\rho,t_1\otimes..\otimes t_k\rangle=\langle\nabla^{k-2}\widetilde{\rho}^{-1}\Phi(\widetilde{x})\widetilde{\rho},1_E\widetilde{t}_1\otimes\widetilde{t}_2\otimes..\otimes\widetilde{t}_k\rangle,\ k\ge 2, \tag{10.28}$$

where again the index 1 can be replaced by any other index in $\{1,..,k\}$. As before, we see that $\rho^{-1}\Phi_\Lambda(x)\rho$ is 2 standard uniformly for $\rho\in W_\Lambda$, so ϕ_Λ satisfies $\widetilde{(\mathrm{H.1})}$ with W_Λ as above, for every fixed $C_0>0$.

We want to apply the discussion of section 9, and we assume (A.1–8) from now on. Since we now have adopted (A.7), completely, we see that $\phi_\Lambda(0)$ is a convolution matrix. We assume that $\Phi(0)$ is ferromagnetic in the sense that

$$\Phi_{j,k}(0)\le 0,\ \ j\ne k. \tag{A.9}$$

Then (H.9) holds for ϕ_Λ and we have

$$\Phi(0)=I-\widetilde{v}_0*,\quad \phi''_\Lambda(0)=(1+o(1))I_\Lambda-v_0*,\ \ L\to\infty \tag{10.29}$$

and v_0 and $\widetilde{v}_0$ are related by (9.12) and $v_0(\nu)$ is even ≥ 0 and vanishes for $\nu=0$. Again we chose the constant 1 for simplicity, by a dilation in h we can always reduce ourselves to that case.

Assume that there exists a finite set $K\subset\mathbf{Z}^d$ such that

$$\widetilde{v}_0(j)>0,\, j\in K,\ \mathrm{Gr}\,(K)=\mathbf{Z}^d, \tag{A.10}$$

where $\mathrm{Gr}\,(K)$ denotes the smallest subgroup of $\mathbf{Z}^d$ which contains K. This is precisely the assumption (H.10) for the functions ϕ_Λ. Since $\widetilde{v}_0(k)=\mathcal{O}_{C_0}(1)e^{-C_0|k|}$ for every $C_0>0$, the function $F_{\widetilde{v}_0}(\eta)$ in (9.14) is well defined on $\mathbf{R}^d$ and from (A.10) and the fact that $\widetilde{v}_0>0$, we see that $\lim_{|\eta|\to\infty}F_{\widetilde{v}_0}(\eta)=+\infty$. We then have(H.11) and (9.17H) for suitable sets Ω, $\widetilde{\Omega}$. Then the whole discussion of part B of section 9 applies and we have Proposition 9.2 for ϕ_Λ (when L is large enough).

We next look at part C of section 9, where we shall take $V_j=U_j\nearrow\mathbf{Z}^d$, $j\to\infty$ with U_j bounded. For $U\subset\mathbf{Z}^d$ finite, define $\rho_0=\rho_{0,U}:\mathbf{Z}^d\to]0,\infty[$ as in (9.62):

$$\rho_0(\nu)=\exp\theta\mathrm{dist}\,(\nu,\mathbf{Z}^d\setminus U), \tag{10.30}$$

for some fixed $\theta>0$. In order to have (H12) (which implies (H5) in the discussion in section 9) we need an assumption:

$$\big(\rho_{0,U}(j)\rho_{0,U}(k)(\Phi_{j,k}(1_Ux)-\Phi_{j,k}(x))\big)_{j,k\in U}\text{ is 2 standard.} \tag{A.11}$$

Here 1_U is the characteristic function of U, so $(1_Ux)(j)$ is equal to $x(j)$ when $j\in U$ and equal to 0 when $j\in\mathbf{Z}^d\setminus U$. We observe that (A.11) and (A.8) follow from (A.1–3) and the following finite range condition:

$$\exists C_0,\text{ such that }\Phi_{j,k}(x)=0\text{ for }|j-k|>C_0. \tag{A.fr}$$

In fact, if (A.fr) holds, then $\Phi_{j_1,j_2,..,j_m}:=\partial_{x_{j_3}}..\partial_{x_{j_m}}\Phi$ vanishes if $|j_\nu-j_\mu|>C_0$ for some ν,μ, and (A.3) is equivalent to the statement that $\Phi_{j_1,..,j_m}(x)=\mathcal{O}_m(1)$ for $m=2,3,...$ Similarly (A.8) is equivalent to

$$\frac{\rho(j_2)}{\rho(j_1)}\Phi_{j_1,..,j_m}(x)=\mathcal{O}_m(j),$$

for ρ as in (A.8). Since $(\rho(j_2)/\rho(j_1))\Phi_{j_1,..,j_m} = \mathcal{O}(1)\Phi_{j_1,..,j_m}$, we see that (A.8) follows from (A.3) and (A.fr). Moreover, $\partial_{x_\ell}\Phi_{j,k} = \partial_{x_j}\Phi_{\ell,k} = ..$ vanishes if $|j-\ell|$ or $|k-\ell|$ is $> C_0$ and consequently the expression (A.11) vanishes as soon as $\mathrm{dist}\,(j, \mathbf{Z}^d \setminus U)$ or $\mathrm{dist}\,(k, \mathbf{Z}^d \setminus U)$ is larger than C_0. This means that we can replace $\rho_{0,U}$ by some uniformly bounded functions $\widetilde{\rho}_{0,U}$ without changing the expression in (A.11), and (A.11) then follows from the 2 standardness of $\Phi_{j,k}(1_U x)$ and of $\Phi_{j,k}(x)$. More generally (A.11) holds if we assume that there is a $C_0 > 0$ such that $\Phi_{j,k}(x) = \Phi_{j,k}(0)$ whenever $|j-k| > C_0$. Indeed, we again obtain that $\partial_\ell \Phi_{j,k} = 0$ if $|k-\ell| > C_0$ or $|j-\ell| > C_0$.

We add (A.11) to our assumptions from now on, and verify (H.12). If $U \subset \widetilde{U} \subset \mathbf{Z}^d$ are finite, then $(\rho_{0,U} \otimes \rho_{0,U})(\phi_U \oplus 0 - \phi_{\widetilde{U}})''$ is equal to the tensor

$$\rho_0(j)\rho_0(k)(\Phi_{j,k}(1_U x)(1_U j)(1_U k) - \Phi_{j,k}(1_{\widetilde{U}} x)),\ j,k \in \widetilde{U}, \tag{10.31}$$

and we split this tensor into four, according to the cases $j \in U$ or not, $k \in U$ or not. In the three cases where at least one of j,k is in $\widetilde{U} \setminus U$, we know from (A.8) that both $\rho_0(j)\rho_0(k)\Phi_{j,k}(1_U x)$ and $\rho_0(j)\rho_0(k)\Phi_{j,k}(1_{\widetilde{U}} x)$ are 2 standard, and in the remaining case when both j and k belong to U, the tensor (10.31) is 2 standard by (A.11). This means that we have verified (H.12) for $\phi_j = \phi_{U_j}$.

We have already checked $(\widetilde{\mathrm{H}.1})$ (section 7) with the set of weights W above and we next look at $(\widetilde{\mathrm{H}.4})$ (section 8), that we need to check for the new and smaller set of weights in (9.63), of the form $\rho = e^r$, $|\nabla r| \le 2\theta$ a.e. Using the observation after the statement of $(\widetilde{\mathrm{H}.4})$, we only need to check that the inverse $B_U(x)$ of $A_U(x)$ (which exists and is uniformly bounded by (A.6)) remains uniformly bounded after conjugation with a weight ρ as in (9.63), if $\theta > 0$ is sufficiently small. However, using the Shur class remark, we see that $\|\rho^{-1}A_U\rho - A_U\|_{\mathcal{L}(\ell^p,\ell^p)}$, $1 \le p \le \infty$ is as small as we like if θ is small enough (but independent of U), and consequently $(\rho^{-1}A_U\rho)^{-1}$ is uniformly bounded, so we have checked $(\widetilde{\mathrm{H}.4})$ for ϕ_U.

We also need $(\widetilde{\mathrm{H}.4})$ for $\phi_{j,k,t}$ given after (9.66), so that

$$(\phi''_{j,k,t}(x)_{\nu,\mu} = t\Phi_{\nu,\mu}(1_{U_k}x) + (1-t)(1_{U_j}(\nu)1_{U_j}(\mu)\Phi_{\nu,\mu}(1_{U_j}x) + 1_{U_k\setminus U_j}(\mu)\delta_{\nu,\mu}), \tag{10.32}$$
$$\nu,\mu \in U_k,\ x \in \mathbf{R}^{U_k}.$$

The corresponding matrix A becomes

$$A_{j,k,t}(x) = tA_{U_k}(x) + (1-t)(1_{U_j}A_{U_j}(x)1_{U_j} + 1_{U_k\setminus U_j}),$$

and we assume in analogy with (A.6) that

$$\begin{aligned}&A_{j,k,t}(x) \text{ has an inverse } B_{j,k,t}(x) : \ell^p \to \ell^p, \text{ which}\\ &\text{is uniformly bounded for } x \in \mathbf{R}^{U_k},\ 1 \le p \le \infty.\end{aligned} \tag{A.12}$$

Using the Shur class point of view, we see as before that $(\widetilde{\mathrm{H}4})$ is fulfilled with W as in (9.63).

Remark 10.1. Using the maximum principle of [S4] we can get a simple condition which implies (A.5), (A.6), (A.12) and the similar condition (A.13) below. Assume for $A(x) = \int_0^1 \Phi(sx)ds$:

$$\exists\epsilon_0 > 0 \text{ such that for every } x \in \mathbf{R}^{\mathbf{Z}^d},\ A(x) \text{ satisfies } (\mathrm{mp}\,\epsilon_0)\text{: If} \\ t \in \ell^1(\mathbf{Z}^d),\ s \in \ell^\infty(\mathbf{Z}^d), \text{ and } \langle t, s\rangle = |t|_1 |s|_\infty, \text{ then } \langle A(x)t, s\rangle \geq \epsilon_0 |t|_1 |s|_\infty. \tag{A.mp}$$

It is easy to check (first for $p = 1, \infty$ and then by interpolation for intermdiate values of p) that $A(x) : \ell^p \to \ell^p$ has a uniformly bounded inverse $B(x)$, so (A.mp) implies (A.5). Moreover, if $A(x)$ satifies (mpϵ_0), so does $1_U A(x) 1_U^*$ (as a $U \times U$ matrix), so we get (A.6). Finally, the set of matrices which satisfy (mpϵ_0) is convex, so $A_{j,k,t}(x)$ will also satisfy (mpϵ_0) and consequently we will have (A.12).

So if we add the assumption (A.12), or replace (A.5,6) by (A.mp), then ($\widetilde{\mathrm{H}}$4) holds for $\phi_{j,k,t}$, and we get (9.67).

Now let $\Lambda = \Lambda_j = (\mathbf{Z}/L_j\mathbf{Z})^d$ be a sequence of discrete tori with

$$U_j \subset [-\frac{L_j}{4}, \frac{L_j}{4}]^d, \tag{10.33}$$

so that we can view U_j as a subset of Λ_j in the natural way. Let $\widetilde{\phi}_j = \phi_{\Lambda_j}$. We need to check (9.69) with $\rho_0(\nu) = \rho_{0,j}(\nu) = \exp\theta \mathrm{dist}\,(\nu, \Lambda_j \setminus U_j)$, $\nu \in \Lambda_j$. As before, we see that it suffices to check the 2 standardness of

$$\rho_0(\nu)\rho_0(\mu)(\Phi_{\nu,\mu}(1_{U_j}x) - \Phi_{\Lambda_j,\nu,\mu}(x)),\ \nu, \mu \in U_j. \tag{10.34}$$

Recall that U_j is viewed as a subset of Λ_j and let $\widetilde{x}$ denote the $L_j\mathbf{Z}^d$ periodic lift of x. Write (10.34) as the difference of the following two expressions:

$$\rho_0(\nu)\rho_0(\mu)(\Phi_{\nu,\mu}(1_{U_j}\widetilde{x}) - \Phi_{\nu,\mu}(\widetilde{x})), \tag{10.35}$$

and

$$\rho_0(\nu)\rho_0(\mu) \sum_{0 \neq \alpha \in \mathbf{Z}^d} \Phi_{\nu,\mu + L_j\alpha}(\widetilde{x}). \tag{10.36}$$

Thanks to (A.8), the last expression is 2 standard if we replace $\widetilde{x}$ by a general $x \in \mathbf{R}^{\mathbf{Z}^d}$. The same holds for (10.35) by (A.11). As with Φ_Λ, we then see that (10.35), (10.36) are 2 standard, and that completes the verification of (9.69).

We finally need to check that $t\phi_{\Lambda_j} + (1-t)(\phi_{U_j} \oplus \psi_j)$ satisfies ($\widetilde{\mathrm{H.4}}$) of section 8, with $\psi_j(x) = \sum_{\nu \in \Lambda_j \setminus U_j} \frac{1}{2} x_\nu^2$, and as before, we see that we only need the uniform invertibility in $\mathcal{L}(\ell^p, \ell^p)$ of

$$\widetilde{A}_{j,k} = tA_{\Lambda_j}(x) + (1-t)(1_{U_j} A_{U_j}(1_{U_j}x)1_{U_j} + 1_{\Lambda_j \setminus U_j}),\ x \in \mathbf{R}^{\Lambda_j}. \tag{10.37}$$

Assume

$$\widetilde{A}_{j,t}(x) : \ell^p \to \ell^p \text{ is uniformly invertible as in (A.12).} \tag{A.13}$$

Fortunately (A.13) is also a consequence of (A.mp). To see that, it suffices to verify that with $\Lambda=\Lambda_j$, $A_\Lambda(x)$ has the property (mpϵ_0) as we shall now do: Let $t\in\ell^1(\Lambda)$, $s\in\ell^\infty(\Lambda)$ satisfy $\langle t,s\rangle=|t|_1|s|_\infty$. We have the obvious analogue of (10.22):

$$\langle A_\Lambda t,s\rangle=\lim_{R\to\infty}\langle A\widetilde{t}_R,\widetilde{s}_R\rangle, \tag{10.38}$$

with $\widetilde{t}_R$, $\widetilde{s}_R$ defined as after (10.22). Here $\langle\widetilde{t}_R,\widetilde{s}_R\rangle_{\ell^2(\mathbf{Z}^d)}=\langle t,s\rangle_{\ell^2(\Lambda)}$, while

$$|\widetilde{t}_R|_1|\widetilde{s}_R|_\infty=|t|_1|s|_\infty=\langle t,s\rangle=\langle\widetilde{t}_R,\widetilde{s}_R\rangle,$$

so (A.mp) implies that $\langle A\widetilde{t}_R,\widetilde{s}_R\rangle\geq\epsilon_0|\widetilde{t}_R|_1|\widetilde{s}_R|_\infty=\epsilon_0|t|_1|s|_\infty$. Hence by (10.38), $\langle A_\Lambda t,s\rangle\geq\epsilon_0|t|_1|s|_\infty$, and we have checked that A_Λ satisfies (mpϵ_0) and hence that we have (A.13) when (A.mp) holds.

Summing up, we have verified that the assumptions (A.1–13) imply the results of part C in section 9, as will be restated in the main theorem below. We have also seen that the more explicit conditions (A.fr) and (A.mp) permit to reduce the number of conditions and to simplify them in the sense that they only concern $\Phi_{j,k}$ and not the particular choice of sequences U_j and Λ_j. Indeed we have verified the implications:

$$\text{(A.1–3), (A.fr)}\Rightarrow\text{ (A.8), (A.11),}$$

$$\text{(A.1–3), (A.mp)}\Rightarrow\text{ (A.5,6,12),}$$

$$\text{(A.1–3), (A.7), (A.mp)}\Rightarrow\text{ (A.13).}$$

Also notice that (A.4) follows from (A.mp). Especially (A.1–3,7,9,10,fr,mp) imply (A.1–13).

Theorem 10.2. *Let $\Phi_{j,k}(x)\in C^\infty(\mathbf{R}^{\mathbf{Z}^d})$ satisfy (A.1–5,7–10) and define $\phi_U\in C^\infty(\mathbf{R}^U;\mathbf{R})$, $\phi_\Lambda\in C^\infty(\mathbf{R}^\Lambda;\mathbf{R})$ as above, when $U\subset\mathbf{Z}^d$ is finite and $\Lambda=(\mathbf{Z}/L\mathbf{Z})^d$ is a discrete torus. Let $U_j\subset\mathbf{Z}^d$, $j=1,2,..$ be an increasing sequence of finite subsets with $0\in U_1$, and assume that $r_j:=\operatorname{dist}(0,\mathbf{Z}^d\setminus U_j)\to\infty$, $j\to\infty$. Choose $\Lambda_j=(\mathbf{Z}/L_j\mathbf{Z})^d$ with $U_j=[-\frac{L_j}{4},\frac{L_j}{4}]$. Assume also that (A.6, 11–13) hold and recall that the assumptions (A.1–3,7,9,10), (A.fr), (A.mp) imply (A.1–13).*

Then there exist $C_0\geq 1$, $j_0\in\mathbf{N}$, $\theta>0$, $h_0>0$, such that for $j\geq j_0$, $0<h\leq h_0$, we have:

$$\begin{aligned}&\operatorname{Cor}_{\phi_{U_j}}(x_\nu,x_\mu),\ \operatorname{Cor}_{\phi_{\Lambda_j}}(x_\nu,x_\mu)\\&=\mathcal{O}(h)e^{-\theta r_j/4}+he^{-p^\infty_{1,h}(\nu-\mu)}q^\infty(\nu-\mu;h),\ \text{for } |\nu|,|\mu|\leq\frac{r_j}{C_0}.\end{aligned} \tag{10.39}$$

Here, for the statement about $\operatorname{Cor}_{\phi_{\Lambda_j}}$, we view U_j as a subset of Λ_j in the natural way. $p^\infty_{1,h}\in C^\infty(\mathbf{R}^d\setminus\{0\})$ is positively homogenous of degree 1 and has the h asymptotic expansion

$$p^\infty_{1,h}(\nu)\sim\sum_{\ell=0}^\infty p^\infty_{1,\ell}(\nu)h^\nu,\ \ h\to 0, \tag{10.40}$$

in the space of such functions. $p_{1,0}$ *is a norm, strictly convex transversally to the radial direction. Further,*

$$q^{\infty}(\nu;h)=|\nu|^{-\frac{d-1}{2}}e^{-s^{\infty}(\nu;h)},\ \nu\in\mathbf{Z}^d,\ |\nu|\geq C_0, \tag{10.41}$$

where

$$s^{\infty}(\nu;h)\sim\sum_{-\infty}^{0}s^{\infty}_{\alpha}(\nu;h),|\nu|\to\infty, \tag{10.42}$$

uniformly with respect to h, *and* $s^{\infty}_{\alpha}(\cdot;h)\in C^{\infty}(\mathbf{R}^d\setminus\{0\})$ *is positively homogeneous of degree* α. *Here*

$$s^{\infty}_{\alpha}(\nu;h)\sim\sum_{0}^{\infty}s^{\infty}_{\alpha,\beta}(\nu)h^{\beta},\ \ h\to 0, \tag{10.43}$$

in the space of smooth functions on $\mathbf{R}^d\setminus\{0\}$, *positively homogeneous of degree* α.

Finally $p^{\infty}_{1,0}(\nu)$, $s^{\infty}_{0,0}(\nu)$ *were determined in [S1]. They appear in the asymptotics of* $(\Phi(0)^{-1})_{\nu,\mu}$, $\nu-\mu\to\infty$.

A. Non-commutative Taylor expansions.

Assume that that $\phi\in C^{\infty}(\mathbf{R}^{\Lambda})$ satisfies the assumptions (H1-4) (section 1). Let $f=f_j(x)$, $j=(j_1,..,j_P)$, $x\in\mathbf{R}^{\Lambda}$ be a P standard. We recall (cf [S2]) that this means that for every $k\in\mathbf{N}$:

$$\begin{aligned}&\langle\nabla^k f_{(\cdot)}(x),t_1\otimes..\otimes t_{P+k}\rangle=\\&\sum_j\langle t_{P+1},\partial_x\rangle..\langle t_{P+k},\partial_x\rangle f_j(x)t_{1,j_1}..t_{P,j_P}=\mathcal{O}_k(1)|t_1|_{p_1}..|t_{P+k}|_{p_{P+k}},\end{aligned} \tag{A.1}$$

for all $t_1,..,t_{P+k}\in\mathbf{C}^{\Lambda}$, $p_j\in[1,\infty]$, $1=\frac{1}{p_1}+..+\frac{1}{p_{P+k}}$, where the estimate is uniform in p_j (as well as in t_j and Λ).

In the main text we shall use iteratively the following result with $N=1$. For the sake of generality we state it also for larger N.

Proposition A.1. *For every* $N\in\mathbf{N}$, *there exists* $M(N)\in\mathbf{N}$ *such that for* $M(N)\leq M\in\mathbf{N}$

$$f_j(x)=\sum_{\substack{L,R\geq 0\\ L+R\leq N\\ 0\leq\nu\leq M}}h^{\frac{1}{2}(L+R)+\nu}\sum_{\substack{\ell\in\Lambda^L\\ r\in\Lambda^R}}Z^*_{\ell}\circ f^{(\nu)}_{j,\ell,r}(x)\circ Z_r, \tag{A.2}$$

in the sense of differential operators, where $Z_r=\prod_{\nu=1}^{R}Z_{r_\nu}$, $Z^*_{\ell}=\prod_{\nu=1}^{L}Z^*_{\ell_\nu}$. *Here* $f^{(\nu)}_{j,\ell,r}(x)$ *is* $P+L+R$ *standard, and when* $\nu<M$ *and* $L+R<N$ *it is independent of* x.

Proof. It will be convenient to use multiindex notation, with multiindices $\alpha\in\mathbf{N}^{\Lambda}$ of length 1, i.e. with $|\alpha|=|\alpha|_1=1$. In other terms (cf. section 3), we let $\alpha\in\mathbf{N}^{\Lambda}_1\simeq\Lambda$. Then a given P standard tensor can be written as $f_\alpha(x)$, with $\alpha\in(\mathbf{N}^{\Lambda}_1)^P$.

We start by writing

$$f_\alpha(x) = f_\alpha(0) + \sum_{|\gamma|=1} g_{\alpha,\gamma}(x) x^\gamma$$

by means of Taylor's formula, where $g_{\alpha,\gamma}(x)$ is $P+1$ standard. Using (H4) and the fact that the composition of a standard tensor in one of the indices with a 2 standard matrix gives a new standard tensor of the same size ([S2]), we get

$$f_\alpha(x) = f_\alpha(0) + \sum_{|\gamma|=1} k_{\alpha,\gamma}(x) \phi'(x)^\gamma,$$

where $k_{\alpha,\gamma}(x)$ is $P+1$-standard. This can be rewritten as

$$\begin{aligned} f_\alpha(x) &= f_\alpha(0) + \sum_{|\gamma|=1} h^{1/2} k_{\alpha,\gamma}(x)((Z^*)^\gamma + Z^\gamma) \\ &= f_\alpha(0) + h \sum_{|\gamma|=1} \partial_x^\gamma k_{\alpha,\gamma}(x) + \sum_{|\beta|=1} h^{1/2}(Z^*)^\beta \circ k_{\alpha,\beta} + \sum_{|\gamma|=1} h^{1/2} k_{\alpha,\gamma}(x) Z^\gamma. \end{aligned}$$

Here $f_\alpha^{(1)} := \sum_{|\gamma|=1} \partial_x^\gamma f_{\alpha,\gamma}(x)$ is P standard by the trace lemma (cf. [S3,2]), which says that $|\operatorname{tr} A| \le \|A\|_{\mathcal{L}(\ell^\infty,\ell^1)}$ for every finite square matrix A.

We can repeat the procedure with the $k_{\alpha,\beta}$ and $k_{\alpha,\gamma}$ in the last two sums, and eventually we reach:

$$f_\alpha(x) = \sum_{\substack{L+R\le N \\ 0\le\nu\le M}} h^{\frac{1}{2}(L+R)+\nu} \sum_{\substack{|\beta_1|,..,|\beta_L|=1 \\ |\gamma_1|,..,|\gamma_R|=1}} (Z^*)^{\beta_1+..+\beta_L} \widetilde{f}_{\alpha,\beta_1,..,\beta_L,\gamma_1,..,\gamma_R}(x) Z^{\gamma_1+..+\gamma_R}, \quad \text{(A.3)}$$

where $\widetilde{f}_{\alpha,\beta_1,..,\beta_L,\gamma_1,..,\gamma_R}(x)$ is $P+L+R$ standard. This is equivalent to (A.1) . #

B. Hilbert–Schmidt property of tensors.

Lemma B.1. *Let $a = a_{j_1,..,j_N}$, $1 \le j_k \le m_k$, $m_k < +\infty$, be an N tensor and assume that*

$$|\langle a, t_1 \otimes \ldots \otimes t_N \rangle| \le |t_1|_{p_1} \cdots |t_N|_{p_N}, \quad \text{(B.1)}$$

whenever $t_j \in \mathbf{C}^{m_j}$, $1 \le p_j \le \infty$, $\frac{1}{p_1} + \ldots + \frac{1}{p_N} = \frac{1}{2}$. Then

$$|a|_2 \le 1. \quad \text{(B.2)}$$

Proof. This is obvious for $N=1$. Let $N \ge 2$ and assume that we have already proved the lemma with N replaced by $N-1$. Put $At_N = \langle a, t_N \rangle$, where the bracket indicates contraction in the last variable. We may view A as a linear map from $\mathbf{C}^{m_N}$ into the $N-1$ tensors, with matrix $a_{j';j_N}$, $j' = (j_1,..,j_{N-1})$. We have

$$|\langle At_N, t_1 \otimes \ldots \otimes t_{N-1} \rangle| = |\langle a, t_1 \otimes \ldots \otimes t_N \rangle| \le |t_1|_{p_1} \ldots |t_{N-1}|_{p_{N-1}} |t_N|_\infty,$$

if $\frac{1}{p_1}+\ldots+\frac{1}{p_{N-1}}=\frac{1}{2}$. By the induction hypothesis, we get

$$|At_N|_2 \leq |t_N|_\infty,$$

so

$$\|A\|_{\mathcal{L}(\ell^\infty,\ell^2)} \leq 1.$$

Hence by duality, $\|A^*\|_{\mathcal{L}(\ell^2,\ell^1)} \leq 1$, and we conclude that

$$\|A^*A\|_{\mathcal{L}(\ell^\infty,\ell^1)} \leq 1. \tag{B.3}$$

By the trace lemma (stating that for a square matrix the modulus of the trace is bounded by the $\mathcal{L}(\ell^\infty,\ell^1)$ norm, cf. [S3], Lemma 1.2), we conclude that

$$1 \geq \operatorname{tr} A^*A = \|A\|_{\mathrm{HS}}^2 = \sum_{j',j_N} |a_{j',j_N}|^2 = |a|_{\ell^2}^2, \tag{B.4}$$

and we get the lemma for N. #

The following result is an easy consequence of the preceding lemma.

Lemma B.2. *Let a be an N tensor of the same form as above and assume that*

$$|\langle a, t_1\otimes\ldots\otimes t_N\rangle| \leq |t_1|_{p_1}\cdots|t_N|_{p_N}, \tag{B.5}$$

when $t_j\in\mathbf{C}^{m_j}$, $1\leq p_j\leq\infty$, $\frac{1}{p_1}+\ldots+\frac{1}{p_N}=1$ (rather than $\frac{1}{2}$). Then if we decompose the indices: $j=(j',j'',j''')$, $j'=(j_1,\ldots,j_k)$, $j''=(j_{k+1},\ldots,j_{k+\ell})$, $j'''=(j_{k+\ell+1},\ldots,j_N)$, with $k,\ell\geq 1$, we have

$$|\langle a, b'\otimes b''\otimes t_{k+\ell+1}\otimes\ldots\otimes t_N\rangle| \leq |b'|_2|b''|_2\prod_{k+\ell+1}^{N}|t_j|_\infty, \tag{B.6}$$

for all $t_j\in\mathbf{C}^{m_j}$, k tensors b' and ℓ tensors b''.

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