OPERATOR-NORM CONVERGENCE OF THE TROTTER PRODUCT FORMULA FOR SECTORIAL GENERATORS

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Abstract. The operator-norm convergence of the Trotter product formula is known for self-adjoint semigroups with compactness or smallness conditions on the generators involved in this formula. We generalize these two types of results to sectorial generators.

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1 Introduction and main result

Nowadays there are two strategies to establish the operator-norm convergence of the Trotter product formula: with error bounds [15], [1]-[6], [11], [12]; and without them [10], [13]. The present letter belongs to the second class. Among the preceding results, all (except [1]) concern self-adjoint semigroups. The aim of this letter is to generalize to sectorial generators the following results of [13], [11], [12] (where, in addition, the error bounds are given): the first one uses compactness conditions, and the second one smallness conditions.

Proposition 1 *Let A and B be non-negative densely defined self-adjoint operators on a Hilbert space $\mathcal{H}$. If $(I+A)^{-1}$ or $(I+A)^{-1}(I+B)^{-1}$ are compact, then the sequence $\left\{\left(e^{-tA/n}e^{-tB/n}\right)^n\right\}_{n\geq 1}$ converges in the operator-norm to $e^{-tH}P$ locally uniformly for $t>0$. Here $H = A\dot{+}B$ is the form-sum of A and B, and P is the orthogonal projector on the closed subspace $\overline{\mathcal{D}(A^{1/2})\cap\mathcal{D}(B^{1/2})}$.*

Proposition 2 *Let A and B be non-negative densely defined self-adjoint operators on a Hilbert space $\mathcal{H}$, with $A\geq I$, $B\geq 0$, and $\mathcal{D}(A^\alpha)\subset\mathcal{D}(B^\alpha)$ for some $1/2<\alpha\leq 1$, $\|B^\alpha u\|\leq a\|A^\alpha u\|$, $u\in\mathcal{D}(A^\alpha)$ for $a<1$. Then the sequence $\left\{\left(e^{-tA/n}e^{-tB/n}\right)^n\right\}_{n\geq 1}$ converges in the operator-norm to e^{-tH} uniformly in $t\geq 0$. Here $H=A\dot{+}B$ is the form-sum of A and B.*

Now let A and B be m-sectorial operators on a Hilbert space $\mathcal{H}$ with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$. Let a and b be associated closed sectorial forms with positive vertices and semi-angles θ_A and θ_B (here and below we follow notations of [8]). Let S_θ denote the open sector $\{z\in\mathbb{C}, z\neq 0, |\arg z|<\theta\}$. In the present paper we obtain the following generalization of Proposition 1:

Theorem 1 *If A and B are m-sectorial operators on a Hilbert space $\mathcal{H}$ and if $(I+A)^{-1}$ or $(I+\mathrm{Re}A)^{-1/2}(I+\mathrm{Re}B)^{-1/2}$ are compact, then the sequence $\{(e^{-tA/n}e^{-tB/n})^n\}_{n\geq 1}$ converges in operator-norm to $e^{-tH}P$ for any $t \in S_\theta$. Here $H = A\dot{+}B$ is the form-sum of A and B, P is the orthogonal projector on $\overline{\mathcal{D}(a)\cap\mathcal{D}(b)}$, and $\theta = \pi/2 - \max\{\theta_A, \theta_B\}$. The convergence is uniform on the compact subsets of the sector S_θ.*

On the other hand, we generalize Proposition 2 under the assumptions reflecting the smallness of B with respect to A:

$$\begin{array}{rl} (i) & \mathrm{Re}A, \mathrm{Re}B \geq \gamma > 0; \\ (ii) & \mathcal{D}(\mathrm{Re}A) \subset \mathcal{D}(\mathrm{Im}A); \\ (iii) & \mathcal{D}(\mathrm{Re}B) \subset \mathcal{D}(\mathrm{Im}B); \\ (iv) & \mathcal{D}((\mathrm{Re}A)^\alpha) \subset \mathcal{D}((\mathrm{Re}B)^\alpha) \text{ for some } \alpha \in (1/2, 1]; \\ (v) & \|(\mathrm{Re}B)^\alpha u\| \leq c\|(\mathrm{Re}A)^\alpha u\|,\ u \in \mathcal{D}((\mathrm{Re}A)^\alpha),\ 0 < c < 1, \\ & \text{and for the same numbers } \alpha. \end{array}$$

Theorem 2 *Let A and B be m-sectorial operators on a Hilbert space $\mathcal{H}$, satisfying the assumptions (i)-(v). Then for any complex t in an open sector S_θ, the sequence $\{(e^{-tA/n}e^{-tB/n})^n\}_{n\geq 1}$ converges in the operator-norm to e^{-tH}, uniformly on the compact subsets of the sector. Here $H = A\dot{+}B$ is the form-sum of A and B, and $\theta = \pi/2 - \max\{\theta_A, \theta_B\}$.*

Before we present the proofs, here are some comments about assumptions (i)-(v). From (i) it follows that self-adjoint operators $\mathrm{Re}A$ and $\mathrm{Re}B$ are boundedly invertible, and thus (ii), (iii) imply that there are positive constants λ, μ such that:

$$\|\mathrm{Im}Au\| \leq \lambda\|\mathrm{Re}Au\|,\ u \in \mathcal{D}(\mathrm{Re}A); \tag{1}$$

$$\|\mathrm{Im}Bu\| \leq \mu\|\mathrm{Re}Bu\|,\ u \in \mathcal{D}(\mathrm{Re}B). \tag{2}$$

¿From (iv) it follows that $\mathcal{D}(a) \subset \mathcal{D}(b)$, thus $\overline{\mathcal{D}(a)\cap\mathcal{D}(b)} = \mathcal{H}$ and the form-sum $A\dot{+}B$ is densely defined.

To clarify assumption (ii) we give a sufficient condition which implies (ii). Let us consider the standard representation $A = G(I+iC)G$ of the sectorial operator A. Here $G = (\mathrm{Re}A)^{1/2}$ and C is a self-adjoint operator bounded by $\tan\theta_A$ (see [8, Ch.VI Theorem 3.2]). If $C(\mathcal{D}(G)\cap\mathcal{R}(G)) \subset \mathcal{D}(G)$, where $\mathcal{R}(G)$ is range of G, then (ii) holds. A similar condition provides (iii).

2 Proofs

We start with an abstract couple of m-sectorial operators A and B on a Hilbert space $\mathcal{H}$. Following idea formulated in Addendum of [9], we consider two families of closed forms:

$$a(z) = \mathrm{Re}\ a + z\mathrm{Im}\ a,\ \mathcal{D}(a(z)) = \mathcal{D}(a); \tag{3}$$

$$b(z) = \mathrm{Re}\ b + z\mathrm{Im}\ b,\ \mathcal{D}(b(z)) = \mathcal{D}(b); \tag{4}$$

where a and b are associated with the m-sectorial operators A and B defined above, and $z \in \mathbb{C}$.

Lemma 1 *There exists $\epsilon > 0$ such that the forms $a(z)$ and $b(z)$ are sectorial for any z in the strip $D = \{z \in \mathbb{C} : |\mathrm{Re}\ z| < \epsilon\}$. The families of forms are also holomorphic of type (a) for $z \in \mathbb{C}$.*

Proof.
Let $z = x + iy$ with $x, y \in \mathbb{R}$, and $u \in \mathcal{D}(a)$. Since a is sectorial, we have:

$$|x\mathrm{Im}\ a[u]| \leq |x| \tan\theta_A \mathrm{Re}\ a[u]; \tag{5}$$

hence $\mathrm{Re}\ a(z) = \mathrm{Re}\ a + x\mathrm{Im}\ a$ is positive for $|x|\tan\theta_A < 1$. Moreover, for such x:

$$\mathrm{Re}\ a(z)[u] \geq \mathrm{Re}\ a[u] - |x\mathrm{Im}\ a[u]| \geq (1 - |x|\tan\theta_A)\mathrm{Re}\ a[u]. \tag{6}$$

By (5) and (6) we obtain:

$$\begin{aligned} |\mathrm{Im}\ a(z)[u]| = |y\mathrm{Im}\ a[u]| &\leq |y|\tan\theta_A \mathrm{Re}\ a[u] \\ &\leq \frac{|y|\tan\theta_A}{1 - |x|\tan\theta_A}\mathrm{Re}\ a(z)[u], \end{aligned} \tag{7}$$

which proves that $a(z)$ is sectorial for any z in the strip $|\mathrm{Re}\ z| < \cot\theta_A$. Similarly, $b(z)$ is sectorial for $|\mathrm{Re}\ z| < \cot\theta_B$, and we can set $\epsilon = \min\{\cot\theta_A, \cot\theta_B\}$.

On the other hand, since by (3) and (4) one has:

$$\frac{d}{dz}a(z)[u] = \mathrm{Im}\ a[u],\ \ u \in \mathcal{D}(a), \tag{8}$$

$$\frac{d}{dz}b(z)[u] = \mathrm{Im}\ b[u],\ \ u \in \mathcal{D}(b), \tag{9}$$

the forms $a(z)$, $b(z)$ are holomorphic families of type (a) for $z \in \mathbb{C}$ (cf [8, Ch.VII §4]). □

Corollary 1 *By the first representation theorem [8, Ch. VI, Theorem 2.1], there exist m-sectorial operators $A(z)$ and $B(z)$ associated with the closed sectorial forms $a(z)$ and $b(z)$ ($|\mathrm{Re}z| < \epsilon$) such that these operators form holomorphic families and are locally uniformly sectorial [8, Ch. VII, Theorem 4.2].*

If A is an m-sectorial operator with positive vertex and semi-angle θ_A, then $-A$ generates a holomorphic contraction semigroup (we denote it by e^{-tA}, where t is a complex number with $|\arg t| < \pi/2 - \theta_A$), moreover we have the following result [8, Ch.IX, Theorem 2.6 and footnote]:

Proposition 3 *Let $T(z)$ be a holomorphic family of closed operators in an open domain $D \subset \mathbb{C}$, and for any $z \in D$ operator $-T(z)$ generates a holomorphic semigroup of angle $\theta(z)$. Then the semigroup $e^{-tT(z)}$ is holomorphic in $z \in D$ for t in some open sector containing $t > 0$.*

Proof.
The holomorphic semigroup $e^{-tT(z)}$ can be defined by the Dunford-Taylor integral

$$e^{-tT(z)} = \frac{1}{2\pi i}\int_\Gamma e^{\zeta t}(T(z) + \zeta)^{-1} d\zeta, \tag{10}$$

where Γ is a curve running from $\infty e^{-i(\theta(z)+\pi/2-\delta)}$ to $\infty e^{i(\theta(z)+\pi/2-\delta)}$ and enclosing the spectrum of $-T(z)$. Since $T(z)$ is a holomorphic family, $(T(z)+\zeta)^{-1}$ is holomorphic in z for ζ in the resolvent set of $-T(z)$. Moreover, in virtue of the estimate:

$$\left\| \frac{d}{dz}(T(z)+\zeta)^{-1} \right\| = \left\| \frac{1}{2\pi i} \int_C (T(z')+\zeta)^{-1}(z'-z)^{-2} dz' \right\| \leq \frac{M_\delta}{|\zeta| r} \tag{11}$$

for $|\arg \zeta| \leq \pi/2+\theta(z)-\delta$, the representation (10) can be differentiated under the integral sign. Here C is a small circle around z of radius r, and

$$M_\delta = \sup_{\substack{|\arg\zeta| \leq \pi/2+\theta(z)-\delta \\ z' \in C}} \|\zeta (T(z')+\zeta)^{-1}\|.$$

Therefore, (10) and (11) imply the assertion.

□

Applying this result to our case, we find that the semigroups $e^{-tA(z)}$, $e^{-tB(z)}$ are holomorphic in z with $|\mathrm{Re}\ z| < \epsilon$ and in t in some (depending on z) open sector containing $t > 0$. For real $t \geq 0$, these semigroups are contractive.

Notice that for holomorphic functions with bounded operator-values there is no distinction between uniform, strong and weak analyticity [8, Ch.III, Theorem 3.12]. For that reason the following result, known as the theorem of Vitali [7, Theorem 3.14.1], will achieve the proof of our Theorems:

Proposition 4 *Let f_n be a family of holomorphic functions on an open $D \subset \mathbb{C}$, with values in a Banach space $\mathcal{X}$. Suppose that $\|f_n(z)\| \leq M$ for any $n \in \mathbb{N}$ and $z \in D$, and $f_n(z)$ converges for z in a subset of D having a limit point in D. Then $f_n(z)$ converges for any $z \in D$, uniformly on compact subsets of D, and the limit function is holomorphic in D.*

Our method of generalization of results [11], [12], and [13], consists in applying this statement for some fixed $t > 0$ to the family of functions

$$f_n(z) = \left(e^{-tA(z)/n} e^{-tB(z)/n}\right)^n, \ n = 1, 2, \dots,$$

with values in the Banach space of bounded operators on $\mathcal{H}$. By Lemma 1, Corollary 1, and Proposition 3, these functions are holomorphic in $D = \{z \in \mathbb{C} : |\mathrm{Re}\ z| < \epsilon\}$, and $\|f_n(z)\| \leq 1$ for any $z \in D$. We will show that, under conditions formulated in Theorems 1 (or 2), Proposition 1 (or 2) can be applied to the self-adjoint operators $A(x)$ and $B(x)$ for real and small x. Thus the operator-norm convergence will be extended to D. In particular, for $z = i \in D$ we find the Trotter product formula for initial operators $A \equiv A(z=i)$ and $B \equiv B(z=i)$, and for $t > 0$.

Proof of Theorem 1

Let $(I+A)^{-1}$ be compact. Since by Lemma 1 and Corollary 1, $I+A(z)$ is a holomorphic family of operators invertible for $z \in D$, the operators $(I+A(z))^{-1}$ are also compact for any $z \in D$ by [8, Ch.VII, Theorem 4.3]. In particular it is true for $z = x$ from the real interval $(-\epsilon, \epsilon)$. Then combining Proposition 1 for the self-adjoint families $A(x)$ and $B(x)$, $x \in (-\epsilon, \epsilon)$ and Proposition 4, one gets the first part of Theorem 1.

Now let the product $(I+\mathrm{Re}A)^{-1/2}(I+\mathrm{Re}B)^{-1/2}$ be compact. Consider holomorphic families of invertible m-sectorial operators $I+A(z)$ and $I+B(z)$, where $z\in D$, cf Corollary 1. By the standard representation [8, Ch.VI Theorem 3.2] one gets:

$$\begin{aligned} I+A(z) &= (I+\mathrm{Re}A)^{1/2}(I+z\tilde{C}_A)(I+\mathrm{Re}A)^{1/2}, &(12)\\ I+B(z) &= (I+\mathrm{Re}B)^{1/2}(I+z\tilde{C}_B)(I+\mathrm{Re}B)^{1/2}, &(13)\end{aligned}$$

where $\tilde{C}_A$ and $\tilde{C}_B$ are self-adjoint operators bounded respectively by $\tan\theta_A$ and $\tan\theta_B$. Since for real $x\in D$ we obtain:

$$\begin{aligned}(I+A(x))^{-1}(I+B(x))^{-1} = (I+\mathrm{Re}A)^{-1/2}(I+x\tilde{C}_A)^{-1}\\ (I+\mathrm{Re}A)^{-1/2}(I+\mathrm{Re}B)^{-1/2}(I+x\tilde{C}_B)^{-1}(I+\mathrm{Re}B)^{-1/2},\end{aligned}$$

the family $(I+A(x))^{-1}(I+B(x))^{-1}$ is compact. Combining as above Proposition 1 for $A(x)$ and $B(x)$, $x\in(-\epsilon,\epsilon)$ with Proposition 4, one finishes Theorem 1.
Proof of Theorem 2

We begin by proving a technical Lemma 2 which is based on the following result known as the Heinz-Kato Theorem [17, §2.3.3]:

Proposition 5 *Let A and B be positive self-adjoint operators in a Hilbert space $\mathcal{H}$, and suppose $\mathcal{D}(A)\subset\mathcal{D}(B)$. For all α satisfying $0<\alpha<1$, we have $\mathcal{D}(A^\alpha)\subset\mathcal{D}(B^\alpha)$. If, in addition, there is a certain number M such that $\|Bu\|\le M\|Au\|$ for all $u\in\mathcal{D}(A)$, then $\|B^\alpha u\|\le M^\alpha\|A^\alpha u\|$ for all $u\in\mathcal{D}(A^\alpha)$.*

Lemma 2 *If A and B satisfy the assumptions (i)-(v), then for any real x with $|x|<\min\{\epsilon,1/\lambda\}$ one has:*

$$\|B(x)^\alpha u\|\le\left(\frac{1+\mu|x|}{1-\lambda|x|}\right)^\alpha c\|A(x)^\alpha u\|,\ u\in\mathcal{D}(A(x)^\alpha). \tag{14}$$

Here ϵ is as in Lemma 1, λ and μ are defined by (1) and (2), whereas c and $0<\alpha\le 1$ are the same as in (v).

Proof.

By (2), which follows from (iii), we have

$$\|B(x)u\|\le\|\mathrm{Re}Bu\|+|x|\|\mathrm{Im}Bu\|\le(1+\mu|x|)\|\mathrm{Re}Bu\|, \tag{15}$$

for any $u\in\mathcal{D}(\mathrm{Re}B)\subset\mathcal{D}(B(x))$. Since $B(x)$ is self-adjoint, this inequality implies by the Heinz-Kato Theorem that

$$\|B(x)^\alpha u\|\le(1+\mu|x|)^\alpha\|(\mathrm{Re}B)^\alpha u\|,\ u\in\mathcal{D}((\mathrm{Re}B)^\alpha). \tag{16}$$

In virtue of (1) and perturbation theory of self-adjoint operators, the sum $\mathrm{Re}(A)+x\mathrm{Im}(A)$ is self-adjoint on the domain $\mathcal{D}(\mathrm{Re}A)$ for $|x|<1/\lambda$. Since by construction (Corollary 1) operator $A(x)$ is a self-adjoint extension of this sum, we have $\mathcal{D}(A(x))=\mathcal{D}(\mathrm{Re}A)$. By (1) for any u in this domain we obtain:

$$\|\mathrm{Re}Au\|\le\|A(x)u\|+|x|\|\mathrm{Im}Au\|\le\|A(x)u\|+\lambda|x|\|\mathrm{Re}Au\|. \tag{17}$$

Since $|x| < 1/\lambda$, one gets:

$$\|\mathrm{Re}Au\| \leq \frac{1}{1-\lambda|x|}\|A(x)u\|, \tag{18}$$

and:

$$\|(\mathrm{Re}A)^{\alpha}u\| \leq \frac{1}{(1-\lambda|x|)^{\alpha}}\|A(x)^{\alpha}u\|,\ u \in \mathcal{D}(A(x)^{\alpha}) \tag{19}$$

by the Heinz-Kato Theorem.

Finally the inequalities (16), (19), together with assumption (v) give the annonced estimate (14). $\square$

Since $c < 1$, there exists $0 < \eta < \min\{\epsilon, 1/\lambda\}$ such that for $|x| < \eta$,

$$\left(\frac{1+\mu|x|}{1-\lambda|x|}\right)^{\alpha} c < 1. \tag{20}$$

Thus the conditions of Proposition 2 are satisfied for families of self-adjoint operators $A(x)$ and $B(x)$ for any $x \in (-\eta, \eta)$. Since the interval $(-\eta, \eta)$ has a limit point in the domain $D = \{z \in \mathbb{C}, |\mathrm{Re}\ z| < \epsilon\}$, the theorem of Vitali accomplishes the proof of Theorem 2 for real $t > 0$ and $z = i$.

To extend the result to $t \in S_{\theta}$ we use the theorem of Vitali once again. Since $A = A(z = i)$ and $B = B(z = i)$ are sectorial of semi-angles θ_A, θ_B, we put $\theta = \pi/2 - \max\{\theta_A, \theta_B\}$. Then semigroups e^{-tA} and e^{-tB} are holomorphic and bounded by 1 on the open sector $S_{\theta} = \{z \in \mathbb{C}, z \neq 0, |\arg z| < \theta\}$. Thus by the theorem of Vitali the operator-norm convergence of the Trotter formula for $t > 0$ implies the convergence in the same topology for any $z \in S_{\theta}$, uniformly on compact subsets. Since the limit function is holomorphic, it is the analytic continuation of the semigroup $e^{-tH}P$ from the real semi-axis to the sector S_{θ}.

3 Conclusion

Since the first annonced result [15] about the operator-norm convergence of the Trotter product formula, it appeared number of papers considered self-adjoint contraction semigroups: first for Schrödinger semigroups [2], [3], [4], [6]; then on the abstract level [5], [10], [11], [12], [13], [14]. Recently it was proven [16] that the result of [12] is in fact optimal in the following sense: the error bound of [12] is the best possible for $\alpha \in (1/2, 1)$, and for fractional powers $\alpha \leq 1/2$, the operator-norm convergence does not hold without supplementary conditions (see Proposition 2 and discussion in [13]).

In this context, the present letter shows that self-adjointness is not an essential ingredient for the operator-norm convergence of the Trotter formula. It takes place for sectorial generators (insuring analyticity of such semigroups) under compactness or smallness conditions similar to [11], [12], and [13]. Notice that similar to [13] we establish the operator-norm convergence of the Trotter formula without error estimates known for the self-adjoint semigroups, see [11], [12], and [14]. Finally, smallness conditions and corresponding results of this letter are different from those of [1]. We consider here a certain class of sectorial generators on a Hilbert space (see (i)-(v)), instead of general holomorphic semigroups on a Banach space as in [1]. In [1] we proved the Trotter formula for a holomorphic contraction semigroup generated by A via an error bound estimate, but only for those B which have relative bound zero with respect to the operator A.

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