Scattering Theory for Sparce Random Potentials

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Abstract: We prove the existence of wave operators and thus of absolutely continuous spectrum for a huge class of random Schrödinger operators with randomness decaying at infinity in dimension $d \geq 3$.

1 Introduction

In scattering theory one usually considers potentials that decay at infinity with a specified rate. A typical result is the following: If the potential V decays faster then the Coulomb potential (i.e. $|V(x)| \leq C\,(1+|x|)^{-(1+\varepsilon)}$) and is locally not too singular, then the wave operators $\Omega_{\pm} = s - \lim e^{itH}e^{-itH_0}$ exist. Here, as usual, we set $H_0 = -\Delta$ and $H = H_0 + V$ (see virtually any textbook on scattering theory, for example [6], [4] or [7]). For definiteness let's assume that $(1+|x|)^{1+\varepsilon}V(H_0+1)^{-1}$ is a compact operator. The existence of the wave operators implies that $\sigma_{ac}(H_0) \subset \sigma_{ac}(H)$ (with $[0,\infty) = \sigma(H_0) = \sigma_{ac}(H_0)$ in this case). The above assumptions imply that V is relatively compact, so $\sigma_{ess}(H) = \sigma_{ess}(H_0) = [0,\infty)$. Similar results hold for quantum mechanics on the lattice $\mathbb{Z}^d$, i.e. for $H_0 = -\Delta$ on $L^2(\mathbb{R}^d)$ replaced by the finite difference Laplacian on $\ell^2\left(\mathbb{Z}^d\right)$. In this case H_0 is given by $H_0u\,(n) = -\sum_{|i|=1}\left(u(n+i) - u(n)\right)$ (see [5] or [4]).

It was Krishna ([1] and [2]) who observed that for a *random* potential V_ω on $\ell^2\left(\mathbb{Z}^d\right)$ it is sufficient that a certain *average* of V_ω goes to zero at infinity rather than V_ω itself. This leaves place for potentials which are not relatively compact, as was observed in [3]. In fact, there examples of random potentials with "decaying randomness" are constructed, which exhibit a transition from pure point to absolutely continuous spectrum.

In this paper we extend Krishna's result about the existence of absolutely continuous spectrum to a large class of potentials with decaying randomness giving a very simple argument.

2 Main Result

Denote by H_0 the free Schrödigeroperator on $L^2(\mathbf{R}^d)$ (resp. $\ell\left(\mathbf{Z}^d\right)$). Suppose $V_\omega(x)$ is a random potential on $\mathbf{R}^d$ (resp. on $\mathbf{Z}^d$) such that the function

$$W(x) := E\left(V_\omega(x)^2\right)^{\frac{1}{2}}$$

is finite for (Lebesque-) almost all $x \in \mathbf{R}$ (resp. all $x \in \mathbf{Z}^d$).

Definition 1 $V_\omega(x)$ *satisfies the* averaged Cook condition *if*

$$\int_T^\infty \left\| W\ e^{\mp\ itH_0}\ \varphi \right\| dt < \infty$$

for a $T \geq 0$ *and for all* φ *in a dense set* D_0 *in* L^2.

Theorem 2 *If* V_ω *satisfies the averaged Cook condition then the wave operators* $\Omega_\pm$ *for the random potential* V_ω *exist for almost all* ω. *Consequently*

$$\sigma(H_0) \subset \sigma_{ac}(H_\omega)$$

almost surely.

Proof. By Cook's method (see e.g.[4]) is suffices to prove that

$$\int_T^\infty \left\| V_\omega\ e^{\mp\ itH_0}\ \varphi \right\| dt < \infty$$

for almost all ω, wich is certainly true if the expectation of this quantity is finite.

$$\begin{aligned}
& E\left(\int_T^\infty \left\| V_\omega\ e^{\mp\ itH_0}\ \varphi \right\| dt\right) \\
&= \int_T^\infty E\left(\int \left|V_\omega(x)\right|^2\ \left|e^{\mp\ itH_0}\ \varphi(x)\right|^2 dx\right)^{\frac{1}{2}} dt \\
&\leq \int_T^\infty \left(\int \left(E\left(V_\omega(x)^2\right)\right)\ \left|e^{\mp\ itH_0}\ \varphi(x)\right|^2 dx\right)^{\frac{1}{2}} dt \\
&= \int_T^\infty \left\| W\ e^{\mp\ itH_0}\ \varphi \right\| dt < \infty
\end{aligned}$$

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3 Examples

We give a few examples which illustrate the theorem.

a) Set

$$V_n = a_n \; q_n$$

with q_n independent, identically distributed random variables with $E(q_n^2) < \infty$ and $a_n \geq 0$ a (deterministic) sequence with $a_n < C_1 \, (1 + |n|)^{-\alpha}$ for an $\alpha > 1$.

Then

$$W_n = E \left(V_n^2 \right)^{\frac{1}{2}} \leq C_2 \, (1 + |n|)^{-\alpha} \quad .$$

By a result of [5] W_n satisfies the Cook condition.

b) Consider

$$V_\omega (x) = \sum q_n(\omega) f_n(x) \; .$$

Here again q_n are independent, identically distributed random variables with $E(q_n^2) < \infty$ and f_n are functions such that

$$\sum_{n \in \mathbf{Z}^d} |f_n(x)| \leq C \, (1 + |x|)^{-\alpha} \quad , \quad \alpha > 1 \; .$$

Then

$$V_0(x) = \sum E(q_n) \; f_n(x)$$

satisfies the Cook criterion. To prove that V_ω satisfies this criterion, we may therefore assume that $E(q_n) = 0$.

Then, due to the independence of the q_n:

$$\begin{aligned} W(x) &= E \left(V_\omega (x)^2 \right)^{\frac{1}{2}} \\ &= \left(\sum E \left(q_n^2 \right) \; f_n(x)^2 \right)^{\frac{1}{2}} \\ &\leq E(q_0^2)^{\frac{1}{2}} \sum |f_n(x)| \\ &\leq C' \, (1 + |x|)^{-\alpha} \quad . \end{aligned}$$

c) Let V_n be independent random variables with

$$P \left(V_n = 1 \right) = p_n \; , \; P \left(V_n = 0 \right) = 1 - p_n \; ,$$

$$W_n = \left(E \left(V_n^2 \right) \right)^{\frac{1}{2}} = p_n^{\frac{1}{2}} \; .$$

So V_n satisfies the averaged Cook criterion if

$$p_n \leq C \, (1 + |n|)^{-\beta} \text{ for } \beta > 2 \; .$$

d) Let ξ_n be random variables as in (c) and

$$V_n = q_n \ \xi_n$$

with q_n independent, identically distributed random variables independent of the ξ_n and $E(q_n^2) < \infty$. Then V_n satisfies the averaged Cook criterion if for some $\beta > 2$

$$p_n \leq C\,(1 + |n|)^{-\beta}$$

Similar models can be considered on $\mathbf{R}^d$.

e) Let μ_ω be a Poisson random measure with intensity measure ρ. This means that

$$P(\mu_\omega(A) = k) = \frac{\rho(A)^k}{k!} \ e^{-\rho(A)} \ .$$

Set

$$V_\omega(x) = \int f(x-y) \ \mu_\omega\,(dy) \ ,$$

then

$$\begin{aligned} & E\left(\int |f(x-y)| \ \mu_\omega\,(dy)\right) \\ & = \int f(x-y) \ \rho\,(dy) \ . \end{aligned}$$

Hence $V_\omega(x)$ exists almost surely if the latter integral is finite for almost all x.

Moreover

$$\begin{aligned} & E\left(\left(\int f(x-y) \ \mu_\omega\,(dy)\right)^2\right) \\ & = \int |f(x-y)|^2 \ d\rho\,(dy) + \left(\int f(x-y) \ \rho\,(dy)\right)^2 \end{aligned}$$

So, the wave operators exist almost surely if

$$\int |f(x-y)|^2 \,\rho(dy) + \left(\int f(x-y) \ \rho(dy)\right)^2 \leq C\,(1+|x|)^{-\alpha} \ .$$

4 Discussion and Extensions

It can be shown that for the Anderson model as well as for the alloy type models b) and d) there is pure point spectrum outside the spectrum of the free Hamiltonian provided the distribution of the appropriate random values is smooth enough. For the Anderson model (i.e. on the lattice) this can be done as in [3]. For the continuous models this will be done by a suitable modification of the multiscale analysis in a subsequent publication [8].

While the above argument proves the existence of absolutely continuous spectrum we can not exclude whether this part of the spectrum is mixed with pure point or singular continuous spectrum. The remarkable paper by Jaksic and Last [9] settles this question for the Anderson model. These authors prove that the spectrum is purely absolutely continuous in the region under consideration. Unfortunately their methods relies on the technique of rank one perturbations and is therefore not applicable to the continuous case.

Above we assumed that our random potential is decaying - in an averaged sense - in all directions. However, it is enough to ensure absolutely continuous spectrum that the potential "decays" in "enough" directions [8]. This will allow us to treat potentials living in a half space thus proving and extending results of [10] as well as potentials concentrated near a surface (see [11], [12] and references therein)..

Finally we remark that the statements of this paper are valid "almost everywhere", thus giving no hint for a given deterministic potential. However such deterministic results will be given in [8] as well.

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