Hydrodynamics for totally asymmetric k-step exclusion processes

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Abstract

We describe the hydrodynamic behavior of the k-step exclusion process. Since the flux appearing in the hydrodynamic equation for this particle system is neither convex nor concave, the set of possible solutions include in addition to entropic shocks and continuous solutions those with contact discontinuities. We finish with a limit theorem for the tagged particle.

1 Introduction and Notation

In his paper Liggett (1980) introduced a Feller non conservative approximation of the long range exclusion process to study the latter. A conservative version of this dynamics, called k-step exclusion process was defined and studied in Guiol (1999). It is described in the following way.

Let $k \in \mathbb{N}^* := \{1, 2, ...\}$, $\mathbf{X} := \{0,1\}^{\mathbb{Z}}$ be the state space, and let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain on $\mathbb{Z}$ with transition matrix $p(.,.)$ and $\mathbf{P}^x(X_0 = x) = 1$. Under the mild hypothesis $\sup_{y\in\mathbb{Z}} \sum_{x\in\mathbb{Z}} p(x,y) < +\infty$, L_k, defined below, is an infinitesimal pregenerator: For all cylinder function f,

$$L_k f(\eta) = \sum_{\eta(x)=1,\eta(y)=0} q_k(x,y,\eta)\left[f(\eta^{x,y}) - f(\eta)\right], \tag{1}$$

where $q_k(x,y,\eta) = \mathbf{E}^x\left[\prod_{i=1}^{\sigma_y-1}\eta(X_i), \sigma_y \le \sigma_x, \sigma_y \le k\right]$ is the intensity for moving from x to y on configuration η, $\sigma_y = \inf\{n \ge 1 : X_n = y\}$ is the first (non

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zero) arrival time to site y of the chain starting at site x and $\eta^{x,y}$ is configuration η where the states of sites x and y were exchanged.

In words if a particle at site x wants to jump it may go to the first empty site encountered before returning to site x following the chain X_n (starting at x) provided it takes less than k attempts; otherwise the movement is cancelled.

By Hille-Yosida's theorem, the closure of L_k generates a continuous Markov semi-group $S_k(t)$ on $C(\mathbf{X})$ which corresponds to the k-step exclusion process $(\eta_t)_{t\geq 0}$. Notice that when $k=1$, $(\eta_t)_{t\geq 0}$ is the simple exclusion process. An important property of k-step exclusion is that it is an *attractive process*.

Let $\mathcal{I}_k$ be the set of invariant measures for $(\eta_t)_{t\geq 0}$ and let $\mathcal{S}$ be the set of translation invariant measures on $\mathbf{X}$. If $p(x,y)=p(0,y-x)$ for all $x,y\in\mathbb{Z}$ and $p(.,.)$ is irreducible then

$$(\mathcal{I}_k \cap \mathcal{S})_e = \{\nu_\alpha : \alpha \in [0,1]\}\,,$$

where the index e mean extremal and ν_α is the Bernoulli product measure with constant density α, *i.e.* the measure with marginal

$$\nu_\alpha\{\eta \in \mathbf{X} : \eta(x)=1\} = \alpha.$$

In this paper we prove conservation of local equilibrium for the totally asymmetric process in the Riemann case *i.e.*: $p(x,x+1)=1$ for all $x\in\mathbb{Z}$ and the initial distribution is a product measure with densities λ to the left of the origin and ρ to its right, we denote it by $\mu_{\lambda,\rho}$. The derived equation involves a flux which is neither concave nor convex and appears for the first time as a hydrodynamic limit of an interacting particle system. Up to now the "constructive" proofs for hydrodynamics relied on the concavity of the flux, see Andjel & Vares (1987) or the papers by Seppäläinen (*e.g.* Seppäläinen (1998)), whose key tool is the Lax-Hopf formula. The entropy solution for the type of equation we consider was first studied in Ballou (1970). Such solutions can have entropy shocks as well as contact discontinuities. Our aim is to take advantage of Ballou's result to deduce conservation of local equilibrium also in a constructive way, in the spirit of Andjel & Vares (1987). However we explain in the last section how to derive hydrodynamics for general initial profiles, and nearest neighbor dynamics.

In section 3 we prove a law of large numbers for a tagged particle in a k-step exclusion process.

2 The hydrodynamic equation

2.1 Heuristic derivation of the equation

Since the process is (totally) asymmetric we take Euler scaling. For every $r\in\mathbb{R}$ define $\eta_t^\varepsilon(r) := \eta_{\varepsilon^{-1}t}([\varepsilon^{-1}r])$, where $[\varepsilon^{-1}r]$ is the integer part of $\varepsilon^{-1}r$, for $\varepsilon>0$.

Given a continuous function $u^0(x), x\in\mathbb{R}$ (initial density profile), we define a family of Bernoulli product measures $\{\nu_{u^0}^\varepsilon\}_{\varepsilon>0}$ on $\mathbf{X}$, by: For all $x\in\mathbb{Z}$,

$$\nu_{u^0}^\varepsilon\{\eta\in\mathbf{X} : \eta(x)=1\} = u^0(\varepsilon x).$$

We call $\{\nu^\varepsilon_{u^0}\}$ the family of measures determined by the profile u^0. Let $u^\varepsilon(r,t) := \int (S_k(\varepsilon^{-1}t)\eta^\varepsilon_0(r))d\nu^\varepsilon_{u^0}(\eta_0)$. Then for all $r \in \mathbb{R}$, $u^\varepsilon(r,0)$ converges to $u^0(r)$ when ε goes to 0; applying the generator (1) to $\eta^\varepsilon_t(r)$ we have

$$\frac{d}{dt}S_k(t\varepsilon^{-1})\,(\eta^\varepsilon_0(r)) = \varepsilon^{-1}S_k(t\varepsilon^{-1})\left[-\sum_{i=0}^{k-1}\prod_{j=0}^{i}\eta^\varepsilon_0(r+j\varepsilon)\,[1-\eta^\varepsilon_0\,(r+(i+1)\varepsilon)]\right.$$

$$\left.+\sum_{i=0}^{k-1}\prod_{j=0}^{i}\eta^\varepsilon_0(r-j\varepsilon)\,[1-\eta^\varepsilon_0(r)]\right].$$

Assuming that local equilibrium is preserved (thus expectation of products factor), and taking expectations with respect to $\nu^\varepsilon_{u^0}$, we obtain

$$\frac{\partial u^\varepsilon}{\partial t}(r,t) = \varepsilon^{-1}\left[-\sum_{i=0}^{k-1}\prod_{j=0}^{i}u^\varepsilon(r+j\varepsilon,t)\,[1-u^\varepsilon\,(r+(i+1)\varepsilon,t)]\right.$$

$$\left.+\sum_{i=0}^{k-1}\prod_{j=0}^{i}u^\varepsilon(r-j\varepsilon,t)\,[1-u^\varepsilon(r,t)]\right].$$

If we now let ε converge to 0, $u(r,t) := \lim_{\varepsilon\to 0} u^\varepsilon(r,t)$ should satisfy

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial G_k(u)}{\partial x} = 0 \\ u(x,0) = u^0(x), \end{cases} \tag{2}$$

where G_k represents the flux of particles:

$$G_k(u) = \sum_{j=1}^{k} ju^j(1-u).$$

This is a non standard form because G_k is neither convex nor concave, thus equation (2) is no longer “a genuinely nonlinear conservation law”, using the language of Lax (1973). To deal with this equation, we have to use an extended version of non linear Cauchy problems treated by Ballou (1970).

Remark 2.1

Let k go to infinity and denote by G_∞ the limiting flux function:

$$G_\infty(u) = \frac{u}{1-u}.$$

That case corresponds to the totally asymmetric long range exclusion process. The resulting equation is simpler because the flux function G_∞ is strictly convex. The hydrodynamics in this case should follow from the arguments of Aldous & Diaconis (1995) for the Hammersley's process.

For notational simplicity, from now on we restrict ourselves to the case $k = 2$. However our arguments can be easily extended for all k.

2.2 Hydrodynamics in the Riemann case

2.2.1 Notation and result

Our main theorem characterizes the hydrodynamic (Euler) limit of the 2-step exclusion process at points of continuity, when the family of initial measures is determined by a step function profile. From the heuristic derivation we would expect that in the hydrodynamic limit the density profile would satisfy equation (2). We show that the limiting density profile at time t is the entropy solution of equation (2) starting with the initial value u^0, a step function profile. We now give a brief summary of results concerning the solution of equation (2), due to D.P. Ballou (1970), when u^0 is a step function. This will motivate the formulation of the theorem as well as some aspects of the proof.

Existence of weak solution to the Cauchy problem given by equation (2) with bounded measurable initial condition was proved in Ballou (1970), under the assumptions:

1. $G_k \in C^2(\mathbb{R})$.
2. G_k'' vanishes at a finite number of points and changes sign at these points.

In order to obtain uniqueness further conditions are needed. We require our solutions to satisfy the:

Condition E: (O.A. Oleĭnik)

Let $x(t)$ be any curve of discontinuity of the weak solution $u(t,x)$, and let v be any number lying between $u^- := u(t,x(t)-0)$ and $u^+ := u(t,x(t)+0)$. Then except possibly for a finite number of t,

$$S[v;u^-] \geq S[u^+;u^-],$$

where

$$S[v;w] := \frac{G_k(w)-G_k(v)}{w-v}.$$

It is known (Ballou (1970)) that the following two conditions are necessary and sufficient for a piecewise smooth function $u(x,t)$ to be a weak solution of equation (2):

1. $u(x,t)$ solves equation (2) at points of smoothness.

2.If $x(t)$ is a curve of discontinuity of the solution then the Rankine-Hugoniot condition (*i.e.* $d(x(t))/dt = S[u^+;u^-]$) holds along $x(t)$.

Moreover condition E is sufficient to ensure the uniqueness of piecewise smooth solutions, which are the entropy solutions to the equation. Hereafter we only deal with the case $k=2$. We denote $G(u) := G_2(u)$.

If G were convex (concave) only two types of solutions would be possible. We now describe these two types of solutions.

Let $u^0(x) = \lambda 1_{\{x<0\}} + \rho 1_{\{x\geq 0\}}$.

If $\lambda > \rho$ ($\rho > \lambda$), then the speed of characteristics which start from $x \leq 0$ (given by G') is greater than speed of characteristics which start from $x > 0$. If the intersection of characteristics occurs along a curve $x(t)$, then since

$$S[u^+;u^-] = \frac{G(\lambda)-G(\rho)}{\lambda-\rho} = S[\lambda;\rho]$$

Rankine-Hugoniot condition will be satisfied if $x'(t) = S[\lambda;\rho]$. Thus

$$u(x,t) = \begin{cases} \lambda, & x \leq S[\lambda;\rho]t; \\ \rho, & x > S[\lambda;\rho]t. \end{cases}$$

is a weak solution. The convexity of G implies that condition E is satisfied across $x(t)$. Therefore $u(x,t)$ defined above is the unique entropic solution in this case and will be referred to as a shock in the sense of Lax (1973).
If $\lambda < \rho \quad (\rho < \lambda)$, then the characteristics starting respectively from $x \leq 0$ and from $x > 0$ never meet. Moreover they never enter the space-time wedge between lines $x = \lambda t$ and $x = \rho t$. We can choose values in this region to obtain a continuous solution, the so-called *continuous solution with a rarefaction fan*: Let h be the inverse of $H := G'$,

$$u(x,t) = \begin{cases} \lambda, & x \leq H(\lambda)t; \\ h(x/t), & H(\lambda)t < x \leq H(\rho)t; \\ \rho, & H(\rho)t < x. \end{cases}$$

It is possible to define piecewise smooth weak solutions with a jump occurring in the wedge satisfying the Rankine-Hugoniot condition. But the convexity of G prevents such solutions to satisfy condition E. Thus the continuous solution with a rarefaction fan is the unique entropic solution in this case.

For the 2-step exclusion process, the flux function G is neither concave nor convex. Instead $G(u) = u + u^2 - 2u^3$ is convex for $u < 1/6$ and concave for $u > 1/6$. In this case in addition to the shock and continuous solution with a rarefaction fan it is possible to have solutions for which the curve of discontinuities never enters the region of intersecting characteristics. The quotation in boldface is the original number of lemmas, prop... in Ballou (1970), but the notation refers to 2-step exclusion:

Definition 2.2 [**B def2.1**]
For any $u < 1/6$, define $u^* := u^*(u)$ as

$$u^* = \sup\{\eta > u : S[u;\eta] > S[v;u] \ \forall v \in (u,\eta)\}.$$

For any $u > 1/6$, define $u_* := u_*(u)$ as

$$u_* = \inf\{\eta < u : S[u;\eta] > S[v;u] \ \forall v \in (\eta,u)\}.$$

In other words, for $u < 1/6$, if we consider the upper convex envelope G^c of G on $(u,+\infty)$, then u^* is the first point where G^c coincides with G. In the same way when $u > 1/6$, u_* is the first point where the lower convex envelope G_c of G on $(-\infty,u)$ coincides with G. For $\eta < 1/6$, $\eta^* = (1-2\eta)/4$, and for $\eta > 1/6$, $\eta_* = (1-2\eta)/4$.

Let h_1 and h_2 be the inverses of H respectively restricted to $(-\infty,1/6)$ and to $(1/6,+\infty)$, *i.e.* $h_1(x) = (1/6)(1-\sqrt{7-6x})$ and $h_2(x) = (1/6)(1+\sqrt{7-6x})$ for $x \in (-\infty,7/6)$.

The following lemmas are taken from Ballou (1970).

Lemma 2.3 **[B lem2.2]** *Let $\eta < 1/6$ be given, and suppose that $\eta^* < \infty$. Then $S[\eta;\eta^*] = H(\eta^*)$.*

Lemma 2.4 **[B lem2.4]** *Let $\eta < 1/6$ be given, and suppose that $\eta^* < \infty$. Then η^* is the only zero of $S[u;\eta] - H(u)$, $u > \eta$.*

If $\lambda < \rho < 1/6$, the relevant part of the flux function is convex and the unique entropic weak solution is the continuous solution with a rarefaction fan.

If $\rho < \lambda < \rho^*$ ($\rho < 1/6$), then $H(\eta) > H(\rho)$ if $\rho < \eta \leq 1/6$, and $H(\eta) > H(\rho^*) > H(\rho)$ if $1/6 < \eta < \rho^*$ since H is decreasing in this region. Thus $H(\lambda) > H(\rho)$, which implies an intersection of characteristics: The unique entropic weak solution is the shock.

Let $\rho < \rho^* < \lambda$ ($\rho < 1/6$): Lemma 2.4 applied to ρ suggests that a jump from ρ^* to ρ along the line $x = H(\rho^*)t$ will satisfy the Rankine-Hugoniot condition. Since ρ^* is specially defined for this, a solution with such a jump will also satisfy condition E. Therefore if we can construct a solution with the jump described above it will be the unique entropic weak solution in this case. Notice that since $H(\lambda) < H(\rho^*)$, no characteristics intersect along the line of discontinuity $x = H(\rho^*)t$. We call this case *contact discontinuity*, following Ballou. The solution is defined by

$$u(x,t) = \begin{cases} \lambda, & x \leq H(\lambda)t; \\ h_2(x/t), & H(\lambda)t < x \leq H(\rho^*)t; \\ \rho, & H(\rho^*)t < x. \end{cases}$$

Corresponding cases on the concave side of G are treated similarly.

Let τ denote the shift operator. We are able now to state our result.

Theorem 2.5 *Let $v \in \mathbb{R}$, $\lambda, \rho \neq 1/6$, and $\mu_{\lambda,\rho}$ the Bernoulli product measure on $\mathbb{Z}$ with densities λ for $x \leq 0$ and ρ for $x > 0$. Then*

$$\lim_{t\to\infty} \mu_{\lambda,\rho}\tau_{[vt]}S_2(t) = \nu_{u(v,1)}$$

at every continuity point of $u(.,1)$, where $\nu_{u(v,1)}$ denotes the product measure with density $u(v,1)$ defined by:

Case 1. $\lambda < \rho < 1/6$: continuous solution, with a rarefaction fan

$$u(x,1) = \begin{cases} \lambda, & x \leq H(\lambda); \\ h_1(x), & H(\lambda) < x \leq H(\rho); \\ \rho, & H(\rho) < x. \end{cases}$$

Case 2. $\rho < \lambda < \rho^*$, ($\rho < 1/6$): entropy shock

$$u(x,1) = \begin{cases} \lambda, & x \leq S[\lambda;\rho]; \\ \rho, & x > S[\lambda;\rho]. \end{cases}$$

Case 3. $\rho < \rho^* < \lambda$, ($\rho < 1/6$): contact discontinuity

$$u(x,1) = \begin{cases} \lambda, & x \leq H(\lambda); \\ h_2(x), & H(\lambda) < x \leq H(\rho^*); \\ \rho, & H(\rho^*) < x. \end{cases}$$

Case 4. $1/6 < \rho < \lambda$*: continuous solution, with a rarefaction fan*

$$u(x,1) = \begin{cases} \lambda, & x \leq H(\lambda); \\ h_2(x), & H(\lambda) < x \leq H(\rho); \\ \rho, & H(\rho) < x. \end{cases}$$

Case 5. $\rho > \lambda > \rho_*$, *(*$\rho > 1/6$*): entropy shock*

$$u(x,1) = \begin{cases} \lambda, & x \leq S[\lambda;\rho]; \\ \rho, & x > S[\lambda;\rho]. \end{cases}$$

Case 6. $\rho > \rho_* > \lambda$, *(*$\rho > 1/6$*): contact discontinuity*

$$u(x,1) = \begin{cases} \lambda, & x \leq H(\lambda); \\ h_1(x), & H(\lambda) < x \leq H(\rho_*); \\ \rho, & H(\rho_*) < x. \end{cases}$$

Remark 2.6

For any $k \geq 2$ the profiles will be of the same kind, because G_k has only one inflection point between 0 and 1 and is first convex then concave.

Remark 2.7

Comparing with hydrodynamics of simple exclusion we observe that k-step exclusion ($k \geq 2$) has not only a stable increasing shock (Case 5) and a decreasing continuous solution (Case 1) but also a stable decreasing shock (Case 2), an increasing continuous solution (Case 4) and two contact discontinuities (Cases 3 and 6).

2.2.2 Proof of Theorem 2.5

It follows the scheme introduced in Andjel & Vares (1987), where the authors obtained the hydrodynamic limit for the one-dimensional zero-range process in the Riemann case, i.e. the hydrodynamic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial \phi(u)}{\partial x} = 0 \\ \\ u(x,0) = u^0(x) = \lambda 1_{\{x<0\}} + \rho 1_{\{x \geq 0\}} \end{cases}$$

was derived. There ϕ, the mean flux of particles through the origin, was a concave function. Therefore, their proof used both the monotonicity of the process, still valid here, and the concavity of the flux, that we have to replace by an *ad hoc* use of the properties of the solution of (2).

Informally speaking, they first showed that a weak Cesáro limit of (the measure of) the process is an invariant and translation invariant measure. Then they showed that the (Cesáro) limiting density inside a macroscopic box is equal to the difference of the edge values of a flux function. These propositions were based on monotonicity, and on the characterization of invariant and translation invariant measures (both valid for k-step as well),thus we can quote them (with appropriate notation for the k-step), and take them for granted.

Lemma 2.8 [**AV 3.1**] *Let μ be a probability measure on $\{0,1\}^{\mathbb{Z}}$ such that*
(a) $\nu_\rho \le \mu \le \nu_\lambda$ for some $0 \le \rho < \lambda \le 1$, (b) either $\mu\tau_1 \le \mu$ or $\mu\tau_1 \ge \mu$. Then any sequence $T_n \to \infty$ has a subsequence T_{n_k} for which there exists D dense (countable) subset of $\mathbb{R}$ such that for each $v \in D$,

$$\lim_{k\to\infty} \frac{1}{T_{n_k}} \int_0^{T_{n_k}} \mu\tau_{[vt]} S_2(t)dt = \mu_v$$

for some $\mu_v \in \mathcal{I}_2 \cap \mathcal{S}$.

Lemma 2.9 [**AV 3.2**] *For $v \in D$, we can write $\mu_v = \int \nu_\alpha \gamma_v(d\alpha)$, where γ_v is a probability on $[\rho, \lambda]$. Also, if $u < v$ are in D,*

$$\lim_{k\to\infty} \mu S_2(T_{n_k}) \left(\frac{1}{T_{n_k}} \sum_{[uT_{n_k}]}^{[vT_{n_k}]} \eta(x) \right) = F(v) - F(u) \tag{3}$$

with, for $w \in D$, $F(w) = \int [w\alpha - G(\alpha)]\gamma_w(d\alpha)$.

The difficult part is then to prove that γ_v is in fact the Dirac measure concentrated on $u(x,1)$. They did it in Lemma [AV 3.3] and Theorem [AV 2.10] using the concavity of their flux function.

For k-step exclusion, at this point we will have to look separately at the six cases given in the theorem.

To conclude, Andjel & Vares had to prove that the Cesáro limit implies a weak limit, through the following propositions, based on monotonicity. We will use these results also without proof.

Proposition 2.10 [**AV 3.4**] *Let $\mu = \mu_{\lambda,\rho}$. If*

$$\mu_v = \begin{cases} \nu_\lambda, & \text{if } v \in D,\ v < S[\lambda;\rho] \\ \nu_\rho, & \text{if } v \in D,\ v > S[\lambda;\rho] \end{cases}$$

then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mu\tau_{[vt]} S_2(t)dt = \mu_v.$$

Proposition 2.11 [**AV 3.5**] *If μ satisfies*
(a) $\mu \le \nu_\lambda$, (b) $\mu\tau_1 \ge \mu$, (c) there exists v_0 finite so that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mu\tau_{[vt]} S_2(t)dt = \nu_\lambda$$

for all $v > v_0$. Then

$$\lim_{t\to\infty} \mu\tau_{[vt]} S_2(t) = \nu_\lambda \quad \text{for all } v > v_0.$$

Proof of Theorem 2.5 in Cases 2 and 3. In case 1, the proof is not different from the one given in Andjel & Vares (1987) since G is convex in the relevant region; thus we omit it. In case 2, G is not convex in the relevant region, therefore we supply a proof, though it is quite close to the original one. Case 3 uses ideas from cases 1 and 2 and introduces some new ideas to deal with complications arising from the non-convexity of G. Cases 4-6 are symmetric to cases 1-3 in the sense that the roles played by convexity and concavity are exchanged.

The proof has 3 steps. The main ingredients are monotonicity, and inequalities relying on the properties of $S[.;.]$, G, H given before.
First step.

Using monotonicity of the 2-step exclusion, we can proceed as in the beginning of the proof of Lemma [AV3.3], and get two finite values $\underline{v}$ and $\bar{v}$ so that: If $v \in D$ and $v > \bar{v}$, then $\gamma_v = \delta_\rho$, while $\gamma_v = \delta_\lambda$ if $v < \underline{v}$.

Second step, preliminary.

Let $u < v$, both in D. Attractiveness of the process (since $\nu_\rho \leq \mu_{\lambda,\rho} \leq \nu_\lambda$) and (3) imply

$$(v-u)\rho \leq v\int \alpha\gamma_v(d\alpha) - \int G(\alpha)\gamma_v(d\alpha) - u\int \alpha\gamma_u(d\alpha) + \int G(\alpha)\gamma_u(d\alpha) \leq (v-u)\lambda \tag{4}$$

(i) Taking $u < \underline{v}$, the first step gives $\gamma_u = \delta_\lambda$, so the second inequality of (4) is simplified in

$$v\int \alpha\gamma_v(d\alpha) - \int G(\alpha)\gamma_v(d\alpha) - u\lambda + G(\lambda) \leq (v-u)\lambda$$

which can be written

$$\int_{[\rho,\lambda]} (G(\lambda) - G(\alpha))\gamma_v(d\alpha) \leq v\int_{[\rho,\lambda]} (\lambda - \alpha)\gamma_v(d\alpha) \tag{5}$$

(ii) Similarly, for $\bar{v} < v$, $\gamma_v = \delta_\rho$, and the first inequality of (4) reads

$$u\int_{[\rho,\lambda]} (\alpha - \rho)\gamma_u(d\alpha) \leq \int_{[\rho,\lambda]} (G(\alpha) - G(\rho))\gamma_u(d\alpha) \tag{6}$$

Proof for Case 2: $\rho < \lambda < \rho^*$, $\rho < 1/6$.

The definition of ρ^* implies that for every $\alpha \in (\rho, \lambda)$

$$\frac{G(\lambda) - G(\alpha)}{\lambda - \alpha} \geq \frac{G(\lambda) - G(\rho)}{\lambda - \rho} = S[\lambda;\rho],$$

so that inequality (5) for $\underline{v} \leq v < S[\lambda;\rho]$ ($v \in D$) yields

$$v\int_{[\rho,\lambda]} (\lambda - \alpha)\gamma_v(d\alpha) \geq S[\lambda;\rho]\int_{[\rho,\lambda]} (\lambda - \alpha)\gamma_v(\alpha).$$

Since $v < S[\lambda;\rho]$ we conclude that $\gamma_v = \delta_\lambda$.

Starting from inequality (6) with $\bar v \geq u > S[\lambda;\rho]$ ($u \in D$) and proceeding in a similar manner we can show that $\gamma_u = \delta_\rho$.

Proof for case 3: $\rho < \rho^* < \lambda$, $\rho < 1/6$.

Second step, part 1.

Let $u < v$, both in D, $u < \underline{v}$, and $v < H(\lambda)$. On $[\rho^*, \lambda)$, G is concave, thus for every $\alpha \in [\rho^*, \lambda]$

$$H(\lambda) < S[\alpha;\lambda] \leq H(\rho^*). \tag{7}$$

If $\alpha \in [\rho, \rho^*)$ then from the definition of ρ^* it follows that

$$S[\alpha;\rho^*] \geq H(\rho^*). \tag{8}$$

We decompose $[\rho,\lambda] = [\rho,\rho^*] \cup (\rho^*,\lambda]$, and (5) becomes

$$\begin{aligned} v\int_{[\rho,\rho^*]}(\lambda-\rho^*)\gamma_v(d\alpha) + v\int_{[\rho,\rho^*]}(\rho^*-\alpha)\gamma_v(d\alpha) + v\int_{(\rho^*,\lambda]}(\lambda-\alpha)\gamma_v(d\alpha) \geq \\ \int_{[\rho,\rho^*]}(G(\lambda)-G(\alpha))\gamma_v(d\alpha) + \int_{(\rho^*,\lambda]}(G(\lambda)-G(\alpha))\gamma_v(d\alpha). \end{aligned} \tag{9}$$

By (8)

$$\int_{[\rho,\rho^*]}(G(\lambda)-G(\alpha))\gamma_v(d\alpha) = \int_{[\rho,\rho^*]}\left[(G(\lambda)-G(\rho^*)+G(\rho^*)-G(\alpha)\right]\gamma_v(d\alpha)$$

$$\geq \int_{[\rho,\rho^*]}\left[(\lambda-\rho^*)S[\rho^*;\lambda] + H(\rho^*)(\rho^*-\alpha)\right]\gamma_v(d\alpha)$$

and by (7)

$$\int_{(\rho^*,\lambda]}(G(\lambda)-G(\alpha))\gamma_v(d\alpha) \geq H(\lambda)\int_{(\rho^*,\lambda]}(\lambda-\alpha)\gamma_v(d\alpha).$$

Those two inequalities together with (9) give

$$(v - S[\rho^*;\lambda])(\lambda-\rho^*)\int_{[\rho,\rho^*]}\gamma_v(d\alpha) + (v - H(\rho^*))\int_{[\rho,\rho^*]}(\rho^*-\alpha)\gamma_v(d\alpha)$$

$$+(v-H(\lambda))\int_{(\rho^*,\lambda]}(\lambda-\alpha)\gamma_v(d\alpha) \geq 0.$$

Since $v < H(\lambda)$, by (7), the only possibility is $\gamma_v = \delta_\lambda$.

Second step, part 2.

Let $u, v \in D$, such that $v > \bar v$, and $H(\rho^*) < u < v$. We decompose each integral of (6) on the two intervals $[\rho,\rho^*]$ and $(\rho^*,\lambda]$.

$$\int_{(\rho^*,\lambda]}(G(\alpha)-G(\rho))\gamma_u(d\alpha)$$

$$= \int_{(\rho^*,\lambda]} (G(\alpha) - G(\rho^*))\gamma_u(d\alpha) + \int_{(\rho^*,\lambda]} (G(\rho^*) - G(\rho))\gamma_u(d\alpha)$$

By definition of ρ^* and by Lemma 2.3, for all $\alpha \in [\rho, \rho^*]$,

$$G(\alpha) - G(\rho) \leq H(\rho^*)(\alpha - \rho),$$

and G being strictly concave on $(\rho^*, \lambda]$, we have for all $\alpha \in [\rho^*, \lambda]$

$$G(\alpha) - G(\rho^*) \leq H(\rho^*)(\alpha - \rho^*).$$

Thus by (6)

$$\begin{aligned} u \int_{[\rho,\rho^*]} (\alpha - \rho)\gamma_u(d\alpha) + u \int_{(\rho^*,\lambda]} (\alpha - \rho^*)\gamma_u(d\alpha) + u \int_{(\rho^*,\lambda]} (\rho^* - \rho)\gamma_u(d\alpha) \\ \leq \int_{[\rho,\rho^*]} H(\rho^*)(\alpha - \rho)\gamma_u(d\alpha) + \int_{(\rho^*,\lambda]} H(\rho^*)(\alpha - \rho^*)\gamma_u(d\alpha) \\ + \int_{(\rho^*,\lambda]} H(\rho^*)(\rho^* - \rho)\gamma_u(d\alpha) \end{aligned}$$

which can be written

$$\begin{aligned} (u - H(\rho^*)) \Bigg[\int_{[\rho,\rho^*]} (\alpha - \rho)\gamma_u(d\alpha) + \int_{(\rho^*,\lambda]} (\alpha - \rho^*)\gamma_u(d\alpha) \\ + \int_{(\rho^*,\lambda]} (\rho^* - \rho)\gamma_u(d\alpha) \Bigg] \leq 0. \end{aligned}$$

Since $u - H(\rho^*) > 0$, this implies

$$\int_{[\rho,\lambda]} (\alpha - \rho)\gamma_u(d\alpha) = 0$$

Therefore we conclude $\gamma_u = \delta_\rho$.

Second step, conclusion.

Using the two preceding parts, attractivity, Propositions 2.10 and 2.11, we conclude in case 3

$$\lim_{t\to\infty} \mu_{\lambda,\rho}\tau_{[vt]}S_2(t) = \begin{cases} \nu_\lambda, & if\ v < H(\lambda) \\ \nu_\rho, & if\ v > H(\rho^*) \end{cases} \tag{10}$$

and in case 2

$$\lim_{t\to\infty} \mu_{\lambda,\rho}\tau_{[vt]}S_2(t) = \begin{cases} \nu_\lambda, & if\ v < S[\lambda;\rho] \\ \nu_\rho, & if\ v > S[\lambda;\rho] \end{cases} \tag{11}$$

Third step, first part.

Let $u_1, v, v_1 \in D$, $u_1 < H(\lambda) < v < H(\rho^*) < v_1$, such that u_1 and v_1 belong to an interval where G is concave. By attractivity,

$$\limsup_{t\to\infty} \mu_{\lambda,\rho} S_2(t) \left(\frac{1}{t} \sum_{[u_1 t]}^{[vt]} \eta(x) \right) \leq (v - u_1)\lambda \tag{12}$$

The second step and (3) imply

$$\lim_{t\to\infty} \mu_{\lambda,\rho} S_2(t) \left(\frac{1}{t} \sum_{[u_1 t]}^{[v_1 t]} \eta(x) \right) = v_1\rho - G(\rho) - u_1\lambda + G(\lambda)$$

which, combined with (12) gives

$$\liminf_{t\to\infty} \mu_{\lambda,\rho} S_2(t) \left(\frac{1}{t} \sum_{[vt]}^{[v_1 t]} \eta(x) \right) \geq v_1\rho - G(\rho) + G(\lambda) - v\lambda \tag{13}$$

Since $H(\lambda) < v < H(\rho^*)$, there exists some θ, $\rho^* < \theta < \lambda$, with $v = H(\theta)$, so that $\theta = h_2(v)$. Let $\theta < \theta' < \lambda$, we now apply (13) to $\mu_{\theta',\rho}$, and we use attractivity through $\mu_{\theta',\rho} \leq \mu_{\lambda,\rho}$ to get

$$\begin{aligned} \liminf_{t\to\infty} \mu_{\lambda,\rho} S_2(t) \left(\frac{1}{t} \sum_{[vt]}^{[v_1 t]} \eta(x) \right) &\geq \liminf_{t\to\infty} \mu_{\theta',\rho} S_2(t) \left(\frac{1}{t} \sum_{[vt]}^{[v_1 t]} \eta(x) \right) \\ &\geq v_1\rho - G(\rho) + G(\theta') - v\theta' \end{aligned} \tag{14}$$

By (3) again,

$$\liminf_{t\to\infty} \mu_{\lambda,\rho} S_2(t) \left(\frac{1}{t} \sum_{[vt]}^{[v_1 t]} \eta(x) \right) \leq v_1\rho - G(\rho) - v \int_{[\rho,\lambda]} \alpha \gamma_v(d\alpha) + \int_{[\rho,\lambda]} G(\alpha)\gamma_v(d\alpha)$$

which, together with (14), if we make $\theta' \to \theta$, gives

$$G(\theta) - v\theta \leq \int_{[\rho,\lambda]} (G(\alpha) - v\alpha)\gamma_v(d\alpha) \tag{15}$$

Since $G(\alpha)$ is convex when $\alpha \in (\rho, 1/6)$ and concave when $\alpha \in (1/6, \lambda)$, for all v, $G(\alpha) - v\alpha$ has at most two critical points determined by the condition $G'(\alpha) = v$. One satisfies $1/6 < \alpha < \lambda$ and is a local maximum, the other (when it exists) is a local minimum with $\rho < \alpha < 1/6$. We want to conclude that the local maximum is a global maximum when $\rho < \alpha < \lambda$. We know from Lemma (2.3) that

$$G(\rho^*) - G(\rho) = (\rho^* - \rho)H(\rho^*)$$

which implies, since $v = H(\theta) < H(\rho^*)$,

$$G(\rho^*) - v\rho^* > G(\rho) - v\rho$$

Because $G(\alpha) - v\alpha$ is increasing in (ρ^*, θ) (recall that $H(\theta) = v$) we have: For all $\alpha \in (\rho^*, \theta)$

$$G(\alpha) - v\alpha > G(\rho^*) - v\rho^* > G(\rho) - v\rho$$

We conclude that θ is a global maximum:

$$\max_{\rho \leq \alpha \leq \lambda} [G(\alpha) - v\alpha] = G(\theta) - v\theta$$

thus $\gamma_v = \delta_\theta = \delta_{h_2(v)}$.

Third step, second part.

This part follows closely the argument in Andjel & Vares (1987), but we detail it for the sake of completeness. Since the measures $\frac{1}{T}\int_0^T \mu_{\lambda,\rho}\tau_{[vt]}S_2(t)\ dt$ depend monotonically on v and form a relatively compact set, we have for all v,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mu_{\lambda,\rho}\tau_{[vt]}S_2(t)\ dt = \nu_{h_2(v)}.$$

It remains to prove that

$$\lim_{t\to\infty} \mu_{\lambda,\rho}\tau_{[vt]}S_2(t) = \nu_{h_2(v)}$$

when $H(\lambda) < v < H(\rho^*)$ (by continuity of h_2, this result will also be valid at $H(\lambda)$). For this, let $\tilde{\mu}_v$ be a weak limit of $\mu_{\lambda,\rho}\tau_{[vt]}S_2(t)$. It is enough to show

$$(a)\ \tilde{\mu}_v \geq \nu_{h_2(v)}, \qquad (b)\ \tilde{\mu}_v(\eta(0)) = h_2(v).$$

(a) Let $\theta = h_1(v)$, with $H(\lambda) < v < H(\rho^*)$, $v = H(\theta)$. Let $\rho^* < \tilde{\theta} < \theta < \lambda$, then $v < H(\tilde{\theta})$ and $\mu_{\tilde{\theta},\rho}\tau_{[vt]}S_2(t) \leq \mu_{\lambda,\rho}\tau_{[vt]}S_2(t)$. Added to (10), this yields

$$\lim_{t\to\infty} \mu_{\tilde{\theta},\rho}\tau_{[vt]}S_2(t) = \nu_{\tilde{\theta}} \leq \lim_{t\to\infty} \mu_{\lambda,\rho}\tau_{[vt]}S_2(t) = \tilde{\mu}_v.$$

Hence, by continuity, if $\tilde{\theta}$ converges to θ,

$$\tilde{\mu}_v \geq \nu_\theta = \nu_{h_2(v)}.$$

(b) Let $u_1 < H(\lambda) < H(\rho^*) < v_1$. By the definition of $u(x,t)$ in that case,

$$\begin{aligned}\int_{u_1}^{v_1} u(x,1)\ dx &= \int_{u_1}^{H(\lambda)} \lambda\ dx + \int_{H(\lambda)}^{H(\rho^*)} h_2(x)\ dx + \int_{H(\rho^*)}^{v_1} \rho\ dx \\ &= \lambda(H(\lambda) - u_1) + \int_{\lambda}^{\rho^*} \theta H'(\theta)\ d\theta + \rho(v_1 - H(\rho^*)) \\ &= v_1\rho - u_1\lambda - G(\rho) + G(\lambda)\end{aligned}$$

where we have integrated by parts the integral of the second line (derived by a change of variables); therefore, by (3),

$$\lim_{t\to\infty} \mu_{\lambda,\rho}S_2(t)\left(\frac{1}{t}\sum_{[u_1t]}^{[v_1t]} \eta(x)\right) = \int_{u_1}^{v_1} u(x,1)\ dx \tag{16}$$

and (b) follows from (16) and (a), with the same argument as in [2] p.332 (proof of Theorem 3.2), based on monotonicity. ■

3 Asymptotic behavior of a tagged particle

We introduce here an interpretation of the k-step exclusion dynamics valid in the totally asymmetric case. Up to now we considered that a particle might jump from x to the first empty site in $\{x+1,...,x+k\}$. If we want to leave the particles ordered we could equally say that the particle at x pushes the "pack" of ($\leq k$) neighboring particles in front of it, each one moving of one unit to the right. In other words, if there is no particle at site $x+1$ then the particle at site x goes to site $x+1$; if there is one particle at site $x+1$ and no particle at site $x+2$ then the particle at x pushes the particle of site $x+1$ to site $x+2$ and occupies site $x+1$; and so on... Then the generator reads

$$L_k f(\eta) = \sum_{x\in\mathbf{Z}} \sum_{i=0}^{k-1} \prod_{j=0}^{i} \eta(x+j)\,(1-\eta(x+i+1))\,\left[f(\eta^{x,x+1,\dots,x+i+1}) - f(\eta)\right] \tag{17}$$

where

$$\eta^{x_1,x_2,\dots,x_l}(u) = \begin{cases} \eta(x_l) & \text{if } u = x_1, \\ \eta(x_{i-1}) & \text{if } u = x_i,\ i = 2,...,l, \\ \eta(u) & \text{otherwise.} \end{cases}$$

It is easy to see, comparing (17) to (1), that they do correspond in this setting, because we do not label the particles. The interest of (17) w.r.t. (1) is that it keeps track of the particles' order. We will need this interpretation in the next section.

Similarly, we define a Tagged "Pushing" Particle, and the generator of the k-step exclusion process as seen from this Tagged Pushing Particle is

$$\begin{aligned} &\widetilde{L}_k(\eta) = \\ &\sum_{i=0}^{k-1} \sum_{x\neq 0,-1,\dots,-(i+1)} \prod_{j=0}^{i} \eta(x+j)\,(1-\eta(x+i+1))\,\left[f(\eta^{x,x+1,\dots,x+i+1}) - f(\eta)\right] \\ &+\sum_{n=1}^{k} \prod_{m=1}^{n-1} \eta(m)(1-\eta(n))[f(\tau_1\eta^{0,1,\dots,n}) - f(\eta)] \\ &+\sum_{n=1}^{k-1} \sum_{l=1}^{k-n} \prod_{m=-l}^{-1} \eta(m) \prod_{i=1}^{m-1} \eta(i)(1-\eta(n))\,\left[f(\tau_1\eta^{-l,\dots,0,\dots,n}) - f(\eta)\right]. \end{aligned}$$

To be clearer, let us write and comment it for $k=2$.

$$\widetilde{L}_2(\eta) = \sum_{x\neq 0,-1} \eta(x)(1-\eta(x+1))\,\left[f(\eta^{x,x+1}) - f(\eta)\right] \tag{18}$$

$$+(1-\eta(1))\,\left[f(\tau_1\eta^{0,1}) - f(\eta)\right] \tag{19}$$

$$+\sum_{x\neq 0,-1,-2} \eta(x)\eta(x+1)(1-\eta(x+2))\,\left[f(\eta^{x,x+1,x+2}) - f(\eta)\right] \tag{20}$$

$$+\eta(1)(1-\eta(2))\,\left[f(\tau_1\eta^{0,1,2}) - f(\eta)\right] \tag{21}$$

$$+\eta(-1)(1-\eta(1))\left[f(\tau_1\eta^{-1,0,1})-f(\eta)\right]. \tag{22}$$

Part (18) is simple exclusion involving sites away from the origin, part (19) corresponds to the "classical" tagged particle for simple exclusion. Part (20) is a "strictly" 2 steps exclusion involving sites away from the origin. Part (21) describes the "pushing" of the tagged particle. Finally in part (22) the tagged particle is "pushed" by another particle.

A straightforward adaptation of the simple exclusion case (see Ferrari (1986)) gives

Theorem 3.1 *The Palm measure $\widehat{\nu}_\alpha$ of ν_α (i.e. the measure on $\mathbf{X}$ defined by $\widehat{\nu}_\alpha(.)=\nu_\alpha(.|\eta(0)=1)$) is invariant and ergodic for the k-step exclusion process as seen from a tagged pushed particle.*

Sketch of proof: For instance when $k=2$ it is enough to show

$$\mu\{\eta(-1)\eta(0)(1-\eta(1))f(\eta^{-1,0,1})\}=\mu\{\eta(0)\eta(1)(1-\eta(2))f(\tau_1\eta^{0,1,2})\}$$

and

$$\mu\{\eta(-2)\eta(-1)(1-\eta(0))f(\eta^{-2,-1,0})\}=\mu\{\eta(-1)\eta(0)(1-\eta(1))f(\tau_1\eta^{-1,0,1})\}$$

which are obvious for any $\mu\in\mathcal{S}$ (recall that $\mathcal{S}$ is the set of translation invariant measures on $\mathbf{X}$). ■

Theorem 3.2 *Law of large numbers for the Tagged Pushing Particle ($k=2$).*

For a 2-step exclusion process with initial distribution ν_α, if $Y(t)$ denotes the position at time t of a Tagged Pushing Particle starting at the origin then

$$\lim_{t\to\infty}\frac{Y(t)}{t}=(1-\alpha)(1+2\alpha)\ \mathbf{P}_{\widehat{\nu}_\alpha}a.s.$$

Proof: Using the notation of Ferrari (1992a) pp. 41-43 we define the instantaneous increment of the position of the Tagged Pushing Particle by

$$\psi(\eta):=\lim_{h\to0}\frac{\mathbf{E}(Y(t+h)-Y(t)|Y_t=x)}{h}$$

$$=(1-\eta(x+1))+\eta(x-1)(1-\eta(x+1))+\eta(x+1)(1-\eta(x+2)).$$

So

$$\int\psi\ d\nu_\alpha=1-\alpha+2\alpha(1-\alpha).$$

Furthermore $\lim_{h\to0}\mathbf{E}_{\nu_\alpha}(Y(t+h)-Y(t))^2/h<+\infty$ so that the conditions of theorem 9.2 of Ferrari (1992a) are satisfied, which gives the result. ■

Remark 3.3

Referring to results of Guiol (1999), we notice that a Tagged Pushing Particle X_t behaves as a regular tagged particle in the k-step exclusion. Indeed, intuitively, the "regular" tagged particle can make long jumps, so is expected to move faster, but it cannot be pushed; and the rate at which the Tagged Pushing Particle moves compensates exactly those long jumps.

4 Generalizations

1. We obtained conservation of local equilibrium for the Riemann case. It can be generalized to an initial product measure with a non-constant profile of bounded variation following the papers of Rezakhanlou (1991) and Landim (1993) (see also Kipnis & Landim (1999) chapters 8 and 9). Indeed Condition E is equivalent to Kružkov entropy inequality (see Godlewski & Raviart(1991), Lemma 6.1 p.88) which is the key tool of the latter proofs. Moreover, since we are working with a one-dimensional totally asymmetric process, we can equivalently consider its interpretation defined by (17); this way, particles do not jump over each other. We can therefore follow Section 6 of Rezakhanlou (1991) which contains a two-block estimate valid only for one-dimensional nearest neighbor processes, and enables to conclude the derivation of hydrodynamic equation without using Young measures. The key point consists in coupling two versions of the process, to prove that the number of sign changes between them can only decrease, due to attractivity and the reduction to a one-dimensional nearest neighbor case. For k-step exclusion, this result is a straightforward adaptation of Liggett (1976), Lemma 5.1. For the same reason, extension to deterministic initial configurations is possible, using Venkatsubramani (1995).

2. We treated only totally asymmetric k-step exclusion but our proof is also valid for nearest neighbor asymmetric transition rates, which lead to a flux function with the same properties. However one has to compute carefully this function, due to the complexity of the k-step dynamics. For instance for $k = 5$ and transition rates $p(x, x+1) = p$, $p(x, x-1) = q = 1-p$, $p > q$, we obtain

$$\begin{aligned} G_5^{p,q}(u) = \quad & u(1-u)\left[(p-q)\left\{(1+2u) + 3u^2(1-pq)\right.\right. \\ & \left.\left. +4u^3(1-2pq) + 5u^4(1-3pq+p^2q^2)\right\} + 3u^2 2p^4 q\right]. \end{aligned}$$

Indeed the last term corresponds to a "cycle" jump: To go from x to $x+3$ when sites $x+1$ and $x+2$ are occupied, the particle follows the path $(x, x+1, x+2, x+1, x+2, x+3)$.

The extension to an initial product measure with a general profile is still possible, using Rezakhanlou (1991) and Landim (1993), but with the help of Young measures.

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