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ANALYTICITY IN HUBBARD MODELS

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Abstract. The Hubbard model describes a lattice system of quantum particles with local (on-site) interactions. Its free energy is analytic when βt is small, or $\beta t^2/U$ is small; here, β is the inverse temperature, U the on-site repulsion and t the hopping coefficient.

For more general models with Hamiltonian $H = V + T$ where V involves local terms only, the free energy is analytic when $\beta\|T\|$ is small, irrespectively of V. There exists a unique Gibbs state showing exponential decay of spatial correlations. These properties are rigorously established in this paper.

Keywords: Hubbard model, local interactions, analyticity of free energy, uniqueness of Gibbs states.

1. Introduction

Electrons in condensed matter feel an external periodic potential due to the presence of atoms. A natural basis for the Hilbert space describing the states of the electrons consists in Wannier states, that are indexed by the sites of the lattice. The Hamiltonian for this Statistical Physics system can be written in second quantization in terms of Wannier states, and with some simplifications we obtain a lattice model [Hub].

Forgetting the initial physical motivation, we can consider a lattice model as if the particles were really moving on a lattice, and develop a physical intuition in this case. It may help in understanding the behaviour of the system.

The most famous lattice model for the description of quantum particles is the Hubbard model. It consists in a hopping term (discretized Laplacian) representing the kinetic energy, and a Coulomb interaction between the particles. This interaction is local, or on-site, meaning that it is expressed in terms of creation and annihilation operators of a same site. Many interesting properties of the Hubbard model have been rigorously established, see [Lieb] for a review; however, the basic questions about magnetism and superconductivity are still unsolved. The present paper brings a modest contribution in the sense that interesting phenomena are definitively excluded for some values of the thermodynamic parameters.

The general setting is as follows. We consider a class of models that include the Hubbard one, with Hamiltonian

$$H^{\boldsymbol{\mu}} = V^{\boldsymbol{\mu}} + T^{\boldsymbol{\mu}}.$$

The vector $\boldsymbol{\mu}$ represents a finite number of parameters, such as chemical potential, magnetic field, ... $V^{\boldsymbol{\mu}} = \sum_{x\in\mathbb{Z}^\nu} V_x^{\boldsymbol{\mu}}$ is a sum of local operators, and $T^{\boldsymbol{\mu}} = \sum_{A\subset\mathbb{Z}^\nu} T_A^{\boldsymbol{\mu}}$ is a

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finite-range or exponentially decaying quantum "interaction". The free energy is shown to be analytic in $\boldsymbol{\mu}$ and β in the domain

$$\beta \sum_{A \ni x} \|T_A^{\boldsymbol{\mu}}\| \, \mathrm{e}^{c|A|} < \text{const} \tag{1.1}$$

where c is a constant depending on the lattice and on the dimension of the local Hilbert space. What is quite surprising is that the domain does not depend on the local interaction V. The reason is that when T is small with respect to β, the sites of the lattice are almost independent, and the (mean) free energy is essentially that of a model with only one site. Such a zero-dimensional system is free from phase transitions, hence its free energy is analytic. High temperature expansions would yield comparable results; however they do not only require the condition (1.1), but also $\beta\|V\| < \text{const}$.

Concerning the Hubbard model, it is not only true that we have analyticity however strong is the repulsive potential; the latter favours this phase, i.e. the stronger the interactions, the larger the domain. This last result holds at half filling, and if the ratio t/U is small enough. In fact, at half filling the Hubbard model is unitarily equivalent to a series in powers of t/U that starts with V and the Heisenberg model [KS, CSO, MGY]:

$$H_{\text{Hubbard}} \simeq V + \tfrac{t^2}{U} H_{\text{Heisenberg}} + O(\tfrac{t^4}{U^3})$$

(a rigorous statement can be found in [DFF]). Such a model enters our class, with $\|T\| \sim \frac{t^2}{U}$, hence the condition $\beta t^2/U < \text{const}$.

Another example is the Falicov-Kimball model [GM]; it is a Hubbard model where only particles of a given spin have hopping, the others being considered as heavy, static classical particles. The statements for the Hubbard model are also valid in this case, and were proven by Kennedy and Lieb [KL].

Section 2 contains precise definitions, statements and proofs for general systems with on-site interactions. Section 3 is devoted to the Hubbard model; domains of parameters where analyticity can be rigorously proven are proposed with explicit bounds, in the case of the 3D square lattice. Finally, the paper ends with a discussion of the Bose-Hubbard model, for which partial results may be obtained.

2. Models with local interactions

2.1. **General framework.** Let us be more precise and introduce the mathematical framework. Let $\mathbb{L}$ a ν-dimensional lattice; for instance, $\mathbb{L} = \mathbb{Z}^\nu$, but any other periodic lattice can be considered. We denote with Λ a finite subset of $\mathbb{L}$, and the thermodynamic limit $\lim_{\Lambda \nearrow \mathbb{L}} f_\Lambda$ means $\lim_{n\to\infty} f_{\Lambda_n}$ with any sequence of finite volumes (Λ_n) such that $\Lambda_n \subsetneq \Lambda_{n+1}$, and $\lim_{n\to\infty} |\partial\Lambda_n|/|\Lambda_n| = 0$, where $\partial\Lambda_n$ is the boundary of Λ_n. Let Ω a finite set with $|\Omega| = S$; we consider the set of "classical configurations" Ω^Λ. The Hilbert space $\mathcal{H}_\Lambda$ at finite volume Λ is spanned by the classical configurations, i.e. each vector of $\mathcal{H}_\Lambda$ is a linear combination of vectors $|n_\Lambda\rangle$, $n_\Lambda \in \Omega^\Lambda$.

A *quantum interaction* T is a collection $(T_A)_{A \subset \mathbb{L}}$, where T_A is a self-adjoint operator with support A. Its action is defined on each $\mathcal{H}_\Lambda$ with $\Lambda \supset A$, and we have factorization properties

$$\langle n_\Lambda | \, T_A T_{A'} \, | n'_\Lambda \rangle = \langle n_A | \, T_A \, | n'_A \rangle \langle n_{A'} | \, T_{A'} \, | n'_{A'} \rangle \tag{2.1}$$

when $A \cap A' = \emptyset$ (a hopping matrix is an example of a "quantum interaction"). Let us introduce the connected cardinality $\|A\|$ of $A \subset \mathbb{L}$ as the cardinality of the smallest connected set containing A, i.e.

$$\|A\| = \min_{B \supset A,\, \text{connected}} |B|; \tag{2.2}$$

notice that $\|A\| = |A|$ when A is connected, and $\|A\| > |A|$ when it is not. We define the norm of an interaction to be

$$\|T\|_c = \sup_{x \in \mathbb{L}} \sum_{A \ni x} \|T_A\| \, \mathrm{e}^{c\|A\|} \tag{2.3}$$

where c is a positive number. Here, $\|T_A\|$ is the operator norm of T_A. We call V a *local interaction* (or *on-site interaction*) if $V_A = 0$ for all $|A| \geqslant 2$; local interactions are denoted by (V_x) instead of $(V_{\{x\}})$.

Let $\boldsymbol{\mu} \in \mathbb{R}^s$ be thermodynamic parameters. The finite volume Hamiltonian $H_\Lambda^{\boldsymbol{\mu}}$ depends on $\boldsymbol{\mu}$ and is given by

$$H_\Lambda^{\boldsymbol{\mu}} = \sum_{x \in \Lambda} V_x^{\boldsymbol{\mu}} + \sum_{A \subset \Lambda} T_A^{\boldsymbol{\mu}}. \tag{2.4}$$

We suppose here that both $(V_x^{\boldsymbol{\mu}})$ and $(T_A^{\boldsymbol{\mu}})$ are translation invariant, although periodic interactions could be considered with only small modifications. The free energy is given by the limit (whenever it exists)

$$f(\beta, \boldsymbol{\mu}) = -\frac{1}{\beta} \lim_{\Lambda \nearrow \mathbb{L}} \frac{1}{|\Lambda|} \log \operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda^{\boldsymbol{\mu}}}. \tag{2.5}$$

We write f_0 for the "classical free energy"

$$f_0(\beta, \boldsymbol{\mu}) = -\frac{1}{\beta} \log \sum_{n_x \in \Omega} \langle n_x | \, \mathrm{e}^{-\beta V_x^{\boldsymbol{\mu}}} \, | n_x \rangle. \tag{2.6}$$

We notice that f_0 is also given by (2.5) with $T^{\boldsymbol{\mu}} = 0$.

A Gibbs state is a functional that attributes to any bounded local operator K the value

$$\langle K \rangle = \lim_{\Lambda \nearrow \mathbb{L}} \frac{\operatorname{Tr} K \, \mathrm{e}^{-\beta H_\Lambda}}{\operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda}}. \tag{2.7}$$

A Gibbs state is *exponentially clustering* if for any two local operators K and K' there exists $C_{K,K'} < \infty$ (with $C_{K,K'} = C_{t_x K, t_y K'}$ for any translations t_x and t_y, $x, y \in \mathbb{L}$) such that

$$\big| \langle K K' \rangle - \langle K \rangle \langle K' \rangle \big| \leqslant C_{K,K'} \, \mathrm{e}^{-d(K,K')/\xi} \tag{2.8}$$

for some finite constant ξ. Here, $d(K, K')$ is the distance between supports of K and K'.

2.2. **Uniqueness of the Gibbs state.** The Hamiltonians we consider possess many symmetries. For instance, they have translation invariance by assumption; and typical models have further conserved quantities, such as the total number of particles, or total spin...

Usually Gibbs states obtained with free boundary conditions (2.7), or with periodic ones, have same symmetry properties than Hamiltonians. To obtain pure states with symmetry breaking, there are mainly two ways: to introduce boundary conditions, or to perturb the system.

In the quantum case, boundary conditions may be defined by means of a suitable *boundary interaction* $\partial^\Lambda = (\partial_A^\Lambda)_{A \subset \mathbb{L}}$, where operators ∂_A^Λ are non zero only for subsets A that touch the boundary of Λ. The corresponding Gibbs state is defined by the expression (2.7), with H_Λ replaced by $H_\Lambda + \sum_{A \subset \Lambda} \partial_A^\Lambda$.

Ferromagnetic states are associated with operators of the form $(n_{x\uparrow}-n_{x\downarrow})$ applied on the boundary of the volume, while for antiferromagnetism we would use $(-1)^x(n_{x\uparrow}-n_{x\downarrow})$. Of special importance are boundary conditions that break conservation of the total number of particles. A state displaying superfluid behaviour should be sensitive to the operator $\sum(\mathrm{e}^{-\mathrm{i}\theta}\, c_x^\dagger + \mathrm{e}^{\mathrm{i}\theta}\, c_x)$ where the sum is over sites touching the boundary. The order parameter for superfluidity is $c_0^\dagger$ (creation operator of a particle at site 0) [PO], and with above boundary conditions we may have $\langle c_x^\dagger \rangle^\theta = \alpha\, \mathrm{e}^{\mathrm{i}\theta}$ with $\alpha > 0$, revealing the presence of superfluidity.

For a superconductor with Cooper pairs described by a Hubbard-like model, relevant boundary conditions are of the form $\sum(\mathrm{e}^{-\mathrm{i}\theta}\, c_{x\uparrow}^\dagger c_{y\downarrow}^\dagger + \mathrm{e}^{\mathrm{i}\theta}\, c_{x\uparrow} c_{y\downarrow})$ with the sum taken on sites of the boundary, close to each other. This allows expectation values of the form $\langle c_{0\uparrow}^\dagger c_{x\uparrow}^\dagger \rangle$ to be non zero, and this should be the indication of superconductivity [Yang].

The second way to obtain states with less symmetry than the Hamiltonian is to add a perturbation, that is then set to zero. As for superfluidity, a good perturbation to consider is $h \sum_{x\in\Lambda} \big(\mathrm{e}^{-\mathrm{i}\theta}\, c_x^\dagger + \mathrm{e}^{\mathrm{i}\theta}\, c_x\big)$ (see e.g. [Hua]), and the question is whether $\lim_{h\to 0} \langle c_0^\dagger \rangle^{\theta,h}$ differs from zero.

Here we shall speak of *uniqueness of the Gibbs state* if it is insensitive to both boundary conditions and to external perturbations. In the range of parameters we consider, systems are described by Gibbs states sharing the two properties

- $\lim_{\Lambda\nearrow\mathbb{L}} \langle K \rangle_\Lambda^{\partial^\Lambda}$ does not depend on the boundary conditions ∂^Λ, provided $\|\partial^\Lambda\|$ is small enough (independently of Λ);
- for all quantum perturbation P with exponential decay, $\|P\|_c < \infty$ for a large enough c, and all local observable K,

$$\langle K \rangle = \lim_{\alpha\to 0} \lim_{\Lambda\nearrow\mathbb{L}} \frac{\mathrm{Tr}\, K\, \mathrm{e}^{-\beta(H_\Lambda + \alpha \sum_{A\subset\Lambda} P_A)}}{\mathrm{Tr}\ \mathrm{e}^{-\beta(H_\Lambda + \alpha \sum_{A\subset\Lambda} P_A)}}. \tag{2.9}$$

Notice that P is not necessarily translation invariant, it may even not be periodic.

Remark: the stability against perturbations can be given a simpler, however more abstract definition. Let us consider $\mathcal{Q}$, the Banach space of interactions with finite norm (2.3), and $\mathcal{G}$ the space of Gibbs states obtained with periodic boundary conditions; $\mathcal{G}$ is a topological space with the weak topology. Let g denote the corresponding mapping $\mathcal{Q} \to \mathcal{G}$. It is continuous at $H \in \mathcal{Q}$ provided $g^{-1}(G)$ is a neighbourhood of H if G is a neighbourhood of $g(H)$.

Then the stability of a Gibbs state with respect to perturbations amounts to saying that g is continuous at H.

Indeed, we can see *ab absurdo* that (2.9) implies the continuity of g: suppose G is a neighbourhood of $g(H)$ such that $g^{-1}(G)$ is not a neighbourhood of H; since $\mathcal{Q}$ is a metric space, there exists a sequence (H_n), $H_n \to H$, with $H_n \notin g^{-1}(G)$; by (2.9), $g(H_n) \to g(H)$, then $g(H_n) \in G$ for n sufficiently large, and therefore $H_n \in g^{-1}(G)$. Conversely, for any open set G that contains $g(H)$, $g^{-1}(G)$ is a neighbourhood of H; then if $H_n \to H$, we have $H_n \in g^{-1}(G)$ for n sufficiently large, therefore $g(H_n) \in G$.

2.3. **Result.** In order to state the result, we let $\beth$ be a constant that depends only on the lattice, such that

$$\#(A \ni x,\ \text{connected}, |A| = k) \ \leqslant\ \beth^k. \tag{2.10}$$

A possible choice, probably not optimal, is $\daleth = (2\nu)^2$ for the ν-dimensional square lattice. The Golden Ratio appears here, that we write $\phi = \frac{\sqrt{5}+1}{2}$ following a standard convention.[2]

Theorem 2.1 (Analyticity in models with local interactions).
Assume that $V^{\boldsymbol{\mu}}$ and $T^{\boldsymbol{\mu}}$ are smooth, i.e. that all matrix elements of $V_x^{\boldsymbol{\mu}}$ and $T_A^{\boldsymbol{\mu}}$ are analytic in $\boldsymbol{\mu}$ for all x and all A. Let $c \geqslant c_0 = \log S + \log 2\daleth + \phi + 2\log\phi$. Then in the domain

$$\beta \|T^{\boldsymbol{\mu}}\|_c < 1,$$

(i) *the free energy exists in the thermodynamic limit and is analytic in β and $\boldsymbol{\mu}$;*
(ii) *the Gibbs state converges weakly in the thermodynamic limit;*
(iii) *the Gibbs state is exponentially clustering with a correlation length bounded by $\xi = 4(c-c_0)^{-1}$.*

And the Gibbs state is unique, i.e.

(iv) *the Gibbs state is stable with respect to boundary conditions ∂^Λ with $\beta\|T^{\boldsymbol{\mu}} + \partial^\Lambda\|_c < 1$ for all Λ;*
(v) *the Gibbs state is stable with respect to all external perturbations P with $\|P\|_c < \infty$.*

Remark: the bound $4(c-c_0)^{-1}$ for the correlation length is rather arbitrary and could certainly be improved.

The stability against boundary conditions should hold for any bounded ∂^Λ, not only small ones. However, such a statement is difficult to prove in quantum systems, where we have to deal with negative or complex numbers.

Proof of Theorem 2.1 (i). The idea of the proof is to expand the operator $\mathrm{e}^{-\beta H_\Lambda}$ with Duhamel formula; it allows next to express the partition function as the one of a polymer model. After having shown that the weights of polymers have exponential decay with respect to their size, the analyticity of the free energy is a result of cluster expansions.

The Duhamel formula (very similar to the Trotter formula) yields

$$\operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda^{\boldsymbol{\mu}}} = \operatorname{Tr} \mathrm{e}^{-\beta \sum_{x\in\Lambda} V_x^{\boldsymbol{\mu}}} + \sum_{m\geqslant 1} (-1)^m \sum_{A_1,\dots,A_m \subset \Lambda} \int_{0<\tau_1<\dots<\tau_m<\beta} \mathrm{d}\tau_1 \dots \mathrm{d}\tau_m$$
$$\operatorname{Tr} \mathrm{e}^{-\tau_1 \sum_{x\in\Lambda} V_x^{\boldsymbol{\mu}}} T_{A_1}^{\boldsymbol{\mu}} \mathrm{e}^{-(\tau_2-\tau_1)\sum_{x\in\Lambda} V_x^{\boldsymbol{\mu}}} \dots T_{A_m}^{\boldsymbol{\mu}} \mathrm{e}^{-(\beta-\tau_m)\sum_{x\in\Lambda} V_x^{\boldsymbol{\mu}}}. \quad (2.11)$$

For given $A_1, \dots, A_m$, we construct the graph $\mathcal{G}$ of m vertices, with an edge between i and j whenever $A_i \cap A_j \neq \emptyset$. Decomposing $\mathcal{G}$ into connected subgraphs, it induces a partition of $\{A_1,\dots,A_m\}$ into ℓ subsets ($\ell \leqslant m$). We let $\mathcal{A}_1,\dots,\mathcal{A}_\ell \subset \mathbb{Z}^\nu$ to be the unions of sets $A_1,\dots,A_m$ for each partition. As a result, to each sequence $A_1,\dots,A_m$ corresponds a unique set $\{\mathcal{A}_1,\dots,\mathcal{A}_\ell\}$ of subsets of $\mathbb{Z}^\nu$, such that

$$\begin{cases} \cup_{i=1}^m A_i = \cup_{i=1}^\ell \mathcal{A}_i, & \\ \mathcal{A}_i \cap \mathcal{A}_j = \emptyset & \text{if } i \neq j. \end{cases} \quad (2.12)$$

We call $\mathcal{A}_1,\dots,\mathcal{A}_\ell$ polymers and define their weight

$$\rho(\mathcal{A}) = \mathrm{e}^{\beta f_0(\beta,\boldsymbol{\mu})|\mathcal{A}|} \sum_{m\geqslant 1} (-1)^m \sum_{A_1,\dots,A_m} \sum_{n_\mathcal{A}\in\Omega^\mathcal{A}} \int_{0<\tau_1<\dots<\tau_m<\beta} \mathrm{d}\tau_1\dots\mathrm{d}\tau_m$$
$$\langle n_\mathcal{A}| \, \mathrm{e}^{-\tau_1\sum_{x\in\mathcal{A}} V_x^{\boldsymbol{\mu}}} T_{A_1}^{\boldsymbol{\mu}} \mathrm{e}^{-(\tau_2-\tau_1)\sum_{x\in\mathcal{A}} V_x^{\boldsymbol{\mu}}} \dots T_{A_m}^{\boldsymbol{\mu}} \mathrm{e}^{-(\beta-\tau_m)\sum_{x\in\mathcal{A}} V_x^{\boldsymbol{\mu}}} |n_\mathcal{A}\rangle. \quad (2.13)$$

[2]It is a pleasure to welcome here the Golden Ratio. Its presence is however fortuitous and does not involve any of its special and beautiful properties.

The sum is over sets $A_1, \dots, A_m$ satisfying two restrictions: (i) $\cup_{i=1}^m A_i = \mathcal{A}$, (ii) the graph $\mathcal{G}$ defined above is connected. The partition function can then be rewritten as

$$\operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda^{\boldsymbol{\mu}}} = \mathrm{e}^{-\beta f_0(\beta,\boldsymbol{\mu})|\Lambda|} \sum_{\substack{\{\mathcal{A}_1,\dots,\mathcal{A}_\ell\} \\ \mathcal{A}_i \cap \mathcal{A}_j = \emptyset}} \prod_{j=1}^{\ell} \rho(\mathcal{A}_j). \tag{2.14}$$

We have now to bound $\rho(\mathcal{A})$; first the matrix element:

$$\begin{aligned} \big|\langle n_{\mathcal{A}}| \cdot |n_{\mathcal{A}}\rangle\big| &\leqslant \big\| \mathrm{e}^{-\tau_1 \sum_{x\in\mathcal{A}} V_x^{\boldsymbol{\mu}}} T_{A_1}^{\boldsymbol{\mu}} \, \mathrm{e}^{-(\tau_2-\tau_1)\sum_{x\in\mathcal{A}} V_x^{\boldsymbol{\mu}}} \dots T_{A_m}^{\boldsymbol{\mu}} \, \mathrm{e}^{-(\beta-\tau_m)\sum_{x\in\mathcal{A}} V_x^{\boldsymbol{\mu}}} \big\| \\ &\leqslant \big\| \mathrm{e}^{-\beta V_x^{\boldsymbol{\mu}}} \big\|^{|\mathcal{A}|} \prod_{j=1}^{m} \|T_{A_j}^{\boldsymbol{\mu}}\|. \end{aligned} \tag{2.15}$$

Let $e_0^{\boldsymbol{\mu}}$ be the lowest eigenvalue of $V_x^{\boldsymbol{\mu}}$; since $f_0(\beta,\boldsymbol{\mu}) \leqslant e_0^{\boldsymbol{\mu}}$, we have

$$\big\| \mathrm{e}^{-\beta V_x^{\boldsymbol{\mu}}} \big\| = \mathrm{e}^{-\beta e_0^{\boldsymbol{\mu}}} \leqslant \mathrm{e}^{-\beta f_0(\beta,\boldsymbol{\mu})}. \tag{2.16}$$

Furthermore $|\Omega^{\mathcal{A}}| = S^{|\mathcal{A}|}$ and the integral over "times" $\{\tau_j\}$ brings a factor $\beta^m/m!$; using $\|\mathcal{A}\| \leqslant \sum_{j=1}^m \|A_j\|$, we obtain

$$\begin{aligned} |\rho(\mathcal{A})| &\leqslant S^{|\mathcal{A}|} \, \mathrm{e}^{-c\|\mathcal{A}\|} \sum_{m \geqslant 1} \frac{\beta^m}{m!} \sum_{A_1,\dots,A_m \subset \mathcal{A}} \prod_{j=1}^{m} \|T_{A_j}^{\boldsymbol{\mu}}\| \, \mathrm{e}^{c\|A_j\|} \\ &\leqslant S^{\|\mathcal{A}\|} \, \mathrm{e}^{-c\|\mathcal{A}\|} \sum_{m \geqslant 1} \frac{1}{m!} \Big(\beta|\mathcal{A}| \sup_{x\in\mathbb{Z}^\nu} \sum_{A \ni x} \|T_A^{\boldsymbol{\mu}}\| \, \mathrm{e}^{c\|A\|} \Big)^m \\ &\leqslant \mathrm{e}^{-(c-\log S-1)\|\mathcal{A}\|}. \end{aligned} \tag{2.17}$$

Results of cluster expansions are summarized in Proposition 2.2 below. From this we obtain the following expression for the free energy

$$f(\beta,\boldsymbol{\mu}) = f_0(\beta,\boldsymbol{\mu}) - \frac{1}{\beta} \sum_{C, \operatorname{supp} C \ni x} \frac{\Phi^{\mathrm{T}}(C)}{|\operatorname{supp} C|}. \tag{2.18}$$

It does not depend on x, because the Hamiltonian is translation invariant. Since $\Phi^{\mathrm{T}}(C)$ is analytic in β and $\boldsymbol{\mu}$, and the series converges uniformly, the free energy $f(\beta,\boldsymbol{\mu})$ is an analytic function by Vitali theorem. □

Proposition 2.2 (Cluster expansions).

Let us recall that we can choose $\daleth = (2\nu)^2$ *for the* ν*-dimensional square lattice and* $\phi = \frac{\sqrt{5}+1}{2}$ *is the Golden Ratio.*

Assume that a function $z^{\beta,\boldsymbol{\mu}} : \mathcal{P}(\mathbb{L}) \to \mathbb{C}$ *is given and such that for all* $A \subset \mathbb{L}$,

- $|z^{\beta,\boldsymbol{\mu}}(A)| \leqslant \mathrm{e}^{-\tau\|A\|}$ *with* $\tau \geqslant \tau_0 = \log 2\daleth + \phi - 1 + 2\log\phi$;
- $z^{\beta,\boldsymbol{\mu}}(A)$ *is analytic in* $\beta, \boldsymbol{\mu}$.

Then there exists an analytic function $\Phi^{\mathrm{T}} : \mathcal{P}(\mathcal{P}(\mathbb{L})) \to \mathbb{C}$ *such that*

$$\log \sum_{\substack{\{A_1,\dots,A_k\} \\ A_j \subset \Lambda, A_i \cap A_j = \emptyset}} \prod_{j=1}^{k} z^{\beta,\boldsymbol{\mu}}(A_j) = \sum_{\substack{C = \{A_1,\dots,A_k\} \\ A_j \subset \Lambda}} \Phi^{\mathrm{T}}(C);$$

Let $\operatorname{supp} C = \cup_{A \in C} A$*;* $\Phi^{\mathrm{T}}(C) = 0$ *if* C *is not a cluster, i.e. if* $C = C_1 \cup C_2$ *with* $\operatorname{supp} C_1 \cap \operatorname{supp} C_2 = \emptyset$*. This function has exponential decay:*

$$\sum_{\substack{C=\{A_1,\dots,A_k\} \\ \cup_j A_j \ni x}} |\Phi^{\mathrm{T}}(C)| \, \mathrm{e}^{(\tau-\tau_0)\|C\|} \leqslant \phi - 1$$

for all $x \in \mathbb{L}$*. Here, we set* $\|C\| = \sum_{A \in C} \|A\|$*.*

This proposition is an immediate corollary of Kotecký and Preiss theorem on cluster expansions [KP] (see [Dob] for an elegant and simpler proof). To make the link between our notation and theirs:

Here	[KP]
A	γ
$2\beth$	K
$(\phi-1)\|A\|$	$a\|\gamma\|$
$(\tau-\tau_0)\|A\|$	$d(\gamma)$

Our polymers are not necessarily connected; but we note that their entropy satisfies

$$\begin{aligned} \#(A \ni x, \|A\| = k) &\leqslant 2^k \, \#(A \ni x, \text{ connected}, |A| = k) \\ &\leqslant (2\beth)^k. \end{aligned}$$

Hence the factor 2 in front of $\beth$.

A useful consequence of this proposition is the existence of the thermodynamic limit of the free energy of a gas of polymers; if the weight $z^{\beta,\boldsymbol{\mu}}$ is periodic with respect to lattice translations, then the limit

$$f(\beta,\boldsymbol{\mu}) = -\frac{1}{\beta} \lim_{\Lambda \nearrow \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log \sum_{\substack{\{A_1,\dots,A_k\} \\ A_i \subset \Lambda, A_i \cap A_j = \emptyset}} \prod_{j=1}^{k} z^{\beta,\boldsymbol{\mu}}(A_j)$$

exists and is analytic in β and $\boldsymbol{\mu}$.

Proof of Theorem 2.1 (ii). Having the expansion (2.14) for the partition function, the treatment of expectation values of local operators is standard. The expectation value of K involves the quantity $\operatorname{Tr} K \, \mathrm{e}^{-\beta H^{\boldsymbol{\mu}}_\Lambda}$, that we expand as before with Duhamel formula.

$$\begin{aligned} \operatorname{Tr} K \, \mathrm{e}^{-\beta H^{\boldsymbol{\mu}}_\Lambda} = \operatorname{Tr} K \, \mathrm{e}^{-\beta \sum_{x \in \Lambda} V^{\boldsymbol{\mu}}_x} + \sum_{m \geqslant 1} (-1)^m \sum_{A_1,\dots,A_m \subset \Lambda} \int_{0<\tau_1<\dots<\tau_m<\beta} \mathrm{d}\tau_1 \dots \mathrm{d}\tau_m \\ \operatorname{Tr} K \, \mathrm{e}^{-\tau_1 \sum_{x \in \Lambda} V^{\boldsymbol{\mu}}_x} T^{\boldsymbol{\mu}}_{A_1} \mathrm{e}^{-(\tau_2-\tau_1)\sum_{x \in \Lambda} V^{\boldsymbol{\mu}}_x} \dots T^{\boldsymbol{\mu}}_{A_m} \mathrm{e}^{-(\beta-\tau_m)\sum_{x \in \Lambda} V^{\boldsymbol{\mu}}_x}. \end{aligned} \tag{2.19}$$

We construct the graph $\mathcal{G}$ of $(m+1)$ vertices, for the sets $\operatorname{supp} K, A_1, \dots, A_m$, and we look for $(\ell+1)$ connected components. One of these components contains $\operatorname{supp} K$, and we denote it by $\mathcal{A}_K$; others are denoted by $\mathcal{A}_1, \dots, \mathcal{A}_\ell$, as before. The weight of $\mathcal{A}_K$ is modified by the operator K; namely,

$$\rho_K(\mathcal{A}_K) = \mathrm{e}^{\beta f_0(\beta,\boldsymbol{\mu})|\mathcal{A}_K|} \Big\{ \operatorname{Tr} K \, \mathrm{e}^{-\beta \sum_{x \in \mathcal{A}_K} V_x^{\boldsymbol{\mu}}} + \sum_{m \geqslant 1} (-1)^m \sum_{A_1,\dots,A_m} \sum_{n_{\mathcal{A}_K} \in \Omega^{\mathcal{A}_K}}$$

$$\int_{0<\tau_1<\dots<\tau_m<\beta} \mathrm{d}\tau_1 \dots \mathrm{d}\tau_m \langle n_{\mathcal{A}_K} | \, K \, \mathrm{e}^{-\tau_1 \sum_{x \in \mathcal{A}_K} V_x^{\boldsymbol{\mu}}} \, T_{A_1}^{\boldsymbol{\mu}} \, \mathrm{e}^{-(\tau_2-\tau_1) \sum_{x \in \mathcal{A}_K} V_x^{\boldsymbol{\mu}}} \dots$$

$$\dots T_{A_m}^{\boldsymbol{\mu}} \, \mathrm{e}^{-(\beta-\tau_m) \sum_{x \in \mathcal{A}_K} V_x^{\boldsymbol{\mu}}} \, | n_{\mathcal{A}_K} \rangle \Big\}. \tag{2.20}$$

The expectation value of K takes form

$$\langle K \rangle = \lim_{\Lambda \nearrow \mathbb{L}} \frac{1}{Z_\Lambda(\beta,\boldsymbol{\mu})} \sum_{\{\mathcal{A}_K,\mathcal{A}_1,\dots,\mathcal{A}_\ell\}} \rho_K(\mathcal{A}_K) \prod_{j=1}^{\ell} \rho(\mathcal{A}_j) \tag{2.21}$$

where the sum is over non intersecting sets $\mathcal{A}_K, \mathcal{A}_1, \dots, \mathcal{A}_\ell$, $\ell \geqslant 0$. The weight ρ_K has also exponential decay:

$$|\rho_K(\mathcal{A}_K)| \leqslant \|K\| \, \mathrm{e}^{c\|\operatorname{supp} K\|} \, \mathrm{e}^{-(c-\log S-1)\|\mathcal{A}_K\|}. \tag{2.22}$$

From cluster expansions, we obtain an expression for $\langle K \rangle$:

$$\langle K \rangle = \lim_{\Lambda \nearrow \mathbb{L}} \sum_{\substack{\mathcal{A}_K \subset \Lambda \\ \mathcal{A}_K \supset \operatorname{supp} K}} \rho_K(\mathcal{A}_K) \exp\Big\{ - \sum_{\substack{C, \operatorname{supp} C \subset \Lambda \\ \operatorname{supp} C \cap \mathcal{A}_K \neq \emptyset}} \Phi^{\mathrm{T}}(C) \Big\}. \tag{2.23}$$

The limit exists, because the sums converge uniformly in the volume Λ. This proves the weak convergence of the Gibbs state in the thermodynamic limit.

□

Proof of Theorem 2.1 (iii). Using cluster expansion, this is standard stuff. Expanding the expectation value $\langle KK' \rangle$ as before, we get

$$\langle KK' \rangle = \sum_{\mathcal{A}_{KK'} \supset \operatorname{supp} K \cup \operatorname{supp} K'} \rho_{KK'}(\mathcal{A}_{KK'}) \exp\Big\{ - \sum_{C, \operatorname{supp} C \cap \mathcal{A}_{KK'} \neq \emptyset} \Phi^{\mathrm{T}}(C) \Big\}$$

$$+ \sum_{\substack{\mathcal{A}_K \supset \operatorname{supp} K \\ \mathcal{A}_{K'} \supset \operatorname{supp} K' \\ \mathcal{A}_K \cap \mathcal{A}_{K'} = \emptyset}} \rho_K(\mathcal{A}_K) \rho_{K'}(\mathcal{A}_{K'}) \exp\Big\{ - \sum_{C, \operatorname{supp} C \cap [\mathcal{A}_K \cup \mathcal{A}_{K'}] \neq \emptyset} \Phi^{\mathrm{T}}(C) \Big\}. \tag{2.24}$$

The weights ρ_K, $\rho_{K'}$, $\rho_{KK'}$ are given by (2.20) when substituting K with K, K', KK' respectively.

We define $\langle KK' \rangle^{\mathrm{short}}$ to be as (2.24), but with sums only over polymers and clusters of connected cardinality less than $\frac{1}{4} d(K,K')$ (we call such polymers *short*, they are *big* otherwise). We denote $\langle KK' \rangle^{\mathrm{big}} = \langle KK' \rangle - \langle KK' \rangle^{\mathrm{short}}$.

When the expectation values are restricted to short polymers and clusters, correlation functions are zero:

$$\langle KK' \rangle^{\mathrm{short}} = \langle K \rangle^{\mathrm{short}} \langle K' \rangle^{\mathrm{short}}. \tag{2.25}$$

Therefore

$$\langle KK' \rangle - \langle K \rangle \langle K' \rangle = \langle KK' \rangle^{\mathrm{big}} - \langle K \rangle^{\mathrm{big}} \langle K' \rangle - \langle K \rangle^{\mathrm{short}} \langle K' \rangle^{\mathrm{big}}. \tag{2.26}$$

Expectation values $\langle\cdot\rangle^{\rm big}$ involve sums over polymers of connected cardinality bigger than $\frac14 d(K,K')$, and this has exponential decay. There are also terms

$$\begin{aligned}
&\exp\Big\{-\sum_{C,\operatorname{supp} C\cap\mathcal{A}\neq\emptyset}\Phi^{\rm T}(C)\Big\}-\exp\Big\{-\sum_{\substack{C,\operatorname{supp} C\cap\mathcal{A}\neq\emptyset\\ |\operatorname{supp} C|\leqslant\frac14 d(K,K')}}\Phi^{\rm T}(C)\Big\}\\
&\quad=\exp\Big\{-\sum_{\substack{C,\operatorname{supp} C\cap\mathcal{A}\neq\emptyset\\ |\operatorname{supp} C|\leqslant\frac14 d(K,K')}}\Phi^{\rm T}(C)\Big\}\Big[\exp\Big\{-\sum_{\substack{C,\operatorname{supp} C\cap\mathcal{A}\neq\emptyset\\ |\operatorname{supp} C|>\frac14 d(K,K')}}\Phi^{\rm T}(C)\Big\}-1\Big]. \qquad (2.27)
\end{aligned}$$

We know from Proposition 2.2 that the sum over clusters has exponential decay; more precisely, the quantity between brackets is bounded by $C(\mathcal{A})\,\mathrm{e}^{-\frac14 d(K,K')(c-c_0)}$, where $C(\mathcal{A})$ depends on $|\mathcal{A}|$ only.

Exponential clustering is now clear.

□

Proof of Theorem 2.1 (iv). The proof is rather standard, so we content ourselves by outlining it.

Expanding $\operatorname{Tr}\,\mathrm{e}^{-\beta H^{\boldsymbol\mu}_\Lambda-\beta\sum_{A\subset\Lambda}\partial^\Lambda_A}$ with Duhamel formula, we obtain an expression very similar to (2.19). The difference is that operators ∂^Λ_A now appear in the second line of (2.19). We define $\hat\rho_K(\mathcal{A}_K)$ to be as in (2.20), except that at least one boundary operator ∂^Λ_A shows up. Similarly, let $\hat\rho(\mathcal{A})$ be like (2.13) but with at least one ∂^Λ_A; an important property of these weights is that they are zero if the polymer does not touch the boundary of Λ. As a result we get

$$\langle K\rangle^{\partial^\Lambda}_\Lambda=\sum_{\substack{\mathcal{A}_K\subset\Lambda\\ \mathcal{A}_K\supset\operatorname{supp} K}}\Big(\rho_K(\mathcal{A}_K)+\hat\rho_K(\mathcal{A}_K)\Big)\exp\Big\{-\sum_{\substack{C,\operatorname{supp} C\subset\Lambda\\ \operatorname{supp} C\cap\mathcal{A}_K\neq\emptyset}}\big(\Phi^{\rm T}(C)+\hat\Phi^{\rm T}(C)\big)\Big\} \qquad (2.28)$$

where $\hat\Phi^{\rm T}(C)$ is such that $[\Phi^{\rm T}(C)+\hat\Phi^{\rm T}(C)]$ is the truncated function for polymers with weights $[\rho(\mathcal{A})+\hat\rho(\mathcal{A})]$.

A few more developments lead to an expression for $\langle K\rangle^{\partial^\Lambda}_\Lambda$ that is equal to $\langle K\rangle_\Lambda$, plus terms that connect $\operatorname{supp} K$ with the boundary of Λ, and that decay exponentially quickly. In the thermodynamic limit, this correction vanishes, and therefore for all ∂^Λ:

$$\lim_{\Lambda\nearrow\mathbb{L}}\langle K\rangle^{\partial^\Lambda}_\Lambda=\langle K\rangle. \qquad (2.29)$$

□

Proof of Theorem 2.1 (v). Once we have an expansion in terms of clusters, the proof of stability with respect to external perturbations is actually fairly simple.

First we note that if $\sum_{n\geqslant 0}|g^\alpha_n|<\infty$ uniformly in α, and $g^\alpha_n\to g_n$ when $\alpha\to 0$, then

$$\sum_{n\geqslant 0}g^\alpha_n\to\sum_{n\geqslant 0}g_n.$$

Indeed, for any $\varepsilon>0$ there exists m such that

$$\sum_{n\geqslant m}\big(|g^\alpha_n|+|g_n|\big)\;\leqslant\;\frac{\varepsilon}{2}.$$

Furthermore, for all n there exists $\bar{\alpha}_n > 0$ such that $|g_n^\alpha - g_n| \leqslant \frac{\varepsilon}{2m}$ when $\alpha \leqslant \bar{\alpha}_n$. Choosing $\bar{\alpha} = \min_{n<m} \bar{\alpha}_n$, we have for all $\alpha \leqslant \bar{\alpha}$

$$\Big| \sum_{n \geqslant 0} g_n^\alpha - \sum_{n \geqslant 0} g_n \Big| \leqslant \sum_{n<m} |g_n^\alpha - g_n| + \sum_{n \geqslant m} \big(|g_n^\alpha| + |g_n| \big) \leqslant \varepsilon.$$

We add to the Hamiltonian a new quantum interaction P with $\|P\|_c < \infty$. Since $\|T^{\boldsymbol{\mu}}\|_c < 1$, there exists $\bar{\alpha} > 0$ such that for all $\alpha \leqslant \bar{\alpha}$,

$$\|T^{\boldsymbol{\mu}} + \alpha P\|_c < 1. \tag{2.30}$$

Retracing the steps above, we obtain

$$\langle K \rangle^\alpha = \sum_{\mathcal{A}_K \supset \operatorname{supp} K} \rho_K^\alpha(\mathcal{A}_K) \exp\Big\{ - \sum_{C, \operatorname{supp} C \cap \mathcal{A}_K \neq \emptyset} \Phi_\alpha^{\mathrm{T}}(C) \Big\} \tag{2.31}$$

where $\rho_K^\alpha(\mathcal{A}_K)$ is obtained by replacing $T_A^{\boldsymbol{\mu}}$ with $(T_A^{\boldsymbol{\mu}} + \alpha P_A)$ in (2.20); similarly, Φ_α^{T} is constructed with weights ρ^α that we get by the same substitution in (2.13).

The weights ρ^α and ρ_K^α are absolutely convergent series in matrix elements of $\{T_A^{\boldsymbol{\mu}} + \alpha P_A\}$, therefore $\rho^\alpha \to \rho$ and $\rho_K^\alpha \to \rho_K$ as $\alpha \to 0$. Hence $\Phi_\alpha^{\mathrm{T}} \to \Phi^{\mathrm{T}}$, and also

$$\sum_{C, \operatorname{supp} C \cap \mathcal{A}_K \neq \emptyset} \Phi_\alpha^{\mathrm{T}}(C) \to \sum_{C, \operatorname{supp} C \cap \mathcal{A}_K \neq \emptyset} \Phi^{\mathrm{T}}(C). \tag{2.32}$$

Since the expression (2.31) for $\langle K \rangle^\alpha$ is absolutely convergent, this implies that $\langle K \rangle^\alpha \to \langle K \rangle$.

□

3. The Hubbard model

The phase space of the Hubbard model is the Fock space of antisymmetric wave functions on $\Lambda \subset \mathbb{L}$. A convenient basis is the one in occupation numbers of position operators. It is given by $\{ |n_\Lambda\rangle \}_{n_\Lambda \in \Omega^\Lambda}$ where $\Omega = \{0, \uparrow, \downarrow, 2\}$. The Hamiltonian is

$$H_\Lambda = -t \sum_{\substack{<x,y> \subset \Lambda \\ \sigma \in \{\uparrow,\downarrow\}}} c_{x\sigma}^\dagger c_{y\sigma} + U \sum_{x \in \Lambda} n_{x\uparrow} n_{x\downarrow} - \mu \sum_{x \in \Lambda} (n_{x\uparrow} + n_{x\downarrow}) \tag{3.1}$$

where the first sum is over nearest neighbours $x, y \in \Lambda$. The first term represents the kinetic energy, the second one is the local repulsion between electrons, and the last term is the chemical potential multiplying the total number of electrons.

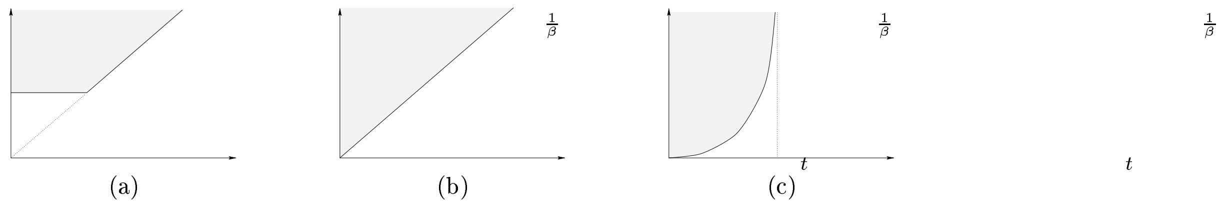

Figure 1. Domains of analyticity, stemming from (a) high temperature expansions, (b) domain $\mathcal{D}_1$ of Theorem 3.1, and (c) domain $\mathcal{D}_2$ of the same theorem.

High temperature expansions yield analyticity of the free energy for all β such that

$$\beta t < \text{const} \quad \text{and} \quad \beta U < \text{const},$$

see Fig. 1; the domain of analyticity may be extended, as we see now. Let χ be the maximum coordination number of $\mathbb{L}$ ($\chi = 2\nu$ for ν-dimensional square lattice), and $\epsilon = (2\chi \beth \, \mathrm{e}^{\phi} \, \phi^2)^{-1}$.

Theorem 3.1 (Analyticity in the Hubbard model)**.**
Let $\Delta = \min(\mu, U - \mu)$. Thermodynamic limits of the free energy and of the Gibbs state exist in the domain $\mathcal{D}_1 \cup \mathcal{D}_2$, where

$$\mathcal{D}_1 = \big\{ (\beta, t, \mu) : \beta t < \epsilon \big\}$$

and

$$\mathcal{D}_2 = \big\{ (\beta, t, \mu) : 0 < \mu < U \text{ and } \beta \tfrac{t^2}{\Delta} < 2\chi\epsilon^2 (1 - 2t/\epsilon\Delta) \big\}.$$

The free energy is analytic, and the Gibbs state is exponentially clustering and unique (that is, stable against boundary conditions ∂^Λ, $\|\partial^\Lambda\|_c < 1$, and external perturbations P, $\|P\|_c < \infty$ for a sufficiently big c).

Remark that the domain $\mathcal{D}_2$ is meaningful only if t is small enough, namely $\frac{t}{\Delta} < \frac{1}{2}\epsilon$. For the 3-dimensional square lattice, we find for $\mathcal{D}_1$ the condition $\beta t < 1.75... \cdot 10^{-4}$, and for $\mathcal{D}_2$, $\frac{\beta t^2}{\Delta} < 3.68... \cdot 10^{-7}(1 - 1.14... \cdot 10^4 \frac{t}{\Delta})$. Of course, the domain of analyticity is much larger than these domains, where analyticity is proven to hold.

These properties likely hold for all $\beta < \infty$ in dimension 1, and possibly also in dimension 2. When $\nu \geqslant 3$, a domain with antiferromagnetic phase is expected for $t \ll U$ and $\beta t^2/U >$ const. Such a phase can be proven in the *asymmetric* Hubbard model, where electrons of different spins are assumed to have different hopping parameters [KL, LM, MM, DFF] (see also [DFFR] and [KU] for two general methods to study rigorously such situations). Assuming this to be true in the standard Hubbard model, we observe that the condition for the domain $\mathcal{D}_2$ is qualitatively correct; this is illustrated in Fig. 2.

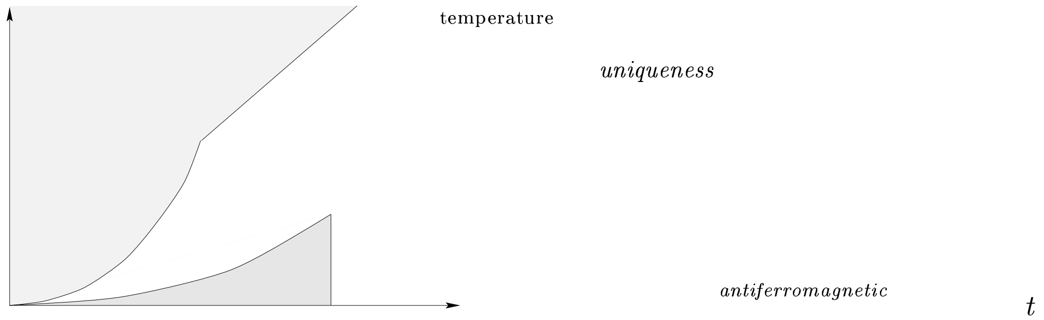

FIGURE 2. Phase diagram of the Hubbard model. Antiferromagnetic phase is expected for dimension $\nu \geqslant 3$; it can be proven when $\nu \geqslant 2$ for the *asymmetric* model.

Proof of Theorem 3.1, domain $\mathcal{D}_1$. The classical free energy of the Hubbard model is easily computed and is given by

$$f_0(\beta, \mu) = -\frac{1}{\beta} \log\big[1 + 2\,\mathrm{e}^{\beta\mu} + \mathrm{e}^{-\beta U + 2\beta\mu}\big]. \tag{3.2}$$

Let us write the kinetic operator $-t \sum_{\boldsymbol{A} \subset \Lambda} T_{\boldsymbol{A}}$ where $\boldsymbol{A} = (<x,y>, \sigma)$ and the notation $\boldsymbol{A} \subset \Lambda$ means $x, y \in \Lambda$; $T_{\boldsymbol{A}} = c^\dagger_{x\sigma} c_{y\sigma}$.

The expression (2.13) for $\rho(\mathcal{A})$ takes the following form

$$\rho(\mathcal{A}) = \mathrm{e}^{\beta f_0(\beta,\mu)|\mathcal{A}|} \sum_{m \geqslant 1} t^m \sum_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_m} \sum_{n_{\mathcal{A}} \in \Omega^{\mathcal{A}}} \int_{0<\tau_1<\dots<\tau_m<\beta} \mathrm{d}\tau_1 \dots \mathrm{d}\tau_m$$

$$\langle n_{\mathcal{A}}| \, \mathrm{e}^{-\tau_1 \sum_{x\in\mathcal{A}} V_x^\mu} \, T_{\boldsymbol{A}_1} \, \mathrm{e}^{-(\tau_2-\tau_1)\sum_{x\in\mathcal{A}} V_x^\mu} \dots T_{\boldsymbol{A}_m} \, \mathrm{e}^{-(\beta-\tau_m)\sum_{x\in\mathcal{A}} V_x^\mu} \, |n_{\mathcal{A}}\rangle \quad (3.3)$$

with a restriction on the sum over $\boldsymbol{A}_1, \dots, \boldsymbol{A}_m$, namely their union yields $\mathcal{A}$ and they are connected in the sense of the graph $\mathcal{G}$ described above. Notice that here $\rho(\mathcal{A}) = 0$ if $\mathcal{A}$ is not connected. The expression for $\rho_K(\mathcal{A}_K)$ is similar, compare with (2.20).

We obtain the domain $\mathcal{D}_1$ of Theorem 3.1 by proceeding as before. Namely, we bound the matrix element with $\mathrm{e}^{-\beta e_0^\mu |\mathcal{A}|} \leqslant \mathrm{e}^{-\beta f_0(\beta,\mu)|\mathcal{A}|}$. A few observations allow to slightly optimize the bound for $\rho(\mathcal{A})$. First, there are no more than $\frac{\chi}{2}|\mathcal{A}|$ sets of nearest neighbours in $\mathcal{A}$. Second, if we choose $\boldsymbol{A}_1, \dots, \boldsymbol{A}_m$ (such that they cover $\mathcal{A}$), then there is at most one configuration $n_{\mathcal{A}}$ such that $\langle n_{\mathcal{A}}| \, T_{\boldsymbol{A}_1} \dots T_{\boldsymbol{A}_m} \, |n_{\mathcal{A}}\rangle$ differs from 0.

Therefore we obtain the bound

$$\begin{aligned} |\rho(\mathcal{A})| &\leqslant \mathrm{e}^{-(c+1)|\mathcal{A}|} \sum_{m \geqslant 0} \frac{(\beta t)^m}{m!} (\frac{\chi}{2}|\mathcal{A}|)^m 4^m \, \mathrm{e}^{(c+1)m} \\ &\leqslant \mathrm{e}^{-c|\mathcal{A}|}, \end{aligned} \quad (3.4)$$

assuming that

$$2\chi \, \mathrm{e}^{c+1} \, \beta t \leqslant 1. \quad (3.5)$$

Here the polymers are connected sets; in this case, Proposition 2.2 holds with $2\daleth$ replaced by $\daleth$. Therefore the condition (3.5) must be fulfilled with $c_0 = \log\daleth + \phi - 1 + 2\log\phi$. When this inequality is strict, it also holds with $c > c_0$, so that we obtain exponential clustering and stability against perturbations or boundary interactions.

□

Proof of Theorem 3.1, domain $\mathcal{D}_2$. Domain $\mathcal{D}_2$ benefits from the following geometric representation (see Fig. 3 for intuition). First we let $n_{\mathcal{A}}^m = n_{\mathcal{A}}$, then $|n_{\mathcal{A}}^{m-1}\rangle = \pm T_{\boldsymbol{A}_m} |n_{\mathcal{A}}^m\rangle$, $\dots$, $|n_{\mathcal{A}}^1\rangle = \pm T_{\boldsymbol{A}_2} |n_{\mathcal{A}}^2\rangle$, $|n_{\mathcal{A}}^m\rangle = \pm T_{\boldsymbol{A}_1} |n_{\mathcal{A}}^1\rangle$. The last condition follows by cyclicity of the trace.

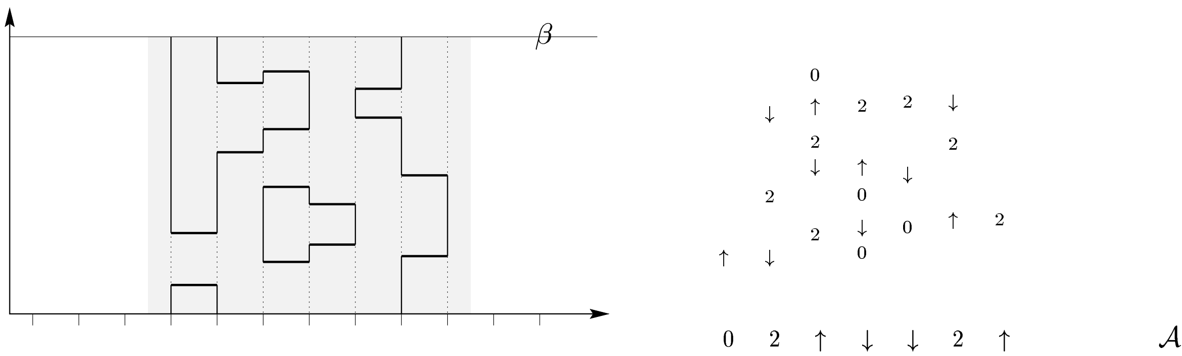

FIGURE 3. Three loops.

Next, if $\boldsymbol{A}_j = (<x_j, y_j>, \sigma)$, we define horizontal bonds $\mathcal{B} \subset \mathbb{R}^\nu \times [0,\beta]_{\text{per}}$

$$\mathcal{B} = \bigcup_{j=1}^m \overline{x_j y_j} \times \{\tau_j\}$$

where $\overline{x_j y_j} \subset \mathbb{R}^\nu$ is the segment joining x_j and y_j. We consider vertical segments

$$\mathcal{S} = \bigcup_{j=0}^{m} \big\{ x \in \mathbb{L} : n_x^j \in \{0,2\} \big\} \times [\tau_j, \tau_{j+1}],$$

where we set $\tau_0 = 0$, $\tau_{m+1} = \beta$ and $n^0 = n^m$. Actually, $\mathcal{S}$ is a subset of $\mathbb{L} \times [0,\beta]_{\text{per}}$; but with a small abuse of notation, we consider $\mathcal{S} \subset \mathbb{R}^\nu \times [0,\beta]_{\text{per}}$.

The set $\mathcal{B} \cup \mathcal{S}$ decomposes into a finite number of closed circuits that we call *loops*.[3] To be precise, a loop ℓ is a pair $(\operatorname{supp}\ell, \boldsymbol{A}(\ell))$ where $\operatorname{supp}\ell \subset \mathbb{R}^\nu \times [0,\beta]_{\text{per}}$ is the support of ℓ, and $\boldsymbol{A}(\ell) = (\boldsymbol{A}_1, \dots, \boldsymbol{A}_{m(\ell)})$ are successive applications of operators $T_{\boldsymbol{A}_1}, \dots, T_{\boldsymbol{A}_{m(\ell)}}$; here $m(\ell)$ is the number of horizontal segments ("jumps") in ℓ that we mark out $1, \dots, m$ in increasing vertical coordinates, and $T_{\boldsymbol{A}_j}$ is the operator associated with the segment j.

The weight $\rho(\mathcal{A})$ can be written as an integral over sets of loops, with many restrictions. In particular, each vertical line $\{x\} \times [0,\beta]_{\text{per}}$, $x \in \mathcal{A}$, must intersect at least one loop. As a consequence, to a given set of loops corresponds at most one sequence of configurations $(n_{\mathcal{A}}^1, \dots, n_{\mathcal{A}}^m)$.

$$\rho(\mathcal{A}) = \mathrm{e}^{\beta f_0(\beta,\mu)|\mathcal{A}|} \, \mathrm{e}^{\beta\mu|\mathcal{A}|} \sum_{k \geqslant 1} \frac{1}{k!} \int \mathrm{d}\ell_1 \dots \mathrm{d}\ell_k \, \varepsilon(\ell_1, \dots, \ell_k) \prod_{j=1}^{k} z(\ell_j) \tag{3.6}$$

where

$$\varepsilon(\ell_1, \dots, \ell_k) = \langle n_{\mathcal{A}} | \prod_{\boldsymbol{A} \in (\ell_1, \dots, \ell_k)} T_{\boldsymbol{A}} \, | n_{\mathcal{A}} \rangle \tag{3.7}$$

and

$$z(\ell) = t^{m(\ell)} \, \mathrm{e}^{-\mu|\ell|_0 - (U-\mu)|\ell|_2} \,. \tag{3.8}$$

Let us explain these notations. The configuration $n_{\mathcal{A}}$ is defined by $(\ell_1, \dots, \ell_k)$; namely, if $\{x\} \times \{0\} \in \operatorname{supp}\ell_j$, we know that $n_x \in \{0,2\}$; looking at the first occurrence of an operator $T_{\boldsymbol{A}}$, $\boldsymbol{A} \ni x$, we can check whether a particle is created or annihilated at x, in which case $n_x = 0$ or $n_x = 2$ respectively. Similarly, if $\{x\} \times \{0\} \notin \cup_j \operatorname{supp}\ell_j$, we have $n_x \in \{\uparrow, \downarrow\}$; if the first operator $T_{\boldsymbol{A}}$ such that $\boldsymbol{A} \ni x$ creates an $\uparrow$ electron, or annihilate a $\downarrow$ electron, we have $n_x = \downarrow$; otherwise $n_x = \uparrow$.

The product is over all operators $T_{\boldsymbol{A}}$ that occur in the loops, ordered in decreasing vertical coordinate of the corresponding horizontal segment. Notice that $\varepsilon(\cdot) \in \{-1, 0, 1\}$.

The vertical length of a loop is $|\ell| = |\ell|_0 + |\ell|_2$, where $|\ell|_j$ denotes the length of all vertical segments where the configuration takes value j. We bound

$$z(\ell) \leqslant t^{m(\ell)} \, \mathrm{e}^{-\Delta|\ell|} \,. \tag{3.9}$$

We have the following bound for $\rho(\mathcal{A})$ (we use $f_0(\beta,\mu) \leqslant \mu$):

$$|\rho(\mathcal{A})| \leqslant \mathrm{e}^{-(c+1)|\mathcal{A}|} \sum_{k \geqslant 1} \frac{1}{k!} \Big[\int_{\operatorname{supp}\ell \subset \mathcal{A} \times [0,\beta]_{\text{per}}} \mathrm{d}\ell (t \, \mathrm{e}^{c+1})^{m(\ell)} \, \mathrm{e}^{-\Delta|\ell|} \Big]^k . \tag{3.10}$$

The integral over one loop with m jumps may be evaluated in the following way.

1. We choose two nearest neighbour sites in $\mathcal{A}$ (there are less than $\chi|\mathcal{A}|$ possibilities), we integrate over a number τ in $[0,\beta]$, and we choose a spin; we obtain the first jump of the loop.

[3] This representation has many similarities with that of [MM], introduced for the Falicov-Kimball model.

2. We decide whether the loop is going up or down in the vertical dimension, we integrate over the vertical distance, we choose a neighbour of our site and a spin; integration over the vertical distance is bounded by
$$\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\Delta\tau} = \frac{1}{\Delta}.$$
We repeat this procedure until the last jump but one.
3. For the last jump, we decide whether the vertical direction is up or down, and we integrate over the distance, yielding a factor $1/\Delta$. Then the loop completes itself in a unique way (provided there is a way); therefore there are no sums over nearest neighbour and spin.

Since the first jump is arbitrary, we can divide by m the contribution of loops with m jumps. Notice that the second step is superfluous when $m = 2$. We obtain
$$\begin{aligned}\sum_{m\geqslant 2} \int_{\substack{\operatorname{supp}\ell\subset\mathcal{A}\times[0,\beta]_{\text{per}}\\ m(\ell)=m}} \mathrm{d}\ell (t\,\mathrm{e}^{c+1})^m \mathrm{e}^{-\Delta|\ell|} &\leqslant \sum_{m\geqslant 2} \frac{1}{m}(t\,\mathrm{e}^{c+1})^m \chi|\mathcal{A}|\beta 2[2\frac{1}{\Delta}\chi 2]^{m-2} 2\frac{1}{\Delta} \\ &\leqslant |\mathcal{A}|\frac{\beta t^2}{\Delta} 2\chi\,\mathrm{e}^{2(c+1)} \frac{1}{1-4\chi\,\mathrm{e}^{c+1}\frac{t}{\Delta}}. \end{aligned} \tag{3.11}$$
From the conditions $\frac{t}{\Delta} < \frac{1}{4\chi}\mathrm{e}^{-(c+1)}$ and $\frac{\beta t^2}{\Delta} < \frac{1}{2\chi}\mathrm{e}^{-2(c+1)}(1-4\chi\,\mathrm{e}^{c+1}\frac{t}{\Delta})$, we finally have
$$|\rho(\mathcal{A})| \leqslant \mathrm{e}^{-(c+1)|\mathcal{A}|} \sum_{k\geqslant 1} \frac{1}{k!}|\mathcal{A}|^k \leqslant \mathrm{e}^{-c|\mathcal{A}|} \tag{3.12}$$
with $c = \log\beth + \phi - 1 + 2\log\phi$. Since the weights of polymers are analytic functions of β, μ, so is the free energy in the thermodynamic limit.

Existence and properties of the Gibbs state are readily obtained by repeating the proofs of Section 2, using above estimates.

□

4. The Bose-Hubbard model

The Bose-Hubbard model describes a lattice system of interacting bosons. In a finite volume $\Lambda \in \mathbb{L}$, the phase space is the Hilbert space with basis $\{\,|n_\Lambda\rangle : n_\Lambda \in \mathbb{N}^\Lambda\}$; the Hamiltonian consists in a kinetic operator and a local repulsive interaction:
$$H_\Lambda = t\sum_{<x,y>\subset\Lambda} c_x^\dagger c_y + U\sum_{x\in\Lambda}(\hat{n}_x^2 - \hat{n}_x) - \mu\sum_{x\in\Lambda}\hat{n}_x. \tag{4.1}$$
The first term is a standard hopping operator between nearest-neighbours; the second term describes the local repulsion between bosons (each pair of particles at a given site contributes for $2U$); the chemical potential $\mu \in \mathbb{R}$ controls the density of the system.

It has been introduced in [FWGF] and despite its simplicity, it has very interesting phase diagram, see Fig. 4. A phase transition insulator-superfluid is expected when the hopping coefficient increases. Of course one would like to have mathematical statements to support this, but the superfluid phase of interacting particles is hard to study.[4] On the other hand, one can tame the insulating phase much more easily. When t/U is small, and $2U(k-1) < \mu < 2Uk$, it is possible to show that the Gibbs state exists at low temperature

[4]The best rigorous statements concern the hard-core Bose-Hubbard model, where off-diagonal long-range order can be proven using reflection positivity for special value of μ [DLS]; this constitutes a beautiful result, although it does not allow to study pure states.

and $\frac{1}{Z}\,\mathrm{e}^{-\beta H}$ is close to the projector onto the configuration $n_x = k$ for all $x \in \mathbb{L}$; moreover, the density of the (quantum) ground state is not only close, but equal to k [BKU].

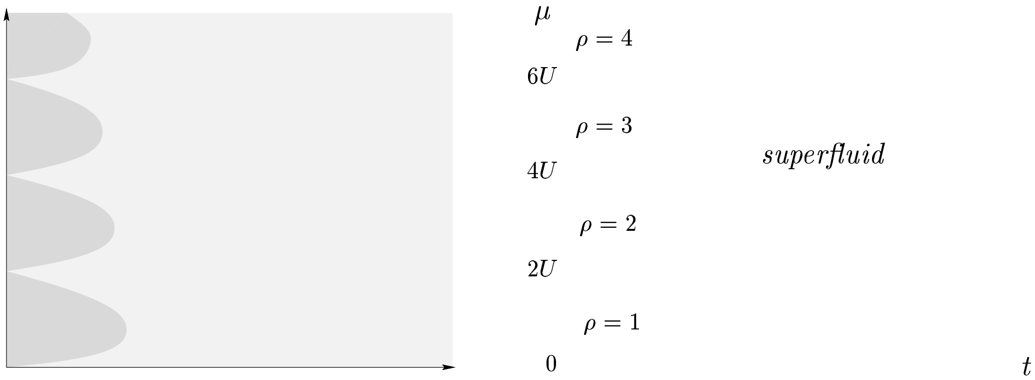

FIGURE 4. Zero temperature phase diagram for the Bose-Hubbard model. Lobes are incompressible phases with integer densities.

We show here that these phases can be reached from the high temperatures without phase transition (Fig. 5). Theorem 2.1 does not apply here, because the single site phase space Ω is infinite. Actually, boson systems and unbounded spin systems present some difficulties at high temperatures, since partition functions diverge at $\beta = 0$. Results for small β have been obtained in [PY]. We can prove analyticity of the free energy and existence of Gibbs state when βt is small, but we are unable to show stability against perturbations, or against boundary interactions, that do not conserve the total number of particles (see discussion in Section 2.2).

Because the phase space has infinite dimension, we need to define the Gibbs state as a functional over possibly unbounded operators, as for instance $c_x^\dagger$, or number operators; but not all operators can be considered. In order to define a suitable class of local operators, let $\tilde{N}_A$ be the number operator in A with minimum eigenvalue 1, i.e.

$$\tilde{N}_A \,|n_A\rangle = \begin{cases} |n_A\rangle & \text{if } n_x = 0 \text{ for all } x \in A \\ \left(\sum_{x\in A} n_x\right) |n_A\rangle & \text{otherwise;} \end{cases} \tag{4.2}$$

this operator has an inverse which is defined everywhere. Defining the *boson norm* of a local operator K by

$$\|K\|^{\mathrm{boson}} \doteqdot \sup_{n,n' \in \mathbb{N}^{\operatorname{supp} K}} \left| \langle n|\, \tilde{N}_{\operatorname{supp} K}^{-\frac{1}{2}} K \tilde{N}_{\operatorname{supp} K}^{-\frac{1}{2}} \,|n'\rangle \right|, \tag{4.3}$$

we consider the class $\mathcal{K}$ of local operators with finite boson norm. It is not hard to check that $\|\tilde{N}_{\{x,y\}}^{-\frac{1}{2}} c_x^\dagger c_y \tilde{N}_{\{x,y\}}^{-\frac{1}{2}}\| \leqslant 1$, and thus $c_x^\dagger c_y \in \mathcal{K}$.

Theorem 4.1 (Analyticity in the Bose-Hubbard model).
There exists a function $\epsilon(U,\mu) > 0$ such that for all β with $\beta t < \epsilon$,

- *the free energy $f(\beta,\mu)$ exists in the thermodynamic limit and is analytic in β and μ;*
- *the Gibbs state $\langle\cdot\rangle : \mathcal{K} \to \mathbb{C}$, with free or periodic boundary conditions, converges weakly in the thermodynamic limit and is exponentially clustering.*

Remark: adding a hard-core condition on the model, e.g. by limiting the number of bosons at a given site to $M < \infty$, Theorem 4.1 becomes a consequence of Theorem 2.1

(with moreover uniqueness, and absence of superfluidity). Actually, we expect Theorem 2.1 to hold when T has finite *boson interaction norm* $\|T\|_c^{\text{boson}}$

$$\|T\|_c^{\text{bosons}} \doteq \sup_{x\in\mathbb{L}} \sum_{A\ni x} \|\tilde{N}_A^{-\frac{1}{2}} T_A \tilde{N}_A^{-\frac{1}{2}}\| \, \mathrm{e}^{c\|A\|}\,. \tag{4.4}$$

However, we are unable to handle such a general situation; the Bose-Hubbard model conserves the total number of particles, and this property plays a crucial technical role.

The present result, together with [BKU], shows that the free energy is analytic in a domain that includes low and high temperatures, corresponding to insulating phase; see Fig. 5.

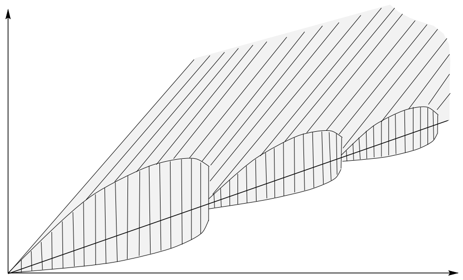

FIGURE 5. Analyticity of the free energy and existence of the Gibbs state can be proven on the left of the grey frontier.

Proof of Theorem 4.1. Expanding $\operatorname{Tr} K\,\mathrm{e}^{-\beta H_\Lambda}$, we obtain (2.20) and (2.21). We show now exponential decay for $\rho_K(\mathcal{A}_K)$; similar (and simpler) considerations lead to exponential decay for $\rho(\mathcal{A})$. Analyticity of the free energy and existence of Gibbs state are then consequences of Proposition 2.2 for cluster expansion.

The condition $\cup_{j=1}^m A_j \cup \operatorname{supp} K = \mathcal{A}_K$ implies

$$|\mathcal{A}_K| \leqslant |\operatorname{supp} K| + m \tag{4.5}$$

(recall that $|A_j| = 2$ for all $1 \leqslant j \leqslant m$). Then

$$\begin{aligned}
|\rho_K(\mathcal{A}_K)| \leqslant\ & \mathrm{e}^{-c|\mathcal{A}_K|}\,\mathrm{e}^{c|\operatorname{supp} K|}\,\mathrm{e}^{\beta f_0(\beta,\mu)|\mathcal{A}_K|} \\
& \sum_{m\geqslant 0} t^m \mathrm{e}^{cm} \sum_{<x_1,y_1>,\dots,<x_m,y_m>\subset\mathcal{A}_K} \int_{0<\tau_1<\dots<\tau_m<\beta} \mathrm{d}\tau_1\dots\mathrm{d}\tau_m \sum_{n_{\mathcal{A}_K}} \\
& \langle n_{\mathcal{A}_K}|\,K\,\mathrm{e}^{-\tau_1\sum_{x\in\mathcal{A}_K}V_x^\mu}\,c_{x_1}^\dagger c_{y_1}\,\mathrm{e}^{-(\tau_2-\tau_1)\sum_{x\in\mathcal{A}_K}V_x^\mu}\dots c_{x_m}^\dagger c_{y_m}\,\mathrm{e}^{-(\beta-\tau_m)\sum_{x\in\mathcal{A}_K}V_x^\mu}\,|n_{\mathcal{A}_K}\rangle.
\end{aligned} \tag{4.6}$$

For given $|n_{\mathcal{A}_K}\rangle$, let $n_{\mathcal{A}_K}^1, \dots, n_{\mathcal{A}_K}^m \in \mathbb{N}^{\mathcal{A}_K}$ such that

$$\begin{aligned}
|n_{\mathcal{A}_K}^m\rangle &\sim c_{x_m}^\dagger c_{y_m}\,|n_{\mathcal{A}_K}\rangle \\
|n_{\mathcal{A}_K}^{m-1}\rangle &\sim c_{x_{m-1}}^\dagger c_{y_{m-1}}\,|n_{\mathcal{A}_K}\rangle \\
&\ \ \vdots \\
|n_{\mathcal{A}_K}^1\rangle &\sim c_{x_1}^\dagger c_{y_1}\,|n_{\mathcal{A}_K}^2\rangle.
\end{aligned}$$

Then the matrix element in the above equation takes form

$$\begin{aligned}&\langle n_{\mathcal{A}_K}|\,K\,|n^1_{\mathcal{A}_K}\rangle\,\mathrm{e}^{-\tau_1\sum_{x\in\mathcal{A}_K}\langle n^1_{\mathcal{A}_K}|\,V_x\,|n^1_{\mathcal{A}_K}\rangle}\,\langle n^1_{\mathcal{A}_K}|\,c^\dagger_{x_1}c_{y_1}\,|n^2_{\mathcal{A}_K}\rangle\dots\\ &\qquad\qquad\dots\langle n^m_{\mathcal{A}_K}|\,c^\dagger_{x_m}c_{y_m}\,|n_{\mathcal{A}_K}\rangle\,\mathrm{e}^{-(\beta-\tau_m)\sum_{x\in\mathcal{A}_K}\langle n_{\mathcal{A}_K}|\,V_x\,|n_{\mathcal{A}_K}\rangle}\\ &\leqslant \|K\|^{\text{boson}}\big(1+\sum_{x\in\mathcal{A}_K}n_x\big)\prod_{j=1}^{m+1}\mathrm{e}^{-(\tau_j-\tau_{j-1})\sum_{x\in\mathcal{A}_K}\langle n^j_{\mathcal{A}_K}|\,V_x\,|n^j_{\mathcal{A}_K}\rangle}\prod_{j=1}^{m}(n^{j+1}_{x_j}+n^{j+1}_{y_j});\end{aligned}\tag{4.7}$$

in the last line we set $\tau_0 = 0$, $\tau_{m+1} = \beta$ and $n^{m+1}_{\mathcal{A}_K} = n_{\mathcal{A}_K}$. We used the inequality $\langle n'|\,c^\dagger_x c_y\,|n\rangle \leqslant n_x + n_y$. Now

$$\prod_{j=1}^{m+1}\mathrm{e}^{-(\tau_j-\tau_{j-1})\sum_{x\in\mathcal{A}_K}\langle n^j_{\mathcal{A}_K}|\,V_x\,|n^j_{\mathcal{A}_K}\rangle} \leqslant \sum_{j=1}^{m+1}\mathrm{e}^{-\beta\sum_{x\in\mathcal{A}_K}\langle n^j_{\mathcal{A}_K}|\,V_x\,|n^j_{\mathcal{A}_K}\rangle}.\tag{4.8}$$

The sum over $n_{\mathcal{A}_K}$ can be replaced by a sum over $n^j_{\mathcal{A}_K}$; since this last bound does not depend any more on $n^i_{\mathcal{A}_K}$, $i\neq j$, we can sum over $<x_i,y_i>\subset\mathcal{A}_K$, and since $\sum_{x\in\mathcal{A}_K}n^i_x = \sum_{x\in\mathcal{A}_K}n^j_x$ for all i, the last product of (4.7) is bounded by

$$\prod_{i=1}^{m}\sum_{<x_i,y_i>\subset\mathcal{A}_K}(n^{i+1}_{x_i}+n^{i+1}_{y_i}) \leqslant \big(\chi\sum_{x\in\mathcal{A}_K}n_x\big)^m.\tag{4.9}$$

Collecting these estimates, we get

$$\begin{aligned}|\rho_K(\mathcal{A}_K)| \leqslant{}& \mathrm{e}^{-c|\mathcal{A}_K|}\,\|K\|^{\text{boson}}\,\mathrm{e}^{c|\operatorname{supp}K|}\,\mathrm{e}^{\beta f_0(\beta,\mu)|\mathcal{A}_K|}\\ &\sum_{m\geqslant 0}(m+1)\,t^m\,\mathrm{e}^{cm}\frac{\beta^m}{m!}\chi^m\sum_{n_{\mathcal{A}_K}}\big(1+\sum_{x\in\mathcal{A}_K}n_x\big)^{m+1}\mathrm{e}^{-\beta\sum_{x\in\mathcal{A}_K}\langle n_{\mathcal{A}_K}|\,V_x\,|n_{\mathcal{A}_K}\rangle}.\end{aligned}\tag{4.10}$$

When βt is small, we have $(\beta t\,\mathrm{e}^c\,\chi)^m \leqslant (\sqrt{\beta t\,\mathrm{e}^c\,\chi})^{m+1}$; therefore

$$\sum_{m\geqslant 0}\frac{m+1}{m!}\Big[\big(1+\sum_{x\in\mathcal{A}_K}n_x\big)\sqrt{\beta t\,\mathrm{e}^c\,\chi}\Big]^{m+1} \leqslant \exp\Big\{\sqrt{\beta t\,\mathrm{e}^{c+2}\,\chi}\big(1+\sum_{x\in\mathcal{A}_K}n_x\big)\Big\}.\tag{4.11}$$

We obtain finally

$$|\rho_K(\mathcal{A}_K)| \leqslant \mathrm{e}^{-c|\mathcal{A}_K|}\,\|K\|^{\text{boson}}\,\mathrm{e}^{c|\operatorname{supp}K|}\Big\{\mathrm{e}^{\beta f_0(\beta,\mu)}\sum_{n=0}^{\infty}\mathrm{e}^{n\sqrt{\beta t\,\mathrm{e}^{c+2}\,\chi}}\,\mathrm{e}^{-\beta[U(n^2-n)-\mu n]}\Big\}^{|\mathcal{A}_K|}.\tag{4.12}$$

The sum over n is absolutely convergent, so that the quantity between brackets converge to 1 when $\beta t\to 0$. Having chosen c sufficiently large to ensure validity of cluster expansion, we see that ρ_K decays exponentially when βt is small enough.

□

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