Anderson Localization for a multidimensional model including long range potentials and displacements

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Abstract

We give a short summary on how to combine and extend results of Combes and Hislop [CH1] (short range Anderson model with additional displacements), Kirsch, Stollmann and Stolz [KSS1] and [KSS2] (long range Anderson model without displacements) to get localization in an energy interval above the infimum of the almost sure spectrum for a continuous multidimensional Anderson model including long range potentials and displacements.

1 Introduction

We treat the multidimensional Anderson model including displacements as in [CH1], using a long range single site potential $\wp$ as in [KSS2]. To be more precise, we consider a continuous function $\wp : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ and two independent families $(\lambda_j : (\Omega', P') \longrightarrow \mathbb{R})_{j\in\mathbb{Z}^d}$ and $(\xi_j : (\Omega'', P'') \longrightarrow \mathbb{R}^d)_{j\in\mathbb{Z}^d}$ of independent identically distributed random variables satisfying the following conditions:

1. The joint probability distribution

$$\begin{aligned} F = F_{P'\circ\lambda_j^{-1}} : \mathbb{R} &\longrightarrow [0,1] \\ t &\longmapsto P'\circ\lambda_j^{-1}(]-\infty,t[) = \int_{-\infty}^{t} g(x)dx \end{aligned}$$

   has a density $g \in L^\infty(\mathbb{R})$ with respect to Lebesgue measure μ, and there is a finite interval $[0, g_0]$ containing the support of g.

2. There is an $R \in]0, \frac{1}{2}[$, such that $|\xi_j(\omega'')| < R$, for every $j \in \mathbb{Z}^d$ and $\omega'' \in \Omega''$. Here $|\cdot|$ denotes the maximum norm in $\mathbb{R}^d$.

3. We further assume that there is a $c_0 > 0$ such that $c_0 1^0_{1+2R} \leq \wp$ holds, where 1^x_l denotes the characteristic function of the box $\Lambda(x,l)$ with sidelength $l > 0$ centered at a point $x \in \mathbb{R}^d$. We consider two conditions for $\wp(x)$, as $|x|$ tends to ∞:

(a) the "long range case": There is a $c_1 > 0$ and $\varsigma > 2d$ such that

$$\wp(x) \le c_1 \langle x \rangle^{-\varsigma} := c_1(1+|x|^2)^{-\frac{\varsigma}{2}}$$

holds, for every $x \in \mathbb{R}^d$. As a special case we consider

(b) the "short range case": $\wp$ has compact support.

Using these properties of $\wp$, $(\lambda_j)_{j\in\mathbb{Z}^d}$ and $(\xi_j)_{j\in\mathbb{Z}^d}$, we define a bounded potential

$$V(\omega',\omega'',x) := \sum_{j\in\mathbb{Z}^d} \lambda_j(\omega')\wp(x-j-\xi_j(\omega''))$$

and arrive at the selfadjoint random Schrödinger operator $H(\omega',\omega'') := -\triangle + V(\omega',\omega'')$ in $L^2(\mathbb{R}^d)$ on the domain $\mathcal{D}(H(\omega',\omega'')) = H^2(\mathbb{R}^d)$. Thus the operator H is measurable (a proof can be found in [CL] Prop. V.3.1) in the sense, that $\omega \longmapsto \langle \varphi, (H(\omega)-z)^{-1}\Psi\rangle$ is measurable for all $\varphi, \Psi \in L^2(\mathbb{R}^d)$ and all $z \in \mathbb{C}\backslash\mathbb{R}$. If we denote the path of $(\lambda_j)_{j\in\mathbb{Z}^d}$ as

$$\begin{array}{rcl} \lambda_\bullet : \Omega' & \longrightarrow & \mathbb{R}^{\mathbb{Z}^d}, \\ \omega' & \longmapsto & \lambda_\bullet(\omega') = (\lambda_j(\omega'))_{j\in\mathbb{Z}^d} \end{array}$$

we get an equivalent stochastic process

$$\left(\begin{array}{rcl} \mathrm{pr}_j^{(1)} : (\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d}), P'\circ\lambda_\bullet^{-1}) & \longrightarrow & \mathbb{R} \\ (x_k)_{k\in\mathbb{Z}^d} & \longmapsto & x_j \end{array} \right)_{j\in\mathbb{Z}^d},$$

for the family $(\lambda_j)_{j\in\mathbb{Z}^d}$ and another process $(\mathrm{pr}_j^{(2)} : ((\mathbb{R}^d)^{\mathbb{Z}^d}, \mathbb{P}''\circ\xi_\bullet^{-1}) \to \mathbb{R}^d)_{j\in\mathbb{Z}^d}$ equivalent to $(\xi_j)_{j\in\mathbb{Z}^d}$. Using independence and identical distribution one easily verifies $\mathbb{P}'\circ\lambda_\bullet^{-1} = \mathbb{P}_0^{\otimes\mathbb{Z}^d} := (P'\circ\lambda_0^{-1})^{\otimes\mathbb{Z}^d}$ and $\mathbb{P}''\circ\xi_\bullet^{-1} = \mathbb{P}_1^{\otimes\mathbb{Z}^d} := (P''\circ\xi_0^{-1})^{\otimes\mathbb{Z}^d}$ for all cylinder sets in $\mathcal{B}(\mathbb{R}^{\mathbb{Z}^d})$ resp. $\mathcal{B}((\mathbb{R}^d)^{\mathbb{Z}^d})$, hence for all elements of $\mathcal{B}(\mathbb{R}^{\mathbb{Z}^d})$ resp. $\mathcal{B}((\mathbb{R}^d)^{\mathbb{Z}^d})$. As $(\lambda_j)_{j\in\mathbb{Z}^d}$ and $(\xi_j)_{j\in\mathbb{Z}^d}$ are independent families, we construct a new probability space $(\Omega,\mathcal{B}(\Omega),\mathbb{P}) = (\mathbb{R}^{\mathbb{Z}^d}\times(\mathbb{R}^d)^{\mathbb{Z}^d}, \mathcal{B}(\Omega), \mathbb{P}_0^{\otimes\mathbb{Z}^d}\otimes\mathbb{P}_1^{\otimes\mathbb{Z}^d})$, where $\mathcal{B}(\Omega)$ is the completion of the product-σ-algebra on Ω with respect to the measure $\mathbb{P}$ containing all $\mathbb{P}$-ze rosets, and an equivalent stochastic process (the canonical representant)

$$\left(\begin{array}{rcl} \mathrm{pr}_j : \Omega & \longrightarrow & \mathbb{R}\times\mathbb{R}^d \\ \omega = (\omega_k)_{k\in\mathbb{Z}^d} & \longmapsto & \omega_j \end{array} \right)_{j\in\mathbb{Z}^d}$$

for $((\lambda_j,\xi_j) : \Omega'\times\Omega'' \to \mathbb{R}\times\mathbb{R}^d)_{j\in\mathbb{Z}^d}$, see e.g. [HT] for more details about equivalent stochastic processes. In this way we identify $\omega = (\omega_j)_{j\in\mathbb{Z}^d} = (\omega_j',\omega_j'')_{j\in\mathbb{Z}^d} \equiv (\lambda_j(\omega'),\xi_j(\omega'')) \in \Omega$ with $(\omega',\omega'') \in \Omega'\times\Omega''$ and thus the random potential reads as

$$V(\omega,x) = \sum_{j\in\mathbb{Z}^d} \omega_j'\wp(x-j-\omega_j'').$$

The product structure of Ω also immediately shows $\lim_{n\to\infty}\mathbb{P}(\theta_j^{-n}A\cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for all $j \in \mathbb{Z}^d\backslash\{0\}$ and all cylinder sets A, B in $\mathcal{B}(\Omega)$, where $\theta_j : \Omega \to \Omega$ is the measure preserving transformation defined by $\theta_j((\omega_k)_{k\in\mathbb{Z}^d}) = (\omega_{j+k})_{k\in\mathbb{Z}^d}$. This equality extends to all $A, B \in \mathcal{B}(\Omega)$, cf. [Wa], Theorem 1.17, hence shows the mixing property and especially the ergodicity of the system $(\Omega,\mathbb{P},(\theta_j)_{j\in\mathbb{Z}^d})$. As $V(\theta_x\omega,y) = V(\omega,x+y)$, the existence of subsets

$\Sigma, \Sigma_c, \Sigma_s, \Sigma_{ac}, \Sigma_{sc}, \Sigma_{ess}, \Sigma_{pp} \subseteq \mathbb{R}$ such that $\Sigma = \sigma(H(\omega))$, $\Sigma_c = \sigma_c(H(\omega))$, etc. holds for $\mathbb{P}$-almost all $\omega \in \Omega$, is a standard result in the theory of random operators; see e.g. [CL] Prop. V.2.4. Adapting an idea of [KSS1] for the proof of equation (1.1) therein, we obtain $\inf \Sigma = 0$ under the assumptions specified above. Now we state the main theorem of this paper:

Theorem 1.1. *Let $H(\omega)$ be the random Schrödinger operator specified above. There exists an $E_X > \inf \Sigma$, such that $\sigma(H(\omega)) \cap [\inf \Sigma, E_X]$ is pure point spectrum for $\mathbb{P}$-almost all $\omega \in \Omega$. In the short range case the corresponding eigenfunctions are exponentially decaying; in the long range case, the eigenfunctions decay faster than every inverse polynomial.*

The general scheme for a proof of such a theorem includes a Wegner estimate, multiscale analysis, spectral averaging and a criterion to detect pure point spectrum. The first three techniques originate from papers in the eighties, dealing with a discrete model in $l^2(\mathbb{Z}^d)$. In the fundamental work [FS], Fröhlich and Spencer proved exponential decay of the Green's function for large disorder and fixed energy. Then Fröhlich, Martinelli, Scoppola and Spencer [FMSS], Delyon, Levy and Souillard [DLS] and Simon and Wolff [SW] gave three independent proofs for the existence of dense pure point spectrum. Later on these methods were extended and simplified, see e.g., in [DK] or [CL], but these proofs seemed to be restricted to the discrete model. Combes and Hislop extended these ideas to continuous random Schrödinger operators in [CH1]. They treat the continuous Anderson model with displacements for compact supported single site potential $\wp$ and in the case $\wp(x) \leq c_1 \langle x \rangle^{-\varsigma}$, $\varsigma > \frac{9}{2}d$. But the proof of this long range case is considered to be "incomplete" by the authors in [CH2]. Kirsch, Stollmann and Stolz studied a model without random displacements but with an additional periodic potential in [KSS1], discussing effects of the random perturbation on the band structure. In [KSS2] they extended their model to single site potentials $\wp$ satisfying $\varsigma > 2d$. In [KSS3] the case $\varsigma > 4d$ is considered; under this stronger requirement on $\wp$ they proved, that the eigenfunctions are exponentially decaying. As Theorem **??** also covers the long range case, it is an improvement of the results in [CH1].

In Section 2 we start with a proof of the Wegner estimate on the resolvent $(H_\Lambda(\omega) - E - i\varepsilon)^{-1}$ of operators $H_\Lambda(\omega)$, where $H_\Lambda(\omega)$ is obtained as the restriction of $H(\omega)$ to a bounded open box Λ with Dirichlet boundary conditions. Using the geometric resolvent equation (4), we may view the resolvent of $H_\Lambda(\omega)$ as the restriction of the resolvent of $H_{\Lambda'}(\omega)$, with $\Lambda' \supseteq \Lambda$, to functions with support in Λ modulo a boundary term. So provided we control these boundary terms, the geometric resolvent equation enables us to pass from $H_\Lambda(\omega)$ to $H_{\Lambda'}(\omega)$, i.e. to enlarge the boxes under consideration. The multiscale analysis in Section 4 consists of constructing series $(l_n)_{n\in\mathbb{N}} \overset{n\to\infty}{\longrightarrow} \infty$ of length scales and events $(A_n)_{n\in\mathbb{N}}$ with $(1 - \mathbb{P}(A_n))_{n\in\mathbb{N}} \in l^1$ and in finding a lower bound $E_X > \inf \Sigma$, such that we get enough controll on all the boundary terms arrising from the geometrical resolvent equation, when we pass inductively from the operators $(H_{\Lambda(\mathfrak{z},l_n)}(\omega) - E - i\varepsilon)^{-1}$, $\omega \in A_n$, $\mathfrak{z} \in \mathbb{R}^d$, $E \in [\inf \Sigma, E_X]$ to those restricted to boxes of the next length scale $l_{n+1} > l_n$. This result of multiscale analysis allows us to use the Borel-Cantelli Lemma, yielding a key step of the proof in Section 5: We obtain estimates on the decay of the norm of some localized versions $1^y(H(\omega) - E - i\varepsilon)^{-1}1^x$ and $1^x(H(\omega) - E - i\varepsilon)^{-1}\wp(\cdot - \mathfrak{z})$ of the resolvent for $\mathbb{P}$-almost all $\omega \in \Omega$ and for μ-almost all $E \in [\inf \Sigma, E_X]$ as $\varepsilon \searrow 0$ and $|x| \to \infty$. Using these estimates we establish the decay properties of Theorem 1.1 for distributional solutions $\Psi \in H^1_{-\frac{5}{4}}(\mathbb{R}^d)$ of the "perturbed" eigenvalue equation $H(\omega)\Psi + \alpha\wp(\cdot - \mathfrak{z})\Psi = E\Psi$ for $\alpha \neq 0$, $\mathbb{P}$-almost all $\omega \in \Omega$ and μ-almost all $E \in [\inf \Sigma, E_X]$. Due to the fact, that with respect to a spectral measure $\nu_{\alpha,\mathfrak{z},\omega}$ of $H(\omega) + \alpha\wp(\cdot - \mathfrak{z})$ for almost every $E \in \mathbb{R}$ such a distributional solution Ψ exists, we get pure point spectrum and the decay properties of the eigenfunctions for those "perturbed" operators $H(\omega) + \alpha\wp(\cdot - \mathfrak{z})$. The proof of Theorem 1.1 follows in Section 6.4, where we write "unperturbed" operators $H(\tilde{\omega})$ for $\mathbb{P}$-almost all $\tilde{\omega} \in \Omega$ as appropriate "perturbed" operators $H(\omega) + \alpha\wp(\cdot - \mathfrak{z})$, using several relations between sets of $\mathbb{P}$, μ, respectively, $\nu_{\alpha,\mathfrak{z},\omega}$ measure 0.

2 Spectral Averaging and Wegner Estimate

In this Section we make use of the product structure of $\mathbb{P}$ to estimate expectation values via Fubini's theorem. We fix $j \in \mathbb{Z}^d$ and perform the integration with respect to ω'_j first by some inequalities from [CH1] and then integrate the remaining variables. We begin with the following Lemma from [CH1], cf. Lemma 4.1 and Corollary 4.2 there.

Lemma 2.1. *Let $\mathcal{H}$ be a separable Hilbert space and H_0 a selfadjoint operator in $\mathcal{H}$. Let $B, V \geq 0$ be bounded operators on $\mathcal{H}$, such that there is a constant $c_2 > 0$ satisfying $0 \leq c_2 B^2 \leq V$. For $\alpha \in \mathbb{R}$, let π_α be the spectral decomposition of the operator $H(\alpha) := H_0 + \alpha V$ and fix a bounded function $h : \mathbb{R} \to \mathbb{R}$ of compact support. Then for every measurable set $M \subseteq \mathbb{R}$ of finite Lebesgue measure, $\mu(M) < \infty$, one has:*

1. $\left\| \int_{\mathbb{R}} \frac{1}{1+t\alpha^2} \langle u, B\pi_\alpha(M)Bu \rangle d\alpha \right\| \leq \frac{\mu(M)\|u\|^2}{c_2}$, *for $t > 0$ and $u \in \mathcal{H}$,*

2. $\left\| \int_{\mathbb{R}} h(\alpha) \langle u, B\pi_\alpha(M)Bu \rangle d\alpha \right\| \leq \frac{\mu(M)\|u\|^2\|h\|_\infty}{c_2}$, *for $u \in \mathcal{H}$.*

Theorem 2.2 (Wegner estimate). *Let $\Lambda \subseteq \mathbb{R}^d$ be an open finite box and let $H_\Lambda(\omega) := -\triangle + V(\omega)|_\Lambda$ be the restriction of $H(\omega)$ to $L^2(\Lambda)$ with Dirichlet boundary conditions. Then there is a function $C_W : \mathbb{R} \to \mathbb{R}_+$ satisfying*

- *for each $E_0 \in \mathbb{R}$ the function C_W is bounded on $]-\infty, E_0]$*
- *$\mathbb{P}\{\omega \in \Omega : \mathrm{dist}(\sigma(H_\Lambda(\omega)), E_0) < \eta\} \leq C_W(E_0)\eta l^d$ holds for every box Λ of sidelength $l \geq 1$.*

Proof. For a box $\Lambda \subseteq \mathbb{R}^d$ we fix $J := \{j \in \mathbb{Z}^d : \Lambda_j := \Lambda(j,1) \cap \Lambda \neq \emptyset\}$. Let χ^j be the characteristic function of Λ_j and define $H_{N,\Lambda} := -\bigoplus_{j\in J} \triangle_N^{\Lambda_j} \leq H_\Lambda(\omega)$ as the direct sum of the Laplacians on $L^2(\Lambda_j)$ with Neumann boundary conditions. Then we use the proof of Proposition 4.5 of [CH1] to get

$$\mathbb{P}\left\{\mathrm{dist}\left(\sigma(H_\Lambda(\omega)), E_0\right) < \eta\right\} \leq e^{E_0+\eta}\mathbb{E}\left(\mathrm{tr}(\pi_\Lambda(]E_0 - \eta, E_0 + \eta[)e^{-H_{N,\Lambda}})\right) \leq$$

$$\leq e^{E_0+\eta} \sum_{n\in\mathbb{N}} \sum_{j\in J} \|\mathbb{E}(\chi^j \pi_\Lambda(]E_0 - \eta, E_0 + \eta[)\chi^j)\| e^{-E_{n,j}} \tag{1}$$

where π_Λ is the spectral decomposition of H_Λ and $(E_{n,j})_{n\in\mathbb{N}}$ are the eigenvalues of $-\triangle_N^{\Lambda_j}$. These eigenvalues are explicitly known (see e.g. [RS4]), so one gets

$$\sum_{n\in\mathbb{N}}\sum_{j\in J} e^{-E_{n,j}} \leq Cl^d \tag{2}$$

For a given $j \in J$, we now estimate $\|\mathbb{E}(\chi^j\pi_\Lambda(]E_0-\eta,E_0+\eta[)\chi^j)\|$. In order to do so, we choose $\varphi \in L^2(\Lambda)$ satisfying $\|\varphi\| = 1$ and decompose $(\Omega,\mathbb{P})$ as a product of $(\hat{\Omega}_j,\hat{\mathbb{P}}) = (\mathbb{R}^{\mathbb{Z}^d\setminus\{j\}}\times(\mathbb{R}^d)^{\mathbb{Z}^d}, \mathbb{P}_0^{\otimes(\mathbb{Z}^d\setminus\{j\})}\otimes\mathbb{P}_1^{\otimes\mathbb{Z}^d})$ and $(\mathbb{R},\mathbb{P}_0)$. Writing $\omega = (\hat{\omega},\alpha)$ Fubini's theorem yields

$$\langle\varphi,\mathbb{E}(\chi^j\pi_\Lambda(]E_0-\eta,E_0+\eta[)\chi^j)\varphi\rangle = \int\limits_{\hat{\Omega}_j}\int\limits_{\mathbb{R}}\langle\varphi,\chi^j\pi_\Lambda(]E_0-\eta,E_0+\eta[,(\hat{\omega},\alpha))\chi^j\varphi\rangle d\mathbb{P}_0(\alpha)d\hat{\mathbb{P}}(\hat{\omega})$$

$$= \int\limits_{\hat{\Omega}_j}\int\limits_{\mathbb{R}} g(\alpha)\langle\varphi,\chi^j\pi_\Lambda(]E_0-\eta,E_0+\eta[,(\hat{\omega},\alpha))\chi^j\varphi\rangle d\alpha d\hat{\mathbb{P}}(\hat{\omega}).$$

Defining $\widehat{V}_j(\hat{\omega},x) := \sum\limits_{\substack{i\in\mathbb{Z}^d\\ i\neq j}}\omega_j'\wp(x-i-\omega_i'')$ one gets

$$V(\omega,x) = V((\hat{\omega},\alpha),x) = \widehat{V}_j(\hat{\omega},x) + \alpha\wp(x-j-\omega_j'')$$

and $H_\Lambda(\omega) = (-\triangle^\Lambda + \widehat{V}_j(\hat{\omega}))|_\Lambda + \alpha\wp(\cdot-j-\omega_j'')|_\Lambda$. As the assumptions on the model imply that g is a bounded function of compact support, $\wp(\cdot-j-\omega_j'')|_\Lambda \geq 0$, and $c_0(\chi^j)^2 \leq \wp(\cdot-j-\omega_j'')|_\Lambda$, we may apply Lemma 2.1 with $h = g$, $c_2 = c_0$, $\mathcal{H} = L^2(\Lambda)$, $H_0 = (-\triangle^\Lambda + \widehat{V}_j(\omega))|_\Lambda$, $V = \wp(\cdot-j-\omega_j'')|_\Lambda$ and $B = \chi^j$. This application yields

$$\int\limits_{\hat{\Omega}_j}\int\limits_{\mathbb{R}} g(\alpha)\langle\varphi,\chi^j\pi_\Lambda(]E_0-\eta,E_0+\eta[,(\hat{\omega},\alpha))\chi^j\varphi\rangle d\alpha d\hat{\mathbb{P}}(\hat{\omega}) \leq$$

$$\leq \int\limits_{\hat{\Omega}_j}\frac{\mu(]E_0-\eta,E_0+\eta[)\|g\|_\infty}{c_0}d\hat{\mathbb{P}}(\hat{\omega}) = \frac{2\eta\|g\|_\infty}{c_0} \tag{3}$$

Combining (1),(2) and (3), we arrive at the desired result. □

For $E_0 = \inf\Sigma = 0$ we also use Proposition 4.2 from [KSS1]:

Proposition 2.3. *For any $\zeta > 0$ and $b \in]0,2[$ there is a $l_1^* = l_1^*(\zeta,b)$, such that for all boxes Λ of sidelength $l \geq l_1^*$ the following inequality is valid:*

$$\mathbb{P}\{\omega\in\Omega : \operatorname{dist}(\sigma(H_\Lambda(\omega)),\inf\Sigma) \leq l^{b-2}\} \leq l^{-\zeta}.$$

3 Geometric Resolvent Equation

The results of Section 2 enable us to control the resolvent for the operator $H_\Lambda(\omega)$ restricted to a finite box Λ. We state the geometric resolvent equations and some consequences, which were implicitly stated in [CH1], [KSS1] and [KSS2]. These observations allow us to increase the size of the box Λ.

Theorem 3.1. *Let $\emptyset \neq \Lambda \subseteq \mathbb{R}^d$ be a finite, open box and $\chi \in C_0^\infty(\Lambda, \mathbb{R})$ a smooth function of compact support in Λ, and choose as Λ' another finite, open box $\Lambda' \supseteq \Lambda$ or $\Lambda' = \mathbb{R}^d$. For a potential $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$ consider the Schrödinger operators*

$$H_{\Lambda'} := -\triangle^{\Lambda'} + V|_{\Lambda'} \qquad \text{and} \qquad H_\Lambda := -\triangle^{\Lambda} + V|_{\Lambda}$$

with Dirichlet boundary conditions for Λ and for Λ' whenever Λ' is finite. Suppose that $W_\chi^{\Lambda'}$ is the operator in $L^2(\Lambda')$ with domain $\mathcal{D}(W_\chi^{\Lambda'}) = H^1(\Lambda')$ satisfying

$$W_\chi^{\Lambda'} f := -(\triangle\chi)f - 2\nabla\chi \cdot \nabla f \qquad \forall f \in H^1(\Lambda').$$

Then for all $z \in \mathbb{C}\backslash\mathbb{R}$, the following geometric resolvent equations hold true for all $u \in L^2(\Lambda')$ and $v \in L^2(\Lambda)$:

$$\chi(H_{\Lambda'} - z)^{-1}u \;=\; (H_\Lambda - z)^{-1}\chi u + (H_\Lambda - z)^{-1} W_\chi^{\Lambda'} (H_{\Lambda'} - z)^{-1} u \tag{4}$$

$$(H_{\Lambda'} - z)^{-1}\chi v \;=\; \chi(H_\Lambda - z)^{-1} v - (H_{\Lambda'} - z)^{-1} W_\chi^{\Lambda} (H_\Lambda - z)^{-1} v \tag{5}$$

Proof. We notice that $W_\chi^{\Lambda'} f = -\triangle^{\Lambda'}(\chi f) + \chi \triangle^{\Lambda'} f$ for all $f \in \mathcal{D}(\triangle^{\Lambda'})$ and $-\triangle^{\Lambda'}(\chi f) = -\triangle^{\Lambda}(\chi f)$ for all $f \in \mathcal{D}(-\triangle^{\Lambda})$ or $f \in \mathcal{D}(-\triangle^{\Lambda'})$. Hence $(H_\Lambda - z)(\chi f) = \chi(H_{\Lambda'} - z)f + W_\chi^{\Lambda'} f$ for all $f \in \mathcal{D}(\triangle^{\Lambda'})$. Taking $(H_\Lambda - z)^{-1}$ on both sides of this equation and considering $f = (H_{\Lambda'} - z)^{-1}u$ gives (4) and similarly from $\chi(H_\Lambda - z)f = (H_{\Lambda'} - z)(\chi f) - W_\chi^{\Lambda} f$ for $f = (H_\Lambda - z)^{-1}v$ one gets (5). $\square$

Given a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(-\triangle^{\Lambda'})$ then $f_n \to f$, $-\triangle^{\Lambda'} f_n \to h$ and $W_\chi^{\Lambda'} f_n \to g$ altogether imply $W_\chi^{\Lambda'} f = g$. So, we can state some properties of the operator W_χ^{Λ} resulting from the closed graph theorem.

Lemma 3.2. *The operator W_χ^{Λ} has the following properties:*

- $W_\chi^{\Lambda}(H_\Lambda - z)^{-1} : L^2(\Lambda) \to L^2(\Lambda) \subseteq L^2(\Lambda')$ *is bounded for every* $z \in \mathbb{C}\backslash\mathbb{R}$.
- $(H_\Lambda - z)^{-1} W_\chi^{\Lambda'} : H^1(\Lambda') \to L^2(\Lambda)$ *has a continuous extension on $L^2(\Lambda')$ with norm* $\|(H_\Lambda - z)^{-1} W_\chi^{\Lambda'}\| = \|W_\chi^{\Lambda}(H_\Lambda - \bar{z})^{-1}\|$

In the multiscale analysis procedure we fix $\delta > 0$ and consider boxes $\Lambda(\mathfrak{z}, l)$ centered at $\mathfrak{z}$ of sidelength $l > 3\delta$. We also use an Urysohn function φ for $\Lambda(\mathfrak{z}, l)$, which is a function $\varphi \in C_0^\infty(\Lambda(\mathfrak{z}, l - \frac{2}{3}\delta), [0,1])$ with $\varphi = 1$ on $\overline{\Lambda(\mathfrak{z}, l - \frac{4}{3}\delta)}$. In this situation, for every $\tau \in C^\infty(\mathbb{R}^d)$ satisfying $\tau(x) = 1$, for $x \in \Lambda(\mathfrak{z}, l - \frac{2}{3}\delta) \backslash \overline{\Lambda(\mathfrak{z}, l - \frac{4}{3}\delta)}$ one has $W_\varphi^{\Lambda'} = W_\varphi^{\Lambda'} \circ \tau$ for all $\Lambda' \supseteq \Lambda(\mathfrak{z}, l)$. To perform the multiscale analysis we note:

Lemma 3.3. *Let φ be an Urysohn function for $\Lambda = \Lambda(\mathfrak{z}, L)$ and let $\Lambda' \supseteq \Lambda$ be a finite box. Given $\eta > 0$ and $z \in \mathbb{C}\backslash\mathbb{R}$ with $\operatorname{dist}(\Re z, \sigma(H_\Lambda)) \geq \eta$, The following estimate is true for $V \in L^\infty(\mathbb{R}^d, \mathbb{R}_+)$:*

$$\|W_\varphi^\Lambda (H_\Lambda - z)^{-1} 1^{\mathfrak{z}}_{\frac{L}{3}}\| \leq \|1_{\tilde{\Lambda}} (H_{\Lambda'} - z)^{-1} 1^{\mathfrak{z}}_{\frac{L}{3}}\| c_3(\eta, \Re z)$$

where $\tilde{\Lambda} = \{x \in \Lambda(\mathfrak{z}, L) : \frac{L}{2} - \delta < |\mathfrak{z} - x| < \frac{L}{2}\}$ is the enlarged δ-edge of Λ and the function c_3 is given as:

$$\begin{aligned} c_3(\eta, \Re z) &= \| \triangle \varphi\|_\infty + 2\sqrt{\frac{2}{3}[(1 + 2\Re z)\|\nabla\varphi\|_\infty^2 + 8\| \triangle \varphi\|_\infty^2]} + \\ &\quad + \sqrt{2\| \triangle \varphi\|_\infty^2 \frac{1}{\eta^2} + 8\|\nabla\varphi\|_\infty^2 \left(\frac{1}{\eta} + \|\Re z - V\|_\infty \frac{1}{\eta^2}\right)} \\ &\quad \left(\| \triangle \tau_1\|_\infty + 2\sqrt{\frac{2}{3}[(1 + 2\Re z)\|\nabla\tau_1\|_\infty^2 + 8\| \triangle \tau_1\|_\infty^2]}\right) \end{aligned}$$

using a function $\tau_1 \in C_0^\infty(\Lambda(\mathfrak{z}, L))$ with $\tau_1(x) = 1$ for all $x \in \Lambda(\mathfrak{z}, L - \frac{2}{3}\delta)$.

Proof. Taking τ_1 specified above, we have $\tau_1 1^{\mathfrak{z}}_{\frac{L}{3}} = 1^{\mathfrak{z}}_{\frac{L}{3}}$. Thus applying the geometric resolvent equation (4) to $\tau_1 (H_{\Lambda'} - z)^{-1} 1^{\mathfrak{z}}_{\frac{L}{3}} v$ and then applying W_φ^Λ we obtain

$$W_\varphi^\Lambda \tau_1 (H_{\Lambda'} - z)^{-1} 1^{\mathfrak{z}}_{\frac{L}{3}} v = W_\varphi^\Lambda (H_\Lambda - z)^{-1} 1^{\mathfrak{z}}_{\frac{L}{3}} v + W_\varphi^\Lambda (H_\Lambda - z)^{-1} W_{\tau_1}^{\Lambda'} (H_{\Lambda'} - z)^{-1} 1^{\mathfrak{z}}_{\frac{L}{3}} v.$$

Taking also $(\triangle\varphi)\tau_1 = \triangle\varphi$ and $\nabla\varphi \cdot \nabla\tau_1 = 0$ into account, this equation leads to:

$$W_\varphi^\Lambda (H_\Lambda - z)^{-1} 1^{\mathfrak{z}}_{\frac{L}{3}} v = W_\varphi^\Lambda \tau_1 (H_{\Lambda'} - z)^{-1} 1^{\mathfrak{z}}_{\frac{L}{3}} v - W_\varphi^\Lambda (H_\Lambda - z)^{-1} W_{\tau_1}^{\Lambda'} (H_{\Lambda'} - z)^{-1} 1^{\mathfrak{z}}_{\frac{L}{3}} v =$$

$$= -(\triangle\varphi)(H_{\Lambda'} - z)^{-1} 1^{\mathfrak{z}}_{\frac{L}{3}} v - 2(\nabla\varphi)\cdot\nabla((H_{\Lambda'} - z)^{-1} 1^{\mathfrak{z}}_{\frac{L}{3}} v) - W_\varphi^\Lambda (H_\Lambda - z)^{-1} W_{\tau_1}^{\Lambda'} (H_{\Lambda'} - z)^{-1} 1^{\mathfrak{z}}_{\frac{L}{3}} v \quad (6)$$

For $\chi \in C^1(\Lambda', \mathbb{R}^d)$, $u \in \mathfrak{D}(-\triangle^{\Lambda'})$ and $z \in \mathbb{C}\backslash\mathbb{R}$ we remark

$$\frac{1}{4}\|\chi \cdot \operatorname{grad} u\|^2 + 4\|u \operatorname{div}\chi\|^2 \geq -2\Re\langle \chi \cdot \operatorname{grad} u, u \operatorname{div}\chi\rangle$$

$$\begin{aligned} \|\chi(H_{\Lambda'} - z)u\|^2 + \|\chi u\|^2 &\geq 2\Re\langle \chi(H_{\Lambda'} - z)u, \chi u\rangle \geq \\ &\geq \frac{3}{2}\|\chi \cdot \operatorname{grad} u\|^2 - 8\|u \operatorname{div}\chi\|^2 + 2\langle V\chi u, \chi u\rangle - 2\Re z\|u\chi\|^2. \end{aligned}$$

Due to the positivity of the potential V, we obtain

$$\|\chi \operatorname{grad} u\|^2 \leq \frac{2}{3}\left[\|\chi(H_{\Lambda'} - z)u\|^2 + (1 + 2\Re z)\|\chi u\|^2 + 8\|u \operatorname{div}\chi\|^2\right]$$

and furthermore using the assumption on $\operatorname{dist}(\Re z, \sigma(H_\Lambda))$

$$\begin{aligned}\|\nabla(H_{\Lambda'}-z)^{-1}v\|^2 &= \langle (H_{\Lambda'}-z)^{-1}v, (-\triangle^{\Lambda'})(H_{\Lambda'}-z)^{-1}v\rangle = \\ &= \Re\langle (H_{\Lambda'}-z)^{-1}v, v\rangle + \langle (H_{\Lambda'}-z)^{-1}v, (\Re z - V|_{\Lambda'})(H_{\Lambda'}-z)^{-1}v\rangle \\ &\le \frac{1}{\eta}\|v\|^2 + \|\Re z - \mathrm{V}\|_\infty \frac{1}{\eta^2}\|v\|^2.\end{aligned}$$

Expressing W_φ^Λ and $W_{\tau_1}^{\Lambda'}$ in terms of $\triangle\varphi$ and $\nabla\varphi\cdot\nabla$ resp. $\triangle\tau_1$ and $\nabla\tau_1\cdot\nabla$ in (6) and inserting these estimates, the lemma is proven. $\square$

4 Multiscale Analysis

In this Section we give a slight modification of the multiscale analysis, which has been used in [CH1], [KSS1] and [KSS2]. We state the results in a different way to the papers mentioned above, stressing the deterministic or probabilistic nature of the statements. In this way, we can take into account the displacements and the long range case for the single site potential more easily. For the multiscale analysis procedure we now fix a $\delta > 0$ and further on, we only consider boxes of sidelength $l > 3\delta$. An Urysohnfunction of such a box is then always chosen with respect to this f ixed δ in accordance with the conventions made before Lemma 3.3.

Definition 4.1. *Given an energy E, a configuration $\omega \in \Omega$ and $\gamma > 0$, we call a box $\Lambda = \Lambda(\mathfrak{z}, l)$ a γ-good box (for E and ω), if*

$$\sup_{\varepsilon\neq 0} \|W_\varphi^\Lambda (H_\Lambda(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l}{3}}^{\mathfrak{z}}\| \le e^{-\gamma l} \tag{7}$$

holds for an Urysohn function φ of Λ. Otherwise Λ is called γ-bad (for E and ω).

We remark, that with the help of the first resolvent equation we can change $\sup_{\varepsilon\neq 0}$ in (7) into $\sup_{\varepsilon\in\mathbb{Q},\varepsilon\neq 0}$. Using measurability of H_Λ and the geometric resolvent equation to express $W_\varphi^\Lambda(H_\Lambda(\omega) - E - i\varepsilon)^{-1}$ in terms of φ and $H_\Lambda(\omega)$, we get $\{\omega \in \Omega : \Lambda \text{ is } \gamma \text{ good for } \omega\} \in \mathcal{B}(\Omega)$. The aim of the multiscale analysis is to find sequences $(l_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ and $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\Omega)$, such that $l_n \to \infty$, $\Lambda(\mathfrak{z}, l_n)$ is γ-good for E and $\omega \in A_n$ and $\mathbb{P}(A_n) \ge 1 - l_n^{-\zeta}$ for a $\zeta > 0$. In order to perform this "induction process" there are these steps to do:

4.1 Finding γ-good Boxes for an Initial Length Scale l_0

Lemma 4.2. *Let V be a potential on $\mathbb{R}^d$ which is locally uniformly in L^p, $p = 2$ for $d \le 3$ and $p > \frac{d}{2}$ for $d \ge 4$. Let Λ be a finite box and $H_\Lambda = -\triangle^\Lambda + V|_\Lambda$ be the Schrödinger operator with Dirichlet boundary conditions. Let $]r, s[$ be a spectral gap of H_Λ and take $\chi, \tilde\chi \in L^\infty(\Lambda)$ satisfying $\|\chi\|_\infty, \|\tilde\chi\|_\infty \le 1$ and $\operatorname{dist}(\operatorname{supp}\chi, \partial\Lambda), \operatorname{dist}(\operatorname{supp}\tilde\chi, \partial\Lambda) \ge \delta_0 > 0$. Then there are positive constants C_0, C_1 and C_2 only depending on δ_0 and $\sup_{x\in\mathbb{R}^d} \|V\|_{L^p(\Lambda(x,1))}$, such that for each $E \in]r, s[$, each $j \in \{1, ..., d\}$, and $\eta := \operatorname{dist}(E, \sigma(H_\Lambda))$ the following estimates hold:*

$$\|\tilde\chi(H_\Lambda - E - i\varepsilon)^{-1}\chi\| \le \frac{C_2(C_0+s)}{\eta} e^{-\frac{C_1}{C_0+s}\eta \operatorname{dist}(\operatorname{supp}\chi, \operatorname{supp}\tilde\chi)}$$

$$\|\tilde{\chi}\partial_j\left((H_\Lambda - E - i\varepsilon)^{-1}\chi\right)\| \leq \frac{C_2(C_0+s)}{\eta}e^{-\frac{C_1}{C_0+s}\eta \mathrm{dist}(\mathrm{supp}\chi, \mathrm{supp}\tilde{\chi})}$$

A proof of this lemma can be found in [KSS1] Appendix A; using this lemma, we prove the "initial length scale estimate":

Theorem 4.3. *For $\xi \in]0,1[$ there is a minimal length scale $l_2^* = l_2^*(\xi)$, such that for all $l_0 > l_2^*$ there is a $\gamma_0 = \gamma_0(l_0,\xi) > 0$ such that for all $\mathfrak{z} \in \mathbb{R}^d$, for all Urysohn functions $\varphi_{l_0}^{\mathfrak{z}}$ of $\Lambda(\mathfrak{z},l_0)$ with $\delta \geq 1$ and for all $E \in I := [\inf\Sigma - \frac{l_0^{-\xi}}{2}, \inf\Sigma + \frac{l_0^{-\xi}}{2}]$ and all configurations $\omega \in A := \{\omega \in \Omega : \mathrm{dist}(\inf\Sigma, \sigma(H_{\Lambda(\mathfrak{z},l_0)}(\omega))) \geq l_0^{-\xi}\}$ the box $\Lambda(\mathfrak{z},l_0)$ is γ_0-good (for $E \in I$ and $\omega \in A$).*

Proof. We take $\tilde{\chi} = 1_{\tilde{\Lambda}(\mathfrak{z},l_0)}$ and $\chi = 1^{\mathfrak{z}}_{\frac{l_0}{3}}$ to get $\mathrm{dist}(\mathrm{supp}1_{\tilde{\Lambda}(\mathfrak{z},l_0)}, \mathrm{supp}1^{\mathfrak{z}}_{\frac{l_0}{3}}) = \frac{l_0}{3} - \delta$. If we consider that $I \subseteq \rho(H_{\Lambda(\mathfrak{z},l_0)}(\omega))$ for each $\omega \in A$ and $\eta = \mathrm{dist}(E, \sigma(H_{\Lambda(\mathfrak{z},l_0)}(\omega))) \geq \frac{l_0^{-\xi}}{2}$ for all $E \in I$ and $\omega \in A$ using this spectral gap I, we get from Lemma 4.2:

$$\begin{aligned}
\|W^{\Lambda(\mathfrak{z},l_0)}_{\varphi^{\mathfrak{z}}_{l_0}}(H_{\Lambda(\mathfrak{z},l_0)}(\omega) - E - i\varepsilon)^{-1}1^{\mathfrak{z}}_{\frac{l_0}{3}}\| &\leq \|\triangle\varphi^{\mathfrak{z}}_{l_0}\|_\infty \| \|1_{\tilde{\Lambda}(\mathfrak{z},l_0)}(H_{\Lambda(\mathfrak{z},l_0)}(\omega) - E - i\varepsilon)^{-1}1^{\mathfrak{z}}_{\frac{l_0}{3}}\| \\
&\quad +2\|\nabla\varphi^{\mathfrak{z}}_{l_0}\|_\infty \|1_{\tilde{\Lambda}(\mathfrak{z},l_0)}\nabla((H_{\Lambda(\mathfrak{z},l_0)}(\omega) - E - i\varepsilon)^{-1}1^{\mathfrak{z}}_{\frac{l_0}{3}})\| \leq \\
&\leq (\|\triangle\varphi^{\mathfrak{z}}_{l_0}\|_\infty + 2\|\nabla\varphi^{\mathfrak{z}}_{l_0}\|_\infty)\, C_2[2l_0^{\xi}(C_0 + \inf\Sigma) + 1]e^{-l_0^{-\xi}\frac{C_1(\frac{l_0}{3}-\delta)}{2(C_0+\inf\Sigma+\frac{l_0^{-\xi}}{2})}}
\end{aligned}$$

As the norms $\|\triangle\varphi^{\mathfrak{z}}_{l_0}\|_\infty$ and $\|\nabla\varphi^{\mathfrak{z}}_{l_0}\|_\infty$ are bounded with respect to l_0, and $\frac{l_0^{-\xi}}{2} \leq C_0 + \inf\Sigma$ if l_0 is large enough, there are constants C_3, C_4 and C_5 such that

$$\begin{aligned}
\|W^{\Lambda(\mathfrak{z},l_0)}_{\varphi^{\mathfrak{z}}_{l_0}}(H_{\Lambda(\mathfrak{z},l_0)}(\omega) - E - i\varepsilon)^{-1}1^{\mathfrak{z}}_{\frac{l_0}{3}}\| &\leq C_3 l_0^{\xi} e^{-l_0^{-\xi}(C_4 l_0 - C_5)} = \\
&= \left[C_3 l_0^{\xi} e^{C_5 l_0^{-\xi}} e^{-(\frac{C_4}{2}l_0^{-\xi})l_0}\right] e^{-(\frac{C_4}{2}l_0^{-\xi})l_0}
\end{aligned}$$

holds, provided l_0 is big enough. As $C_3 l_0^{\xi} e^{C_5 l_0^{-\xi}} e^{-(\frac{C_4}{2}l_0^{-\xi})l_0} \overset{l_0\to\infty}{\longrightarrow} 0$ there is a $l_2^* = l_2^*(\xi)$ with $C_3 l_0^{\xi} e^{C_5 l_0^{-\xi}} e^{-(\frac{C_4}{2}l_0^{-\xi})l_0} \leq 1$ for $l_0 \geq l_2^*$. If we take $\gamma_0 = \gamma_0(l_0,\xi) := \frac{C_4}{2}l_0^{-\xi}$ and choose any $l_0 \geq l_2^*(\xi)$, we get

$$\|W^{\Lambda(\mathfrak{z},l_0)}_{\varphi^{\mathfrak{z}}_{l_0}}(H_{\Lambda(\mathfrak{z},l_0)}(\omega) - E - i\varepsilon)^{-1}1^{\mathfrak{z}}_{\frac{l_0}{3}}\| \leq e^{-\gamma_0 l_0}$$

$\square$

4.2 Box Counting or how to get a bigger good Box

In this Section we are looking for fixed E how to get for some typical realizations ω from a γ-good box $\Lambda(\mathfrak{z},l)$ to a γ'-good bigger box $\Lambda(\mathfrak{z},l')$. For fixed $\mathfrak{z} \in \mathbb{R}^d$ and $l < l'$ we consider the lattice $\Gamma_{\mathfrak{z},\frac{l}{3}} := \{\mathfrak{z} + \frac{l}{3}y : y \in \mathbb{Z}^d\}$ and the cluster $Q(\omega,\mathfrak{z},\gamma,l,l') := \{x \in \Gamma_{\mathfrak{z},\frac{l}{3}} : \Lambda(x,l) \subseteq \Lambda(\mathfrak{z},l')$ and $\Lambda(x,l)$ is γ-bad for $\omega\}$ of γ-bad boxes centered at a point of $\Gamma_{\mathfrak{z},\frac{l}{3}}$ contained in $\Lambda(\mathfrak{z},l')$.

Lemma 4.4. *Fix an energy E, let l be big enough, $l' > 13l$, $\beta > 0$ and choose a configuration $\omega \in \Omega$ satisfying $\delta Q(\omega, \mathfrak{z}, \gamma, l, 3l') \leq \frac{11}{3} l$ for the diameter of the cluster of bad boxes,* $\operatorname{dist}(E, \sigma(H_{\Lambda(\mathfrak{z},l')}(\omega))) \geq (l')^{-\beta}$ *and* $\operatorname{dist}(E, \sigma(H_{\Lambda(\mathfrak{z},3l')}(\omega))) \geq (3l')^{-\beta}$*. If we choose $\gamma' := (1 - \frac{13l}{l'})(\gamma - \frac{\log(3^d-1)}{l}) - \frac{(2d+2\beta)\log l'}{l'}$, then $\Lambda(\mathfrak{z}, l')$ is a γ'-good box for ω and E.*

Proof. Taking $w, z \in \Lambda(\mathfrak{z}, 3l')$ with $|w - z| \geq \frac{2}{3} l$ and using the notation φ_l^w for an Urysohn-function of $\Lambda(w, l)$ we get $\varphi_l^w 1_{\frac{l}{3}}^z = 0$ and $1_{\frac{l}{3}}^w \varphi_l^w = 1_{\frac{l}{3}}^w$, so

$$1_{\frac{l}{3}}^w (H_{\Lambda(\mathfrak{z},3l')}(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l}{3}}^z = 1_{\frac{l}{3}}^w (H_{\Lambda(w,l)}(\omega) - E - i\varepsilon)^{-1} W_{\varphi_l^w}^{\Lambda(\mathfrak{z},3l')} (H_{\Lambda(\mathfrak{z},3l')}(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l}{3}}^z \quad (8)$$

is a special case of the geometric resolvent equation (4). Choosing $w \in \Gamma_{\mathfrak{z},\frac{l}{3}} \cap \Lambda(\mathfrak{z}, 3l')$ we take the $3^d - 1$ boxes of sidelength $\frac{l}{3}$ centered at $\gamma_w := (\Gamma_{\mathfrak{z},\frac{l}{3}} \cap \Lambda(w, l)) kslash\{w\}$ to get $\tau = \tau \cdot \sum_{w' \in \gamma_w} 1_{\frac{l}{3}}^{w'}$ as an equality in $L^2(\Lambda(w, l))$ for every function $\tau \in C_0^\infty(\tilde{\Lambda}(w, l))$ with $\tau|_{\Lambda(w,l-\frac{2}{3}\delta)\setminus\Lambda(w,l-\frac{4}{3}\delta)} = 1$. As such a function τ satisfies $W_{\varphi_l^w}^{\Lambda(\mathfrak{z},3l')} = W_{\varphi_l^w}^{\Lambda(\mathfrak{z},3l')} \circ \tau$ and due to Lemma 3.2 we have

$$\|1_{\frac{l}{3}}^w (H_{\Lambda(w,l)}(\omega) - E - i\varepsilon)^{-1} W_{\varphi_l^w}^{\Lambda(\mathfrak{z},3l')}\| = \|W_{\varphi_l^w}^{\Lambda(w,l)} (H_{\Lambda(w,l)}(\omega) - E + i\varepsilon)^{-1} 1_{\frac{l}{3}}^w\|,$$

for $w_1 \in \gamma_w$ with $\|1_{\frac{l}{3}}^{w_1} (H_{\Lambda(\mathfrak{z},3l')}(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l}{3}}^z\| = \max_{w' \in \gamma_w} \|1_{\frac{l}{3}}^{w'} (H_{\Lambda(\mathfrak{z},3l')}(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l}{3}}^z\|$, we obtain from (8)

$$\begin{aligned}
&\|1_{\frac{l}{3}}^w (H_{\Lambda(\mathfrak{z},3l')}(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l}{3}}^z\| = \\
&= \|1_{\frac{l}{3}}^w (H_{\Lambda(w,l)}(\omega) - E - i\varepsilon)^{-1} W_{\varphi_l^w}^{\Lambda(\mathfrak{z},3l')} \tau (\sum_{w' \in \gamma_w} 1_{\frac{l}{3}}^{w'}) (H_{\Lambda(\mathfrak{z},3l')}(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l}{3}}^z\| \leq \\
&\leq (3^d - 1) \|W_{\varphi_l^w}^{\Lambda(w,l)} (H_{\Lambda(w,l)}(\omega) - E + i\varepsilon)^{-1} 1_{\frac{l}{3}}^w\| \max_{w' \in \gamma_w} \|1_{\frac{l}{3}}^{w'} (H_{\Lambda(\mathfrak{z},3l')}(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l}{3}}^z\| = \\
&= (3^d - 1) \|W_{\varphi_l^w}^{\Lambda(w,l)} (H_{\Lambda(w,l)}(\omega) - E + i\varepsilon)^{-1} 1_{\frac{l}{3}}^w\| \|1_{\frac{l}{3}}^{w_1} (H_{\Lambda(\mathfrak{z},3l')}(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l}{3}}^z\|.
\end{aligned}$$

If $|w_1 - z| \geq \frac{2}{3} l$, we can iterate this procedure to get a sequence $w = w_0, w_1, ..., w_n$ in $\Gamma_{\mathfrak{z},\frac{l}{3}} \cap \Lambda(\mathfrak{z}, 3l')$ with $|w_j - z| \geq \frac{2}{3} l$ for $j = 0, ..., n-1$ and

$$\begin{aligned}
\|1_{\frac{l}{3}}^w (H_{\Lambda(\mathfrak{z},3l')}(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l}{3}}^z\| \leq (3^d - 1)^n \prod_{j=0}^{n-1} \|W_{\varphi_l^{w_j}}^{\Lambda(w_j,l)} (H_{\Lambda(w_j,l)}(\omega) - E + i\varepsilon)^{-1} 1_{\frac{l}{3}}^{w_j}\| \\
\|1_{\frac{l}{3}}^{w_n} (H_{\Lambda(\mathfrak{z},3l')}(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l}{3}}^z\|. \quad (9)
\end{aligned}$$

Restricting our considerations to the case $w_0, w_1, ..., w_{n-1} \notin Q(\omega, \mathfrak{z}, \gamma, l, 3l')$, we get from (9)

$$\|1_{\frac{l}{3}}^w (H_{\Lambda(\mathfrak{z},3l')}(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l}{3}}^z\| \leq (3^d - 1)^n (e^{-\gamma l})^n \|1_{\frac{l}{3}}^{w_n} (H_{\Lambda(\mathfrak{z},3l')}(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l}{3}}^z\| \quad (10)$$

This procedure which allows us to get further exponential factors $e^{-\gamma l}$ in (10) stops, if either $|w_n - z| = \frac{l}{3}$ or $\Lambda(w_{n+1}, l) \nsubseteq \Lambda(\mathfrak{z}, 3l')$ or $w_n \in Q(\omega, \mathfrak{z}, \gamma, l, 3l')$ is satisfied. Provid ed we get a pair $w_i = w_j$ with $0 \leq i < j$ the procedure can be iterated as many times we like, by setting $w_{j+1} := w_{i+1}$, "running through this path $w_i, ..., w_j$" once more. Take $x \in S_1 := \{x \in \Gamma_{\mathfrak{z}, \frac{l}{3}} : \Lambda(x, \frac{l}{3}) \cap \Lambda(\mathfrak{z}, \frac{l'}{3}) \neq \emptyset\}$ and $y \in S_2 := \{y \in \Gamma_{\mathfrak{z}, \frac{l}{3}} : \Lambda(y, \frac{l}{3}) \cap \tilde{\Lambda}(\mathfrak{z}, l') \neq \emptyset\}$, then we have $|x - y| \geq \frac{l' - l}{3} - \delta > \frac{11}{3} l$ and consequently $\Lambda(x, l)$ or $\Lambda(y, l)$ is a γ-good box. So we can start the construction above with $w = x$ and $z = y$ whenever $\Lambda(x, l)$ is γ-good and with $w = y$, $z = x$ otherwise. In the case where $\Lambda(x, l)$ and $\Lambda(y, l)$ are γ-good and we end up with $w_n \in Q(\omega, \mathfrak{z}, \gamma, l, 3l')$, we increase the number of exponential factors $e^{-\gamma l}$ by starting the abov e construction once again with $w = y$ and $z = w_n$. Now we have to estimate the minimum number of $e^{-\gamma l}$-factors in terms of $|x - y|$ for each possible combination of these stopping conditions to get

$$\begin{aligned} \|1^x_{\frac{l}{3}} (H_{\Lambda(\mathfrak{z}, 3l')}(\omega) - E - i\varepsilon)^{-1} 1^y_{\frac{l}{3}}\| \quad &\leq \quad e^{-(\gamma - \frac{\log(3^d - 1)}{l}) 3(|x-y| - \max\{\frac{4l}{3}, \delta Q(\omega, \mathfrak{z}, \gamma, l, 3l')\})} \\ &\quad \|(H_{\Lambda(\mathfrak{z}, 3l')}(\omega) - E - i\varepsilon)^{-1}\| \end{aligned}$$

for all lattice sites $x \in S_1$ and $y \in S_2$, where $\delta Q(\omega, \mathfrak{z}, \gamma, l, 3l')$ denotes the diameter of the cluster $Q(\omega, \mathfrak{z}, \gamma, l, 3l')$. As we have $3|x - y| - \max\{\frac{4l}{3}, \delta Q(\omega, \mathfrak{z}, \gamma, l, 3l')\} \geq l' - 13l > 0$ and

$$\|1_{\tilde{\Lambda}(\mathfrak{z}, l')} (H_{\Lambda(\mathfrak{z}, 3l')}(\omega) - E - i\varepsilon)^{-1} 1^{\mathfrak{z}}_{\frac{l'}{3}}\| \leq \mathrm{Card} S_1 \mathrm{Card} S_2 \max_{\substack{x \in S_1 \\ y \in S_2}} \|1^y_{\frac{l}{3}} (H_{\Lambda(\mathfrak{z}, 3l')}(\omega) - E - i\varepsilon)^{-1} 1^x_{\frac{l}{3}}\|$$

the last task is to count the elements of S_1 and S_2 and then to combine this with the estimate of Lemma 3.3, using $c_3((l')^{-\beta}, E) \leq c_4 (l')^{\beta}$ to get a $c \geq 0$ with

$$\|W^{\Lambda(\mathfrak{z}, l')}_{\varphi^{\mathfrak{z}}_{l'}} (H_{\Lambda(\mathfrak{z}, l')}(\omega) - E - i\varepsilon)^{-1} 1^{\mathfrak{z}}_{\frac{l'}{3}}\| \leq c l^{-2d} e^{(l' - 13l)(\frac{\log(3^d - 1)}{l} - \gamma) + (2d + 2\beta) \log l'}$$

yielding the proposed γ', provided l is big enough. $\square$

4.3 An Important Probabilistic Trick in Multiscale Analysis for Long Range Single Site Potentials

Taking a finite set $\emptyset \neq \Upsilon \subset \mathbb{Z}^d$ and $A \subseteq \Omega$, we find a set $B \supseteq \mathrm{pr}_\Upsilon(A)$ in the σ-algebra $\mathcal{B}(\mathbb{R}^{d+1})^\Upsilon$ with $(\mathbb{P}_0 \otimes \mathbb{P}_1)^{\otimes \Upsilon}(B) = [(\mathbb{P}_0 \otimes \mathbb{P}_1)^{\otimes \Upsilon}]^*(\mathrm{pr}_\Upsilon(A))$. Consequently $B \times (\mathbb{R}^{d+1})^{\mathbb{Z}^d \setminus \Upsilon} \in \mathcal{B}(\Omega)$ satisfies $B \times (\mathbb{R}^{d+1})^{\mathbb{Z}^d \setminus \Upsilon} \supseteq \mathrm{pr}_\Upsilon^{-1}(\mathrm{pr}_\Upsilon(A))$ and $\mathbb{P}^*(B \times (\mathbb{R}^{d+1})^{\mathbb{Z}^d \setminus \Upsilon} \setminus A) = 0$, so due to the completeness of $\mathcal{B}(\Omega)$ we obtain $\mathrm{pr}_\Upsilon^{-1}(\mathrm{pr}(A)) \in \mathcal{B}(\Omega)$. Fixing $A, B \in \mathcal{B}(\Omega)$ and finite sets $\emptyset \neq silon_1, \Upsilon_2 \subseteq \mathbb{Z}^d$, $\Upsilon_1 \cap \Upsilon_2 = \emptyset$ then the two events $pr_{\Upsilon_1}^{-1}(pr_{\Upsilon_1}(A))$ and $pr_{\Upsilon_2}^{-1}(pr_{\Upsilon_2}(B))$ are independent. Due to this fact in [KSS2] one finds the following definition.

Definition 4.5. *A box $\Lambda(\mathfrak{z}, l)$ is called γ-super good for ω and E, if $\Lambda(\mathfrak{z}, l)$ is γ-good for E and for all $\overline{\omega} \in \Omega$ satisfying $pr_{\Upsilon(\mathfrak{z}, 4l)}(\omega) = pr_{\Upsilon(\mathfrak{z}, 4l)}(\overline{\omega})$ with $\Upsilon(\mathfrak{z}, 4l) := \mathbb{Z}^d \cap \Lambda(\mathfrak{z}, 4l)$.*

Using γ-super good boxes instead of γ-good boxes eliminates all the problems, we would have, if we want to estimate for noncompact $\wp$ the probability $\mathbb{P}\{\omega \in \Omega : \Lambda(x, l)$ and $\Lambda(y, l)$ are not γ-good for E and $\omega\}$ in terms of $\mathbb{P}\{\omega \in \Omega : \Lambda(x, l)$ is not γ- good for E and $\omega\}$, even if we assume large $|x-y|$. In order to proceed, we need some $pr_{\Upsilon}^{-1}pr_{\Upsilon}$-versions of earlier statements.

Lemma 4.6. *There is a c_5, such that for L large enough we have the inclusions:*

$$pr_{\Upsilon(\mathfrak{z},4L)}^{-1}\left(pr_{\Upsilon(\mathfrak{z},4L)}\{\omega\in\Omega:\mathrm{dist}(E,\sigma(H_{\Lambda(\mathfrak{z},L)}(\omega)))<\eta\}\right)\subseteq$$

$$\subseteq\{\omega\in\Omega:\mathrm{dist}(E,\sigma(H_{\Lambda(\mathfrak{z},L)}(\omega)))<\eta+c_5L^{-(\varsigma-d)}\}$$

$$pr_{\Upsilon(\mathfrak{z},4L)}^{-1}\left(pr_{\Upsilon(\mathfrak{z},4L)}\{\omega\in\Omega:\mathrm{dist}(E,\sigma(H_{\Lambda(\mathfrak{z},L)}(\omega)))>\eta\}\right)\subseteq$$

$$\subseteq\{\omega\in\Omega:\mathrm{dist}(E,\sigma(H_{\Lambda(\mathfrak{z},L)}(\omega)))>\eta-c_5L^{-(\varsigma-d)}\}$$

Proof. We take $A=\{\omega\in\Omega:\mathrm{dist}(E,\sigma(H_{\Lambda(\mathfrak{z},L)}(\omega)))<\eta\}$ for the first inclusion and similarly $A=\{\omega\in\Omega:\mathrm{dist}(E,\sigma(H_{\Lambda(\mathfrak{z},L)}(\omega)))>\eta\}$ for the second. Then we take $\overline{\omega}\in pr_{\Upsilon(\mathfrak{z},4L)}^{-1}(pr_{\Upsilon(\mathfrak{z},4L)}(A))$ and choose $\omega\in A$ satisfying $pr_{\Upsilon(\mathfrak{z},4L)}(\omega)=pr_{\Upsilon(\mathfrak{z},4L)}(\overline{\omega})$. As the ω_j and $\overline{\omega}_j$ coincide for all sites $j\in\Upsilon(\mathfrak{z},4L)$ and $\wp$ decreases at least polynomial as $c_1\langle x\rangle^{-\varsigma}$, one gets $|V(\omega,x)-V(\overline{\omega},x)|\leq c_5L^{-(\varsigma-d)}$. So the Minimax theorem shows, that the absolute value of the differences of the n-th eigenvalues of $H_{\Lambda(\mathfrak{z},L)}(\omega)$ and $H_{\Lambda(\mathfrak{z},4L)}(\overline{\omega})$ is bounded above by $c_5L^{-(\varsigma-d)}$ and this lemma follows. $\square$

Lemma 4.7. *If $l'>13l$ and $\mathbb{P}\{\omega\in\Omega:\Lambda(x,l)$ is not γ-super good for $\omega\}\leq\eta$ for each $x\in S(\mathfrak{z},l,l'):=\{x\in\Gamma_{z,\frac{l}{3}}:\Lambda(x,l)\subseteq\Lambda(\mathfrak{z},3l')\}$ are satisfied, one gets:*

$$\mathbb{P}\left(pr_{\Upsilon(\mathfrak{z},4l')}^{-1}(pr_{\Upsilon(\mathfrak{z},4l')}\{\omega\in\Omega:\delta Q(\omega,\mathfrak{z},\gamma,l,3l')>\frac{11}{3}l\})\right)\leq 9^{2d}\left(\frac{l'}{l}\right)^{2d}\eta^2$$

Proof. We remark, that for finite sets $\emptyset\neq\Upsilon_1\subseteq\Upsilon_2\subset\mathbb{Z}^d$ and every $A\subseteq\Omega$ we get the "reversed" $\mathrm{pr}^{-1}\mathrm{pr}$-relation $\mathrm{pr}_{\Upsilon_1}^{-1}(\mathrm{pr}_{\Upsilon_1}(A))\supseteq\mathrm{pr}_{\Upsilon_2}^{-1}(\mathrm{pr}_{\Upsilon_2}(A))$. So

$$pr_{\Upsilon(\mathfrak{z},4l')}^{-1}\left(pr_{\Upsilon(\mathfrak{z},4l')}\{\omega\in\Omega:\delta Q(\omega,\mathfrak{z},\gamma,l,3l')>\frac{11}{3}l\}\right)=$$

$$=\bigcup_{\substack{x,y\in S(\mathfrak{z},l,l')\\|x-y|>\frac{11}{3}l}}pr_{\Upsilon(\mathfrak{z},4l')}^{-1}(pr_{\Upsilon(\mathfrak{z},4l')}\{\omega\in\Omega:\Lambda(x,l)\text{ and }\Lambda(y,l)\text{ are }\gamma\text{-bad for }\omega\})\subseteq$$

$$\subseteq\bigcup_{\substack{x,y\in S(\mathfrak{z},l,l')\\|x-y|>\frac{11}{3}l}}\Big[pr_{\Upsilon(x,4l)}^{-1}(pr_{\Upsilon(x,4l)}\{\omega\in\Omega:\Lambda(x,l)\text{ is }\gamma\text{-bad for }\omega\})\cap$$

$$pr_{\Upsilon(y,4l)}^{-1}(pr_{\Upsilon(y,4l)}\{\omega\in\Omega:\Lambda(y,l)\text{ is }\gamma\text{-bad for }\omega\})\Big]$$

and as $\Upsilon(x,4l) \cap \Upsilon(y,4l) = \emptyset$ for all $x, y \in S(\mathfrak{z}, l, l')$ with $|x-y| > \frac{11}{3}l$, we can use the independence of the two events $pr^{-1}_{\Upsilon(x,4l)}(pr_{\Upsilon(x,4l)}\{\omega \in \Omega : \Lambda(x,l) \text{ is } \gamma\text{-bad for } \omega\})$ and $pr^{-1}_{\Upsilon(y,4l)}(pr_{\Upsilon(y,4l)}\{\omega \in \Omega : \Lambda(y,l) \text{ is } \gamma\text{-bad for } \omega\})$ to estimate

$$\mathbb{P}\left(pr^{-1}_{\Upsilon(\mathfrak{z},4l')}(pr_{\Upsilon(\mathfrak{z},4l')}\{\omega \in \Omega : \delta Q(\omega, \mathfrak{z}, \gamma, l, 3l') > \frac{11}{3}l\})\right) \leq$$

$$\leq \sum_{\substack{x,y \in S(\mathfrak{z},l,l') \\ |x-y| > \frac{11}{3}l}} \mathbb{P}\left(\{\omega \in \Omega : \Lambda(x,l) \text{ not } \gamma\text{-super good for } \omega\}\right)$$

$$\mathbb{P}\left(\{\omega \in \Omega : \Lambda(y,l) \text{ not } \gamma\text{-super good for } \omega\}\right) \leq$$

$$\leq (\mathrm{Card} S(\mathfrak{z}, l, l'))^2 \eta^2 \leq 9^{2d} \left(\frac{l'}{l}\right)^{2d} \eta^2.$$

□

Theorem 4.8. *Let l be big enough and $l' > 13l$ and suppose that for each $x \in S(\mathfrak{z}, l, l')$ the estimate $\mathbb{P}\{\omega \in \Omega : \Lambda(x,l)$ is γ-super good for $\omega\} \geq 1 - \eta$ holds, then there is a c_6, such that*

$$\mathbb{P}\{\omega \in \Omega : \Lambda(\mathfrak{z}, l') \text{ is } \gamma'\text{-super good for } \omega\} \geq 1 - \eta'$$

where we can choose γ' and η' as follows:

$$\gamma' = \left(1 - \frac{13l}{l'}\right)\left(\gamma - \frac{\log(3^d - 1)}{l}\right) - \frac{(2d + 2\beta)\log l'}{l'}$$

$$\eta' = c_6 \left[\left(\frac{l'}{l}\right)^{2d} \eta^2 + (l')^{d-\beta} + (l')^{-(\varsigma - 2d)}\right].$$

Proof. We only look for configurations $\omega \in A_1$ with

$$\begin{aligned} A_1 \quad := \quad & \left[\Omega \backslash pr^{-1}_{\Upsilon(\mathfrak{z},4l')}\left(pr_{\Upsilon(\mathfrak{z},4l')}\{\omega \in \Omega : \delta Q(\omega, \mathfrak{z}, \gamma, l, 3l') > \frac{11}{3}l\}\right)\right] \cap \\ & \cap \left[\Omega \backslash pr^{-1}_{\Upsilon(\mathfrak{z},4l')}\left(pr_{\Upsilon(\mathfrak{z},4l')}\{\omega \in \Omega : \mathrm{dist}(E, \sigma(H_{\Lambda(\mathfrak{z},3l')}(\omega))) < (3l')^{-\beta}\}\right)\right] \cap \\ & \cap \left[\Omega \backslash pr^{-1}_{\Upsilon(\mathfrak{z},4l')}\left(pr_{\Upsilon(\mathfrak{z},4l')}\{\omega \in \Omega : \mathrm{dist}(E, \sigma(H_{\Lambda(\mathfrak{z},l')}(\omega))) < (l')^{-\beta}\}\right)\right]. \end{aligned}$$

Fixing $\omega \in A_1$ and $\overline{\omega} \in \Omega$ with $pr_{\Upsilon(\mathfrak{z},4l')}(\omega) = pr_{\Upsilon(\mathfrak{z},4l')}(\overline{\omega})$, one easiliy verifies that this $\overline{\omega}$ satisfies the conditions of Lemma 4.4, so one obtains the γ' and that $\Lambda(\mathfrak{z}, l')$ is a γ'-good box for $\overline{\omega}$. A combination of Lemma 4.6, Theorem 2.2 and 4.7 gives the estimate $\mathbb{P}(A_1) \geq 1 - \eta'$. □

4.4 The Induction Process

Theorem 4.9. *There are $l^*, \zeta > 0$ and $E_X > \inf \Sigma$, such that if we choose $l_0 > l^*$ and $\alpha \in]1, \frac{2d+2\zeta}{2d+\zeta}[$ and define $l_n := l_0^{\alpha^n}$ there is a $\gamma > 0$, such that*

$$\mathbb{P}\{\omega \in \Omega : \Lambda(\mathfrak{z}, l_n) \text{ is not } \gamma\text{-super good for } \omega \text{ and } E\} \leq l_n^{-\zeta} \tag{11}$$

for each $\mathfrak{z} \in \mathbb{R}^d$ and $E \in [\inf \Sigma, E_X]$.

Proof. Choosing $b > 1$, we get $\xi \in]2-b, 1[$ such that $l_0^{b-2} - c_5 l_0^{-(\varsigma-d)} \geq l_0^{-\xi}$ if l_0 is large enough. Now we conclude from Lemma 4.6 that

$$pr^{-1}_{\Upsilon(\mathfrak{z},4l_0)}(pr_{\Upsilon(\mathfrak{z},4l_0)}\{\omega \in \Omega : \operatorname{dist}(\inf \Sigma, \sigma(H_{\Lambda(\mathfrak{z},l_0)}(\omega))) > l_0^{b-2}\}) \subseteq$$

$$\subseteq \{\omega \in \Omega : \operatorname{dist}(\inf \Sigma, \sigma(H_{\Lambda(\mathfrak{z},l_0)}(\omega))) > l_0^{b-2} - c_5 l_0^{-(\varsigma-d)}\},$$

so the initial length scale estimate shows that for all $l_0 > l_2^*(\xi)$ the box $\Lambda(\mathfrak{z}, l_0)$ is $\gamma_0 = \gamma_0(l_0, \xi) = \frac{C_4}{2} l_0^{-\xi}$ good for all $E \in [\inf \Sigma - \frac{l_0^{-\xi}}{2}, \inf \Sigma + \frac{l_0^{-\xi}}{2}]$ and all

$$\omega \in pr^{-1}_{\Upsilon(\mathfrak{z},4l_0)}(pr_{\Upsilon(\mathfrak{z},4l_0)}\{\omega \in \Omega : \operatorname{dist}(\inf \Sigma, \sigma(H_{\Lambda(\mathfrak{z},l_0)}(\omega))) > l_0^{b-2}\}).$$

This implies that for $\omega \in \{\omega \in \Omega : \operatorname{dist}(\inf \Sigma, \sigma(H_{\Lambda(\mathfrak{z},l_0)}(\omega))) > l_0^{b-2}\}$ the box $\Lambda(\mathfrak{z}, l_0)$ is γ_0-super good, and so using Proposition 2.3 we get

$$\mathbb{P}\{\omega \in \Omega : \Lambda(\mathfrak{z}, l_0) \text{ is } \gamma_0\text{-super good for } \omega \text{ and } E\} \geq 1 - l_0^{-\zeta}$$

if we choose a suitable lower bound for l_0. Now the induction works; we define

$$\gamma_{k+1}(l_0, \gamma_0, \alpha) = \gamma_{k+1} := \gamma_k(1 - \frac{13 l_k}{l_{k+1}}) - c_k$$

$$c_k(l_0) = c_k := \frac{\log(3^d - 1)}{l_k}(1 - \frac{13 l_k}{l_{k+1}}) + (2d + 2\beta)\frac{\log l_{k+1}}{l_{k+1}}$$

and use the proof of Lemma A3 in [CH1] to express γ_{k+1} in terms of l_j and c_j, $j = 1, ..., k$ to find $\kappa = \kappa(l_0, \alpha) = \prod_{j \in \mathbb{N}}(1 - 13 l_0^{-(\alpha-1)\alpha^j})$ satisfying $0 < \frac{\gamma_0(l_0,\xi)}{2}\kappa(l_0, \alpha) \leq \gamma_k$ for all $k \in \mathbb{N}$ if we choose l_0 big enough. In order to check the $\mathbb{P}$-estimate (11), we use the expression for η' in Theorem 4.8 and optimize the parameter α, β and ζ as in the proof of Proposition 5.4 in [KSS2], which says: We can choose $\zeta \in]0, \varsigma - 2d[$, then $\alpha \in \left]1, \frac{2d+2\zeta}{2d+\zeta}\right[$ and $\beta > d + \zeta$ such that if we only consider large values of l_0, then for all $k \in \mathbb{N}$, we will get the inequalities:

$$c_6 l_{k+1}^{2d-\varsigma} \leq \frac{1}{3} l_{k+1}^{-\zeta}$$

$$c_6 \left(\frac{l_{k+1}}{l_k}\right)^{2d} \eta_k^2 \leq c_6 l_{k+1}^{2d} \left(l_{k+1}^{\frac{1}{\alpha}}\right)^{-2d} \left(l_{k+1}^{\frac{1}{\alpha}}\right)^{-2\zeta} = c_6 l_{k+1}^{2d(1-\frac{1}{\alpha}) - 2\frac{\zeta}{\alpha}} \leq \frac{1}{3} l_{k+1}^{-\zeta}$$

$$c_6 l_{k+1}^{d-\beta} \leq \frac{1}{3} l_{k+1}^{-\zeta}.$$

With this choice of l_0 and for $E_X := \inf\Sigma + \frac{l_0^{-\zeta}}{2} > \inf\Sigma$ Theorem 4.9 follows. $\square$

5 From the Resolvent on Finite Boxes to the Localized Resolvent on the Whole Space

Theorem 5.1. *Let E_X be as in Theorem 4.9, then for every $E \in [\inf\Sigma, E_X]$ there is a measurable set $B_1 = B_1(E)$ with $\mathbb{P}(B_1) = 1$, such that for every bounded set $K \subseteq \mathbb{R}^d$ and each $\omega \in B_1$ there is $r_1(\omega, E, K) \in]0,\infty[$ such that*

$$\sup_{\varepsilon\neq 0} \|(H(\omega) - E - i\varepsilon)^{-1} 1_K\| = \sup_{\varepsilon\neq 0} \|1_K (H(\omega) - E - i\varepsilon)^{-1}\| \le r_1(\omega, E, K).$$

Proof. Setting $\Lambda_k := \Lambda(0, l_k)$, $\varepsilon_k := e^{-\gamma l_k}$ and $W_k := W_{\varphi_k}^{\Lambda_k}$ for an Urysohn function φ_k of Λ_k and $l_k := l_0^{\alpha^k}$ as in Theorem 4.9, we define

$$M_k = M_k(E) := \{\omega \in \Omega : \sup_{\varepsilon\neq 0} \|W_k (H_{\Lambda_k}(\omega) - E - i\varepsilon)^{-1} 1_{\frac{l_k}{3}}\| \le \varepsilon_k\}$$

$$N_k = N_k(E) := \{\omega \in \Omega : \operatorname{dist}(\sigma(H_{\Lambda_k}(\omega)), E) > \varepsilon_{k-1}\}.$$

Combining Theorem 4.9 and 2.2 we get $(1 - \mathbb{P}(M_k))_{k\in\mathbb{N}} \in l^1$ and $(1 - \mathbb{P}(N_k))_{k\in\mathbb{N}} \in l^1$. Hence the Borel-Cantelli-Lemma implies that $B_1 := B_1(E) = \liminf_{k\to\infty} M_k(E) \cap N_k(E)$ has measure 1. For each $\omega \in B_1(E)$ we have a lower bound $k_0(\omega)$, such that $\omega \in M_k(E) \cap N_k(E)$ for all $k \ge k_0(\omega)$. For the given bounded set K we choose m with $1_K \subseteq \Lambda(0, \frac{l_m}{3})$ and then we treat indices $j \ge \max\{k_0(\omega), m\} =: n$. If we take the Urysohn functions φ_k of Λ_k and use $\varphi_{k+1} W_k = W_k$, we get for $v = 1_K v$ applying the geometric resolvent equation (5) several times

$$
\begin{aligned}
&(H(\omega) - E - i\varepsilon_j)^{-1} 1_K v = (H(\omega) - E - i\varepsilon_j)^{-1} \varphi_n v = \\
&= \varphi_n (H_{\Lambda_n}(\omega) - E - i\varepsilon_j)^{-1} v - (H(\omega) - E - i\varepsilon_j)^{-1} W_n (H_{\Lambda_n}(\omega) - E - i\varepsilon_j)^{-1} v = \\
&= \varphi_n (H_{\Lambda_n}(\omega) - E - i\varepsilon_j)^{-1} v - \varphi_{n+1} (H_{\Lambda_{n+1}}(\omega) - E - i\varepsilon_j)^{-1} W_n (H_{\Lambda_n}(\omega) - E - i\varepsilon_j)^{-1} 1_{\frac{l_n}{3}} v \\
&\quad + (H(\omega) - E - i\varepsilon_j)^{-1} W_{n+1} (H_{\Lambda_{n+1}}(\omega) - E - i\varepsilon_j)^{-1} W_n (H_{\Lambda_n}(\omega) - E - i\varepsilon_j)^{-1} 1_{\frac{l_n}{3}} v \\
&= \varphi_n (H_{\Lambda_n}(\omega) - E - i\varepsilon_j)^{-1} v + \\
&\quad + \sum_{s=0}^{j-n-1} (-1)^{s+1} \varphi_{n+s+1} (H_{\Lambda_{n+s+1}}(\omega) - E - i\varepsilon_j)^{-1} (\bigcirc_{t=0}^{s} W_{n+t} (H_{\Lambda_{n+t}}(\omega) - E - i\varepsilon_j)^{-1} 1_{\frac{l_{n+t}}{3}}) v \\
&\quad + \left[(-1)^{n-j+1} (H(\omega) - E - i\varepsilon_j)^{-1} \bigcirc_{t=0}^{j-n} W_{n+t} (H_{\Lambda_{n+t}}(\omega) - E - i\varepsilon_j)^{-1} 1_{\frac{l_{n+t}}{3}} \right] v. \qquad (12)
\end{aligned}
$$

As $\omega \in M_k(E) \cap N_k(E)$ we estimate $\|(H_{\Lambda_k}(\omega) - E - i\varepsilon_j)^{-1}\| \leq \frac{1}{\operatorname{dist}(E, \sigma(H_{\Lambda_k}(\omega)))} \leq \frac{1}{\varepsilon_{k-1}}$ and $\|W_k(H_{\Lambda_k}(\omega) - E - i\varepsilon_j)^{-1} 1_{\frac{l_k}{3}}\| \leq \varepsilon_k$ to get

$$\|(H(\omega) - E - i\varepsilon_j)^{-1} 1_K v\| \leq \|v\| \left(\frac{1}{\varepsilon_{n-1}} + \sum_{s=0}^{j-n-1} \frac{1}{\varepsilon_{n+s}} \prod_{t=0}^{s} \varepsilon_{n+t} + \frac{1}{\varepsilon_j} \prod_{t=0}^{j-n} \varepsilon_{n+t} \right) \leq$$

$$\leq \|v\| \left(e^{\gamma l_{n-1}} + \sum_{k \in \mathbb{N}} e^{-\gamma l_k} \right).$$

Using the first resolvent equation we pass from $(\varepsilon_j)_{j \in \mathbb{N}}$ to $\varepsilon \neq 0$, if we take $r_1(\omega, E, K) := 2 \left(e^{\gamma l_{n-1}} + \sum_{k \in \mathbb{N}} e^{-\gamma l_k} \right)$. As $\varphi_n|_K = 1$ the proof is complete. □

Lemma 5.2. *Let E_X and γ be given as in Theorem 4.9.*

1. *Taking a bounded function $\mathcal{Q}$ of compact support and an $E \in [\inf \Sigma, E_X]$, there is a $B_2 = B_2(E, \mathcal{Q}) \in \mathcal{B}(\Omega)$ of full measure $\mathbb{P}(B_2) = 1$ and for each $\omega \in B_2$ there are constants $s_2 = s_2(\omega, E, \mathcal{Q})$, $s_3 = s_3(\omega, E, \mathcal{Q})$ and $s_4(\omega, E, \mathcal{Q})$, such that for all $x \in \mathbb{R}^d$, $|x| \geq s_3(\omega, E, \mathcal{Q})$ and for all $\omega \in B_2(E, \mathcal{Q})$ one gets:*

$$\sup_{\varepsilon \neq 0} \|(H(\omega) - E - i\varepsilon)^{-1} \mathcal{Q}\| \leq s_2(\omega, E, \mathcal{Q}) < \infty \tag{13}$$

$$\sup_{\varepsilon \neq 0} \|1^x (H(\omega) - E - i\varepsilon)^{-1} \mathcal{Q}\| \leq s_4(\omega, E, \mathcal{Q}) e^{-\frac{\gamma}{5}|x|} \tag{14}$$

2. *Taking $d < D < r$ and an $E \in [\inf \Sigma, E_X]$, there is a set $B_2 = B_2(E, D, r) \in \mathcal{B}(\Omega)$ with $\mathbb{P}(B_2(E, D, r)) = 1$ and for each $\omega \in B_2(E, D, r)$ there are $t_2 = t_2(\omega, E, r)$, $t_3 = t_3(\omega, E, D, r)$ and $t_4 = t_4(\omega, E, D, r) \in \mathbb{R}_+$, such that for all $x \in \mathbb{R}^d$ satisfying $|x| \geq t_3(\omega, E, D, r)$ and for all $\omega \in B_2(E, D, r)$ the following inequalities are valid:*

$$\sup_{\varepsilon \neq 0} \|(H(\omega) - E - i\varepsilon)^{-1} \langle \cdot \rangle^{-r}\| \leq t_2(\omega, E, r) < \infty \tag{15}$$

$$\sup_{\varepsilon \neq 0} \|1^x (H(\omega) - E - i\varepsilon)^{-1} \langle \cdot \rangle^{-r}\| \leq t_4(\omega, E, D, r) \langle x \rangle^{D-r} \tag{16}$$

Proof. The first part is proven like Theorem 6.1 in [KSS1]; the necessary modifications to get the second part are listed in the proof of Theorem 6.1 in [KSS2]. □

Theorem 5.3. *Choosing E_X and γ as in Theorem 4.9 there is a $Z_0 \in \mathcal{B}(\Omega)$ with $\mathbb{P}(Z_0) = 1$, such that for each $\omega \in Z_0$ there is an $I(\omega) \in \mathcal{B}([\inf \Sigma, E_X])$ with $\mu(I(\omega)) = \mu([\inf \Sigma, E_X])$, such that the following statements hold:*

1. *For all $y \in \mathbb{Q}^d$, $\omega \in Z_0$ and $E \in I(\omega)$, there are constants $r_3(\omega,E,y), r_4(\omega,E,y) \in \mathbb{R}_+$, such that for all $x \in \mathbb{R}^d$ satisfying $|x| \geq r_3(\omega,E,y)$ one gets:*

$$\sup_{\varepsilon\neq 0} \|1^y(H(\omega)-E-i\varepsilon)^{-1}1^x\| \leq r_4(\omega,E,y)e^{-\frac{\gamma}{5}|x|}$$

2. *Is $\wp$ of compact support, then for all $\omega \in Z_0$ and $E \in I(\omega)$ there are constants $r_3(\omega,E,\wp)$, $r_4(\omega,E,\wp)$ such that for all $x,\mathfrak{z} \in \mathbb{R}^d$ with $|x| \geq r_3(\omega,E,\wp)$ and $|\mathfrak{z}| < R$ we have:*

$$\sup_{\varepsilon\neq 0} \|1^x(H(\omega)-E-i\varepsilon)^{-1}\wp(\cdot-\mathfrak{z})\| \leq r_4(\omega,E,\wp)e^{-\frac{\gamma}{5}|x|}$$

3. *For $n \in \mathbb{N}$, $\omega \in Z_0$ and $E \in I(\omega)$ there are $r_3(\omega,E,\varsigma,n)$, $r_4(\omega,E,\varsigma,n) \in \mathbb{R}_+$, such that for all $x,\mathfrak{z} \in \mathbb{R}^d$ with $|x| \geq r_3(\omega,E,\varsigma,n)$ and $|\mathfrak{z}| < R$ the following inequality is valid:*

$$\sup_{\varepsilon\neq 0} \|1^x(H(\omega)-E-i\varepsilon)^{-1}\langle\cdot-\mathfrak{z}\rangle^{-\frac{\varsigma}{2}-n\frac{\varsigma}{4}}\| \leq r_4(\omega,E,\varsigma,n)\langle x\rangle^{-n\frac{\varsigma}{4}}$$

Proof. Using $\mathcal{Q} = 1^y$ in Lemma 5.2 we obtain $B_2(E,y) \in \mathcal{B}(\Omega)$ with $\mathbb{P}(B_2(E,y)) = 1$ and $r_3(\omega,E,y)$, $r_4(\omega,E,y) \in \mathbb{R}_+$ for each $\omega \in B_2(E,y)$, such that for $|x| \geq r_3(\omega,E,y)$ we get

$$\sup_{\varepsilon\neq 0} \|1^y(H(\omega)-E-i\varepsilon)^{-1}1^x\| \leq r_4(\omega,E,y)e^{-\frac{\gamma}{5}|y|}.$$

In the case that $\wp$ has compact support, we take $M := \bigcup_{|\mathfrak{z}|<R} \mathrm{supp}(\wp(\cdot-\mathfrak{z}))$ and $\mathcal{Q} := \|\wp\|_\infty 1_M$. Using Lemma 5.2 again we get

$$\sup_{\varepsilon\neq 0} \|1^x(H(\omega)-E-i\varepsilon)^{-1}\wp(\cdot-\mathfrak{z})\| = \sup_{\varepsilon\neq 0} \|\wp(\cdot-\mathfrak{z})(H(\omega)-E-i\varepsilon)^{-1}1^x\| \leq$$

$$\leq \sup_{\varepsilon\neq 0} \|\mathcal{Q}(H(\omega)-E-i\varepsilon)^{-1}1^x\| = \sup_{\varepsilon\neq 0} \|1^x(H(\omega)-E-i\varepsilon)^{-1}\mathcal{Q}\| \leq r_4(\omega,E,\wp)e^{-\frac{\gamma}{5}|x|}$$

for $\omega \in B_2(E,\mathcal{Q})$, $|x| \geq r_3(\omega,E,\wp)$ with $\mathbb{P}(B_2(E,\mathcal{Q})) = 1$ and $r_3(\omega,E,\wp)$, $r_4(\omega,E,\wp) \in \mathbb{R}_+$. As $1+|x-\mathfrak{z}|^2 \geq \frac{1}{2}(1+|x|^2)$ for $|x| \geq 4R$, we estimate $\langle\cdot-\mathfrak{z}\rangle^{-r} \leq c\langle\cdot\rangle^{-r}$ for all $|\mathfrak{z}| < R$. Taking $D = \frac{\varsigma}{2}$ and $r = \frac{\varsigma}{2} + n\frac{\varsigma}{4}$ we get sets $B_2(E,\frac{\varsigma}{2},\frac{\varsigma}{2}+n\frac{\varsigma}{4}) \in \mathcal{B}(\Omega)$ with $\mathbb{P}(B_2(E,\frac{\varsigma}{2},\frac{\varsigma}{2}+n\frac{\varsigma}{4})) = 1$ and $r_3(\omega,E,\varsigma,n)$, $r_4(\omega,E,\varsigma,n) \in \mathbb{R}_+$ such that for $|\mathfrak{z}| < R$ and $|x| \geq r_3(\omega,E,\varsigma,n)$:

$$\sup_{\varepsilon\neq 0} \|1^x(H(\omega)-E-i\varepsilon)^{-1}\langle\cdot-\mathfrak{z}\rangle^{-\frac{\varsigma}{2}-n\frac{\varsigma}{4}}\| = \sup_{\varepsilon\neq 0} \|\langle\cdot-\mathfrak{z}\rangle^{-\frac{\varsigma}{2}-n\frac{\varsigma}{4}}(H(\omega)-E-i\varepsilon)^{-1}1^x\| \leq$$

$$\leq r_4(\omega,E,\varsigma,n)\langle x\rangle^{-n\frac{\varsigma}{4}}$$

Hence for all $E \in [\inf\Sigma, E_X]$ the set

$$B_3(E) := \bigcap_{n\in\mathbb{N}} B_2(E,\frac{\varsigma}{2},\frac{\varsigma}{2}+\frac{n\varsigma}{4}) \cap \bigcap_{y\in\mathbb{Q}^d} B_2(E,y) \text{ or } B_3(E) := \bigcap_{y\in\mathbb{Q}^d} B_2(E,y) \cap B_2(E,\mathcal{Q})$$

has full measure $\mathbb{P}(B_3(E)) = 1$. So

$$Y := \bigcup_{E\in[\inf\Sigma, E_X]} B_3(E) \times \{E\} \subseteq \Omega \times [\inf\Sigma, E_X]$$

has full outer measure $(\mathbb{P}\otimes\mu)^*(Y) = E_X - \inf\Sigma$, because for all $X \in \mathcal{B}(\Omega)\otimes\mathcal{B}([\inf\Sigma, E_X])$ with $X \supseteq Y$ every section $X(E) := \{\omega \in \Omega : (\omega, E) \in X\}$ contains $B_3(E)$, so

$$E_X - \inf\Sigma \geq (\mathbb{P}\otimes\mu)(X) = \int\limits_{\inf\Sigma}^{E_X} \mathbb{P}(X(E))dE \geq \int\limits_{\inf\Sigma}^{E_X} \mathbb{P}(B_3(E))dE = E_X - \inf\Sigma.$$

This calculation provides us with examples of sets $X \in \mathcal{B}(\Omega)\otimes\mathcal{B}([\inf\Sigma, E_X])$ with $X \supseteq Y$ and $(\mathbb{P}\otimes\mu)(X) = (\mathbb{P}\otimes\mu)^*(Y)$. Fixing such a set X and due to $(\mathbb{P}\otimes\mu)^*(X\backslash Y) = 0$, we can choose an $N \in \mathcal{B}(\Omega)\otimes\mathcal{B}([\inf\Sigma, E_X])$ with $N \supseteq X\backslash Y$ and $(\mathbb{P}\otimes\mu)(N) = 0$. Defining $I := X\backslash N \subseteq Y$ the sections $I(\omega) := \{E \in [\inf\Sigma, E_X] : (\omega, E) \in I\}$ and the function $\begin{array}{ccc}\Omega & \longrightarrow & \mathbb{R}_+ \\ \omega & \longmapsto & \mu(I(\omega))\end{array}$ are measurable. Due to Fubini's Theorem we get

$$\int\limits_{\Omega} \mu(I(\omega))d\mathbb{P}(\omega) = (\mathbb{P}\otimes\mu)(I) = E_X - \inf\Sigma,$$

so $\mathbb{P}\{\omega \in \Omega : \mu(I(\omega)) < E_X - \inf\Sigma\} = 0$ and $Z_0 := \Omega\backslash\{\omega \in \Omega : \mu(I(\omega)) < E_X - \inf\Sigma\} \in \mathcal{B}(\Omega)$ has full $\mathbb{P}$-measure, which completes the proof. $\square$

6 Proof of Localization

Following the ideas of [KSS1], [KSS2] and [CH1], we note three results used for the proof of localization, before turning to the proof of our main theorem.

6.1 "$\mathbb{P}$-almost sure Absolute Continuity"

Theorem 6.1. *Let $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$ and $\mathcal{Q} \in L^\infty(\mathbb{R}^d, \mathbb{R}_+)$ and suppose that there is a nonempty open set $K \subseteq \mathbb{R}^d$ with $\mathcal{Q}\cdot 1_K \geq 1_K$. Define $H_0 := -\triangle + V$ and for each $\alpha \in \mathbb{R}$ take $H(\alpha) := H_0 + \alpha\mathcal{Q}$ and let π_α be the spectral decomposition of $H(\alpha)$. Then for each $M \in \mathcal{B}(\mathbb{R})$ with $\mu(M) = 0$ there is an $N \in \mathcal{B}(\mathbb{R})$ with $\mu(N) = 0$, such that $\pi_\alpha(M) = 0$ for all $\alpha \in \mathbb{R}\backslash N$.*

Proof. Choosing $V = \mathcal{Q}$ and $B = \sqrt{\mathcal{Q}}$ in part 1 of Lemma 2.1 we get

$$\int\limits_{\mathbb{R}} \frac{1}{1+\alpha^2}\langle\Phi, \sqrt{\mathcal{Q}}\pi_\alpha(M)\sqrt{\mathcal{Q}}\Phi\rangle d\alpha \leq \mu(M)\|\Phi\|^2$$

for all $\Phi \in L^2(\mathbb{R}^d)$. As $\mu(M) = 0$ we get $N(M, \Phi) \in \mathcal{B}(\mathbb{R})$ with $\mu(N(M, \Phi)) = 0$ and $\langle\Phi, \sqrt{\mathcal{Q}}\pi_\alpha(M)\sqrt{\mathcal{Q}}\Phi\rangle = \|\pi_\alpha(M)\sqrt{\mathcal{Q}}\Phi\|^2 = 0$ for all $\alpha \in \mathbb{R}\backslash N(M, \Phi)$. Taking a countable dense subset $F \subseteq L^2(\mathbb{R}^d)$, we have $\pi_\alpha(M)\sqrt{\mathcal{Q}}\Phi = 0$ and $0 = g(H(\alpha))\pi_\alpha(M)\sqrt{\mathcal{Q}}\Phi =$ $= \pi_\alpha(M)g(H(\alpha))\sqrt{\mathcal{Q}}\Phi$ for all $\Phi \in L^2(\mathbb{R}^d)$, $\alpha \in \mathbb{R}\backslash \bigcup\limits_{\phi\in F} N(M, \phi)$ and $g \in L^\infty(\pi_\alpha)$. As

$\mathcal{Q} \cdot 1_K \geq 1_K$ the subspace $\mathrm{lin}\{g(H(\alpha))\sqrt{\mathcal{Q}}\Phi : g \in L^\infty(\pi_\alpha), \Phi \in L^2(\mathbb{R}^d)\}$ is dense in $L^2(\mathbb{R}^d)$ due to Proposition A2.2 in [CH1] and Theorem 6.1 is therefore proven. $\square$

6.2 Distributional Eigenfunctions

Theorem 6.2. *Let $s > \frac{d}{2}$ and $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$. For the Schrödinger operator $H = -\triangle + V$ we take a spectral control measure ν. Then for ν-almost every $E \in \mathbb{R}$ there is a distributional solution Ψ of the eigenvalue equation $H\Psi = E\Psi$ with $\Psi \in H^1_{-s}(\mathbb{R}^d)\backslash\{0\} = \{f : \mathbb{R}^d \to \mathbb{C} : f \neq 0 \text{ measurable }, \langle\cdot\rangle^{-s} f \in H^1(\mathbb{R}^d)\}$.*

Proof. With $S = \sqrt{-\triangle + 1}$, $T = \langle\cdot\rangle^s$ and $\gamma(\alpha) := (\alpha - z)^{-m}, \alpha \in \mathbb{R}$, for a given $z \in \mathbb{C}\backslash\mathbb{R}$, $m \in \mathbb{N}$, $m > \frac{1}{2} + \frac{d}{4}$, Theorem 3.6 of [PS] shows, that the assumptions of Theorem 1 and 2 of [PSW] are satisfied, so this Theorem follows. $\square$

6.3 Perturbation Theory

Theorem 6.3. *Let Z_0 be the set constructed in Theorem 5.3 and for $\omega \in Z_0$ take $E \in J(\omega) := \{E \in I(\omega) : E \text{ is not an eigenvalue of } H(\omega)\}$, $|\mathfrak{z}| < R$, $\alpha \in \mathbb{R}\backslash\{0\}$ and a distributional solution $\Psi \in H^1_{-\frac{5}{4}}(\mathbb{R}^d)$ of the equation*

$$H(\omega)\Psi + \alpha\wp(\cdot - \mathfrak{z})\Psi = E\Psi. \tag{17}$$

Then Ψ decays as follows:

- *If $\wp$ is of compact support and γ as in Theorem 4.9, then there is a $C = C(\Psi, \alpha) > 0$, such that for all $y \in \mathbb{R}^d$ one gets:*

  $$|\Psi(y)| \leq Ce^{-\frac{\gamma}{5}|y|}$$

- *In the long range case for all $t > 0$ there is a $C_t = C_t(\Psi, \alpha) > 0$ such that*

  $$|\Psi(y)| \leq C_t\langle y\rangle^{-t}$$

  for all $y \in \mathbb{R}^d$.

Proof. All potentials are bounded, hence the subsolution estimate (see [CFKS] Theorem 2.4) implies for distributional solutions $\Psi \in H^1_{-\frac{5}{4}}(\mathbb{R}^d)$ of (17), that ψ is a continuous function and there is a constant c independent of y, such that

$$|\Psi(y)| \leq c \int\limits_{|y-x|<\frac{1}{2}} |\Psi(x)|dx.$$

Application of the Hölder inequality shows $|\Psi(y)| \leq c\|1^y\Psi\|_{L^2}$. To estimate $\|1^y\Psi\|_{L^2}$ we note that the spectral calculus implies

$$\lim_{\varepsilon\to 0}(H(\omega) - E - i\varepsilon)^{-1}(H(\omega) - E)\Phi = \Phi - \pi(\omega, \{E\})\Phi$$

for all $E \in \mathbb{R}$ and $\Phi \in \mathcal{D}(H(\omega))$, where $\pi(\omega,\cdot)$ is the spectral decomposition of $H(\omega)$. Next we choose an Urysohn function φ_n of the box $\Lambda(0,2n+1)$ with $\varphi_n|_{\Lambda(0,2n-1)} = 1$. Due to $\Psi \in H^1_{-\frac{\varsigma}{4}}(\mathbb{R}^d)$ we get $\langle\cdot\rangle^{-\frac{\varsigma}{4}}\partial_j\Psi \in L^2(\mathbb{R}^d)$, hence $\nabla\varphi_n \cdot \nabla\Psi \in L^2(\mathbb{R}^d)$. Inserting the distributional eigenvalue equation $-\triangle\Psi = E\Psi - \mathrm{V}(\omega)\Psi - \alpha\wp(\cdot - \mathfrak{z})\Psi$ in the Leibniz product rule

$$\triangle(\varphi_n\Psi) = \sum_{k=1}^{d}[\Psi(\partial_k^2\varphi_n) + \varphi_n\partial_k^2\Psi + 2\partial_k\varphi_n\partial_k\Psi] = (\triangle\varphi_n)\Psi + \varphi_n\triangle\Psi + 2\nabla\varphi_n\cdot\nabla\Psi$$

implies $\varphi_n\Psi \in \mathcal{D}(-\triangle)$. As $E \in J(\omega)$ is not an eigenvalue of $H(\omega)$, we conclude

$$\lim_{\varepsilon\to 0}(H(\omega) - E - i\varepsilon)^{-1}(H(\omega) - E)\varphi_n\Psi = \varphi_n\Psi.$$

Given $y \in \mathbb{Q}^d$ we choose $n \in \mathbb{N}$ with $\Lambda(y,1) \subseteq \Lambda(0,2n-1)$, then inserting $(H(\omega)-E)\varphi_n\Psi$:

$$\begin{aligned}
\|1^y\Psi\| &= \|1^y\varphi_n\Psi\| = \|1^y\lim_{\varepsilon\to 0}(H(\omega) - E - i\varepsilon)^{-1}(H(\omega) - E)(\varphi_n\Psi)\| = \\
&= \|1^y\lim_{\varepsilon\to 0}(H(\omega) - E - i\varepsilon)^{-1}(-(\triangle\varphi_n)\Psi - 2\nabla\varphi_n\cdot\nabla\Psi - \alpha\wp(\cdot - \mathfrak{z})\varphi_n\Psi)\| \le \\
&\le \sup_{\varepsilon\neq 0}\|1^y(H(\omega) - E - i\varepsilon)^{-1}(\triangle\varphi_n)\Psi\| + 2\sup_{\varepsilon\neq 0}\|1^y(H(\omega) - E - i\varepsilon)^{-1}\nabla\varphi_n\cdot\nabla\Psi\| + \\
&\quad + |\alpha|\sup_{\varepsilon\neq 0}\|1^y(H(\omega) - E - i\varepsilon)^{-1}\wp(\cdot - \mathfrak{z})\varphi_n\Psi\| \qquad (18)
\end{aligned}$$

As $\operatorname{supp}\partial^\beta\varphi_n \subseteq \Lambda(0,2n+1)\backslash\Lambda(0,2n-1) =: M(n)$ for all $\beta \in \mathbb{N}_0^d$ with $|\beta| \ge 1$ we get

$$\|1^y(H(\omega) - E - i\varepsilon)^{-1}\triangle\varphi_n\Psi\| \le \|\triangle\varphi_n\|_\infty \sum_{\substack{x\in\mathbb{Z}^d\\|x|=n}} \|1^y(H(\omega) - E - i\varepsilon)^{-1}1^x\|\,\|1_{M(n)}\Psi\|$$

$$\|1^y(H(\omega) - E - i\varepsilon)^{-1}\nabla\varphi_n\cdot\nabla\Psi\| \le d\|\nabla\varphi_n\|_\infty \sum_{\substack{x\in\mathbb{Z}^d\\|x|=n}} \|1^y(H(\omega) - E - i\varepsilon)^{-1}1^x\|\,\|1_{M(n)}\partial^k\Psi\|.$$

As $\|1_{M(n)}\partial^k\Psi\|, \|1_{M(n)}\Psi\| \le (1 + (n+\frac{1}{2})^2)^{\frac{\varsigma}{8}}\|\Psi\|_{H^1_{\frac{\varsigma}{4}}}$, we obtain a polynom $p(n)$, such that after inserting the estimates of Theorem 5.3 for $|x| = n \ge r_3(\omega, E, y)$

$$\|1^y\Psi\| \le p(n)\|\Psi\|_{H^1_{-\frac{\varsigma}{4}}} e^{-\frac{\gamma}{5}n} + |\alpha|\sup_{\varepsilon\neq 0}\|1^y(H(\omega) - E - i\varepsilon)^{-1}\varphi_n\wp(\cdot - \mathfrak{z})\Psi\| \qquad (19)$$

Estimating the last term of (19) we consider first the short range case for $|y| \ge r_3(\omega, E, \wp)$ and for n big enough, such that $\varphi_n\wp(\cdot - \mathfrak{z}) = \wp(\cdot - \mathfrak{z})$

$$\|1^y(H(\omega) - E - i\varepsilon)^{-1}\varphi_n\wp(\cdot - \mathfrak{z})\Psi\| \le \|1^y(H(\omega) - E - i\varepsilon)^{-1}\wp(\cdot - \mathfrak{z})\|\,\|1_{\operatorname{supp}\wp(\cdot - \mathfrak{z})}\Psi\|$$

$$\le r_4(\omega, E, \wp)e^{-\frac{\gamma}{5}|y|}\|1_{\operatorname{supp}\wp(\cdot - \mathfrak{z})}\Psi\|.$$

Combining the results for $n \to \infty$ we get

$$|\Psi(y)| \le c\|1^y\Psi\| \le c|\alpha|r_4(\omega, E, \wp)\|1_{\operatorname{supp}\wp(\cdot - \mathfrak{z})}\Psi\|e^{-\frac{\gamma}{5}|y|} \qquad (20)$$

and this result extends to all $y \in \mathbb{R}^d$ by continuity of Ψ. In the long range case we have to do a sort of induction process. First note that $\langle \cdot - \mathfrak{z} \rangle^{-\frac{\varsigma}{4}} \Psi$, $\wp(\cdot - \mathfrak{z})\Psi \in L^2(\mathbb{R}^d)$ hence

$$\begin{aligned}
&\|1^y (H(\omega) - E - i\varepsilon)^{-1} \varphi_n \wp(\cdot - \mathfrak{z})\Psi\| \leq \\
&\quad \leq \|1^y (H(\omega) - E - i\varepsilon)^{-1} (\wp(\cdot - \mathfrak{z}) \langle \cdot - \mathfrak{z} \rangle^{\frac{\varsigma}{4}})\| \, \|\langle \cdot - \mathfrak{z} \rangle^{-\frac{\varsigma}{4}} \varphi_n \Psi\| \\
&\quad \leq c_1 \|\langle \cdot - \mathfrak{z} \rangle^{-\frac{3}{4}\varsigma} (H(\omega) - E + i\varepsilon)^{-1} 1^y\| \, \|\Psi\|_{H^1_{-\frac{\varsigma}{4}}}
\end{aligned}$$

Using Theorem 5.3 we estimate

$$\sup_{\varepsilon \neq 0} \|1^y (H(\omega) - E - i\varepsilon)^{-1} \langle \cdot - \mathfrak{z} \rangle^{-\frac{3}{4}\varsigma}\| \leq r_4(\omega, E, \varsigma, 1) \langle y \rangle^{-\frac{\varsigma}{4}}$$

for $|y| \geq r_3(\omega, E, \varsigma, 1)$. Thus for $n \to \infty$ equation (19) implies

$$|\Psi(y)| \leq c\|1^y \Psi\| \leq c c_1 |\alpha| r_4(\omega, E, \varsigma, 1) \|\Psi\|_{H^1_{-\frac{\varsigma}{4}}} \langle y \rangle^{-\frac{\varsigma}{4}}. \tag{21}$$

This estimate yields $\Psi \in L^2(\mathbb{R}^d)$, so we are able to improve the estimate

$$\begin{aligned}
&\|1^y (H(\omega) - E - i\varepsilon)^{-1} \varphi_n \wp(\cdot - \mathfrak{z})\Psi\| \leq \|1^y (H(\omega) - E - i\varepsilon)^{-1} \wp(\cdot - \mathfrak{z})\| \, \|\Psi\| \leq \\
&\quad \leq c_1 \|\langle \cdot - \mathfrak{z} \rangle^{-\varsigma} (H(\omega) - E + i\varepsilon)^{-1} 1^y\| \, \|\Psi\| = c_1 \|1^y (H(\omega) - E - i\varepsilon)^{-1} \langle \cdot - \mathfrak{z} \rangle^{-\varsigma}\| \, \|\Psi\|
\end{aligned}$$

in order to get

$$|\Psi(y)| \leq c\|1^y \Psi\| \leq c|\alpha| c_1 r_4(\omega, E, \varsigma, 2) \|\Psi\| \langle y \rangle^{-2\frac{\varsigma}{4}}$$

for $|y| \geq r_3(\omega, E, \varsigma, 2)$. Continuing we end at

$$|\Psi(y)| \leq c|\alpha| c_1 \|\langle \cdot \rangle^{(n-2)\frac{\varsigma}{4}} \Psi\| r_4(\omega, E, \varsigma, n) \langle y \rangle^{-n\frac{\varsigma}{4}}$$

for $|y| \geq r_3(\omega, E, \varsigma, n)$ and for $n\frac{\varsigma}{4} \geq t$. Choosing $C_t(\Psi, \alpha) \geq c|\alpha| c_1 r_4(\omega, E, \varsigma, n) \|\langle \cdot \rangle^{(n-2)\frac{\varsigma}{4}} \Psi\|$, we get $|\Psi(y)| \leq C_t(\Psi, \alpha) \langle y \rangle^{-t}$ for all $y \in \mathbb{R}^d$. □

6.4 Proof of the Main Theorem

Proof. We first gather some facts from the last Sections, which imply pure point spectrum for operators "perturbed" at site 0: As it was seen in the construction of Z_0, for each $\omega \in Z_0$ we have $\mu([\inf \Sigma, E_X] \setminus J(\omega)) = 0$. So due to Theorem 6.1 we get $S(\omega, \mathfrak{z}) \subseteq \mathbb{R}$ with $\mu(\mathbb{R} \setminus S(\omega, \mathfrak{z})) = 0$ and $\nu_{\alpha,\mathfrak{z},\omega}([\inf \Sigma, E_X] \setminus J(\omega)) = 0$ for every $\omega \in Z_0$, $|\mathfrak{z}| < R$ and $\alpha \in S(\omega, \mathfrak{z})$, where $\nu_{\alpha,\mathfrak{z},\omega}$ is a spectral control measure of the operator $H(\omega) + \alpha\wp(\cdot - \mathfrak{z})$. On the other hand for $\nu_{\alpha,\mathfrak{z},\omega}$-almost all $E \in \mathbb{R}$ there is a distributional solution $\Psi \in H^1_{-\frac{\varsigma}{4}}(\mathbb{R}^d)$ of the eigenvalue equation $(H(\omega) + \alpha\wp(\cdot - \mathfrak{z}))\Psi = E\Psi$. Applying Theorem 6.3 for $|\mathfrak{z}| < R$ and $E \in J(\omega)$ these distributional eigensolutions are L^2-eigenfunctions, decaying as stated in Theorem 6.3. Hence the interval $[\inf \Sigma, E_X]$ consists $\nu_{\alpha,\mathfrak{z},\omega}$-almost sure of eigenvalues of $H(\omega) + \alpha\wp(\cdot - \mathfrak{z})$ and $\sigma(H(\omega) + \alpha\wp(\cdot - \mathfrak{z})) \cap [\inf \Sigma, E_X]$ is therefore pure point. In the second step of the proof, we construct a set Z_1 of full $\mathbb{P}$-measure, such that for each $\overline{\omega} \in Z_1$ the operator $H(\overline{\omega})$ equals such an operator perturbed at site 0 of pure point spectrum as we have just constructed. To do this we consider Ω as a product of the probability spaces $\mathbb{R} = (\mathbb{R}, \mathbb{P}_0)$, $\mathbb{R}^d = (\mathbb{R}^d, \mathbb{P}_1)$ and

$(\hat\Omega,\hat{\mathbb{P}}) = ((\mathbb{R}\times\mathbb{R}^d)^{\otimes(\mathbb{Z}^d\setminus\{0\})}, (\mathbb{P}_0\otimes\mathbb{P}_1)^{\otimes(\mathbb{Z}^d\setminus\{0\})})$ writing $\omega = (\omega_0', \omega_0'', \hat\omega) \in \Omega$ with $\omega_0' \in \mathbb{R}$, $\omega_0'' \in \mathbb{R}^d$ and $\hat\omega \in \hat\Omega$. For each $\alpha \in \mathbb{R}$ let $M(\alpha) := \{(\hat\omega, \mathfrak{z}) \in \hat\Omega \times \mathbb{R}^d : (\alpha, \mathfrak{z}, \hat\omega) \in Z_0\}$ denote the section through Z_0 at α. As

$$1 = \mathbb{P}(Z_0) = \int_{\mathbb{R}} (\hat{\mathbb{P}}\otimes\mathbb{P}_1)(M(\alpha))d\mathbb{P}_0(\alpha)$$

we choose $\alpha_0 \in \mathbb{R}$ with $(\hat{\mathbb{P}}\otimes\mathbb{P}_1)(M(\alpha_0)) = 1$. For $(\hat\omega,\mathfrak{z}) \in \hat\Omega\times\mathbb{R}^d$ resp. $(\omega,\mathfrak{z}) \in \Omega\times\mathbb{R}^d$ define $\overline{\omega}(\mathfrak{z}) = (\overline{\omega}(\mathfrak{z})_j)_{j\in\mathbb{Z}^d}$ and $\delta(\alpha,\mathfrak{z}) = (\delta(\alpha,\mathfrak{z})_j)_{j\in\mathbb{Z}^d} \in \Omega$ taking

$$\overline{\omega}(\mathfrak{z})_j := \begin{cases} (\alpha_0,\mathfrak{z}) & : \ j = 0 \\ \omega_j & : \ j \neq 0 \end{cases} \quad \text{and} \quad \delta(\alpha,\mathfrak{z})_j := \begin{cases} (\alpha,\mathfrak{z}) & : \ j = 0 \\ 0 & : \ j \neq 0 \end{cases} \tag{22}$$

Then we define

$$Z_1 := \{(\overline{\omega}(\mathfrak{z}) + \delta(\alpha,\mathfrak{z})) : \overline{\omega}(\mathfrak{z}) \in Z_0, |\mathfrak{z}| < R, \alpha \in S(\overline{\omega}(\mathfrak{z}),\mathfrak{z})\} \subseteq \Omega$$

$$Z_2 := \{\omega \in \Omega : H(\omega) \text{ is pure point in } [\inf\Sigma, E_X] \text{ and the eigenfunctions}$$

$$\text{decay as in Theorem 6.3 } \} \subseteq \Omega$$

and for $\hat\omega \in \hat\Omega$, $|\mathfrak{z}| < R$ the section through Z_1 at $(\hat\omega,\mathfrak{z})$

$$S_1(\hat\omega,\mathfrak{z}) := \{\alpha \in \mathbb{R} : \overline{\omega}(\mathfrak{z}) + \delta(\alpha,\mathfrak{z}) \in Z_1\}.$$

Using the definition of Z_1

$$\begin{aligned} S_1(\hat\omega,\mathfrak{z}) &= \{\alpha \in \mathbb{R} : (\overline{\omega}(\mathfrak{z}) + \delta(\alpha,\mathfrak{z})) \in Z_1\} = \{\alpha \in \mathbb{R} : \overline{\omega}(\mathfrak{z}) \in Z_0, \alpha \in S(\overline{\omega}(\mathfrak{z}),\mathfrak{z})\} = \\ &= \begin{cases} \emptyset & : \ \overline{\omega}(\mathfrak{z}) \notin Z_0 \\ S(\overline{\omega}(\mathfrak{z}),\mathfrak{z}) & : \ \overline{\omega}(\mathfrak{z}) \in Z_0 \end{cases} = \begin{cases} \emptyset & : \ (\hat\omega,\mathfrak{z}) \notin M(\alpha_0) \\ S(\overline{\omega}(\mathfrak{z}),\mathfrak{z}) & : \ (\hat\omega,\mathfrak{z}) \in M(\alpha_0) \end{cases} \end{aligned}$$

and as $\hat{\mathbb{P}}(M(\alpha_0)) = 1$ and $\mathbb{P}_0(S(\overline{\omega}(\mathfrak{z})),\mathfrak{z})) = 1$ for all $(\hat\omega,\mathfrak{z}) \in M(\alpha_0)$ we compute

$$\begin{aligned} \mathbb{P}(Z_1) &= \int_{\hat\Omega\times\mathbb{R}^d} \int_{\mathbb{R}} 1_{Z_1}(\alpha_0+\alpha,\mathfrak{z},\hat\omega)d\mathbb{P}_0(\alpha)d\mathbb{P}_1(\mathfrak{z})d\hat{\mathbb{P}}(\hat\omega) = \\ &= \int_{M(\alpha_0)} \int_{\mathbb{R}} 1_{S_1(\hat\omega,\mathfrak{z})}(\alpha)d\mathbb{P}_0(\alpha)d\mathbb{P}_1(\mathfrak{z})d\hat{\mathbb{P}}(\hat\omega) = \\ &= \int_{M(\alpha_0)} \int_{\mathbb{R}} 1_{S(\overline{\omega}(\mathfrak{z}),\mathfrak{z})}(\alpha)d\mathbb{P}_0(\alpha)d\mathbb{P}_1(\mathfrak{z})d\hat{\mathbb{P}}(\hat\omega) = \\ &= \int_{M(\alpha_0)} \mathbb{P}_0(S(\overline{\omega}(\mathfrak{z}),\mathfrak{z}))d\mathbb{P}_1(\mathfrak{z})d\hat{\mathbb{P}}(\hat\omega) = \int_{M(\alpha_0)} d\hat{\mathbb{P}}(\hat\omega)d\mathbb{P}_1(\mathfrak{z}) = 1, \end{aligned}$$

In view of

$$H(\overline{\omega}(\mathfrak{z})) + \alpha\wp(\cdot-\mathfrak{z}) = -\triangle + \sum_{\substack{j\in\mathbb{Z}^d \\ j\neq 0}} \omega_j'\wp(\cdot - j - \omega_j'') + (\alpha_0+\alpha)\wp(\cdot-\mathfrak{z}) = H(\overline{\omega}(\mathfrak{z}) + \delta(\alpha,\mathfrak{z}))$$

we get $Z_1 \subseteq Z_2$ and hence the conclusion. □

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