LEBOWITZ INEQUALITIES FOR ASHKIN–TELLER SYSTEMS

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Dedicated to Joel Lebowitz who has, throughout the years, maintained the cohesion of the Statistical Mechanics community.

Abstract. We consider the Ashkin–Teller model with negative four-spin coupling but still in the region where the ground state is ferromagnetic. We establish the standard Lebowitz inequality as well as the extension that is necessary to prove a divergent susceptibility.

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1. Introduction. The Lebowitz inequality was first established for ferromagnetic Ising systems in [L]. It was later shown [B],[GJ] that in these and related systems such an inequality implies a continuous transition. The inequality itself states that the (untruncated) four point correlation function is bounded above by the sum of the products of the two point functions paired in all possible ways. From this one can show that the derivative of the susceptibility[1] is bounded above by a constant times the square of the susceptibility. And from this one can show that if at some β^* the susceptibility, $\mathcal{X}$, is infinite, then for any $\beta < \beta^*$ we have $\mathcal{X} \geq [\text{const.}][\beta^* - \beta]^{-1}$. This necessarily implies that the susceptibility has to diverge continuously at some point.[2]

In spin systems, the derivations of these sorts of inequalities usually follow the same course: Duplicate the system, rearrange variables, expand out all terms and hope everything comes out positive. (Although in [L], some FKG properties of the duplicated system were also exploited.) The model we consider, introduced in [AT], is described by the Hamiltonian

$$-\mathcal{H} = \sum_{\langle i,j\rangle} [J(\sigma_i\sigma_j + \tau_i\tau_j) - U\sigma_i\sigma_j\tau_i\tau_j] \tag{1.1}$$

where σ and τ represent Ising variables and J and U are both positive. Although a proof along the above described lines may be possible, it would be extraordinarily complicated due to the presence of the negative couplings. (Not to mention the number of variables that would be required.) Our strategy – more akin to the derivation in [L] – involves two distinct steps. First: get a bound on the pure four-point function (involving only the σ-variables) in terms of the various mixed four-point functions. Second: use the repulsive nature of the interaction between the two types of variables to bound the mixed four-point functions by products of the corresponding two-point functions. In short, no duplication. In point of fact, we will present these steps in the reverse order. In Section 2, we describe a random cluster representation for this system (introduced in [$\mathrm{CM_I}$], [PfV]) that provides the bounds described in the second step. In Section 3 we present a variant of the so-called loop expansion which gets us through the first step. We remark, without understanding, that both representations have their advantages and limitations. In particular, a unified proof using either representation alone seems to be completely hopeless. We remark without proof that this general strategy can be implemented in other systems as well, e.g. the usual O(2) and O(3) ferromagnets. However the system described by the Hamiltonian in Eq.(1.1) is the first *new* system for which a Lebowitz inequality can be proved by these methods.

It is clear that the behavior of the system depends drastically on the magnitude of U. E.g. if $U < J$, the ground state is ferromagnetic and if $U > J$, it is not. We will work in the ferromagnetic domain which, it seems, is determined by the condition

$$\operatorname{th}^2\beta J \geq \operatorname{th}\beta U. \tag{1.2}$$

The fact that systems described by Equations (1.1) and (1.2) have a continuous transition is, perhaps, not too surprizing. Nevertheless, there is only a small handful of models for which a continuous transition can be rigorously established. Now it is slightly larger handful.

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[1]Throughout this work, we will use the word "susceptibility" to denote the sum of the *untruncated* 2-point function. For the systems under consideration, this coincides with the thermodynamic susceptibility in the single phase regime and it is the divergence of *this* object that we wish to study.

[2]This is easily seen on a formal level in infinite volume; a rigorous proof often involves finite-volume cutoffs. Such derivations are routine and will not be made explicit in this note. For example the method of cutoffs described in [G] for percolation (with suitable modifications for boundary conditions if necessary) can be applied to every (uniform) system known to the authors.

2. The Random Cluster Representation. To implement the random cluster representation, we rewrite the Hamiltonian in Potts form:

$$-\beta\mathcal{H} = \sum_{\langle i,j\rangle}[D(\delta_{\sigma_i\sigma_j} + \delta_{\tau_i\tau_j}) - V\delta_{\sigma_i\sigma_j}\delta_{\tau_i\tau_j}] \tag{2.1}$$

with $D = 2\beta(J+U)$ and $V = 4\beta U$. For each bond, we may expand as in the usual random cluster model, e.g. $\exp\left(D\delta_{\sigma_i\sigma_j}\right) = \left(1 + R_D\delta_{\sigma_i\sigma_j}\right)$, etc. with $R_D = \mathrm{e}^D - 1$ but when we get to the four-spin term, we are forced to acknowledge that $\exp\left(-V\delta_{\sigma_i\sigma_j}\delta_{\tau_i\tau_j}\right) = \left(1 - r_V\delta_{\sigma_i\sigma_j}\delta_{\tau_i\tau_j}\right)$ with $-r_V = \left(\mathrm{e}^{-V} - 1\right)$ a negative number. This could ostensibly lead to negative numbers in the graphical representation – and hence no representation with a *probabilistic* interpretation. However the situation is not nearly as bad as it looks since, on each bond, all we are interested in is the product of all three terms. Indeed, it is seen that there are there are several mechanisms for generating a $\delta_{\sigma_i\sigma_j}\delta_{\tau_i\tau_j}$ term and we get

$$\exp\left\{D\left(\delta_{\sigma_i\sigma_j} + \delta_{\tau_i\tau_j}\right) - V\delta_{\sigma_i\sigma_j}\delta_{\tau_i\tau_j}\right\} = 1 + R_D\delta_{\sigma_i\sigma_j} + R_D\delta_{\tau_i\tau_j} + \mathfrak{R}_{VD}\delta_{\sigma_i\sigma_j}\delta_{\tau_i\tau_j}; \tag{2.2}$$

with $\mathfrak{R}_{VD} = \left[R_D^2 - r_V(1+R_D)^2\right]$. Thus, to get started on a decent graphical representation all we need is that $\mathfrak{R}_{VD} \geq 0$; i.e. $\left(1 - \mathrm{e}^{-D}\right)^2 \geq 1 - \mathrm{e}^{-V}$. After some small amount of work, this is seen to be precisely the condition (1.2) mentioned in the introduction.

There is an obvious way to develop a graphical representation. Following closely the derivation in [FK] for the Potts model, we could define the $R_D\delta_{\sigma_i\sigma_j}$ and $R_D\delta_{\tau_i\tau_j}$ terms as single bonds in the σ-layer and τ-layer respectively while the $\mathfrak{R}_{VD}$-terms represent double bonds. Then all the different types of bonds are treated as separate entities. This was done in [$\mathrm{CM_I}$] and is well suited for certain purposes but none that are related to the present work. An alternative approach, used in both [PfV] and [$\mathrm{CM_I}$], is to define bond configurations Ω_σ and Ω_τ for the σ and τ layers that do not care where the bonds came from. I.e. $\Omega_\sigma(b) = 1$ means that b got a single σ-bond or a maybe a double bond. In this approach, the resulting random cluster measure is defined by the weights

$$W(\Omega_\sigma, \Omega_\tau) = A^{[N(\Omega_\sigma\vee\Omega_\tau)]}B^{[N(\Omega_\sigma\wedge\Omega_\tau)]}2^{C(\Omega_\sigma)}2^{C(\Omega_\tau)} \tag{2.3}$$

with $A = R_D$ and $B = \mathfrak{R}_{VD}/R_D$. As usual: $N(-)$ is the number of bonds of the specified type while $C(\Omega_\sigma)$ and $C(\Omega_\tau)$ individually count the number of connected components in the σ-layer and τ-layers respectively. (Of course the latter objects – as well as possible additional constraints – are not completely defined until one pays heed to the *boundary conditions.*)

It is also convenient to write the weights in the form where each occupied bond gets its own factor of A and then an additional factor of B/A for the number of overlapping bonds. Then, as is is clear, if $B > A$, then the bonds in the separate layers attract while if $B < A$, they repel. And notice that the dividing line is determined by the sign of U. What has impeded progress so far is the (straightforward) result concerning the FKG-properties of these measures that was proved in [$\mathrm{CM_I}$], [PfV] (see also [BC]). For future purposes, we state the result in general terms (c.f. the remark following Proposition 2.1)

Proposition 2.1. *Let $\mathcal{G}$ denote a finite graph. For $r, s \geq 1$ let $\mu_{\mathbb{P}}^{\mathrm{RC}}(-)$ be the measure on pairs of bond configurations $(\Omega_\sigma, \Omega_\tau)$ with weights given by*

$$W_{\mathbb{P}}(\Omega_\sigma, \Omega_\tau) = r^{C(\Omega_\sigma)} s^{C(\Omega_\tau)} \prod_b [A_\sigma(b)]^{\Omega_\sigma(b)} [A_\tau(b)]^{\Omega_\tau(b)} \lambda_b^{[\Omega_\sigma(b)][\Omega_\tau(b)]}$$

where $\Omega_\sigma(b) = 0$ or 1 corresponds to the σ-bond being vacant or occupied and similarly for $\Omega_\tau(b)$ and $\mathbb{P}$ is notation for the various parameters in the weights. Then the sufficient – and the necessary – condition for the measure $\mu_{\mathbb{P}}^{\mathrm{RC}}(-)$ to be strong FKG is that $\lambda_b \geq 1$ for all b.

Proof. This is a reasonably straightforward FKG lattice condition to check; see [$\mathrm{CM_I}$], [PfV] (and also [BC]). □

Remark. The definition of the measures – for the benefit of Propositions 2.1 and 2.2 – is a little unwieldy. The case in hand is simply $A_\sigma(b) = A_\tau(b) \equiv A$ and $\lambda_b \equiv B/A$. (Thus the theorem fails for models of interest in this paper; we will return to this point after we finish apologizing for the statement of the proposition.) The primary reason we use "the most general graphs" is as follows: to complete the argument that gets us from a Lebowitz inequality to a divergent susceptibility we need to consider these measures in the context of sensible graphs – like finite pieces of $\mathbb{Z}^2$ – in the presence of boundary conditions e.g. "wired". On a given finite graph most boundary conditions that lead to FKG measures for random cluster problems can be realized as free boundary conditions on a related graph.

As just mentioned: with the usual ordering, the FKG property for the systems described by the Hamiltonian in Eq.(2.1) will only hold in case $-V \geq 0$. However, there is an alternative ordering which is described in [PfV] that is well suited for the situation at hand. This alternative exploits the competitive nature of the interaction between the layers. In particular, for any bond b, we can define the local order $\Omega_\sigma(b) = 1 > \Omega_\sigma(b) = 0$ and $\Omega_\tau(b) = 0 > \Omega_\tau(b) = 1$. Notice that with this ordering, there is no relation between the configuration with $(\Omega_\sigma(b) = 1, \Omega_\tau(b) = 1)$ and $(\Omega_\sigma(b) = 0, \Omega_\tau(b) = 0)$.

As obvious as all this may seem in hindsight, it is nevertheless surprizing: If, for example, $U < J$, the low temperature states are ordered in *both* layers: Despite the competition, there is ultimately some cooperation. In any case, it is equally not hard to show that for $B < A$, the measures have the FKG property with respect to this alternative partial ordering.

Proposition 2.2. *Let $\mu_{\mathbb{P}}^{\mathrm{RC}}(-)$ be a random cluster measure as described in the statement of Proposition 2.1 with $\lambda_b \leq 1$ on each bond. Then $\mu_{\mathbb{P}}^{\mathrm{RC}}(-)$ is FKG with respect to the partial order where $\Omega_\sigma(b)$ and $1 - \Omega_\tau(b)$ are (separate) increasing coordinates.*

Proof. See [PfV] Lemma 4.1. □

The significance of this representation for the present work is exemplified by the following (which already appears in [PfV]):

Corollary. *For the spin systems described in Proposition 2.2 let i, j, k and ℓ denote any sites of the lattice. Then, for $r = s = 2$,*

$$\langle \sigma_i \sigma_j \tau_k \tau_\ell \rangle_{\beta\mathcal{H}} \leq \langle \sigma_i \sigma_j \rangle_{\beta\mathcal{H}} \langle \tau_k \tau_\ell \rangle_{\beta\mathcal{H}}.$$

For the other spin systems of this type, the same holds with $\sigma_i\sigma_j$ *replaced by* $\frac{r}{r-1}\left(\delta_{\sigma_i\sigma_j} - \frac{1}{r}\right)$*, etc.*

Proof. According to the standard translations between spin systems and their random cluster representations ([ACCN] or, better yet, [ES]) the object $\langle \delta_{\sigma_i\sigma_j} - \frac{1}{r}\rangle_{\beta\mathcal{H}}$ is proportional to the probability that the sites i and j belong to the same Ω_σ- cluster. Let us define the various connectivity events: $\{i \underset{\sigma}{\longleftrightarrow} j\}$ is the event that i and j are connected in the σ-layer etc. Then we have, for $r = s = 2$

$$\langle \sigma_i\sigma_j\tau_k\tau_\ell\rangle_{\beta\mathcal{H}} = \mu^{\mathrm{RC}}_{\beta\mathcal{H}}\Big(\{i \underset{\sigma}{\longleftrightarrow} j\} \cap \{k \underset{\tau}{\longleftrightarrow} \ell\}\Big) \tag{2.5}$$

(and similarly for the other values of r and s). However

$$\mu^{\mathrm{RC}}_{\beta\mathcal{H}}\Big(\{i \underset{\sigma}{\longleftrightarrow} j\} \cap \{k \underset{\tau}{\longleftrightarrow} \ell\}\Big) \leq \mu^{\mathrm{RC}}_{\beta\mathcal{H}}\Big(\{i \underset{\sigma}{\longleftrightarrow} j\}\Big)\mu^{\mathrm{RC}}_{\beta\mathcal{H}}\Big(\{k \underset{\tau}{\longleftrightarrow} \ell\}\Big)$$

by the FKG property and the desired result is established. □

We will use the above inequality – as well as some related inequalities – when the time comes to derive our principal results.

3. The Loop Representation. We now start down an entirely different track namely that of loop expansions. The primitive versions of these expansions are well known (e.g. they are described in [PfV]) and, in the context of the present model, actually date back to [AT]. However, the usual versions will only take us so far and some small modifications will eventually have to be inserted. The unfortunate *general* limitation is that these expansions are heavily tied with the Ising nature of the spin variables (or, more precisely the ability to represent the spin variables in terms of Ising variables). Thus, for the remainder of this paper, we will confine attention to the standard ($r = s = 2$) Ashkin–Teller models.

We start with the famous Ising–spin identity: $\mathrm{e}^{\kappa\sigma_i\sigma_j} \propto 1+(\operatorname{th}\kappa)\sigma_i\sigma_j$. In any Ising-type system, this can be applied to any set that has an Ising-type interaction. But we remark that schemes based on this identity usually require the coefficients of the Ising spin terms to be positive. For the case at hand we get

$$\mathrm{e}^{\beta[J(\sigma_i\sigma_j+\tau_i\tau_j)-U\sigma_i\sigma_j\tau_i\tau_j]} \propto (1+\mathbb{J}\sigma_i\sigma_j)(1+\mathbb{J}\tau_i\tau_j)(1-\mathbb{U}\sigma_i\sigma_j\tau_i\tau_j) \equiv \Phi_{i,j} \tag{3.1}$$

With $\mathbb{J} = \operatorname{th}\beta J$ and $\mathbb{U} = \operatorname{th}\beta U$. If we multiply these out, the coefficient of the four-spin term is seen to be proportional to $\mathbb{J}^2 - \mathbb{U}$ which we want to be positive. This is, again, the condition (1.2). Multiplication of all terms leads to a couple of two-spin terms, a four-spin term and a constant (that gets rescaled to one). The result is

$$\Phi_{i,j} \propto (1 + \mathbb{L}\sigma_i\sigma_j + \mathbb{L}\tau_i\tau_j + \mathbb{K}\sigma_i\sigma_j\tau_i\tau_j) \tag{3.2}$$

with $\mathbb{L} = [\mathbb{J} - \mathbb{U}]/[1 - \mathbb{J}^2\mathbb{U}]$ and $\mathbb{K} = [\mathbb{J}^2 - \mathbb{U}]/[1 - \mathbb{J}^2\mathbb{U}]$. The further development of the loop expansion is somewhat obscured by the presence of the four-spin term so to simplify our arguments, we will start with the treatment of the case $\mathbb{K} = 0$ – saturation of the condition (1.2). (As a matter of fact, this very model was discussed in [CPS] where, for example, it was shown to have a *random surface* interpretation.) We assume that we are on some finite graph that has free boundary conditions. Let us start with the partition function: We get to pick one term from $\Phi_{i,j}$ for each bond of the lattice. One such choice for each bond leads to a graphical configuration (or diagram) which we will call η: a red bond at $\langle i,j\rangle$ if we choose the $\mathbb{L}\tau_i\tau_j$-term, a blue bond if we pick $\mathbb{L}\sigma_i\sigma_j$ and vacant for the "1". When we perform the trace (normalized for convenience) the only diagrams that survive are those where an even number – possibly zero – of bonds of *both* types are incident at each vertex. When these constraints are satisfied, the weight of the configuration is just $\mathbb{L}^{|\eta|}$. It is of some use to consider Ξ, a colorless configuration, that represents all the configurations η that have the same bonds regardless of color. Although it is tempting proclaim that the weight of Ξ is given $\mathbb{L}^{|\Xi|}\times 2^{\#\text{of loops in }\Xi}$, the number of loops in a configuration is not very well defined – certainly not in any way that makes this formula true. (Notwithstanding, we will still call this the loop representation.) In the end, the coefficient of $\mathbb{L}^{|\Xi|}$ is just the number of ways that the loop clusters can be consistently colored red and blue in such a way that the constraints are satisfied at each vertex. The trivial, but important observation is that this counting procedure factors over the individual clusters of Ξ.

The spin–spin correlation function is not particularly pretty in this representation – it is not the probability of anything in the loop measure (i.e. the measure defined by the weights of the loop configurations).[3] When we calculate the correlation between σ_i and σ_j the best description is to say that there is a numerator and a denominator. The denominator is just the partition function – the sum of all the weights described above. The numerator contains similar sorts of configurations except that now the two sites i and j have to have an *odd* number of incident (blue) bonds. Since every other site has an even number of bonds, this implies that there is a walk from i to j. Following the language of [A] we say there are *sources* at the sites i and j.

Let us now consider the four-point function $\langle\sigma_i\sigma_j\sigma_k\sigma_\ell\rangle$. Here we have sources at i, j, k and ℓ. When we look at the contributing terms, we see that there are two distinct possibilities: All four sites are part of the same connected cluster (regardless of color) or the sites pair up forming two distinct clusters – and there are three ways of doing this. We denote the sum of the corresponding terms by $\Gamma_{i,j,k,\ell}$ for the case where the four points are tied up in the same cluster, $\mathfrak{g}_{[i,j|k,\ell]}$ for i and j in a separate cluster from k and ℓ etc. Thus we may write

$$\mathscr{Z}_{\mathcal{R}}\times\langle\sigma_i\sigma_j\sigma_k\sigma_\ell\rangle_{\mathcal{R}}=[\Gamma_{i,j,k,\ell}+\mathfrak{g}_{[i,j|k,\ell]}+\mathfrak{g}_{[i,k|j,\ell]}+\mathfrak{g}_{[i,\ell|j,k]}] \tag{3.3}$$

where $\mathcal{R}$ is the interaction that has $\mathbb{J}^2=\mathbb{U}$.

Now let us consider a mixed four-point function, say $\langle\sigma_i\sigma_j\tau_k\tau_\ell\rangle_{\mathcal{R}}$. Several things are immediately clear: First, there are no terms like the last two for this correlation function – we need the proper sources to be paired. Second, there *will* be terms like the first two – a term with all four points connected and one with (i & j) and (k & ℓ) separately paired. Thus we may tentatively write $\mathscr{Z}_{\mathcal{R}}\langle\sigma_i\sigma_j\tau_k\tau_\ell\rangle_{\mathcal{R}}=\tilde{\Gamma}_{i,j,k,\ell}+\tilde{\mathfrak{g}}_{[i,j|k,\ell]}$. Third, it is clear that $\tilde{\mathfrak{g}}_{[i,j|k,\ell]}=\mathfrak{g}_{[i,j|k,\ell]}$. Indeed in any colored graph that contributes to $\mathfrak{g}_{[i,j|k,\ell]}$, one can simply reverse the colors of the cluster of (k & ℓ). This shows that $\tilde{\mathfrak{g}}_{[i,j|k,\ell]}\geq\mathfrak{g}_{[i,j|k,\ell]}$ and the reverse argument shows $\mathfrak{g}_{[i,j|k,\ell]}\geq\tilde{\mathfrak{g}}_{[i,j|k,\ell]}$.

What is not immediately obvious is that the four-point cluster terms are equal. This is the subject of the argument below.

[3] In general the correlation functions can be expressed as the expected value of some function of Ξ – but this does not seem to serve any tangible purpose. For certain correlation functions we can do better. For example we will show that $\langle\sigma_i\sigma_j\tau_i\tau_j\rangle$ is just the probability that i and j belong to the same loop (c.f. Corollary I to Theorem 3.4). Notice that if $\mathbb{U}=0$ the former is equal to $\langle\sigma_i\sigma_j\rangle^2$. This is a disguised version of the famous "switching identities" derived in [A].

Lemma 3.1. *Let* $\mathbf{\Gamma}$ *denote a finite connected graph that contains among others four points* i, j, k *and* ℓ. *All sites in* $\mathbf{\Gamma}$ *are evenly coordinated except the above mentioned which are odd-coordinated. Let* $N^{\Gamma}_{b(i,j,k,\ell)}$ *denote the number of ways that the bonds of this graph can be colored red and blue such that an even number of red and blue bonds are incident at each vertex save for* i, j, k *and* ℓ *which have an odd number of blues. Let* $N^{\Gamma}_{b(i,j);r(k,\ell)}$ *denote the number of similar colorings except this time the sites* k *and* ℓ *have an odd number of reds. Then*

$$N^{\Gamma}_{b(i,j,k,\ell)} = N^{\Gamma}_{b(i,j);r(k,\ell)}.$$

Proof. Let $\mathfrak{p} : k \to \ell$ denote a self-avoiding path in $\mathbf{\Gamma}$ with terminal points k and ℓ. Let $\mathcal{N}^{\Gamma}_{b(i,j,k,\ell)}$ be the *set* of successful colorings of $\mathbf{\Gamma}$ according to the first set of rules (which is observed to be non-empty) and similarly for $\mathcal{N}^{\Gamma}_{b(i,j);r(k,\ell)}$. We will use $\mathfrak{p}$ to define a map from $\mathcal{N}^{\Gamma}_{b(i,j,k,\ell)}$ to $\mathcal{N}^{\Gamma}_{b(i,j);r(k,\ell)}$ which is one-to-one; the map is simply to reverse the color of each bond along the path.

It is clear that the application of this map to an element of $\mathcal{N}^{\Gamma}_{b(i,j,k,\ell)}$ produces a coloring of the desired type: Indeed, at internal sites of the path, the change in the number of blues is ± 2 or zero with a corresponding change in reds of ∓ 2 or zero. Similarly at the endpoints the number of blues changes by ± 1 accompanied by a change in the number of reds by ∓ 1.

Next, consider two distinct colorings of $\mathbf{\Gamma}$ in $\mathcal{N}^{\Gamma}_{b(i,j,k,\ell)}$. Either the two colorings have some difference on the compliment of $\mathfrak{p}$ – in which case the target configurations are surely different, or they must be different along the path itself – in which case they are still different when their paths get reversed.

Hence this map is indeed one-to-one which establishes $N^{\Gamma}_{b(i,j);r(k,\ell)} \geq N^{\Gamma}_{b(i,j,k,\ell)}$. But we can construct a similar sort of map from $\mathcal{N}^{\Gamma}_{b(i,j);r(k,\ell)}$ to $\mathcal{N}^{\Gamma}_{b(i,j,k,\ell)}$ which gives us the reverse inequality. □

As an immediate consequence we get that $\Gamma_{i,j,k,\ell} = \tilde{\Gamma}_{i,j,k,\ell}$. This leads to the following preliminary result:

Theorem 3.2. *Consider any graph* $\mathcal{G}$ *with free boundary conditions and interaction*

$$-\mathcal{R} = \beta \sum_{i,j} [\sigma_i \sigma_j + \tau_i \tau_j] - \alpha \sum_{i,j} [\sigma_i \sigma_j \tau_i \tau_j]$$

with $\operatorname{th} \alpha = \operatorname{th}^2 \beta$. *Then*

$$\langle \sigma_i \sigma_j \sigma_k \sigma_\ell \rangle_{\mathcal{R}} \leq \langle \sigma_i \sigma_j \rangle_{\mathcal{R}} \langle \sigma_k \sigma_\ell \rangle_{\mathcal{R}} + \langle \sigma_i \sigma_k \rangle_{\mathcal{R}} \langle \sigma_j \sigma_\ell \rangle_{\mathcal{R}} + \langle \sigma_i \sigma_\ell \rangle_{\mathcal{R}} \langle \sigma_j \sigma_k \rangle_{\mathcal{R}}.$$

Proof. On the basis of what has transpired we have

$$\langle \sigma_i \sigma_j \sigma_k \sigma_\ell \rangle_{\mathcal{R}} = \langle \sigma_i \sigma_j \tau_k \tau_\ell \rangle_{\mathcal{R}} + \langle \sigma_i \sigma_k \tau_j \tau_\ell \rangle_{\mathcal{R}} + \langle \sigma_i \sigma_\ell \tau_j \tau_k \rangle_{\mathcal{R}} - 2 \frac{\Gamma_{i,j,k,\ell}}{\mathcal{Z}_{\mathcal{R}}}. \tag{3.4}$$

So we neglect the final term and use the FKG property from Section 2. □

Of course the above is of no immediate benefit for the analysis of the susceptibility; in this model, there are additional terms when we differentiate the two-point correlation function with respect to the temperature. However, the extra ingredients required are not particularly difficult:

Theorem 3.3. *Consider the system described in the statement of Theorem 3.2. Then*

$$\langle\sigma_i\sigma_j;[\sigma_k\sigma_\ell+\tau_k\tau_\ell-\alpha'\sigma_k\sigma_\ell\tau_k\tau_\ell]\rangle_{\mathcal{R}}\leq\langle\sigma_i\sigma_k\rangle_{\mathcal{R}}\langle\sigma_j\sigma_\ell\rangle_{\mathcal{R}}+\langle\sigma_i\sigma_\ell\rangle_{\mathcal{R}}\langle\sigma_j\sigma_k\rangle_{\mathcal{R}}.$$

where, in general, $\langle A;B\rangle=\langle AB\rangle-\langle A\rangle\langle B\rangle$ *and* $\alpha'=\mathrm{d}\alpha/\mathrm{d}\beta$.

Proof. We first remark that all we really require is that the coefficient of the four-spin term is not less than -1. We observe that in the present case, $\alpha'=2\,\mathrm{th}\,\beta/[1+\mathrm{th}^2\beta]\leq 1$. In light of Theorem 3.2, all that we need to show is that $\langle\sigma_i\sigma_j;[\tau_k\tau_\ell-\alpha'\sigma_k\sigma_\ell\tau_k\tau_\ell]\rangle_{\mathcal{R}}$ is not positive. Using $\alpha'\leq 1$, we start with $(1-\alpha')\langle\sigma_i\sigma_j;\tau_k\tau_\ell\rangle\leq 0$ by the FKG inequality. Now what is left is to show that $\langle\sigma_i\sigma_j;[\tau_k\tau_\ell-\sigma_k\sigma_\ell\tau_k\tau_\ell]\rangle_{\mathcal{R}}\leq 0$. We go back to the FK representation and consider the untruncated quantity $\langle\sigma_i\sigma_j\tau_k\tau_\ell(1-\sigma_k\sigma_\ell)\rangle_{\mathcal{R}}$. It is clear that this vanishes unless k is connected to ℓ by τ-bonds. But this also vanished unless k is *not* connected to ℓ by σ-bonds. If both of these are satisfied, *and* the site i is σ-connected to j, we get one. There are other possibilities for getting something non-zero but these all come out negative. We thus have

$$\langle\sigma_i\sigma_j[\tau_k\tau_\ell-\sigma_k\sigma_\ell\tau_k\tau_\ell]\rangle_{\mathcal{R}}\leq\mu_{\mathcal{R}}^{\mathrm{RC}}\Big(\{i\underset{\sigma}{\longleftrightarrow}j\}\cap\big[\{k\underset{\tau}{\longleftrightarrow}\ell\}\cap\{k\underset{\sigma}{\leftarrow\!\times\!\rightarrow}\ell\}\big]\Big)\tag{3.5}$$

where $\{k\underset{\sigma}{\leftarrow\!\times\!\rightarrow}\ell\}$ is notation for the the compliment of the event $\{k\underset{\sigma}{\longleftrightarrow}\ell\}$. The second event on the right hand side (the one in the square brackets) is clearly decreasing while the first one is increasing. Thus

$$\langle\sigma_i\sigma_j[\tau_k\tau_\ell-\sigma_k\sigma_\ell\tau_k\tau_\ell]\rangle_{\mathcal{R}}\leq\mu_{\mathcal{R}}^{\mathrm{RC}}\Big(\{i\underset{\sigma}{\longleftrightarrow}j\}\Big)\;\mu_{\mathcal{R}}^{\mathrm{RC}}\Big(\{k\underset{\tau}{\longleftrightarrow}\ell\}\cap\{k\underset{\sigma}{\leftarrow\!\times\!\rightarrow}\ell\}\Big).\tag{3.6}$$

The second factor on the right hand side of Eq.(3.6) is seen to equal $\langle\tau_k\tau_\ell[1-\sigma_k\sigma_\ell]\rangle_{\mathcal{R}}$ and the desired version of the Lebowitz inequality is established. □

The corollary to the above should be that the susceptibility diverges. However before such a claim can be made, it has to be verified that the susceptibility is actually infinite somewhere. Unfortunately with the above interaction, this can only be established for the model on the square lattice; here it seems likely that here the critical value is in fact $\beta=\infty$ (c.f. the discussion in [CPS]).

Corollary. *For the model on* $\mathbb{Z}^2$ *with the interaction as described in the statements of Theorem 3.2, the susceptibility diverges continuously at some value – possibly infinite – of the parameter* β.

Proof. Infinite susceptibility, at $\beta=\infty$, was established for this model in [CPS] using duality, the rest follows from the bound on $\mathrm{d}\mathcal{X}/\mathrm{d}\beta$ that results from the inequality in the statement of Theorem 3.3 □

As emphasized by the above limitations, it is clearly desirable to extend this set of results past the $\mathbb{K}=0$ restrictions. Going back to Eq.(3.2) – with $\mathbb{K}>0$ – the straightforward procedure would be to include one new type of bond, namely a red–blue (double) bond that has weight $\mathbb{K}$. As indicated, these bonds have double occupancy but all the other constraints are pretty much the same as before. Although this is the standard way of doing these loop expansions, it is clear that our analysis will run into trouble.

First, on the æsthetic level, there is now no worthwhile expansion for the model in terms of colorless graphs. Every double bond forces two colors through that bond. This means that under certain circumstances, the color of one bond in a cluster could determine the coloring scheme of the entire cluster – even though the cluster itself could have many "loops". On a more disturbing note: Although an expression like Eq.(3.3) is still valid for the usual four-point function, and similarly for the mixed four-point functions, there is no obvious correspondence between the various Γ-terms. (We remind the reader that these are the terms corresponding to the diagrams where all four points are in the same cluster.) Indeed it is clear, after a moments reflection, that each of these Γ's contain diagrams that are conspicuously absent in the other. (The relevant $\mathfrak{g}$-terms *do* coincide but this is small consolation.) The resolution of all of these difficulties is, perhaps, obvious in hindsight: We will borrow from the one.

What we actually do is as follows: We write $1 = (1 - \mathbb{K}) + \mathbb{K}$ (noting that $\mathbb{K} < 1$) so that

$$\begin{aligned}\Phi_{i,j} &\propto ([1 - \mathbb{K}] + \mathbb{J}\sigma_i\sigma_j + \mathbb{J}\tau_i\tau_j + \mathbb{K}\sigma_i\sigma_j\tau_i\tau_j + \mathbb{K}) \\ &\propto (1 + \mathbf{J}\sigma_i\sigma_j + \mathbf{J}\tau_i\tau_j + \mathbf{K}\sigma_i\sigma_j\tau_i\tau_j + \mathbf{K})\end{aligned} \tag{3.7}$$

with $\mathbf{K} = \mathbb{K}/[1-\mathbb{K}]$ etc. The "new" terms corresponding to the selection of the $\mathbf{K}$ may, perhaps, be called *dud-bonds*. In the colored graphs these have no special significance – they do not change the color parity of any vertex – but in any colorless representation, they have the same appearance, not to mention the same weight, as the red–blue type of double bond.

This is the bottom line, anything else may be regarded as window dressing. Notwithstanding, an interpretation that is pleasing (at least to the authors) is as follows: Each edge of the lattice can potentially be split into two edges, an "upper" edge and a "lower" edge. Let us write the $\mathbf{K}$ terms in Eq.(3.7) as $\mathbf{K} = \frac{1}{2}\mathbf{K}^{\mathrm{rr}} + \frac{1}{2}\mathbf{K}^{\mathrm{bb}}$ and $\mathbf{K}\sigma_i\sigma_j\tau_i\tau_j = \frac{1}{2}\mathbf{K}^{\mathrm{rb}}\sigma_i\sigma_j\tau_i\tau_j + \frac{1}{2}\mathbf{K}^{\mathrm{br}}\sigma_i\sigma_j\tau_i\tau_j$ with (of course) $\mathbf{K} = \mathbf{K}^{\mathrm{bb}} = \mathbf{K}^{\mathrm{rr}} = \mathbf{K}^{\mathrm{rb}} = \mathbf{K}^{\mathrm{br}}$. Now, in a double bond situation (which automatically implements the "bond splitting" option) one of these four terms is selected. The four choices are identified as double blue, double red, upper–red/lower–blue and upper–blue/lower–red i.e. the two edges of the split actually colored. The constraints are exactly as they were before: an even number of colors of each type at each vertex. With this interpretation in mind, it is clear that we once again have a bonified – and useful – colorless representation. Explicitly, for a given configuration, the graph becomes a graph with single edges and split edges. The weight is [a factor of $\mathbf{J}$ for each (occupied) single bond and a factor of $\frac{1}{2}\mathbf{K}$ for each (occupied) double] × [the number of ways that the configuration can be consistently colored]. Notice that in any coloring in which a particular double ends up red/blue, there is another coloring – regarded as distinct – with all other bonds the same but the particular bond a blue/red. Similarly for exchanging a red/red with a blue/blue (but notice that the mono-colored and di-colored doubles cannot be exchanged without forcing other changes in the configuration.) Thus, when all is said and done, we are back to the situation we had at $\mathbb{K} = 0$ – albeit on the peculiar graph defined by the splits.

As a consequence obtain, rather inexpensively,

Theorem 3.4. *Consider a finite graph with the Hamiltonian*

$$-\mathcal{H} = \sum_{\langle i,j\rangle} J_{i,j}[\sigma_i\sigma_j + \tau_i\tau_j] - \sum_{\langle i,j\rangle} U_{i,j}[\sigma_i\sigma_j\tau_i\tau_j]$$

with $\mathrm{th}^2\beta J_{i,j} \geq \mathrm{th}\,\beta U_{i,j} \geq 0$ *on every edge* $\langle i,j\rangle$. *Then*

$$\langle\sigma_i\sigma_j\sigma_k\sigma_\ell\rangle_{\beta\mathcal{H}} \leq \langle\sigma_i\sigma_j\rangle_{\beta\mathcal{H}}\langle\sigma_k\sigma_\ell\rangle_{\beta\mathcal{H}} + \langle\sigma_i\sigma_k\rangle_{\beta\mathcal{H}}\langle\sigma_j\sigma_\ell\rangle_{\beta\mathcal{H}} + \langle\sigma_i\sigma_\ell\rangle_{\beta\mathcal{H}}\langle\sigma_j\sigma_k\rangle_{\beta\mathcal{H}}.$$

and

$$\langle \sigma_i\sigma_j; [\tau_k\tau_\ell - \sigma_k\sigma_\ell\tau_k\tau_\ell]\rangle_{\beta\mathcal{H}} \leq 0$$

Proof. The second inequality is a direct consequence of the FKG property (that was established in Proposition 2.2) by following the argument in Theorem 3.3. Thus we are left with the extension of Theorem 3.2 to non-zero $\mathbb{K}$. Let us start by repeating Eq.(3.3) and the similar equation for the mixed four-point function without notational changes. Obviously $\tilde{\mathfrak{g}}_{[i,j|k,\ell]}$ still equals $\mathfrak{g}_{[i,j|k,\ell]}$ by blatant color-reversal symmetry. Of more significance, we claim that with the dud-bond representation, (and the ensuing interpretation) the extension of Lemma 3.1 is immediate. In particular the argument that was used to show $\Gamma_{i,j,k,\ell} = \tilde{\Gamma}_{i,j,k,\ell}$ (whereby some path connecting k to ℓ gets color reversed) still goes through – the formulation was for a general underlying graph and thus works for the graph defined with the splits. Given the identities $\Gamma_{i,j,k,\ell} = \tilde{\Gamma}_{i,j,k,\ell}$ and $\tilde{\mathfrak{g}}_{[i,j|k,\ell]} = \mathfrak{g}_{[i,j|k,\ell]}$ (as well as the FKG property) the desired result is manifest. □

As a bonus, we obtain a direct relationship between a correlation function and a loop probability:

Corollary I. *Consider the system as described in the statement of Theorem 3.4. Let $\mathcal{L}_{i,j} = \mathcal{L}_{i,j}(\beta\mathcal{H})$ denote the probability, in the loop representation, that the sites i and j belong to the same loop cluster. Then*

$$\mathcal{L}_{i,j} = \langle \sigma_i\tau_i\sigma_j\tau_j\rangle_{\beta\mathcal{H}}.$$

Proof. The numerator of the right-hand side requires a blue walk *and* a red walk between the sites i and j. Applying the color switching techniques described in Lemma 3.1 in the context of the dud-bond representation – as was done in Theorem 3.4 it is seen that each such term is in one-to-one correspondence with a term contributing to $\mathcal{L}_{i,j}$ and vice versa. □

Corollary II. *Consider a periodic graph on which there is Hamiltonian $\mathcal{H}$ (also periodic) of the type described above with $U_{i,j} \equiv \beta^{-1}\mathbf{A}_{i,j}(\beta)$ where the $\mathbf{A}_{i,j}(\beta)$ are smooth functions satisfying $\operatorname{th}\mathbf{A}_{i,j}(\beta) \leq \operatorname{th}^2\beta J_{i,j}$ as well as $\mathbf{A}'_{i,j}(\beta) \leq J_{i,j}$ and $\lim_{\beta\to\infty}\beta^{-1}\mathbf{A}_{i,j}(\beta) < J_{i,j}$. Here $\beta_0 < \beta < \infty$ with β_0 small enough to ensure that at $\beta = \beta_0$, the susceptibility is finite. Then there is a β_c along this trajectory where the susceptibility diverges continuously; in particular at least as fast as a constant times $(\beta_c - \beta)^{-1}$.*

Remark. One can presume that under the stated conditions every ferromagnetic transition in the region $U > 0$ is covered. Although this may not seem evident in the physical variables used above, it is quite transparent in the A–B variables that were featured in Section 2. For example, in the homogeneous case with $J \equiv 1$ and $\beta U \equiv \alpha$, let us set $B = \lambda A$ with $\lambda < 1$ and allow A to range in $[0,\infty)$. Then we find that $r_V = (1-\lambda)[A/(1+A)]^2$ and, in general, $\alpha' = r_V/[1-r_V(1+A)]$. Thus $\alpha' < 1$ because the worst case is the one with the biggest value of r_V (namely $\lambda = 0$) which is the borderline case discussed in Theorem 3.3. Now starting at $A = 0$ there is exponential clustering for all A small enough. But for A large and $\lambda > 0$ the graphical representation discussed in Section 2 has percolation (in both layers) which implies magnetization. (These facts can be verified by simple-minded comparisons to independent percolation.) Thus, along these curves, there is always a point where the susceptibility is infinite. So, for all intents and purposes, this region has a phase boundary separating $\beta = 0$ from $\beta = \infty$. For negative values of B (i.e. $\lambda < 0$)

the ground states are not ferromagnetic and it seems highly unlikely that ferromagnetism could somenow be generated by entropic effects.

Proof. The above hypotheses are sufficient to derive bounds of the form $\mathrm{d}\mathcal{X}/\mathrm{d}\beta \leq [\text{const.}]\mathcal{X}^2$ (suitably regularized in finite volume if necessary). Furthermore, they are sufficient to ensure a ferromagnetic transition at finite β (c.f. the above remark) and hence $\mathcal{X}$ is actually infinite somewhere. The combination of these ingredients implies the divergence of the susceptibility with the stated bound. $\square$

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