The Periodic Schrödinger Operators with Potentials in the C. Fefferman-Phong Class

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Abstract. We consider the periodic Schrödinger operator $-\Delta + V(\mathbf{x})$ in $\mathbb{R}^d$, $d \geq 3$ with potential V in the C. Fefferman-Phong class. Let Ω be a periodic cell for V. We show that, for $p \in ((d-1)/2, d/2]$, there exists a positive constant ε depending only on the shape of Ω, p and d such that, if

$$\limsup_{r\to 0} \sup_{\mathbf{x}\in\Omega} r^2 \left\{ \frac{1}{|B(\mathbf{x},r)|} \int_{B(\mathbf{x},r)} |V(\mathbf{y})|^p d\mathbf{y} \right\}^{1/p} < \varepsilon,$$

then the spectrum of $-\Delta + V$ is purely absolutely continuous. We obtain this result as a consequence of certain weighted L^2 Sobolev inequalities on the d-torus. It improves an early result by the author for potentials in $L^{d/2}$ or weak-$L^{d/2}$ space.

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1. Introduction

Consider the Schrödinger operator

$$-\Delta + V(\mathbf{x}) \quad \text{in} \quad \mathbb{R}^d \tag{1.1}$$

with a real periodic potential V. In a celebrated work [27], L. Thomas showed that, if $d = 3$ and $V \in L^2_{\text{loc}}(\mathbb{R}^3)$, the spectrum of $-\Delta + V$ is purely absolutely continuous. Thomas' result was extended subsequently by M. Reed-B. Simon [20], L. Danilov

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[7], R. Hempel -I.Herbst [10,11], M. Birman -T. Suslina [1,2,3], A. Morame [19], and A. Sobolev [22]. We remark that [7] studied the Dirac operator with a periodic scalar potential and [1,2,3,10,11,19,22] investigated the magnetic Schrödinger operator $(-i\nabla - \mathbf{a}(\mathbf{x}))^2 + V(\mathbf{x})$ with periodic potentials $\mathbf{a}$ and V. In particular, the results in [2,3], pertaining to the case $\mathbf{a} = 0$, give the absolute continuity of $-\Delta + V$ for $V \in L^r_{\mathrm{loc}}(\mathbb{R}^d)$ where $r > 1$ if $d = 2$, $r = d/2$ if $d = 3$ or 4, and $r = d - 2$ if $d \geq 5$. Recently in [21], the author was able to establish the absolute continuity of $-\Delta + V$ for $V \in L^{d/2}_{\mathrm{loc}}(\mathbb{R}^d)$, $d \geq 3$. In the context of L^p spaces, this is best possible. [21] also gives the optimal condition for the absolute continuity of $-\Delta + V$ with periodic potentials in Lorentz spaces. See [Theorem 1.2, 21].

The purpose of this paper is to study the spectral properties of the periodic Schrödinger operators with potentials in the C. Fefferman-Phong class. Our result improves that in [21].

Definition 1.2. Let $g \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for some $p \geq 1$. We say $g \in F^p_{\mathrm{loc}}(\mathbb{R}^d)$ if

$$\sup_{\substack{0<r<1 \\ \mathbf{x}\in K}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |g(\mathbf{y})|^p d\mathbf{y} \right\}^{1/p} < \infty \qquad \text{for any compact } K,$$

where $Q(\mathbf{x}, r)$ denotes the cube centered at $\mathbf{x}$ with side length r.

Remark 1.3. It is not hard to see that

$$\begin{aligned} F^p_{\mathrm{loc}}(\mathbb{R}^d) &= L^p_{\mathrm{loc}}(\mathbb{R}^d) \quad \text{if } p = d/2 \text{ and } d \geq 2, \\ \text{weak} - L^{d/2}_{\mathrm{loc}}(\mathbb{R}^d) &\subset F^p_{\mathrm{loc}}(\mathbb{R}^d) \quad \text{if } 1 < p < d/2 \text{ and } d \geq 3. \end{aligned}$$

Also one may verify easily that

$$g(\mathbf{x}) = \frac{f(\frac{\mathbf{x}}{|\mathbf{x}|})}{|\mathbf{x}|^2} \in F^p_{\mathrm{loc}}(\mathbb{R}^d) \quad \text{if } 1 \leq p < d/2, \ \ d \geq 3 \ \text{ and } f \in L^p(S^{d-1}).$$

This implies that $F^q_{\mathrm{loc}}(\mathbb{R}^d) \subsetneq F^p_{\mathrm{loc}}(\mathbb{R}^d)$ if $1 \leq p < q \leq d/2$ and $d \geq 3$.

Let $\mathbf{D} = -i\nabla$ and $\mathbf{D}A\mathbf{D}^T = \sum_{j,k} D_j a_{jk} D_k$ where $A = (a_{jk})$ is a $d \times d$ matrix. The following is the main result of the paper.

Main Theorem. *Let $A = (a_{jk})_{d\times d}$ be a symmetric, positive-definite matrix with real constant entries. Let $V \in F^p_{\mathrm{loc}}(\mathbb{R}^d)$ be a real periodic function on $\mathbb{R}^d$, $d \geq 3$*

with periodic cell Ω. Suppose $p > (d-1)/2$. Then there exists a positive constant ε depending only on the shape of Ω, A, p and d such that, if

$$\limsup_{r\to 0} \sup_{\mathbf{x}\in\Omega} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V(\mathbf{y})|^p d\mathbf{y} \right\}^{1/p} < \varepsilon, \tag{1.4}$$

then the spectrum of $\mathbf{D}A\mathbf{D}^T + V$ is purely absolutely continuous.

Remark 1.5. Our main theorem contains Theorem 1.2 in [21]. This follows from the estimate,

$$\begin{aligned} \limsup_{r\to 0} \sup_{\mathbf{x}\in\Omega} r^2 &\left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V(\mathbf{y})|^p d\mathbf{y} \right\}^{1/p} \\ &\le C_{p,d} \limsup_{t\to\infty} t |\left\{ \mathbf{x}\in\Omega : |V(\mathbf{x})| > t \right\}|^{2/d} \end{aligned} \tag{1.6}$$

where $1 < p < d/2$. To see (1.6), suppose that

$$\limsup_{t\to\infty} t |\left\{ \mathbf{x}\in\Omega : |V(\mathbf{x})| > t \right\}|^{2/d} < \varepsilon_0.$$

Let $N = \varepsilon_0/r^2$. If r is sufficiently small,

$$\begin{aligned} r^2 &\left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V(\mathbf{y})|^p d\mathbf{y} \right\}^{1/p} \\ &\le r^2 N + r^2 \left\{ \frac{1}{r^d} \int_{\{\mathbf{y}\in\Omega : |V(\mathbf{y})| > N\}} |V(\mathbf{y})|^p d\mathbf{y} \right\}^{1/p} \\ &= \varepsilon_0 + r^2 \left\{ \frac{1}{r^d} \int_N^\infty p\, t^{p-1} |\{\mathbf{y}\in\Omega : |V(\mathbf{y})| > t\}| dt \right\}^{1/p} \\ &\le \varepsilon_0 + r^2 \left\{ \frac{1}{r^d} \int_N^\infty p\, t^{p-1} \cdot \frac{(\varepsilon_0)^{d/2}}{t^{d/2}} dt \right\}^{1/p} \\ &= C_{p,d}\, \varepsilon_0. \end{aligned}$$

Hence,

$$\limsup_{r\to 0} \sup_{\mathbf{x}\in\Omega} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V(\mathbf{y})|^p d\mathbf{y} \right\}^{1/p} \le C_{p,d}\, \varepsilon_0.$$

This gives (1.6). On the other hand, we can construct a periodic potential V which satisfies (1.4) and

$$V(\mathbf{x}) \approx \frac{1}{|\mathbf{x}'|^\alpha} \quad \text{near } \mathbf{x} = 0,$$

where $0 < \alpha < 2$ and $\mathbf{x}' = (x_2, \ldots, x_d)$. Note that $V \notin$ weak-$L^{d/2}(\Omega)$ if $\alpha > 2(d-1)/d$.

Remark 1.7. Under the assumption that V is periodic and satisfies condition (1.4) for some $p > 1$, one may define the self-adjoint operator $-\Delta + V$ by a quadratic form. See section 2. However the problem of absolute continuity of $-\Delta+V$ remains open in the case $d \geq 4$ and $1 < p \leq (d-1)/2$.

Note that, by a change of coordinates, we may assume that V is periodic with respect to the lattice $(2\pi\mathbb{Z})^d$. Our general approach to the main theorem is similar to that used in [21]. The main new ingredient is a weighted L^2 estimate in the place of the crucial $L^p - L^q$ uniform Sobolev estimate in [21]. To describe this new idea, we fix $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^d$ and consider a family of operators

$$\mathbb{H}_V(z) = (\mathbf{D} + z\mathbf{a} + \mathbf{b})A(\mathbf{D} + z\mathbf{a} + \mathbf{b})^T, \quad z \in \mathbb{C} \tag{1.8}$$

on $L^2(\Omega)$ where $\Omega = [0, 2\pi)^d \approx \mathbb{R}^d/(2\pi\mathbb{Z})^d = \mathbb{T}^d$. Using the Floquet decomposition and following Thomas' idea, we reduce the proof of the main theorem to the problem of showing that, for some appropriate chosen weight function ω, we have

$$c_\rho \|\psi\|_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})} \leq \|\mathbb{H}_V(z)\psi\|_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})} \tag{1.9}$$

where $z = \delta + i\rho$, δ is a fixed number depending on $\mathbf{a}$ and $\mathbf{b}$, and $c_\rho \to \infty$ as $\rho \to \infty$. See section 3.

Our choice of the weight function ω will be

$$\omega(\mathbf{x}) = \{M(|V|^q \chi_{\Omega_N})(\mathbf{x})\}^{1/q} \tag{1.10}$$

where $M(f)$ denotes the Hardy-Littlewood maximal function of f on $\mathbb{R}^d$, $\Omega_N = \{\mathbf{x} \in \Omega : |V(\mathbf{x})| > N\}$ (N is a fixed large number), and $1 < q < p$. The main step in the proof of (1.9) is to show that

$$\begin{aligned} &\|\psi\|_{L^2(\mathbb{T}^d, \omega d\mathbf{x})} \\ &\leq C \sup_{\substack{\mathbf{x}\in\Omega \\ 0<r<1}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} w^p(\mathbf{y})\, d\mathbf{y} \right\}^{1/p} \|\mathbb{H}_0(z)\psi\|_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{w})} \end{aligned} \tag{1.11}$$

where $p>(d-1)/2$, $\mathbb{H}_0(z)=(\mathbf{D}+z\mathbf{a}+\mathbf{b})A(\mathbf{D}+z\mathbf{a}+\mathbf{b})^T$, and C depends only on p, A and d.

We remark that (1.11) should be considered as a weighted L^2 uniform Sobolev inequality on the d-torus. In the case of $\mathbb{R}^d$, similar estimates were established by S. Chanillo -E. Sawyer [4] and F. Chiarenza -A. Ruiz [6] in the study of unique continuation problems. The weighted L^2 estimates in [4,6] for the operator $\Delta+\mathbf{a}\cdot\nabla+b$ extend the uniform L^p-L^q estimates obtained in [14] by C. Kenig - A. Ruiz -C. Sogge.

To show (1.11), we will appeal to the localization argument developed in [14]. It reduces (1.11) to the following estimate,

$$\begin{aligned}\|\psi\|_{L^2(\mathbb{T}^d,\omega d\mathbf{x})}\le C(p,c_0,A,d)\sup_{\substack{\mathbf{x}\in\Omega\\0<r<1}} r^2\left\{\frac{1}{r^d}\int_{Q(\mathbf{x},r)}w^p d\mathbf{y}\right\}^{1/p}\\ \cdot\left\|\left\{(\mathbf{D}+\mathbf{b})A(\mathbf{D}+\mathbf{b})^T+\xi\right\}\psi\right\|_{L^2(\mathbb{T}^d,\frac{d\mathbf{x}}{\omega})}\end{aligned}\tag{1.12}$$

where $p>(d-1)/2$, $\xi\in\mathbb{C}$ and $\mathrm{Re}\sqrt{\xi}\ge c_0>0$. (1.12) may be proved by using the Stein complex interpolation theorem, together with a weighted norm inequality for a fractional integral, as in case of $\mathbb{R}^d$ [4]. See Theorem 4.3. However, in the localization process, we will need the following weighted spectral projection estimate for the elliptic operator $\mathbf{D}A\mathbf{D}^T$ on the d-torus,

$$\begin{aligned}&\Big\|\sum_{|\mathbf{n}B|\in[k,k+1)}\hat{\psi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}\Big\|_{L^2(\mathbb{T}^d,\omega d\mathbf{x})}\\ &\le C\,k^{1/2}\left\{\sup_{\substack{\mathbf{x}\in\Omega\\0<r<1}} r^2\left(\frac{1}{r^d}\int_{Q(\mathbf{x},r)}w^p d\mathbf{y}\right)^{1/p}\right\}^{1/2}\|\psi\|_{L^2(\mathbb{T}^d,d\mathbf{x})},\end{aligned}\tag{1.13}$$

where $k\ge 1$ and $B=B_{d\times d}=\sqrt{A}\ge 0$. See Theorem 5.1. (1.13) is the analogue of the weighted restriction theorem established in [4,6]. Clearly, it would follow from the uniform Sobolev inequality

$$\begin{aligned}\|\psi\|_{L^2(\mathbb{T}^d,\omega d\mathbf{x})}\le Ck^{1/2}\left\{\sup_{\substack{\mathbf{x}\in\Omega\\0<r<1}} r^2\left(\frac{1}{r^d}\int_{Q(\mathbf{x},r)}w^p d\mathbf{y}\right)^{1/p}\right\}^{1/2}\\ \cdot\|(\mathbf{D}A\mathbf{D}^T-k^2-i\,k)\psi\|_{L^2(\mathbb{T}^d,d\mathbf{x})}.\end{aligned}\tag{1.14}$$

Finally we establish (1.14) by proving a weighted version of the Stein oscillatory integral theorem [25]. We remark that for a compact manifold without boundary,

C. Sogge obtained the $L^p - L^q$ estimates for the spectral projection associated with a second order elliptic operator in [23]. The observation that certain uniform Sobolev inequality implies the spectral projection estimate is taken from [23].

The paper is organized as follows. In section 2 we study the properties of periodic functions which are locally in the C. Fefferman-Phong class. In particular we show that the self-adjoint operator $\mathbf{D}A\mathbf{D}^T + V$ may be defined by a quadratic form under the assumption that V is periodic and satisfies (1.4) for some $p > 1$. In section 3 we use the Thomas approach to prove that our main theorem follows from the estimate (1.11). Sections 4-6 are devoted to the proof of (1.11) (Theorem 3.10). In section 4 we give the proof of (1.12) (Theorem 4.3). Section 5 contains the proof of the weighted spectral projection estimate (1.13). See Theorem 5.1. Finally in section 6 we use a localization argument to deduce (1.11) from (1.12)-(1.13).

Throughout the rest of this paper, we will assume that $d \geq 3$, $\Omega = [0, 2\pi)^d$, and V is periodic with respect to the lattice $(2\pi\mathbb{Z})^d$. We will use C to denote positive constants which may depend on d, p and the matrix A, which are not necessarily the same at each occurrence.

2. The C. Fefferman-Phong Class

We begin with the definition of the self-adjoint operator $\mathbf{D}A\mathbf{D}^T + V$ on $L^2(\mathbb{R}^d)$.

Let $\psi \in C_0^1(\mathbb{R}^d)$. Then

$$\Big|\psi(\mathbf{y}) - \frac{1}{|\Omega|}\int_\Omega \psi(\mathbf{x})d\mathbf{x}\Big| \leq C\, I_1(|\nabla\psi|\chi_\Omega)(\mathbf{y}) \tag{2.1}$$

where I_α denotes the Riesz potential

$$I_\alpha(f)(\mathbf{x}) = C_{\alpha,d}\int_{\mathbb{R}^d} \frac{f(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^{d-\alpha}} \tag{2.2}$$

(see [Lemma 7.16, 9]). It follows that

$$\int_\Omega |\psi|^2|V|d\mathbf{x} \leq C\int_{\mathbb{R}^d} |I_1(|\nabla\psi|\chi_\Omega)|^2|V|\chi_\Omega d\mathbf{x} + C\int_\Omega |V|\, d\mathbf{x}\cdot\int_\Omega |\psi|^2 d\mathbf{x}.$$

We now use the C. Fefferman-Phong estimate [Lemmas B and C, 8],

$$\int_{\mathbb{R}^d} |I_1(f)|^2|V|\, d\mathbf{x} \leq C \sup_{\substack{r>0\\ \mathbf{x}\in\mathbb{R}^d}} r^2\left\{\frac{1}{r^d}\int_{Q(\mathbf{x},r)} |V(\mathbf{y})|^p d\mathbf{y}\right\}^{1/p}\int_{\mathbb{R}^d} |f|^2 d\mathbf{x} \tag{2.3}$$

where $1 < p \leq d/2$. We obtain

$$\begin{aligned}\int_\Omega |\psi|^2 |V|\, d\mathbf{x} &\leq C \sup_{\substack{r>0\\ \mathbf{x}\in\mathbb{R}^d}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} (|V|\chi_\Omega)^p\, d\mathbf{y} \right\}^{1/p} \int_\Omega |\nabla\psi|^2 d\mathbf{x} \\ &\quad + C \int_\Omega |V|\, d\mathbf{x} \cdot \int_\Omega |\psi|^2 d\mathbf{x} \\ &\leq C \sup_{\substack{0<r<1\\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V|^p d\mathbf{y} \right\}^{1/p} \int_\Omega |\nabla\psi|^2 d\mathbf{x} + C \int_\Omega |V|\, d\mathbf{x} \cdot \int_\Omega |\psi|^2 d\mathbf{x}.\end{aligned}$$

Since V is periodic, it follows that

$$\begin{aligned}\int_{\mathbb{R}^d} |\psi|^2 |V|\, d\mathbf{x} &\leq C \sup_{\substack{0<r<1\\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V|^p d\mathbf{y} \right\}^{1/p} \int_{\mathbb{R}^d} |\nabla\psi|^2 d\mathbf{x} \\ &\quad + C \int_\Omega |V|\, d\mathbf{x} \cdot \int_{\mathbb{R}^d} |\psi|^2 d\mathbf{x}.\end{aligned} \tag{2.4}$$

To define the self-adjoint operator $\mathbf{D}A\mathbf{D}^T + V$, we consider the quadratic form

$$q[f,g] = \int_{\mathbb{R}^d} \{ < (\mathbf{D}f)A, \mathbf{D}g > + < Vf, g > \}\, d\mathbf{x} \tag{2.5}$$

for $f,\ g \in C_0^1(\mathbb{R}^d)$.

Theorem 2.6. *Let V be a real periodic function and $V \in F^p_{\mathrm{loc}}(\mathbb{R}^d)$ for some $p \in (1, d/2]$. Then there exists a positive constant ε depending only on A, p and d such that, if V satisfies the condition (1.4), then there exists a unique self-adjoint operator, which we denote by $\mathbf{D}A\mathbf{D}^T + V$, such that*

$$q[f,g] = \int_{\mathbb{R}^d} < (\mathbf{D}A\mathbf{D}^T + V)f, g > \ d\mathbf{x}$$

for $f \in Domain(\mathbf{D}A\mathbf{D}^T + V)$ and $g \in H^1(\mathbb{R}^d)$.

The following lemma will be needed in the proof of Theorem 2.6.

Lemma 2.7. *Let $1 < p \leq d/2$. Suppose that V satisfies (1.4). Then*

$$\lim_{N\to\infty} \sup_{\substack{0<r<1\\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V_N|^p d\mathbf{y} \right\}^{1/p} < \varepsilon$$

where $V_N(\mathbf{x}) = V(\mathbf{x})$ *if* $|V(\mathbf{x})| > N$ *and* $V_N(\mathbf{x}) = 0$ *if* $|V(\mathbf{x})| \leq N$.

Proof. By (1.4), there exists $r_0 > 0$ such that

$$\sup_{\substack{\mathbf{x}\in\Omega\\ 0<r<r_0}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V|^p d\mathbf{y} \right\}^{1/p} < \varepsilon.$$

Given $Q = Q(\mathbf{x}, r)$. If $r < r_0$,

$$r^2 \left\{ \frac{1}{|Q|} \int_Q |V_N|^p d\mathbf{y} \right\}^{1/p} \leq r^2 \left\{ \frac{1}{|Q|} \int_Q |V|^p d\mathbf{y} \right\}^{1/p} < \varepsilon.$$

If $r \geq r_0$, we have

$$r^2 \left\{ \frac{1}{|Q|} \int_Q |V_N|^p d\mathbf{y} \right\}^{1/p} \leq C_{r_0} \left\{ \int_\Omega |V_N|^p d\mathbf{y} \right\}^{1/p}$$

where we have used the assumption $p \leq d/2$. Thus,

$$\sup_{\substack{0<r<1\\ \mathbf{x}\in\Omega}} \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V_N|^p d\mathbf{y} \right\}^{1/p} < \varepsilon + C_{r_0} \left\{ \int_\Omega |V_N|^p d\mathbf{x} \right\}^{1/p} \to \varepsilon,$$

as $N \to \infty$. From this, the lemma follows easily.

We now give the

Proof of Theorem 2.6. Let $\psi \in C_0^1(\mathbb{R}^d)$. In view of (2.4) and Lemma 2.7, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |\psi|^2 |V|\, d\mathbf{x} &\leq \int_{\mathbb{R}^d} |\psi|^2 |V_N|\, d\mathbf{x} + N \int_{\mathbb{R}^d} |\psi|^2 d\mathbf{x} \\
&\leq C \sup_{\substack{0<r<1\\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V_N|^p d\mathbf{y} \right\}^{1/p} \int_{\mathbb{R}^d} |\nabla\psi|^2 d\mathbf{x} \\
&\qquad + \left\{ C \int_\Omega |V|\, d\mathbf{x} + N \right\} \int_{\mathbb{R}^d} |\psi|^2 d\mathbf{x} \\
&\leq C \sup_{\substack{0<r<1\\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V_N|^p d\mathbf{y} \right\}^{1/p} \int_{\mathbb{R}^d} < A\nabla\psi, \nabla\psi > d\mathbf{x} \\
&\qquad + \left\{ C \int_\Omega |V|\, d\mathbf{x} + N \right\} \int_{\mathbb{R}^d} |\psi|^2 d\mathbf{x} \\
&\leq C\varepsilon \int_{\mathbb{R}^d} < A\nabla\psi, \nabla\psi > d\mathbf{x} + C_{\varepsilon,V} \int_{\mathbb{R}^d} |\psi|^2 d\mathbf{x}
\end{aligned}$$

if N is large. This implies that, if ε is so small that $C\varepsilon < 1/2$, the symmetric quadratic form q is semi-bounded from below and closable on $H^1(\mathbb{R}^d)$. Hence it defines a unique self-adjoint operator.

In the rest of this section we will establish two propositions concerning the periodic functions which are locally in the C. Fefferman-Phong class. They will be used in the subsequent sections.

Proposition 2.8. *Suppose V is periodic with respect to the lattice $(2\pi\mathbb{Z})^d$ and $V \in F^p_{\text{loc}}(\mathbb{R}^d)$ for some $p \in (1, d/2]$. Let $1 < q < p$ and*

$$\omega(\mathbf{x}) = \{M(|V|^q)(\mathbf{x})\}^{1/q}$$

where $M(f)$ denotes the Hardy-Littlewood maximal function of f on $\mathbb{R}^d$. Then $\omega \in F^p_{\text{loc}}(\mathbb{R}^d)$ and

$$\begin{aligned}
&\sup_{\substack{0<r<1\\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} w^p \, d\mathbf{y} \right\}^{1/p} \\
&\qquad \le C \sup_{\substack{0<r<1\\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V|^p \, d\mathbf{y} \right\}^{1/p}
\end{aligned} \tag{2.9}$$

where C depends only on p, q and d.

Remark 2.10. Since V is periodic and $V \in L^q_{\text{loc}}(\mathbb{R}^d)$, we have

$$\int_{|\mathbf{x}|\le R} |V|^q d\mathbf{x} = O(R) \quad \text{as } R \to \infty.$$

It follows that $M(|V|^q)(\mathbf{x}) < \infty$ for a.e. $\mathbf{x} \in \mathbb{R}^d$. Moreover $\omega = \{M(|V|^q)\}^{1/q}$ is an A_1 weight and

$$\frac{1}{r^d} \int_{Q(\mathbf{x},r)} \omega \, d\mathbf{y} \le C \inf_{\mathbf{y}\in Q(\mathbf{x},r)} \omega(\mathbf{y}).$$

See [pp.214-215, 25]. Also note that, by the periodicity of V,

$$\omega(\mathbf{x}) \le C \sup_{0<r\le 2\pi} \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V|^q d\mathbf{y} \right\}^{1/q}. \tag{2.11}$$

Proof of Proposition 2.8. We shall adapt the argument in [8]. Also see [4].

Fix $Q = Q(\mathbf{x}_0, r_0)$ with $\mathbf{x}_0 \in \Omega$ and $0 < r_0 < 1$. Let $V = V_1 + V_2$ where $V_1 = V\chi_{Q(\mathbf{x}_0, 2r_0)}$. Then

$$\omega(\mathbf{y}) \le \{M(|V_1|^q)(\mathbf{y})\}^{1/q} + \{M(|V_2|^q)(\mathbf{y})\}^{1/q} = \omega_1(\mathbf{y}) + \omega_2(\mathbf{y}).$$

Since $p/q > 1$, by the well known estimates for the maximal function [24],

$$\begin{aligned} r_0^2 \left\{ \frac{1}{|Q|} \int_Q \omega_1^p \, d\mathbf{y} \right\}^{1/p} &\le C r_0^2 \left\{ \frac{1}{|2Q|} \int_{2Q} |V|^p d\mathbf{x} \right\}^{1/p} \\ &\le C \sup_{\substack{0<r<1 \\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V|^p d\mathbf{y} \right\}^{1/p}. \end{aligned} \tag{2.12}$$

Recall that $w_2(\mathbf{x}) = \{M(|V_2|^q)(\mathbf{x})\}^{1/q}$ and $V_2 = 0$ on $Q(\mathbf{x}_0, 2r_0)$. It is not hard to see that $\omega_2(\mathbf{x}) \le C\,\omega_2(\mathbf{x}_0)$ for any $\mathbf{x} \in Q$. Thus

$$r_0^2 \left\{ \frac{1}{|Q|} \int_Q \omega_2^p(\mathbf{x}) \, d\mathbf{x} \right\}^{1/p} \le C\, r_0^2 \omega_2(\mathbf{x}_0).$$

Clearly, $\omega_2(\mathbf{x}_0) < \infty$. Hence there exists $t > 2r_0$ such that

$$\omega_2(\mathbf{x}_0) \le C \left\{ \frac{1}{t^d} \int_{Q(\mathbf{x}_0,t)} |V|^q d\mathbf{x} \right\}^{1/q}.$$

In view of (2.11), we may assume that $t \le 2\pi$. We obtain

$$\begin{aligned} r_0^2 \left\{ \frac{1}{|Q|} \int_Q \omega_2^p d\mathbf{x} \right\}^{1/p} &\le C r_0^2 \left\{ \frac{1}{t^d} \int_{Q(\mathbf{x}_0,t)} |V|^q d\mathbf{x} \right\}^{1/q} \\ &\le C\, t^2 \left\{ \frac{1}{t^d} \int_{Q(\mathbf{x}_0,t)} |V|^p d\mathbf{x} \right\}^{1/p} \\ &\le C \sup_{\substack{0<r<1 \\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V|^p d\mathbf{y} \right\}^{1/p}. \end{aligned}$$

This, together with (2.12), gives the estimate (2.9) in the proposition.

Proposition 2.13. *Suppose $\omega \ge 0$ is periodic with respect to $(2\pi\mathbb{Z})^d$ and $\omega \in F^p_{\mathrm{loc}}(\mathbb{R}^d)$ for some $p \in ((d-1)/2, d/2]$. Let*

$$\omega_*(\mathbf{x}) = \omega_*(x_1, x_2, \dots, x_d) = \sup_{t>0} \left\{ \frac{1}{2t} \int_{x_1-t}^{x_1+t} |\omega(s, x_2, \dots, x_d)|^q ds \right\}^{1/q} \tag{2.14}$$

where $(d-1)/2 < q < p$. *Then*

$$
\begin{aligned}
&\sup_{\substack{0<r<1\\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} \omega_*^p d\mathbf{y} \right\}^{1/p} \\
&\qquad \le C \sup_{\substack{0<r<1\\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} \omega^p d\mathbf{y} \right\}^{1/p}
\end{aligned}
\tag{2.15}
$$

where C *depends only on* p, q *and* d.

Remark 2.16. Note that, for a.e. $\mathbf{x}' = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}$, we have

$$
\frac{1}{|I|} \int_I \omega_*(x_1, x_2, \dots, x_d)\, dx_1 \le C \inf_{x_1 \in I} \omega_*(x_1, x_2, \dots, x_d)
$$

where $I = (a, b)$ and C depends only on q (e.g. see [25]). That is, $\omega_*(\cdot, x_2, \dots, x_d)$ is an $A_1(\mathbb{R}^1)$ weight uniformly in $\mathbf{x}' \in \mathbb{R}^{d-1}$.

Proof of Proposition 2.13. We follow the argument in [4].

Fix $Q = Q(\mathbf{x}, r)$ with $0 < r < 1$. Let N be an integer so that $2\pi \le 2^N r < 4\pi$.

Let $R_0 = Q(\mathbf{x}, 2r)$ and

$$
R_j = \left\{ \mathbf{y} = (y_1, \dots, y_d) : \quad 2^{j-1} r \le |y_1 - x_1| < 2^j r,\ |y_\ell - x_\ell| < r, \quad \ell = 2, \dots, d \right\}
$$

for $j = 1, 2, \dots, N$ and

$$
R_{N+1} = \left\{ \mathbf{y} = (y_1, \dots, y_d) : \quad |y_1 - x_1| \ge 2^N r,\ |y_\ell - x_\ell| < r, \quad \ell = 2, \dots, d \right\}.
$$

Write

$$
\omega = \sum_{j=0}^{N+1} \omega \chi_{R_j} + \varphi.
$$

We have

$$
\omega_* \le \sum_{j=0}^{N+1} \left(\omega \chi_{R_j} \right)_* + \varphi_*.
$$

Note that $\varphi_*(\mathbf{y}) = 0$ if $\mathbf{y} \in Q(\mathbf{x}, r)$.

For $j = 0$, we use the L^p estimate for the one-dimensional maximal function to obtain

$$
\begin{aligned}
r^2 \left\{ \frac{1}{r^d} \int_Q (\omega \chi_{R_0})_*^p \, d\mathbf{y} \right\}^{1/p} &\le C\, r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},2r)} \omega^p d\mathbf{y} \right\}^{1/p} \\
&\le C \sup_{\substack{0<t<1\\ \mathbf{y}\in\Omega}} t^2 \left\{ \frac{1}{t^d} \int_{Q(\mathbf{y},t)} \omega^p d\mathbf{z} \right\}^{1/p}.
\end{aligned}
$$

For $1 \le j \le N$, we note that, if $\mathbf{y} \in Q(\mathbf{x}, r)$,

$$\begin{aligned}
\left(\omega\chi_{R_j}\right)_*(\mathbf{y}) &\le C\left\{\frac{1}{2^j r}\int_{|s-x_1|<2^j r} |\omega(s, y_2, \dots, y_d)|^q ds\right\}^{1/q} \\
&\le C\left\{\frac{1}{2^j r}\int_{|s-x_1|<2^j r} \{\omega(s, y_2, \dots, y_d)\}^p\, ds\right\}^{1/p}.
\end{aligned}$$

It follows that

$$\begin{aligned}
& r^2\left\{\frac{1}{|Q|}\int_Q \left(\omega\chi_{R_j}\right)_*^p(\mathbf{y})d\mathbf{y}\right\}^{1/p} \\
&\le C\, r^2\left\{\frac{1}{r^d 2^j}\int_{|y_1-x_1|<2^j r}\int_{|y_2-x_2|<r/2}\cdots\int_{|y_d-x_d|<r/2} \{\omega(y_1, \dots, y_d)\}^p\, d\mathbf{y}\right\}^{1/p} \\
&\le C\, r^2 2^{j(d-1)/p}\left\{\frac{1}{|Q(\mathbf{x}, 2^{j+1}r)|}\int_{Q(\mathbf{x},2^{j+1}r)} \omega^p(\mathbf{y})d\mathbf{y}\right\}^{1/p} \\
&= C(2^j)^{\frac{d-1}{p}-2}(2^{j+1}r)^2\left\{\frac{1}{|Q(\mathbf{x}, 2^{j+1}r)|}\int_{Q(\mathbf{x},2^j r)} \omega^p(\mathbf{y})d\mathbf{y}\right\}^{1/p} \\
&\le C(2^j)^{\frac{d-1}{p}-2}\sup_{\substack{0<t<1\\ \mathbf{y}\in\Omega}} t^2\left\{\frac{1}{|Q(\mathbf{y}, t)|}\int_{Q(\mathbf{y},t)} \omega^p(\mathbf{z})d\mathbf{z}\right\}^{1/p}.
\end{aligned}$$

Finally, for $j = N+1$, since ω is periodic,

$$(\omega\chi_{R_{N+1}})_*(\mathbf{y}) \le C\left\{\int_{|s-x_1|<\pi} \{\omega(s, y_2, \dots, y_d)\}^q\, ds\right\}^{1/q}.$$

We have

$$\begin{aligned}
r^2\left\{\frac{1}{|Q|}\int_Q \left(\omega\chi_{R_{N+1}}\right)_*^p(\mathbf{y})d\mathbf{y}\right\}^{1/p} &\le C\, r^2\left\{\frac{1}{r^{d-1}}\int_\Omega \omega^p d\mathbf{y}\right\}^{1/p} \\
&= C\, r^{2-\frac{d-1}{p}}\left\{\int_\Omega \omega^p d\mathbf{y}\right\}^{1/p} \\
&\le C\sup_{\substack{0<t<1\\ \mathbf{y}\in\Omega}} t^2\left\{\frac{1}{t^d}\int_{Q(\mathbf{y},t)} \omega^p d\mathbf{z}\right\}^{1/p},
\end{aligned}$$

where we used the assumption $p > (d-1)/2$ in the last inequality.

Thus we have proved that

$$
\begin{aligned}
& r^2\left\{\frac{1}{|Q|}\int_Q \omega_*^p(\mathbf{y})d\mathbf{y}\right\}^{1/p}\\
&\le C\left\{1+\sum_{j=1}^{N}(2^j)^{\frac{d-1}{p}-2}\right\}\sup_{\substack{0<t<1\\ \mathbf{y}\in\Omega}} t^2\left\{\frac{1}{t^d}\int_{Q(\mathbf{y},t)}\omega^p(\mathbf{z})d\mathbf{z}\right\}^{1/p}\\
&\le C\sup_{\substack{0<t<1\\ \mathbf{y}\in\Omega}} t^2\left\{\frac{1}{t^d}\int_{Q(\mathbf{y},t)}\omega^p(\mathbf{z})d\mathbf{z}\right\}^{1/p}
\end{aligned}
$$

since $(d-1)/p-2<0$. (2.15) now follows.

3. The Thomas Approach

We begin with the definition of $(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T+V$ on $L^2(\mathbb{T}^d)$ for $\mathbf{k}\in\mathbb{C}^d$.

As in [21], let

$$
H^1(\mathbb{T}^d)=\left\{\psi\in L^2(\Omega):\ \psi(\mathbf{x})=\sum_{\mathbf{n}\in\mathbb{Z}^d}a_{\mathbf{n}}e^{i\mathbf{n}\mathbf{x}}\ \text{ and }\ \sum_{\mathbf{n}\in\mathbb{Z}^d}|\mathbf{n}|^2|a_{\mathbf{n}}|^2<\infty\right\}. \tag{3.1}
$$

For $\mathbf{k}\in\mathbb{C}^d$, we consider the quadratic form

$$
q(\mathbf{k})[\phi,\psi]=\int_\Omega \left\{<[(\mathbf{D}+\mathbf{k})\phi]A,(\mathbf{D}+\overline{\mathbf{k}})\psi>+<V\phi,\psi>\right\}d\mathbf{x} \tag{3.2}
$$

for $\phi,\psi\in H^1(\mathbb{T}^d)$, where $\overline{\mathbf{k}}$ denotes the conjugate of $\mathbf{k}$. It follows from the proof of Theorem 2.6 that, if V satisfies (1.4) for some $p>1$,

$$
\int_\Omega |\psi|^2|V|d\mathbf{x}\le C\,\varepsilon\int_\Omega <(\nabla\psi)A,\nabla\psi>d\mathbf{x}+C_{\varepsilon,V}\int_\Omega|\psi|^2d\mathbf{x} \tag{3.3}
$$

for any $\varepsilon>0$. This implies that, if ε is so small that $C\,\varepsilon<1/2$, the form $q(\mathbf{k})$ is strictly m-sectorial. Hence there exists a unique closed operator, which we denote by $(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T+V$, such that

$$
q(\mathbf{k})[\phi,\psi]=\int_\Omega <\left\{(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T+V\right\}\phi,\psi>d\mathbf{x} \tag{3.4}
$$

for any $\phi\in\text{Domain}\left((\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T+V\right)$ and $\psi\in H^1(\mathbb{T}^d)$ [12]. Also

$$
\begin{aligned}
&\text{Domain}\left((\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T+V\right)\\
&\qquad=\left\{\phi\in H^1(\mathbb{T}^d):\ \left\{(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T+V\right\}\phi\in L^2(\mathbb{T}^d)\right\}\\
&\qquad=\left\{\phi\in H^1(\mathbb{T}^d):\ (\mathbf{D}A\mathbf{D}^T+V)\phi\in L^2(\mathbb{T}^d)\right\},
\end{aligned} \tag{3.5}
$$

is independent of $\mathbf{k}$.

Choose $\mathbf{a} \in \mathbb{R}^d$ such that

$$|\mathbf{a}| = 1, \quad \mathbf{a}A = (s_0, 0, \ldots, 0) \text{ and } s_0 > 0. \tag{3.6}$$

Let

$$L = \left\{ \mathbf{b} \in \mathbb{R}^d : \ |\mathbf{b}| < \sqrt{d} \ \text{ and } \ < \mathbf{b}, \mathbf{a} >= 0 \right\}. \tag{3.7}$$

For a fixed $\mathbf{b} \in L$, we consider the family of operators

$$\mathbb{H}_V(z) = (\mathbf{D} + z\mathbf{a} + \mathbf{b})A(\mathbf{D} + z\mathbf{a} + \mathbf{b})^T, \quad z \in \mathbb{C} \tag{3.8}$$

defined by the form (3.2). Since Domain$(\mathbb{H}_V(z)) \subset H^1(\mathbb{T}^d)$, $\mathbb{H}_V(z)$ has compact resolvent.

Proposition 3.9. *If, for every $\mathbf{b} \in L$, the family of operators $\{\mathbb{H}_V(z) : z \in \mathbb{C}\}$ has no common eigenvalue, then the spectrum of the operator $\mathbf{D}A\mathbf{D} + V$ defined in Theorem 2.6 is purely absolutely continuous.*

Proof. See [21] and [20].

To show that $\{\mathbb{H}_V(z), z \in \mathbb{C}\}$ has no common eigenvalue under the assumptions of the main theorem, we consider $\mathbb{H}_V(\delta_0 + i\rho)$ for $\rho \in \mathbb{R}$, where

$$\delta_0 = \frac{1}{a_1}\left(\frac{1}{2} - b_1\right).$$

We will show that $\{\mathbb{H}_V(\delta_0 + i\rho) : \rho \in \mathbb{R}\}$ has no common eigenvalue. This will be done by establishing certain uniform estimates for

$$\mathbb{H}_0(\delta_0 + i\rho) = (\mathbf{D} + (\delta_0 + i\rho)\mathbf{a} + \mathbf{b})A(\mathbf{D} + (\delta_0 + i\rho)\mathbf{a} + \mathbf{b})^T$$

on the weighted L^2 spaces.

Theorem 3.10. *Let $\omega \in L^p(\mathbb{T}^d)$ for some $p \in ((d-1)/2, d/2]$. Suppose $\omega \geq c_\omega > 0$ and*

$$\|\omega\|_{F^p(\mathbb{T}^d)} \equiv \sup_{\substack{0<r<1 \\ \mathbf{x}\in\Omega}} r^2 \left(\frac{1}{r^d} \int_{Q(\mathbf{x},r)} \omega^p(\mathbf{y}) d\mathbf{y} \right)^{1/p} < \infty. \tag{3.11}$$

Then, if $\psi \in H^1(\mathbb{T}^d)$ and $\mathbb{H}_0(\delta_0 + i\rho)\psi \in L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})$, we have

$$\|\psi\|_{L^2(\mathbb{T}^d, \omega d\mathbf{x})} \leq C\|\omega\|_{F^p(\mathbb{T}^d)} \|H_0(\delta_0 + i\rho)\psi\|_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})} \tag{3.12}$$

where $|\rho| \geq 1$ and C does not depend on ρ and c_ω.

Theorem 3.13. *Let $\omega \in L^p(\mathbb{T}^d)$ where $p \geq 1$ if $d = 3$, $p > (d-2)/2$ if $d \geq 4$. Suppose $\omega \geq c_\omega > 0$. Then, if $\psi \in H^1(\mathbb{T}^d)$ and $\mathbb{H}_0(\delta_0 + i\rho)\psi \in L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})$, we have*

$$c_\rho \cdot (c_\omega)^{1/2} \|\psi\|_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})} \leq \left(\|\omega\|_{L^p(\mathbb{T}^d, d\mathbf{x})}\right)^{1/2} \cdot \|\mathbb{H}_0(\delta_0 + i\rho)\psi\|_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})} \tag{3.14}$$

where $c_\rho \to \infty$ as $|\rho| \to \infty$.

We now give the

Proof of the main theorem, assuming Theorems 3.10 and 3.13.

In view of Proposition 3.9, we suppose that E is an eigenvalue for $\mathbb{H}_V(z)$ for all $z \in \mathbb{C}$. Let $z = \delta_0 + i\rho$. There exists $\psi_\rho \in \text{Domain}(\mathbb{H}_V(z))$ such that

$$\|\psi\|_{L^2(\mathbb{T}^d, d\mathbf{x})} = 1 \quad \text{ and } \quad \mathbb{H}_V(z)\psi_\rho = E\psi_\rho.$$

Suppose V satisfies (1.4). By Lemma 2.7, we may choose N so large that

$$\sup_{\substack{0<r<1 \\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |V_N|^p d\mathbf{y} \right\}^{1/p} < \varepsilon. \tag{3.15}$$

Let

$$\omega(\mathbf{x}) = \{M(|V_N|^q)(\mathbf{x})\}^{1/q} \tag{3.16}$$

where $M(f)$ denotes the Hardy-Littlewood maximal function of f on $\mathbb{R}^d$, and $1 < q < p$. We may assume that $V \notin L^\infty(\Omega)$ for otherwise the absolute continuity of $\mathbf{D}A\mathbf{D}^T + V$ is well known. This implies that $\omega(\mathbf{x}) \geq c_\omega > 0$. Note that ω is periodic with respect to $(2\pi\mathbb{Z})^d$. Also by (2.9) and (3.15),

$$\sup_{\substack{0<r<1 \\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} |\omega|^p d\mathbf{y} \right\}^{1/p} < C\varepsilon. \tag{3.17}$$

Since $\omega \geq |V_N|$ a.e., we have

$$\begin{aligned} \int_\Omega |V\psi_\rho|^2 \frac{d\mathbf{x}}{\omega} &\leq N^2 \int_\Omega |\psi_\rho|^2 \frac{d\mathbf{x}}{\omega} + \int_\Omega |\psi|^2 |V_N|^2 \frac{d\mathbf{x}}{\omega} \\ &\leq N^2 \int_\Omega |\psi_\rho|^2 \frac{d\mathbf{x}}{\omega} + \int_\Omega |\psi_\rho|^2 \omega \, d\mathbf{x}. \end{aligned}$$

This, together with (3.17), the C. Fefferman-Phong estimate (2.3) and the fact that $\psi_\rho \in H^1(\mathbb{T}^d)$, implies that $V\psi_\rho \in L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})$. Since $\omega \geq c_\omega > 0$, we have $\psi_\rho \in L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})$. It follows that

$$\mathbb{H}_0(\delta_0 + i\rho)\psi_\rho \in L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})$$

and

$$\|\mathbb{H}_0(\delta_0 + i\rho)\psi_\rho\|_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})} \leq \{|E| + N\}\|\psi_\rho\|_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})} + \|\psi_\rho\|_{L^2(\mathbb{T}^d, \omega d\mathbf{x})}. \tag{3.18}$$

Now, by Theorem 3.10 and (3.17),

$$\|\psi_\rho\|_{L^2(\mathbb{T}^d, \omega d\mathbf{x})} \leq C_0\, \varepsilon_0 \|\mathbb{H}_0(\delta_0 + i\rho)\psi_\rho\|_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})}$$

where $\rho \geq 1$ and C_0 depends only on p, q, A and d. In view of (3.18), we see that, if ε is chosen so small that $C_0\, \varepsilon \leq 1/2$, we have

$$\|\mathbb{H}_0(\delta_0 + i\rho)\psi_\rho\|_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})} \leq 2\{|E| + N\}\|\psi_\rho\|_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})}.$$

This, together with Theorem 3.13, implies that

$$c_\rho \cdot (c_\omega)^{1/2} \leq \left(\|\omega\|_{L^p(\mathbb{T}^d, d\mathbf{x})}\right)^{1/2} \cdot 2\{|E| + N\}.$$

We obtain a contradiction since $c_\rho \to \infty$ as $\rho \to \infty$. This shows that $\{\mathbb{H}_V(z) : z \in \mathbb{C}\}$ has no common eigenvalue. By Proposition 3.9, the spectrum of $\mathbf{D}A\mathbf{D}^T + V$ on $L^2(\mathbb{R}^d)$ is purely absolutely continuous.

We end this section with the proof of Theorem 3.13. The proof of Theorem 3.10 is much more involved, as we discussed in the introduction. It will be given in sections 4-6.

Proof of Theorem 3.13.

Theorem 3.13 is a consequence of Theorem 2.1 in [21]. Indeed, it follows from Theorem 2.1 in [21] that, if $\psi \in H^1(\mathbb{T}^d)$ and $\mathbb{H}_0(\delta_0 + i\rho)\psi \in L^{p_0}(\mathbb{T}^d, d\mathbf{x})$, we have

$$c_\rho \|\psi\|_{L^2(\mathbb{T}^d, d\mathbf{x})} \leq \|\mathbb{H}_0(\delta_0 + i\rho)\psi\|_{L^{p_0}(\mathbb{T}^d, d\mathbf{x})}$$

where $c_\rho \to \infty$ as $|\rho| \to \infty$, $1 \le p_0 \le 2$ if $d = 3$ and $2(d-2)/d < p_0 \le 2$ if $d \ge 4$.

Note that, by Hölder's inequality,

$$||\mathbb{H}_0(\delta_0 + i\rho)\psi||_{L^{p_0}(\mathbb{T}^d, d\mathbf{x})} \le ||\mathbb{H}_0(\delta_0 + i\rho)\psi||_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})} \left(||\omega||_{L^p(\mathbb{T}^d, d\mathbf{x})}\right)^{1/2}$$

where $p = p_0/(2-p_0)$. It is easy to see that $1 \le p_0 \le 2$ if $1 \le p \le \infty$, and $2(d-2)/d < p_0 \le 2$ if $(d-2)/2 < p \le \infty$. It follows that

$$\begin{aligned} c_\rho ||\psi||_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})} &\le \frac{1}{(c_\omega)^{1/2}} \cdot c_\rho ||\psi||_{L^2(\mathbb{T}^d, d\mathbf{x})} \\ &\le \frac{1}{(c_\omega)^{1/2}} ||\mathbb{H}_0(\delta_0 + i\rho)\psi||_{L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})} \left(||\omega||_{L^p(\mathbb{T}^d, d\mathbf{x})}\right)^{1/2} \end{aligned}$$

where $p \ge 1$ if $d = 3$, $p > (d-2)/2$ if $d \ge 4$. The proof of Theorem 3.13 is finished.

4. A Weighted Uniform Sobolev Inequality

For

$$\psi(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} \hat{\psi}(\mathbf{n}) e^{i\mathbf{n}\mathbf{x}}, \tag{4.1}$$

and $\mathbf{b} \in \mathbb{R}^d$, $|\mathbf{b}| < \sqrt{d}$, we define

$$S\psi(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} \frac{\hat{\psi}(\mathbf{n}) e^{i\mathbf{n}\mathbf{x}}}{(\mathbf{n}+\mathbf{b})A(\mathbf{n}+\mathbf{b})^T + z} \tag{4.2}$$

where $z \in \mathbb{C}$, $\operatorname{Im} z \ne 0$. We choose a branch of $\sqrt{z}$ so that $\operatorname{Re}\sqrt{z} \ge 0$.

This section is devoted to the proof of the following theorem.

Theorem 4.3. *Let $\omega \in L^p(\mathbb{T}^d)$ for some $p \in ((d-1)/2, d/2]$. Suppose that $\omega \ge c_\omega > 0$ and $||\omega||_{F^p(\mathbb{T}^d)} < \infty$. Let $z \in \mathbb{C}$ such that $Im\, z \ne 0$ and $Re\sqrt{z} \ge c_0 > 0$. Then, for any $\psi \in L^2(\mathbb{T}^d, d\mathbf{x})$,*

$$||\omega^{1/2} S(\psi\omega^{1/2})||_{L^2(\mathbb{T}^d, d\mathbf{x})} \le C ||\omega||_{F^p(\mathbb{T}^d)} ||\psi||_{L^2(\mathbb{T}^d, d\mathbf{x})} \tag{4.4}$$

where C depends only on p, d, A, and c_0.

Proof. For $\xi \in \mathbb{C}$, we consider two analytic families of operators,

$$S_\xi \psi(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} \frac{\hat{\psi}(\mathbf{n}) e^{i\mathbf{n}\mathbf{x}}}{[(\mathbf{n}+\mathbf{b})A(\mathbf{n}+\mathbf{b})^T + z]^\xi} \tag{4.5}$$

and

$$T_\xi \psi = \omega^{\xi/2} S_\xi(\omega^{\xi/2}\psi). \tag{4.6}$$

Clearly, if $\operatorname{Re}\xi = 0$,

$$\|T_\xi \psi\|_{L^2(\Omega, d\mathbf{x})} \le C e^{c|\operatorname{Im}\xi|} \|\psi\|_{L^2(\Omega, d\mathbf{x})}. \tag{4.7}$$

If $\operatorname{Re}\xi = (d-1)/2$, we use the estimate obtained in [21] for the integral kernel of S_ξ. We have

$$S_\xi \psi(\mathbf{x}) = \int_\Omega G_\xi(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\, d\mathbf{y} \tag{4.8}$$

and

$$|G_\xi(\mathbf{x})| \le C e^{c|\operatorname{Im}\xi|} \left\{ 1 + \sum_{|\mathbf{x}+2\pi\mathbf{n}|\le C} \frac{1}{|\mathbf{x}+2\pi\mathbf{n}|} \right\}. \tag{4.9}$$

See (6.10) in [21]. It follows that

$$\begin{aligned} |S_\xi \psi(\mathbf{x})| &\le C e^{c|\operatorname{Im}\xi|} \left\{ \int_\Omega |\psi(\mathbf{y})| d\mathbf{y} + \int_\Omega \sum_{|\mathbf{x}-\mathbf{y}+2\pi\mathbf{n}|\le C} \frac{|\psi(\mathbf{y})|\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}+2\pi\mathbf{n}|} \right\} \\ &\le C e^{c|\operatorname{Im}\xi|} \left\{ \int_\Omega |\psi(\mathbf{y})| d\mathbf{y} + I_{d-1}(|\psi|\chi_{\Omega'})(\mathbf{x}) \right\} \end{aligned}$$

where $\Omega' = \cup_{|\mathbf{n}|\le C}(\Omega + 2\pi\mathbf{n})$, and in the last inequality we used the fact that ψ is periodic. Hence, if $\operatorname{Re}\xi = (d-1)/2$,

$$\begin{aligned} &|T_\xi \psi(\mathbf{x})| \\ &\le C e^{c|\operatorname{Im}\xi|} \left\{ (\omega(\mathbf{x}))^{\frac{d-1}{4}} \int_\Omega |\psi(\mathbf{y})| (\omega(\mathbf{y}))^{\frac{d-1}{4}} d\mathbf{y} + (\omega(\mathbf{x}))^{\frac{d-1}{4}} I_{d-1}(|\psi|\chi_{\Omega'}\omega^{\frac{d-1}{4}})(\mathbf{x}) \right\} \end{aligned}$$

This, together with the weighted norm inequality for the fractional integral I_{d-1} [5,15], gives

$$\begin{aligned} &\|T_\xi \psi\|_{L^2(\Omega, d\mathbf{x})} \\ &\le C e^{c|\operatorname{Im}\xi|} \left\{ \int_\Omega \omega^{\frac{d-1}{2}} d\mathbf{x} \cdot \|\psi\|_{L^2(\Omega, d\mathbf{x})} + \|\chi_{\Omega'}\omega^{\frac{d-1}{4}} I_{d-1}(|\psi|\chi_{\Omega'}\omega^{\frac{d-1}{4}})\|_{L^2(\mathbb{R}^d, d\mathbf{x})} \right\} \\ &\le C_q e^{c|\operatorname{Im}\xi|} \sup_{\substack{r>0 \\ \mathbf{x}\in\mathbb{R}^d}} r^{d-1} \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} (\chi_{\Omega'}\omega^{\frac{d-1}{2}})^q d\mathbf{y} \right\}^{1/q} \cdot \|\psi\chi_{\Omega'}\|_{L^2(\mathbb{R}^d, d\mathbf{x})} \end{aligned}$$

where $q > 1$. Note that, since ψ and ω are periodic, we have

$$\|\psi\chi_{\Omega'}\|_{L^2(\mathbb{R}^d,d\mathbf{x})} \le C\|\psi\|_{L^2(\Omega,d\mathbf{x})}$$

and

$$\begin{aligned}
&\sup_{\substack{r>0\\ \mathbf{x}\in\mathbb{R}^d}} r^{d-1}\left\{\frac{1}{r^d}\int_{Q(\mathbf{x},r)}(\omega^{\frac{d-1}{2}}\chi_{\Omega'})^q d\mathbf{y}\right\}^{1/q}\\
&\le C\sup_{\substack{0<r<1\\ \mathbf{x}\in\Omega}} r^{d-1}\left\{\frac{1}{r^d}\int_{Q(\mathbf{x},r)}\omega^{\frac{(d-1)q}{2}}d\mathbf{y}\right\}^{1/q}\\
&\le C\left\{\sup_{\substack{0<r<1\\ \mathbf{x}\in\Omega}} r^2\left\{\frac{1}{r^d}\int_{Q(\mathbf{x},r)}\omega^p d\mathbf{y}\right\}^{1/p}\right\}^{\frac{d-1}{2}}\\
&= C\left(\|\omega\|_{F^p(\mathbb{T}^d)}\right)^{\frac{d-1}{2}}
\end{aligned}$$

if we choose $q = 2p/(d-1) > 1$. Thus we obtain that, if $\operatorname{Re}\xi = (d-1)/2$,

$$\|T_\xi\psi\|_{L^2(\Omega,d\mathbf{x})} \le Ce^{c|\operatorname{Im}\xi|}\left(\|\omega\|_{F^p(\mathbb{T}^d)}\right)^{\frac{d-1}{2}}\|\psi\|_{L^2(\Omega,d\mathbf{x})} \tag{4.10}$$

Finally, we write

$$1 = t\cdot\frac{d-1}{2} + (1-t)\cdot 0 \qquad \text{where } t = \frac{2}{d-1}.$$

It follows from (4.7), (4.10) and the Stein interpolation theorem [p.385, 25] that

$$\|T_1\psi\|_{L^2(\Omega,d\mathbf{x})} \le C\|\omega\|_{F^p(\mathbb{T}^d)}\|\psi\|_{L^2(\Omega,d\mathbf{x})}.$$

Since $T_1\psi = \omega^{1/2}S(\omega^{1/2}\psi)$, (4.4) is proved.

5. A Weighted L^2 Estimate for the Spectral Projection

The goal of this section is to establish the following theorem for the spectral projection associated with the operator $\mathbf{D}A\mathbf{D}^T$ on the d-torus.

Theorem 5.1. *Let $\omega \in L^p(\mathbb{T}^d)$ for some $p \in ((d-1)/2, d/2]$. Suppose that $\omega \ge c_\omega > 0$ and $\|\omega\|_{F^p(\mathbb{T}^d)} < \infty$. Then, for any $\psi \in L^2(\mathbb{T}^d, d\mathbf{x})$ and $k \ge 0$,*

$$\|\sum_{|\mathbf{n}B|\in[k,k+1)}\hat{\psi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}\|_{L^2(\mathbb{T}^d,\omega d\mathbf{x})} \le C(k+1)^{1/2}\left(\|\omega\|_{F^p(\mathbb{T}^d)}\right)^{1/2}\|\psi\|_{L^2(\mathbb{T}^d,d\mathbf{x})} \tag{5.2}$$

where $B = B_{d\times d} = \sqrt{A} \ge 0$ and C depends only on p, d and A.

It is not hard to see that (5.2) is a direct consequence of the weighted L^2 inequality in the next theorem.

Theorem 5.3. *Let $\omega \in L^p(\mathbb{T}^d)$ for some $p \in ((d-1)/2, d/2]$. Suppose that $\omega \geq c_\omega > 0$ and $\|\omega\|_{F^p(\mathbb{T}^d)} < \infty$. Then, for any $\psi \in C^\infty(\mathbb{T}^d)$ and $k \geq 1$,*

$$\|\psi\|_{L^2(\mathbb{T}^d, \omega d\mathbf{x})} \leq Ck^{-1/2} \left(\|\omega\|_{F^p(\mathbb{T}^d)}\right)^{1/2} \|(\mathbf{D}A\mathbf{D}^T - k^2 - 2ik)\psi\|_{L^2(\mathbb{T}^d, d\mathbf{x})}. \tag{5.4}$$

The proof of Theorem 5.3 relies on a weighted version of the Stein oscillatory integral theorem in $\mathbb{R}^d$ [p.380, 25].

Theorem 5.5. *Let $\omega \geq 0$ and $\omega \in F^p_{\text{loc}}(\mathbb{R}^d)$ for some $p \in ((d-1)/2, d/2)$. Suppose that*

$$\|\omega\|_{F^p(\mathbb{R}^d)} \equiv \sup_{\substack{r>0 \\ \mathbf{x} \in \mathbb{R}^d}} r^2 \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x}, r)} \omega^p \, d\mathbf{y} \right\}^{1/p} < \infty. \tag{5.6}$$

Let $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and

$$\mathit{supp}\, a \subset \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^d \times \mathbb{R}^d : 1/2 \leq |\mathbf{x} - \mathbf{y}| \leq 2 \right\}. \tag{5.7}$$

Then, for any $f \in L^2(\mathbb{R}^d, d\mathbf{x})$,

$$\begin{aligned} &\| \int_{\mathbb{R}^d} e^{i\lambda|\mathbf{x}-\mathbf{y}|} a(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y} \|_{L^2(\mathbb{R}^d, \omega d\mathbf{x})} \\ &\qquad \leq C|\lambda|^{\frac{2-d}{2}} \left(\|\omega\|_{F^p(\mathbb{R}^d)}\right)^{1/2} \|f\|_{L^2(\mathbb{R}^d, d\mathbf{x})} \end{aligned} \tag{5.8}$$

where $\lambda \in \mathbb{R}$, $|\lambda| \geq 1$ and C depends only on p, d and the size of finite many derivatives of the function a.

Proof. We adapt an argument of Stein found in [pp.380-386, 25].

First, note that, using partition of unity and (5.7), we may assume that

$$\begin{aligned} \text{supp}\, a \subset \big\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^d \times \mathbb{R}^d : |\mathbf{x} - \mathbf{x}_0| < \delta, \ |\mathbf{y} - \mathbf{y}_0| < \delta \\ \text{and } 1/2 \leq |\mathbf{x} - \mathbf{y}| \leq 2 \big\} \end{aligned} \tag{5.9}$$

for some $\mathbf{x}_0, \ \mathbf{y}_0 \in \mathbb{R}^d$, where δ is sufficiently small. Also, by considering

$$\widetilde{\omega}(\mathbf{x}) = \omega(\mathbf{x}) + \frac{\varepsilon}{|\mathbf{x}|^2}$$

and then letting $\varepsilon \to 0$, we may assume that $\omega > 0$.

Next we observe that, by duality, it suffices to show that

$$\| \int_{\mathbb{R}^d} e^{i\lambda|\mathbf{x}-\mathbf{y}|} a(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d\mathbf{y} \|_{L^2(\mathbb{R}^d, d\mathbf{x})} \leq C|\lambda|^{\frac{2-d}{2}} \left(\|\omega\|_{F^p(\mathbb{R}^d)}\right)^{1/2} \|f\|_{L^2(\mathbb{R}^d, \frac{d\mathbf{x}}{\omega})}. \tag{5.10}$$

By translation, we may assume that $\mathbf{x}_0 = 0$. Since $a(\mathbf{x},\mathbf{y}) = 0$ if $|x_1| > \delta$, we need only to show that, for any fixed $x_1 \in [-\delta, \delta]$,

$$\int_{\mathbb{R}^{d-1}} \Big| \int_{\mathbb{R}^d} e^{i\lambda|\mathbf{x}-\mathbf{y}|} a(x_1, \mathbf{x}', \mathbf{y}) f(\mathbf{y}) d\mathbf{y} \Big|^2 d\mathbf{x}' \leq C|\lambda|^{2-d} \|\omega\|_{F^p(\mathbb{R}^d)} \int_{\mathbb{R}^d} |f(\mathbf{z})|^2 \frac{d\mathbf{z}}{\omega(\mathbf{z})} \tag{5.11}$$

where $\mathbf{x}' = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$.

To this end, we let

$$S_\lambda f(\mathbf{x}') = \int_{\mathbb{R}^d} e^{i\lambda|\mathbf{x}-\mathbf{y}|} a(x_1, \mathbf{x}', \mathbf{y}) f(\mathbf{y}) d\mathbf{y} \tag{5.12}$$

with $\mathbf{x} = (x_1, \mathbf{x}')$ and $x_1 \in [-\delta, \delta]$ fixed. It is not hard to see that (5.11) would follow from

$$\|\omega^{1/2} S_\lambda^* S_\lambda (\omega^{1/2} f)\|_{L^2(\mathbb{R}^d, d\mathbf{y})} \leq C|\lambda|^{2-d} \|\omega\|_{F^p(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d, d\mathbf{y})}. \tag{5.13}$$

Note that

$$S_\lambda^* S_\lambda f(\mathbf{y}) = \int_{\mathbb{R}^d} K_\lambda(\mathbf{y},\mathbf{z}) f(\mathbf{z}) d\mathbf{z} \tag{5.14}$$

where

$$K_\lambda(\mathbf{y},\mathbf{z}) = \int_{\mathbb{R}^d} e^{-i\lambda(|\mathbf{y}-\mathbf{x}|-|\mathbf{z}-\mathbf{x}|)} \overline{a}(x_1, \mathbf{x}', \mathbf{y}) a(x_1, \mathbf{x}', \mathbf{z}) d\mathbf{x}'. \tag{5.15}$$

By [25], there exists an analytic family of operators

$$U_\lambda^s f(\mathbf{y}) = \int_{\mathbb{R}^d} K_\lambda^s(\mathbf{y},\mathbf{z}) f(\mathbf{z}) d\mathbf{z}, \tag{5.16}$$

such that

$$U_\lambda^0 = S_\lambda^* S_\lambda, \tag{5.17}$$

$$\|U_\lambda^s f\|_{L^2(\mathbb{R}^d, d\mathbf{y})} \leq \frac{C}{|\lambda|^d} \|f\|_{L^2(\mathbb{R}^d, d\mathbf{y})} \quad \text{if } \operatorname{Re} s = 1, \tag{5.18}$$

$$|K_\lambda^s(\mathbf{y},\mathbf{z})| \leq \frac{C}{1 + |\lambda|\,|\mathbf{y}-\mathbf{z}|} \qquad \text{if } \operatorname{Re} s = -\frac{d-3}{2}. \tag{5.19}$$

We remark that $K^s_\lambda(\mathbf{y},\mathbf{z})$ is given by

$$\int_{\mathbb{R}^d} e^{-i\lambda[\Phi(u,\mathbf{x}',\mathbf{y})-\Phi(u,\mathbf{x}',\mathbf{z})]}\overline{a}(x_1,\mathbf{x}',\mathbf{y})a(x_1,\mathbf{x}',\mathbf{z})\alpha_s(u)d\mathbf{x}'du \tag{5.20}$$

where

$$\Phi(u,\mathbf{x}',\mathbf{y}) = |\mathbf{y}-(x_1,\mathbf{x}')| + u\,\phi_0(\mathbf{y}), \quad u\in\mathbb{R}, \quad \mathbf{x}'\in\mathbb{R}^{d-1} \tag{5.21}$$

and ϕ_0 is chosen so that

$$\det\left(\frac{\partial^2}{\partial\widetilde{x}_j\partial y_k}\Phi(u,\mathbf{x}',\mathbf{y})\right)\neq 0, \qquad \widetilde{\mathbf{x}} = (u,\mathbf{x}'), \tag{5.22}$$

if $|(x_1,\mathbf{x}')|\le\delta$ and $|\mathbf{y}-\mathbf{y}_0|\le\delta$. Also in (5.20), $\{\alpha_s\}$ is a family of distributions on $\mathbb{R}^1$. When $\operatorname{Re} s>0$,

$$\alpha_s(u) = \begin{cases} \dfrac{e^{s^2}}{\Gamma(s)}u^{s-1}\eta(u), & \text{if } u>0,\\ \qquad 0, & \text{if } u\le 0\end{cases} \tag{5.23}$$

where $\eta\in C_0^\infty(\mathbb{R})$, $\eta(u)=1$ for $|u|\le 1$. For $\operatorname{Re} s\le 0$, α_s is defined by analytic continuation. See [p.381, 25].

By (5.20),

$$K^s_\lambda(\mathbf{y},\mathbf{z}) = K_\lambda(\mathbf{y},\mathbf{z})\hat{\alpha}_s(\lambda\phi_0(\mathbf{y})-\lambda\phi_0(\mathbf{z})). \tag{5.24}$$

(5.19) follows from

$$|K_\lambda(\mathbf{y},\mathbf{z})|\le\frac{C}{\{1+|\lambda|\,|\mathbf{y}-\mathbf{z}|\}^{(d-1)/2}} \tag{5.25}$$

and

$$|\hat{\alpha}_s(u)|\le\frac{C}{(1+|u|)^\sigma}, \qquad \operatorname{Re} s=\sigma \quad\text{and}\quad \sigma\le 1. \tag{5.26}$$

(see [p.382, 25]).

Finally we consider the analytic family of operators

$$W^s_\lambda(f) = \omega^{\frac{1-s}{2}}U^s_\lambda\left(\omega^{\frac{1-s}{2}}f\right). \tag{5.27}$$

It follows from (5.18) that, if $\operatorname{Re} s = 1$,

$$\|W^s_\lambda f\|_{L^2(\mathbb{R}^d, d\mathbf{y})} \le \frac{C}{|\lambda|^d} \|f\|_{L^2(\mathbb{R}^d, d\mathbf{y})}. \tag{5.28}$$

If $\operatorname{Re} s = -(d-3)/2$, by (5.19), we have

$$|W^s_\lambda(f)| \le \frac{C}{|\lambda|} \omega^{\frac{d-1}{4}} I_{d-1}\left(\omega^{\frac{d-1}{4}}|f|\right). \tag{5.29}$$

This, together with the weighted norm inequality for I_{d-1} [5,15], gives

$$\|W^s_\lambda(f)\|_{L^2(\mathbb{R}^d, d\mathbf{y})} \le \frac{C_q}{|\lambda|} \sup_{\substack{r>0 \\ \mathbf{x}\in\mathbb{R}^d}} r^{d-1} \left\{ \frac{1}{r^d} \int_{Q(\mathbf{x},r)} \omega^{\frac{(d-1)q}{2}} d\mathbf{y} \right\}^{1/q} \|f\|_{L^2(\mathbb{R}^d, d\mathbf{y})}$$

where $q > 1$. Letting $q = 2p/(d-1) > 1$, we have

$$\|W^s_\lambda(f)\|_{L^2(\mathbb{R}^d, d\mathbf{x})} \le \frac{C}{|\lambda|} \left(\|\omega\|_{F^p(\mathbb{R}^d)}\right)^{\frac{d-1}{2}} \|f\|_{L^2(\mathbb{R}^d, d\mathbf{x})} \tag{5.30}$$

if $\operatorname{Re} s = -(d-3)/2$.

With (5.28) and (5.30), we apply the Stein interpolation theorem [p.385, 25]. Since

$$0 = t \cdot (-\frac{d-3}{2}) + (1-t)\cdot 1 \qquad \text{where } t = \frac{2}{d-1},$$

we obtain

$$\|W^0_\lambda(f)\|_{L^2(\mathbb{R}^d, d\mathbf{y})} \le \frac{C}{|\lambda|^{d-2}} \|\omega\|_{F^p(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d, d\mathbf{y})}.$$

Note that $W^0_\lambda(f) = \omega^{1/2} U^0_\lambda(\omega^{1/2} f) = \omega^{1/2} S^*_\lambda S_\lambda(\omega^{1/2} f)$. (5.13) is proved.

To prove Theorem 5.3, let $F_z(\cdot)$ be the Fourier transform of $[|\cdot|^2 + z]^{-1}$ in $\mathbb{R}^d$ where $\operatorname{Re}\sqrt{z} \ge 1$. We have

$$\begin{aligned} F_z(\mathbf{x}) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{-i\mathbf{x}\mathbf{y}}}{|\mathbf{y}|^2 + z} d\mathbf{y} \\ &= c_d \left(\frac{z}{|\mathbf{x}|^2}\right)^{\frac{1}{2}(\frac{d}{2}-1)} K_{\frac{d}{2}-1}(\sqrt{z}|\mathbf{x}|) \end{aligned} \tag{5.31}$$

where $K_{d/2-1}$ denotes the modified Bessel function of the third kind of order $d/2-1$ [18]. By a change of variables, it is easy to see that the Fourier transform of $[|\mathbf{y}B|^2 + z]^{-1}$ is given by

$$\begin{aligned} F_{z,B}(\mathbf{x}) &= \frac{1}{\det(B)} F_z(\mathbf{x}B^{-1}) \\ &= \frac{c_d}{\det(B)} \left(\frac{z}{|\mathbf{x}B^{-1}|}\right)^{\frac{1}{2}(\frac{d}{2}-1)} K_{\frac{d}{2}-1}(\sqrt{z}|\mathbf{x}B^{-1}|) \end{aligned} \tag{5.32}$$

where $B = \sqrt{A}$.

Theorem 5.33. *Let $\eta \in C_0^\infty((-2,2))$ such that $\eta(r)=1$ if $|r|\le 1$. Let ω be a nonnegative function on $\mathbb{R}^d$ satisfying $\|\omega\|_{F^p(\mathbb{R}^d)}<\infty$ for some $p\in((d-1)/2,d/2]$. Then*

$$
\begin{aligned}
&\|\int_{\mathbb{R}^d} \eta(|\mathbf{x}-\mathbf{y}|)\, F_{z,B}(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y}\|_{L^2(\mathbb{R}^d,\omega d\mathbf{x})} \\
&\qquad\qquad \le \frac{C}{|z|^{1/4}}\cdot \left(\|\omega\|_{F^p(\mathbb{R}^d)}\right)^{1/2} \|f\|_{L^2(\mathbb{R}^d,d\mathbf{x})}
\end{aligned}
\tag{5.34}
$$

where $Re\sqrt{z}\ge 1$ and C depends only on p, d, B and the size of finite many derivatives of η.

Proof. Theorem 5.33 follows from Theorem 5.5 by using partition of unity and standard rescaling argument. We omit the details. See [pp.134-135, 23] for the case of $L^2\to L^q$ estimates.

We are now ready to give the

Proof of Theorem 5.3.

Let $\psi\in C^\infty(\mathbb{T}^d)$. Fix $\mathbf{x}_0\in\Omega$. We choose $\widetilde{\eta}\in C_0^\infty(Q(\mathbf{x}_0,1/2))$ such that $\widetilde{\eta}\equiv 1$ on $Q(\mathbf{x}_0,1/4)$. Then, for $z=-(k+i)^2$ with $k\ge 1$,

$$
\begin{aligned}
&\psi(\mathbf{x})\widetilde{\eta}(\mathbf{x})^2 \\
&= \widetilde{\eta}(\mathbf{x})\int_{\mathbb{R}^d} F_{z,B}(\mathbf{x}-\mathbf{y})(\mathbf{D}A\mathbf{D}^T+z)(\psi\,\widetilde{\eta})\, d\mathbf{y} \\
&= \widetilde{\eta}(\mathbf{x})\int_{\mathbb{R}^d} F_{z,B}(\mathbf{x}-\mathbf{y})\left\{(\mathbf{D}A\mathbf{D}^T+z)\psi\cdot\widetilde{\eta} - 2\mathbf{D}\psi A(\mathbf{D}\widetilde{\eta})^T - \psi\mathbf{D}A\mathbf{D}^T\widetilde{\eta}\right\} d\mathbf{y} \\
&= \widetilde{\eta}(\mathbf{x})\int_{\mathbb{R}^d} F_{z,B}(\mathbf{x}-\mathbf{y})\eta(|\mathbf{x}-\mathbf{y}|)\cdot \\
&\qquad\qquad \cdot\left\{(\mathbf{D}A\mathbf{D}^T+z)\psi\cdot\widetilde{\eta} - 2\mathbf{D}\psi A(\mathbf{D}\widetilde{\eta})^T - \psi\mathbf{D}A\mathbf{D}^T\widetilde{\eta}\right\} d\mathbf{y}
\end{aligned}
$$

where $\eta\in C_0^\infty((-2,2))$, $\eta(r)=1$ if $|r|\le 1$. It follows from Theorem 5.33 that

$$
\begin{aligned}
&\|\psi\widetilde{\eta}^2\|_{L^2(\mathbb{R}^d,\omega d\mathbf{x})} \\
&\le Ck^{-1/2}\left(\|\omega\widetilde{\eta}^2\|_{F^p(\mathbb{R}^d)}\right)^{1/2}\cdot \\
&\quad \cdot\left\{\|[(\mathbf{D}A\mathbf{D}^T+z)\psi]\widetilde{\eta}\|_{L^2(\mathbb{R}^d,d\mathbf{x})} + \|\mathbf{D}\psi A(\mathbf{D}\widetilde{\eta})^T\|_{L^2(\mathbb{R}^d,d\mathbf{x})} + \|\psi\mathbf{D}A\mathbf{D}^T\widetilde{\eta}\|_{L^2(\mathbb{R}^d,d\mathbf{x})}\right\} \\
&\le Ck^{-1/2}\left(\|\omega\widetilde{\eta}^2\|_{F^p(\mathbb{R}^d)}\right)^{1/2}\cdot \\
&\quad \cdot\left\{\|(\mathbf{D}A\mathbf{D}^T-k^2-2ik)\psi\|_{L^2(\mathbb{T}^d,d\mathbf{x})} + \|\mathbf{D}\psi\|_{L^2(\mathbb{T}^d,d\mathbf{x})} + \|\psi\|_{L^2(\mathbb{T}^d,d\mathbf{x})}\right\}
\end{aligned}
\tag{5.35}
$$

where we have used the fact that ψ is periodic in the last inequality.

Note that, since ω is periodic,

$$\begin{aligned}\|\omega\tilde{\eta}^2\|_{F^p(\mathbb{R}^d)} &\le C \sup_{\substack{0<r\le 1\\ \mathbf{x}\in\Omega}} r^2 \left\{ \frac{1}{r^d}\int_{Q(\mathbf{x},r)} \omega^p d\mathbf{y} \right\}^{1/p} \\ &= C\|\omega\|_{F^p(\mathbb{T}^d)}.\end{aligned} \tag{5.36}$$

Also, using the Fourier series and Parseval's formula, one may show that, for $k \ge 1$,

$$\|\psi\|_{L^2(\mathbb{T}^d,d\mathbf{x})} + \|\mathbf{D}\psi\|_{L^2(\mathbb{T}^d,d\mathbf{x})} \le C\|(\mathbf{D}A\mathbf{D}^T - k^2 - 2ik)\psi\|_{L^2(\mathbb{T}^d,d\mathbf{x})}. \tag{5.37}$$

In view of (5.35)-(5.36), this gives

$$\|\psi\|_{L^2(\mathbb{T}^d,\omega d\mathbf{x})} \le C\, k^{-1/2} \left(\|\omega\|_{F^p(\mathbb{T}^d)}\right)^{1/2} \|(\mathbf{D}A\mathbf{D}^T - k^2 - 2ik)\psi\|_{L^2(\mathbb{T}^d,d\mathbf{x})}.$$

The proof is finished.

Proof of Theorem 5.1

We may assume $k \ge 1$. For any $\psi \in L^2(\mathbb{T}^d, d\mathbf{x})$, let

$$\phi(\mathbf{x}) = \sum_{|\mathbf{n}B|\in[k,k+1)} \hat{\psi}(\mathbf{n}) e^{i\mathbf{n}\mathbf{x}}.$$

Clearly, $\phi \in C^\infty(\mathbb{T}^d)$ and

$$\|(\mathbf{D}A\mathbf{D}^T - k^2 - 2ik)\phi\|_{L^2(\mathbb{T}^d,d\mathbf{x})} \le C\, k\|\psi\|_{L^2(\mathbb{T}^d,d\mathbf{x})}.$$

It follows from Theorem 5.3 that

$$\begin{aligned}\|\phi\|_{L^2(\mathbb{T}^d,\omega d\mathbf{x})} &\le C\, k^{-1/2} \left(\|\omega\|_{F^p(\mathbb{T}^d)}\right)^{1/2} \|(\mathbf{D}A\mathbf{D}^T - k^2 - 2ik)\phi\|_{L^2(\mathbb{T}^d,d\mathbf{x})} \\ &\le C\, k^{1/2} \left(\|\omega\|_{F^p(\mathbb{T}^d)}\right)^{1/2} \|\psi\|_{L^2(\mathbb{T}^d,d\mathbf{x})}.\end{aligned}$$

(5.2) is now proved.

6. The Proof of Theorem 3.10

Let $\omega \in L^p(\mathbb{T}^d)$ for some $p \in ((d-1)/2, d/2]$. Suppose that $\omega \ge c_\omega > 0$ and $\|\omega\|_{F^p(\mathbb{T}^d)} < \infty$. We need to show that, for any $\psi \in L^2(\mathbb{T}^d, \frac{d\mathbf{x}}{\omega})$,

$$\|\mathbb{H}_0(\delta_0 + i\rho)^{-1}\psi\|_{L^2(\mathbb{T}^d,\omega d\mathbf{x})} \le C\|\omega\|_{F^p(\mathbb{T}^d)}\|\psi\|_{L^2(\mathbb{T}^d,\frac{d\mathbf{x}}{\omega})} \tag{6.1}$$

where

$$\mathbb{H}_0(\delta_0 + i\rho)^{-1}\psi(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{\hat{\psi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}}{(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T} \tag{6.2}$$

and $\mathbf{k} = (\delta_0 + i\rho)\mathbf{a} + \mathbf{b}$.

As in [21], we write $\psi = \sum_{j=-\infty}^{\infty} \psi_j$ where

$$\begin{aligned} \psi_j &= \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ n_1\in[2^{j-1},2^j-1]}} \hat{\psi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}, \qquad \text{for } j \geq 1, \\ \psi_j &= \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ n_1\in[-2^{-j}+1,-2^{-j-1}]}} \hat{\psi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}, \qquad \text{for } j \leq -1, \\ \psi_0 &= \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ n_1=0}} \hat{\psi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}. \end{aligned} \tag{6.3}$$

Proposition 6.4. *If estimate (6.1) holds for ψ_j with a constant C independent of j, then it holds for ψ.*

Proof. Let ω_* be defined by (2.14). Note that $\omega_* \geq \omega$. By Proposition 2.13 and Remark 2.16, $\|\omega_*\|_{F^p(\mathbb{T}^d)} \leq C\|\omega\|_{F^p(\mathbb{T}^d)}$ and $\omega_*(\cdot, x_2, \dots, x_d)$, $\omega_*(\cdot, x_2, \dots, d_d)^{-1}$ are $A_1(\mathbb{R})$ weights uniformly in $\mathbf{x}' = (x_2, \dots, x_d) \in \mathbb{T}^{d-1}$. This , together with the weighted one-dimensional Littlewood-Paley theory on $\mathbb{T}^1$ [17], gives

$$\begin{aligned} \|\mathbb{H}_0(\delta_0 + i\rho)^{-1}\psi\|_{L^2(\mathbb{T}^d,\omega d\mathbf{x})} &\leq \|\mathbb{H}_0(\delta_0 + i\rho)^{-1}\psi\|_{L^2(\mathbb{T}^d,\omega_* d\mathbf{x})} \\ &\leq C\| \left(\sum_{j=-\infty}^{\infty} |\mathbb{H}_0(\delta_0 + i\rho)^{-1}\psi_j|^2 \right)^{1/2} \|_{L^2(\mathbb{T}^d,\omega_* d\mathbf{x})} \\ &\leq C\|\omega_*\|_{F^p(\mathbb{T}^d)}\| \left(\sum_{j=-\infty}^{\infty} |\psi_j|^2 \right)^{1/2} \|_{L^2(\mathbb{T}^d,\frac{d\mathbf{x}}{\omega_*})} \\ &\leq C\|\omega\|_{F^p(\mathbb{T}^d)}\| \left(\sum_{j=-\infty}^{\infty} |\psi_j|^2 \right)^{1/2} \|_{L^2(\mathbb{T}^d,\frac{d\mathbf{x}}{\omega_*})} \\ &\leq C\|\omega\|_{F^p(\mathbb{T}^d)}\|\psi\|_{L^2(\mathbb{T}^d,\frac{d\mathbf{x}}{\omega_*})} \\ &\leq C\|\omega\|_{F^p(\mathbb{T}^d)}\|\psi\|_{L^2(\mathbb{T}^d,\frac{d\mathbf{x}}{\omega})}. \end{aligned}$$

We will give the estimate of $\mathbb{H}_0(\delta_0 + i\rho)^{-1}\psi_j$ for $j \geq 1$ in details. The case $j \leq 0$ may be handled in the same manner.

To this end, we note that, by (3.6)-(3.7),

$$
\begin{aligned}
&(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T \\
&= |(\mathbf{n}+\mathbf{b})B|^2 + 2\delta_0(n_1+b_1)s_0 + (\delta^2-\rho^2)a_1 s_0 + 2i\rho(n_1+\frac{1}{2})s_0
\end{aligned}
\tag{6.5}
$$

where $B=\sqrt{A}$. Fix $j\geq 1$, let

$$
z_j = -\rho^2 a_1 s_0 + 2i\rho\cdot 2^j\cdot s_0. \tag{6.6}
$$

Since $\mathrm{Re}\sqrt{z_j}\geq c\min(2^j,\sqrt{|\rho|2^j})\geq c_0>0$ (see [21]), by Theorem 4.3, we have

$$
\begin{aligned}
&\|\sum_{\mathbf{n}\in\mathbb{Z}^d}\frac{\hat{\psi}_j(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}}{|(\mathbf{n}+\mathbf{b})B|^2+z_j}\|_{L^2(\mathbb{T}^d,\omega\,d\mathbf{x})} \\
&\leq C\|\omega\|_{F^p(\mathbb{T}^d)}\|\psi_j\|_{L^2(\mathbb{T}^d,\frac{d\mathbf{x}}{\omega})}.
\end{aligned}
\tag{6.7}
$$

It remains to show that

$$
\begin{aligned}
&\|\mathbb{H}_0(\delta_0+\rho)^{-1}\psi_j - \sum_{\mathbf{n}\in\mathbb{Z}^d}\frac{\hat{\psi}_j(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}}{|(\mathbf{n}+\mathbf{b})B|^2+z_j}\|_{L^2(\mathbb{T}^d,\omega d\mathbf{x})} \\
&\leq C\|\omega\|_{F^p(\mathbb{T}^d)}\|\psi_j\|_{L^2(\mathbb{T}^d,\frac{d\mathbf{x}}{\omega})}.
\end{aligned}
\tag{6.8}
$$

Using the Minkowski's inequality, we can bound the left hand side of (6.8) by

$$\sum_{M=1}^{\infty} \| \sum_{|\mathbf{n}B|\in[M-1,M)} \frac{\hat{\psi}_j(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}\left\{|(\mathbf{n}+\mathbf{b})B|^2+z_j-(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T\right\}}{[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]\cdot[|(\mathbf{n}+\mathbf{b}|)B|^2+z_j]} \|_{L^2(\mathbb{T}^d,\omega d\mathbf{x})}$$

$$\leq C\|\omega\|^{1/2}_{F^p(\mathbb{T}^d)} \sum_{M=1}^{\infty} M^{1/2} \cdot$$

$$\cdot \| \sum_{|\mathbf{n}B|\in[M-1,M)} \frac{\hat{\psi}_j(\mathbf{n})e^{i\mathbf{n}\mathbf{x}}\left\{|(\mathbf{n}+\mathbf{b})B|^2+z_j-(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T\right\}}{[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]\cdot[|(\mathbf{n}+\mathbf{b}|)B|^2+z_j]} \|_{L^2(\mathbb{T}^d,d\mathbf{x})}$$

$$\leq C\|\omega\|^{1/2}_{F^p(\mathbb{T}^d)} \sum_{M=1}^{\infty} M^{1/2} \cdot$$

$$\cdot \sup_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ n_1\in[2^{j-1},2^j-1] \\ |\mathbf{n}B|\in[M-1,M)}} \left| \frac{|(\mathbf{n}+\mathbf{b})B|^2+z_j-(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T}{[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]\cdot[|(\mathbf{n}+\mathbf{b}|)B|^2+z_j]} \right|$$

$$\cdot \| \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ |\mathbf{n}B|\in[M-1,M)}} \hat{\psi}_j(\mathbf{n})e^{i\mathbf{n}\mathbf{x}} \|_{L^2(\mathbb{T}^d,d\mathbf{x})}$$

$$\leq C\|\omega\|_{F^p(\mathbb{T}^d)}\|\psi_j\|_{L^2(\mathbb{T}^d,\frac{d\mathbf{x}}{\omega})} \sum_{M=1}^{\infty} M \cdot$$

$$\cdot \sup_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ n_1\in[2^{j-1},2^j-1] \\ |\mathbf{n}B|\in[M-1,M)}} \left| \frac{|(\mathbf{n}+\mathbf{b})B|^2+z_j-(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T}{[(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T]\cdot[|(\mathbf{n}+\mathbf{b}|)B|^2+z_j]} \right|$$

$$\leq C\|\omega\|_{F^p(\mathbb{T}^d)}\|\psi_j\|_{L^2(\mathbb{T}^d,\frac{d\mathbf{x}}{\omega})}$$

$$\cdot \sum_{M=1}^{\infty} M \sup_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ |\mathbf{n}B|\in[M-1,M)}} \frac{|\rho|2^j}{\left\{\left||(\mathbf{n}+\mathbf{b})B|^2-\rho^2 a_1 s_0\right|+2^j|\rho|\right\}^2}$$

where we have used (5.2) and its dual

$$\| \sum_{|\mathbf{n}B|\in[k,k+1)} \hat{\psi}(\mathbf{n})e^{i\mathbf{n}\mathbf{x}} \|_{L^2(\mathbb{T}^d,d\mathbf{x})} \leq C(k+1)^{1/2}\left(\|\omega\|_{F^p(\mathbb{T}^d)}\right)^{1/2}\|\psi\|_{L^2(\mathbb{T}^d,\frac{d\mathbf{x}}{\omega})} \tag{6.9}$$

in the first and third inequalities respectively.

(6.1) now follows from the estimate

$$\sum_{M=1}^{\infty} M \sup_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ |\mathbf{n}B|\in[M-1,M)}} \frac{|\rho|2^j}{\left\{\left||(\mathbf{n}+\mathbf{b})B|^2-\rho^2 a_1 s_0\right|+2^j|\rho|\right\}^2} \leq C \tag{6.10}$$

which was proved in [21, (5.10)]. The proof of Theorem 3.10 is finally complete.

References

[1] M.Sh. Birman and T.A. Suslina, *Two-dimensional periodic magnetic Hamiltonian is absolutely continuous*, Algebra i Analiz **9(1)** (1997), 32-48; English transl. in St. Petersburg Math. J. 9(1) (1998), 21-32.

[2] ———, *Absolute continuity of the two-dimensional periodic magnetic Hamilton ian with discontinuous vector-valued potential*, Algebra i Analiz **10(4)** (1998); English transl. in St. Petersburg Math. J. 10(4) (1999).

[3] ———, *Periodic magnetic Hamiltonian with a variable metric. The problem of absolute continuity*, Algebra i Analiz **11(2)** (1999); English transl. in St. Petersburg Math. J. 11(2) (2000).

[4] S. Chanillo and E. Sawyer, *Unique Continuation for $\Delta+v$ and the C.Fefferman-Phong Class*, Trans. Amer. Math. Soc. **318(1)** (1990), 275-300.

[5] S. Chanillo and R. Wheeden, *L^p estimates for fractional integrals and Sobolev inequalities with applications to Schrödinger operators*, Comm. Partial Diff. Eqs. **10** (1985), 1077-1116.

[6] F. Chiarenza and A. Ruiz, *Uniform L^2-weighted Sobolev inequalities*, Proc. Amer. Math. Soc. **112(1)** (1991), 53-64.

[7] L. Danilov, *On the spectrum of the Dirac operator with periodic potential in $\mathbb{R}^n$*, Theor. and Math. Phys. **85** (1990), 1039-1048.

[8] C. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. (N.S) **9** (1983), 129-206.

[9] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order, second edition*, Springer-Verlag, 1983.

[10] R. Hempel and I. Herbst, *Strong magnetic fields, Dirichlet boundaries, and spectral gaps*, Comm. Math. Phys. **164** (1995), 237-259.

[11] ———, *Bands and gaps for periodic magnetic Hamiltonians*, in Operator Theory: Advances and Applications **78** (1995), 175-184.

[12] T. Kato, *Perturbation Theory for Linear Operators, Second edition* (1979), Springer-Verlag.

[13] C.E. Kenig, *Restriction theorems, Carleman estimates, uniform Sobolev inequalities and unique continuation*, Lecture Notes in Math. **1384** (1989), 69-90.

[14] C.E. Kenig, A. Ruiz, and C.D. Sogge, *Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators*, Duke Math. J. **55(2)** (1987), 329-347.

[15] R. Kerman and E. Sawyer, *The trace inequality and eigenvalue estimates for Schrödinger operators*, Ann. Inst. Fourier (Grenoble) **36** (1986), 207-228.

[16] P. Kuchment, *Floquet theory for partial differential equations*, Birkhäuser Verlag, 1993.

[17] D. Kurtz, *Littlewood-Paley and multiplier theorems on weighted L^p spaces*, Trans. Amer. Math. Soc. **259** (1980), 235-254.

[18] N.N. Lebedev, *Special Functions and Their Applications*, Dover Publications, New York, 1972.

[19] A. Morame, *Absence of singular spectrum for a perturbation of a two-dimensional Laplace-Beltrami operator with periodic electro-magnetic potential*, J. Phys. **31** (1998), 7593-7601.

[20] M. Reed and B. Simon, *Methods of Modern Mathematical Physics,*, vol. II and IV, Academic Press, 1978.

[21] Z. Shen, *On Absolute Continuity of the Periodic Schrödinger Operators*, Preprint (1998).

[22] A.V. Sobolev, *Absolute continuity of the periodic magnetic Schrödinger operator*, Invent.

Math **137(1)** (1999), 85-112.

[23] C.D. Sogge, *Concerning the L^p norm of spectral clusters of second order elliptic operators on compact manifolds*, J. Funct. Analysis **77(1)** (1988), 123-138.

[24] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.

[25] ______, *Harmonic Analysis, Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, 1993.

[26] E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

[27] L.E. Thomas, *Time dependent approach to scattering from impurities in a crystal*, Comm. Math. Phys. **33** (1973), 335-343.

[28] A. Zygmund, *Trigonometric Series*, vol. II, Cambridge Univ. Press, 1968.