C^*-algebras related to the N-body problem and the self-adjoint operators affiliated to them

by

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Abstract

We point out a class of "many-channel hamiltonians" naturally associated to vector space structures and we study their spectral properties (essential spectrum and Mourre estimate). The N-body dispersive regular systems as well as perturbed multi-stratified media are covered by our formalism.

1 Introduction

Let X be a finite dimensional real vector space, $\mathcal{H}(X)$ the space of square integrable functions on it and X^* the space dual to X. Our purpose is to define and study several C^*-algebras of operators on $\mathcal{H}(X)$ naturally associated to the vector space structure of X and to develop the spectral theory of the self-adjoint operators affiliated to them. The most important of these algebras, denoted $\mathfrak{C}^X$ below, is suggested by the quantum mechanical N-body problem: one can define it as the smallest C^*-algebra to which are affiliated the non-relativistic N-body hamiltonians corresponding to a quadratic form on X^*, X being the configuration space (see [DaG] for a proof of the fact that the so defined algebra is independent of the quadratic form). The class of hamiltonians affiliated to $\mathfrak{C}^X$ is, however, much larger. Examples of physical interest are the N-body dispersive hamiltonians (arbitrary kinetic energies but interactions of the same type as in the non-relativistic case, cf. [De], [G]) as well as those corresponding to stratified or pluri-stratified media (where the higher order term of the free hamiltonian is x-dependent, see [DBP], [DI]).

Before going into the details of our results, let us recall some notations and introduce some terminology. If X is a locally compact space then $C_c(X)$, $C_0(X)$, $C_b(X)$ are the spaces of continuous functions on X which have compact support, tend to zero at infinity, or are bounded, respectively. If X is a vector space, then $C_{bu}(X)$ is the space of bounded uniformly continuous functions on X. A $*$-morphism between two $*$-algebras will be called morphism. If $\mathcal{H}$ is a Hilbert space, then $B(\mathcal{H})$ and $K(\mathcal{H})$ are the C^*-algebras of all bounded and compact operators on $\mathcal{H}$ respectively. If $\mathfrak{C}$ is a C^*-algebra of operators on $\mathcal{H}$, then a self-adjoint operator H on $\mathcal{H}$ is *affiliated* to $\mathfrak{C}$ if there is a complex number $z \notin \sigma(H)$ (the spectrum of H) such that $(H-z)^{-1} \in \mathfrak{C}$; then $\varphi(H) \in \mathfrak{C}$ for all $\varphi \in C_0(\mathbb{R})$.

Section 2 of this paper is devoted to the study of the notion of affiliation in an abstract setting. Our main results in this context are Theorem 2.8 and Theorem 2.11. Theorem 2.8, used in connection with Lemma 2.9, gives a quite powerful criterion for an operator to be affiliated to a C^*-algebra. See Theorem 4.9 for a concrete application, and also the examples that we shall discuss below. In the preliminary subsection 2.1 we discuss the class of self-adjoint operators affiliated to a C^*-algebra $\mathfrak{C}$ which are nondegenerate with respect to $\mathfrak{C}$ (such operators will be called "strictly affiliated" in what follows). This class is especially important due to the Cohen-Hewitt theorem, which plays an important role in the proof of Theorem 2.11, but also because it avoids the use of non-densely defined self-adjoint operators in the context of graded C^*-algebras (see Proposition 2.3).

In Subsection 2.3 we continue the study (began in [DaG]) of C^*-algebras graded by an infinite semilattice (the lattice of all vector subspaces of a vector space being the case of interest for us). The main point here is an "explicit" realization of the quotient of $\mathfrak{C}$ with respect to its natural minimal ideal. This allows us to prove (for infinite semilattices) a general version of the so-called HVZ theorem (see Theorem 2.10 and its consequences Theorems 4.2, 4.3) and later on will give us the main reduction step for the proof of the Mourre estimate (see Theorems 5.3 and 5.6). We stress the fact that in relations (4.2), (5.11) and (5.19) there is no need to take closures, the (infinite) unions being already closed. The final theorem of Section 2, Theorem 2.11, is our main affiliation criterion in the setting of graded C^*-algebras (see Theorem 4.6 for a concrete application).

In Section 3 we introduce and study several C^*-algebras of operators on $\mathcal{H}(X)$, X a real finite dimensional vector space, which are naturally associated to the vector space structure of X. A version of the algebras denoted here $\mathfrak{C}^X(Y)$, Y being a vector subspace of X, first appeared in [PSS] (see also [P]). This version was considered in detail in [ABG] under the assumption that an euclidean structure is given on X, the algebras being apparently dependent on this structure. In a previous paper [DaG] we have shown that $\mathfrak{C}^X(Y)$ are in fact crossed products and we gave a construction independent of any euclidean structure. Our treatment here is quite different, the main advantage being that it makes sense in a much more general setting (that of unitary representations of a locally compact group in a Hilbert space, see [GI]). On a more technical level, the definitions adopted here allow one to check rather easily the affiliation to the algebra for large classes of operators.

Subsection 3.1 contains an elementary description of the algebras $\mathfrak{C}^X(Y)$ and their properties. Strictly speaking, this subsection is not necessary for the later developments, because all these properties are easy consequences of Theorem 3.5, the main result of Section 3. Since the proof of this theorem is not so easy, we prefered however to begin with a simple treatment which suffices for the needs of the N-body problem.

We now define the algebras we shall study. Let $\{\mathrm{e}_x(P)\}_{x\in X}$, $\{\mathrm{e}_k(Q)\}_{k\in X^*}$ be the representations of X and X^* in $\mathcal{H}(X)$ defined by $(\mathrm{e}_x(P)f)(y) = f(x+y)$, $(\mathrm{e}_k(Q)f)(y) = \mathrm{e}^{i\langle y,k\rangle}f(y)$. The largest C^*-algebra considered in this paper is the set $\mathfrak{U}^X$ of $T \in B(\mathcal{H}(X))$ such that $\|[T, \mathrm{e}_k(Q)]\| \to 0$ if $k \to 0$ in X^* and $\left\|[\mathrm{e}_x(P) - 1]T^{(*)}\right\| \to 0$ if $x \to 0$ in X (we use the notation $T^{(*)}$ when a relation holds for T and T^*). Theorem 3.5 says that $\mathfrak{U}^X$ coincides with the norm closed subspace of $B(\mathcal{H}(X))$ generated by the operators of the form $\varphi(Q)\psi(P)$, with $\varphi \in C_{bu}(X)$, $\psi \in C_0(X^*)$. Thus we get a second description of $\mathfrak{U}^X$ as an algebra of pseudo-differential operators. This also proves that $\mathfrak{U}^X$ is the crossed product of $C_{bu}(X)$ by the action of the additive group X.

If Y is a subspace of X we define $\mathfrak{U}^X_Y$ as the set of $T \in \mathfrak{U}^X$ such that $[T, \mathrm{e}_x(P)] = 0$ for all $x \in Y$. Subsections 3.2 and 3.3 are devoted to the study of these C^*-algebras. The main fact is that $\mathfrak{U}^X_Y$ is the crossed product of the algebra $C_{bu}(X/Y)$ by the action of X as group of translations on X/Y, cf. Theorem 3.7. A choice of a supplementary subspace for Y in X determines an isomorphism $\mathfrak{U}^X_Y \simeq C_0(Y^*) \otimes \mathfrak{U}^{X/Y}$, but the isomorphism is not canonical. Of course, one could choose an euclidean structure on X and always take as supplementary space the orthogonal of Y in X. This is natural in the non-relativistic N-body problem and, more generally, in the case of perturbed pluristratified media (cf. Section 6). But this would only complicate matters in the case of dispersive N-body systems, where the freedom in the choice of a supplementary subspace is quite useful (see [G], [Da], and also our Theorem 5.6). In order to bypass such difficulties, we have introduced the notion of covariant symbol for operators in $\mathfrak{U}^X_Y$, which gives a canonical isomorphism of this algebra with the space of sections of class C_0 of a (trivialisable) fiber bundle of C^*-algebras over Y^*.

Finally, let $\mathfrak{C}^X(Y)$ be the set of $T \in \mathfrak{U}^X_Y$ such that $\|[\mathrm{e}_k(Q) - 1]T\| \to 0$ when $k \to 0$ in $Y^\perp$ (this is the orthogonal of Y in X^*). Then $\mathfrak{C}^X$ is the closure in $B(\mathcal{H}(X))$ of the linear space generated by all the $\mathfrak{C}^X(Y)$. As we said before, our main goal is the spectral analysis of the self-adjoint operators affiliated to $\mathfrak{C}^X$, that we consider as a natural generalization of the N-body hamiltonians. $\mathfrak{C}^X$ has a remarkable structure: it is graded by the lattice of all vector subspaces of X (the grassmannian of X). Moreover, the projection morphisms associated to the grading have a compactness property which plays an important rôle in the proof of the relations giving the essential spectrum and the threshold set for an operator affiliated to $\mathfrak{C}^X$. This compactness property is described in Proposition 5.4. Theorem 3.9 gives a description of $\mathfrak{C}^X(Y)$ in terms of pseudo-differential operators, and thus makes the connection with the crossed product formalism from [DaG].

Sections 4 and 5 are devoted to the spectral analysis of the self-adjoint operators strictly affiliated to $\mathfrak{C}^X$. Our point of view is to make no assumptions on the explicit form of H: the fact that it is affiliated to $\mathfrak{C}^X$ should be a sufficiently strong condition. The results concerning the esential spectrum are stated in two versions in the theorems 4.2 and 4.3. We have two results on the Mourre estimate in Section 5. The first one, Theorem 5.3, is an abstract fact which allows one to compute the threshold set of the total hamiltonian in terms of the critical sets of its sub-hamiltonians. In the case of the standard N-body problem such a result is traditionally proved with the help of partitions of unity in configuration space introduced by B. Simon. See [PSS], [FH] for the non-relativistic case, [G] for the dispersive case, and [DI] for the case of pluristratified media. For us the result is a straightforward consequence of the graded structure of the algebra $\mathfrak{C}^X$. A direct proof in the spirit of [FH] would probably be impossible even in the case $X = \mathbb{R}^2$ and $H = \Delta + \sum_Y V_Y(\pi(x))$, where Y runs over a set of lines whose union is dense in $\mathbb{R}^2$.

Our final theorem concerning the Mourre estimate generalizes the results of C. Gérard [G] to our setting. In Theorem 5.6 we give a complete description of the threshold set of H under two rather strong conditions on the covariant symbol of the sub-hamiltonians. These are the positivity conditions of Gérard and we doubt that one can improve them (unless H has a very special form) if the generator of the dilation group is chosen as conjugate operator (and this seems unavoidable if one works with the lattice of all subspaces; a slight improvement could be achieved by working with $D + \langle Q, k\rangle$ for some $k \in X^*,\ k \neq 0$). For explicit examples of kinetic energies verifing conditions (i), (ii) of Theorem 5.6 we refer to [G], [Da]. We treat another example, the case of perturbed pluristratified media, in Section 6.

We do not state explicitly in this paper the well-known consequences of the Mourre estimate (see, for example, chapters 7 and 9 in [ABG]). We only mention that for the validity of the limiting absorption principle a slightly stronger then the $C^1_u(D)$ regularity condition of the hamiltonian with respect to the conjugate operator D is needed, the optimal one on the Besov scale being $\mathcal{C}^{1,1}(D)$. This covers short range and long range interactions with optimal assumptions on their decay at infinity.

In subsections 4.2 and 4.3 we point out explicit classes of operators affiliated to the algebras $\mathfrak{C}^X$ and $\mathfrak{U}^X$. Our main purpose is to show the power of the affiliation criteria from Section 2 and the usefulness of the definitions of the algebras adopted in this paper. Indeed, as we said before, in our study of the spectral properties we do not make assumptions on the form of H. Let us give a very simple example in connection with Theorem 4.9. Let $X = \mathbb{R}^n$ and $H_0 = \Delta$ (the positive laplacian). Assume that V is the operator of multiplication by a real function such that $-\mu\Delta - \delta \leq V \leq \delta(\Delta + 1)$ for some positive numbers $\mu < 1$ and δ. Then the self-adjoint operator $H = \Delta + V$ is strictly affiliated to $\mathfrak{U}^X$. If one thinks of $\mathfrak{U}^X$ as the C^*-algebra generated by the operators of the form $\varphi(Q)\psi(P)$ with $\varphi \in C_{bu}(\mathbb{R}^n)$, $\psi \in C_0(\mathbb{R}^n)$, this is far from obvious.

However, the main feature of theorems 2.8 and 2.11 is that they allow one to treat singular perturbations of the same order as H_0. Our initial motivation for studying these matters came from a conversation that one of us (V.G.) had in May 1997 with Viorel Iftimie. He gave us a preliminary version of his common work [DI], [DI1], [DI2] with Yves Dermenjian on a class of hamiltonians which unifies the theory of non-relativistic N-body hamiltonians and that of stratified media (in the version of [DBP]). It seemed at that moment that the geometric methods they used (e.g. partitions of unity of Simon type) gave better results than the algebraic methods described in [ABG]. In fact, the strongest affiliation criterion to the N-body C^*-algebra we knew at that moment (Proposition 3.6 from [BGS]), when applied to an operator of the form $\nabla^* M \nabla$, required that the function M be a small Hölder continuous perturbation of 1. Instead, they were able to treat the case $M \in L^\infty$. Now our Theorem 2.11 covers this situation and goes further, since we allow non-local operators M and a more general behaviour at infinity. We treat this question in some detail in Section 6. For example, let X be euclidean and let us assume that for each Y a real function $m^Y \in L^\infty(Y)$ is given such that the measure of the set where $|m^Y(y)| \geq \varepsilon$ is finite for each $\varepsilon > 0$. For simplicity, assume also $m^Y \neq 0$ only for a finite number of subspaces Y and that $M(x) = 1 + \sum_Y m^Y(\pi_Y(x))$ is bounded from below by a strictly positive number. Then the self-adjoint operator $\nabla^* M \nabla$ is affiliated to $\mathfrak{C}^X$ (use Proposition 4.8).

We are grateful to Viorel Iftimie for giving us a preliminary version of the preprints [DI], [DI1], [DI2] and for a very useful discussion.

2 Self-Adjoint Operators Affiliated to C^*-Algebras

2.1. Let $\mathcal{H}$ be a Hilbert space, $\mathfrak{C}$ a C^*-algebra of operators on $\mathcal{H}$, and H a self-adjoint operator on $\mathcal{H}$ affiliated to $\mathfrak{C}$. We say that H is *nondegenerate with respect to* $\mathfrak{C}$, or $\mathfrak{C}$-nondegenerate, if the operators of the form $\varphi(H)T$, with $\varphi \in C_0(\mathbb{R})$ and $T \in \mathfrak{C}$, generate a (norm) dense linear subspace in $\mathfrak{C}$. This property holds if and only if the set of $T \in \mathfrak{C}$ such that TH extends to an operator in $\mathfrak{C}$ is a dense subset of $\mathfrak{C}$. An operator affiliated to $\mathfrak{C}$ and $\mathfrak{C}$-nondegenerate will also be called *strictly affiliated to* $\mathfrak{C}$.

Lemma 2.1 *Assume that H is affiliated to $\mathfrak{C}$.*
(a) *If there is $\theta \in C_0(\mathbb{R})$ with $\theta(0) = 1$ such that $\lim_{\varepsilon\to 0} \|\theta(\varepsilon H)T - T\| = 0$ for each $T \in \mathfrak{C}$ then H is $\mathfrak{C}$-nondegenerate.*
(b) *If H is $\mathfrak{C}$-nondegenerate then for all $\varphi \in C_b(\mathbb{R})$ and $T \in \mathfrak{C}$ the operators $\varphi(H)T$ and $T\varphi(H)$ belong to $\mathfrak{C}$. If, moreover, $\varphi(0) = 1$, then $\|\varphi(\varepsilon H)T - T\| \to 0$ and $\|T\varphi(\varepsilon H) - T\| \to 0$ when $\varepsilon \to 0$.*

Proof. Part (a) and the first assertion of part (b) are obvious. To finish the proof note that if $\psi \in C_0(\mathbb{R})$ then one has $\|(\varphi(\varepsilon H) - 1)\psi(H)\| \leq \sup_{x\in\mathbb{R}} |\,(\varphi(\varepsilon x) - 1)\psi(x)\,| \to 0$ when $\varepsilon \to 0$. ■

A convenient choice in (a) is $\theta(\varepsilon H) = (1 + i\varepsilon H)^{-1}$. The next assertion is rather deep: it follows from the Cohen-Hewitt theorem (see V.9.2 in [FD]).

Theorem 2.2 *If an operator H affiliated to $\mathfrak{C}$ is $\mathfrak{C}$-nondegenerate then for each $S \in \mathfrak{C}$ there are $\varphi, \psi \in C_0(\mathbb{R})$ and $T \in \mathfrak{C}$ such that $S = \varphi(H)T\psi(H)$.*

We recall that the C*-algebra $\mathfrak{C}$ is called nondegenerate (on $\mathcal{H}$) if the linear subspace of $\mathcal{H}$ generated by the elements of the form Tf, with $T \in \mathfrak{C}$ and $f \in \mathcal{H}$, is dense in $\mathcal{H}$. Then, by the same theorem of Cohen and Hewitt, for each $g \in \mathcal{H}$ there are $T \in \mathfrak{C}$ and $f \in \mathcal{H}$ such that $g = Tf$.

Note that if there is a self-adjoint operator H in $\mathcal{H}$ which is affiliated to $\mathfrak{C}$ then $\mathfrak{C}$ is nondegenerate on $\mathcal{H}$. Indeed, if $\theta \in C_0(\mathbb{R})$ and $\theta(0) = 1$ then for each $\varepsilon \neq 0$ one has $\theta(\varepsilon H) \in \mathfrak{C}$ and $\theta(\varepsilon H)f \to f$ as $\varepsilon \to 0$ for each $f \in \mathcal{H}$.

Assume that $\mathfrak{A}$, $\mathfrak{C}$ are C*-algebras and that a continuous bilinear map $\mathfrak{A} \times \mathfrak{C} \ni (A, T) \longmapsto AT \in \mathfrak{C}$ has been given such that $(AB)T = A(BT)$ for all $A, B \in \mathfrak{A}$, $T \in \mathfrak{C}$ (i.e. $\mathfrak{C}$ is a left Banach $\mathfrak{A}$-module). For example, $\mathfrak{A}$ could be a subalgebra of $\mathfrak{C}$. Denote $\mathfrak{A} \cdot \mathfrak{C}$ the linear subspace generated by the elements AT, with $A \in \mathfrak{A}$ and $T \in \mathfrak{C}$. Say that $\mathfrak{A}$ *acts nondegenerately on* $\mathfrak{C}$ if $\mathfrak{A} \cdot \mathfrak{C}$ is dense in $\mathfrak{C}$. Then, again by the Cohen-Hewitt theorem, for each $S \in \mathfrak{C}$ there are $A \in \mathfrak{A}, T \in \mathfrak{C}$ such that $S = AT$.

Proposition 2.3 *Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ be nondegenerate C^*-algebras of bounded operators on the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ respectively and let $\mathcal{P} : \mathfrak{C}_1 \to \mathfrak{C}_2$ be a morphism such that $\mathcal{P}(\mathfrak{C}_1)$ acts nondegenerately on $\mathfrak{C}_2$ (e.g. assume $\mathcal{P}$*

is surjective). Then for each self-adjoint operator H_1 on $\mathcal{H}_1$ strictly affiliated to $\mathfrak{C}_1$ there is a unique self-adjoint operator H_2 on $\mathcal{H}_2$ such that $\mathcal{P}[\varphi(H_1)] = \varphi(H_2)$ for each $\varphi \in C_0(\mathbb{R})$. H_2 is strictly affiliated to $\mathfrak{C}_2$.

Proof. For $z \in \mathbb{C} \setminus \mathbb{R}$ set $R_2(z) = \mathcal{P}[(H_1 - z)^{-1}]$. Then $\{R_2(z)\}$ is a self-adjoint resolvent on $\mathcal{H}_2$ and we shall prove below that $\|niR_2(-ni)f - f\| \to 0$ as $n \to \infty$ for each $f \in \mathcal{H}_2$. This implies that there is a unique self-adjoint (densely defined) operator H_2 in $\mathcal{H}_2$ such that $R_2(z) = (H_2 - z)^{-1}$ for all $z \notin \mathbb{R}$. Then $\mathcal{P}[\varphi(H_1)] = \varphi(H_2)$ for all $\varphi \in C_0(\mathbb{R})$ by the Stone-Weierstrass theorem.

Set $\theta(x) = i(x+i)^{-1}$ and $\varepsilon = 1/n$, then $\theta \in C_0(\mathbb{R})$, $\theta(0) = 1$ and $\theta(\varepsilon H_1) = ni(H_1 + ni)^{-1}$. If $\varphi \in C_0(\mathbb{R})$ then $(\theta(\varepsilon x) - 1)\varphi(x) \to 0$ as $\varepsilon \to 0$ uniformly in x on $\mathbb{R}$, hence $\|(\theta(\varepsilon H_1) - 1)\varphi(H_1)\| \to 0$. Since H_1 is $\mathfrak{C}_1$-nondegenerate we get $\|\theta(\varepsilon H_1)T - T\| \to 0$ for each $T \in \mathfrak{C}_1$. $\mathcal{P}(\mathfrak{C}_1)$ acts nondegenerately on $\mathfrak{C}_2$, so we obtain $\|niR_2(-ni)S - S\| \to 0$ for each $S \in \mathfrak{C}_2$. In particular $niR_2(-ni)Sg \to Sg$ in $\mathcal{H}_2$ for $S \in \mathfrak{C}_2$ and $g \in \mathcal{H}_2$. But $\|niR_2(-ni)\| \leq 1$ and $\mathfrak{C}_2$ is nondegenerate on $\mathcal{H}_2$, hence $niR_2(-ni)f \to f$ for each $f \in \mathcal{H}_2$. ■

Remark 2.4 If $\mathfrak{C}$ is an abstract C*-algebra then by *observable affiliated to* $\mathfrak{C}$ we mean a morphism $H : C_0(\mathbb{R}) \to \mathfrak{C}$ and we use the notation $\varphi(H) = H(\varphi)$. The notion of *nondegenerate observable* has an obvious meaning and Theorem 2.2 remains valid. A nondegenerate observable affiliated to $\mathfrak{C}$ is realized as a (densely defined) self-adjoint operator in each nondegenerate representation of $\mathfrak{C}$. It is useful to go further and to define an *X-valued observable affiliated to* $\mathfrak{C}$ for any locally compact space X: this is simply a morphism $C_0(X) \to \mathfrak{C}$. Theorem 2.2 remains valid if $\mathbb{R}$ is replaced by X. Observe that a X-valued observable strictly affiliated to $\mathfrak{C}$ naturally induces a $C_b(X)$-bimodule structure on $\mathfrak{C}$ (cf. Lemma 2.1(b)).

2.2. Our purpose now is to give a general perturbative criterion for an operator $H = H_0 + V$ to be affiliated to a C*-algebra $\mathfrak{C}$ of bounded operators on a Hilbert space $\mathcal{H}$. Here H_0 is a self-adjoint operator in $\mathcal{H}$ affiliated to $\mathfrak{C}$ and V is a form bounded perturbation of H_0. The results in [BGS] and [ABG] are not sufficiently strong in some applications: they do not cover perturbations V of the same order as H_0 having a large positive part (as happens in the case of stratified media).

In the rest of this section H_0 will be a self-adjoint operator in a Hilbert space $\mathcal{H}$ (with norm denoted $\|\cdot\|$) affiliated to a C*-algebra $\mathfrak{C} \subset B(\mathcal{H})$. We denote by $\mathcal{G}$ the form domain of H_0, so $\mathcal{G} = D(|H_0|^{1/2})$ equipped with the graph topology, and we embed $\mathcal{G} \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{G}^*$ in the usual way ($\mathcal{G}^*$ is the space of antilinear continuous functionals). Then for each $z \in \mathbb{C} \setminus \sigma(H_0)$ the operator $H_0 - z$ extends to an isomorphism of $\mathcal{G}$ onto $\mathcal{G}^*$ whose inverse is a continuous extension of $(H_0 - z)^{-1}$ to an operator $\mathcal{G}^* \to \mathcal{G}$. We shall keep the notations $H_0 - z$ and $(H_0 - z)^{-1}$ for these extensions. We recall that the continuous (symmetric) sesquilinear forms on $\mathcal{G}$ are identified with the continuous (symmetric) linear operators $\mathcal{G} \to \mathcal{G}^*$.

Definition 2.5 *We say that V is a standard form perturbation of H_0 if V is a continuous symmetric sesquilinear form on $\mathcal{G}$ and there are numbers $\mu \in [0,1)$ and $\delta \in \mathbb{R}$ such that one of the following conditions is satisfied:*
(i) $\pm V \leq \mu \,|\, H_0 \,| + \delta$ *as forms on $\mathcal{G}$;*
(ii) H_0 *is bounded from below and* $V \geq -\mu H_0 - \delta$ *as forms on $\mathcal{G}$.*

In this paper we are especially interested in the case (ii), when V may have a large positive part. By using the fact that the class of self-adjoint operators affiliated to $\mathfrak{C}$ is stable under norm resolvent limits, one can treat some perturbations of the form $V + W$ where W is a closed positive form.

If V is a standard form perturbation of H_0 then the operator $H = H_0 + V \in B(\mathcal{G}, \mathcal{G}^*)$ induces a self-adjoint operator in $\mathcal{H}$ which we shall denote also by H; this is the so-called form sum of H_0 and V. More precisely, the restriction of H to $D(H) = \{f \in \mathcal{G} \,|\, Hf \in \mathcal{H}\}$ is a self-adjoint operator in $\mathcal{H}$. This known fact is also a consequence of the proof of the Lemma 2.7. Indeed, we show there that there is a complex number z such that the maps $H - z$ and $H - \overline{z}$ are isomorphisms of $\mathcal{G}$ onto $\mathcal{G}^*$, from which the self-adjointness of H on $D(H)$ follows easily. Note also that in case (ii) the operator H is bounded from below.

For the proof of Theorem 2.8, the main result of this section, we need the next two lemmas.

Lemma 2.6 *Let I be a real open interval and let $\{H_\nu\}_{\nu \in I}$ be a family of self-adjoint operators on $\mathcal{H}$ which depend analytically on ν, i.e. the map $\nu \longmapsto (H_\nu - z)^{-1} \in B(\mathcal{H})$ is analytic on I for some (hence for all) $z \in \mathbb{C} \setminus \mathbb{R}$. If there is a not empty open interval $J \subset I$ such that H_ν is affiliated to $\mathfrak{C}$ when $\nu \in J$ then H_ν is affiliated to $\mathfrak{C}$ for all $\nu \in I$.*

Proof. Let $F(\nu) = (H_\nu - z)^{-1}$, then $F : I \longmapsto B(\mathcal{H})$ is analytic and $F(\nu)$ belongs to the closed subspace $\mathfrak{C}$ if $\nu \in J = (a, b)$. Assume $b \in I$, then all the derivatives $F^{(k)}(b)$ belong to $\mathfrak{C}$. Since $F(\nu)$ for ν near b is the sum of its Taylor expansion at $\nu = b$ we see that $F(\nu) \in \mathfrak{C}$ if $b \leq \nu \leq b + \varepsilon$ for some $\varepsilon > 0$. Repeating the argument we get $F(\nu) \in \mathfrak{C}$ for all $\nu \in I$, $\nu > b$. ■

Lemma 2.7 *Let V be a standard form perturbation of H_0 and assume that there is $z \in \mathbb{C} \setminus \sigma(H_0)$ such that all the products of the form*

$$(H_0 - z)^{-1}\, V\, (H_0 - z)^{-1} \,\cdots\, V\, (H_0 - z)^{-1} \tag{2.1}$$

belong to $\mathfrak{C}$. Then the self-adjoint operator $H = H_0 + V$ is affiliated to $\mathfrak{C}$.

Remark: Assume that for each $k \geq 1$ integer there are numbers $z_0, z_1, \ldots, z_k$ outside $\sigma(H_0)$ such that

$$(H_0 - z_0)^{-1}\; V\; (H_0 - z_1)^{-1}\; \cdots\; V\; (H_0 - z_k)^{-1} \in \mathfrak{C}.$$

Then this relation holds for all choices of $z_0, \cdots, z_k$. This follows by induction on k by using the resolvent identity.

Proof. Assume first that we are in the case (i) and denote $\Lambda = ({H_0}^2 + \lambda^2)^{-1/4}$ where λ is a real number $\neq 0$. Then Λ is an isomorphism of $\mathcal{H}$ onto $\mathcal{G}$ and of $\mathcal{G}^*$ onto $\mathcal{H}$, hence $\Lambda V \Lambda$ is a bounded operator on $\mathcal{H}$, and from the estimate $\pm V \leq \mu\,|\,H_0\,| + \delta$ we get

$$\pm\, \Lambda\; V\; \Lambda \;\leq\; \frac{\mu\,|\,H_0\,| + \delta}{({H_0}^2 + \lambda^2)^{1/2}} \;\leq\; \mu \;+\; \frac{\delta}{|\,\lambda\,|} \;<1$$

if $|\,\lambda\,| > \delta(1-\mu)^{-1}$. Fix such a λ and observe that $\|\Lambda V \Lambda\| < 1$. Let S be the unitary operator $S = (H_0 + i\lambda)\Lambda^2$. Then we have

$$H \;+\; i\lambda \;=\; (H_0 + i\lambda) \;+\; V \;=\; \Lambda^{-1}[S \;+\; \Lambda\; V\; \Lambda]\; \Lambda^{-1}.$$

But Λ^{-1} is an isomorphism of $\mathcal{G}$ onto $\mathcal{H}$ and of $\mathcal{H}$ onto $\mathcal{G}^*$ and $\|\Lambda V \Lambda\| < 1$, so $S + \Lambda V \Lambda$ is invertible on $\mathcal{H}$. Hence $H + i\lambda$ is an isomorphism of $\mathcal{G}$ onto $\mathcal{G}^*$ with

$$\begin{aligned}(H + i\lambda)^{-1} \quad &= \quad \Lambda \sum_{k=0}^{\infty} (-S^{-1}\Lambda V \Lambda)^k\; S^{-1}\Lambda \\ &= \quad \sum_{k=0}^{\infty} (H_0 + i\lambda)^{-1}(-V)(H_0 + i\lambda)^{-1} \cdots (-V)(H_0 + i\lambda)^{-1}.\end{aligned}$$

This is a norm convergent series of elements of $\mathfrak{C}$, so $(H + i\lambda)^{-1} \in \mathfrak{C}$, i.e. H is affiliated to $\mathfrak{C}$.

Assume now that we are in case (ii) and let λ be a positive number such that $\lambda > -\inf H_0$. We take $\Lambda = (H_0 + \lambda)^{-1/2}$ and observe that the estimate $V \geq -\mu H_0 - \delta$ implies

$$U := \;\Lambda\; V\; \Lambda \;\geq -\; \frac{\mu H_0 + \delta}{H_0 + \lambda} \;=\; -\;\mu \;+\; \frac{\mu\lambda - \delta}{H_0 + \lambda} \;\geq\; -\;\mu \;-\; \frac{\delta}{\lambda + \inf H_0}.$$

Hence we can choose $\mu_0 \in (\mu, 1)$ and λ such that the bounded symmetric operator U in $\mathcal{H}$ has the property $U \geq -\mu_0$. Let $\varepsilon > 0$ such that $(1+\varepsilon)\mu_0 = \mu_1 < 1$, then for $0 \leq \nu \leq 1+\varepsilon$ we shall have $1 + \nu U \geq 1 - \mu_1 > 0$, hence $1 + \nu U$ is invertible in $B(\mathcal{H})$ with a bound $\left\|(1 + \nu U)^{-1}\right\| \leq (1 - \mu_1)^{-1}$ independent of ν. Let us denote $H_\nu = H_0 + \nu V$, so $H_1 = H$, then we have

$$H_\nu \;+\; \lambda \;=\; (H_0 + \lambda) \;+\; \nu V \;\;=\; \Lambda^{-1}\;[1 \;+\; \nu U]\; \Lambda^{-1}.$$

So $H_\nu+\lambda$ is an isomorphism of $\mathcal{G}$ onto $\mathcal{G}^*$ and $\left\|(H_\nu+\lambda)^{-1}\right\|_{\mathcal{G}^*\to\mathcal{G}}$ is bounded by a constant independent of ν. Moreover, the map $\nu \longmapsto (H_\nu+\lambda)^{-1} \in B(\mathcal{G}^*,\mathcal{G})$ is analytic on the interval $(0,1+\varepsilon)$ (because so is $\nu \to (1+\nu U)^{-1} \in B(\mathcal{H})$). If $0<\nu<\|U\|^{-1}$ then

$$(H_\nu+\lambda)^{-1} = \Lambda\sum_{k=0}^{\infty}(-\nu U)^k\Lambda = \sum_{k=0}^{\infty}(-\nu)^k(H_0+\lambda)^{-1}V(H_0+\lambda)^{-1}\cdots V(H_0+\lambda)^{-1}$$

belongs to $\mathfrak{C}$. From Lemma 2.6 we get that the self-adjoint operator H_ν is affiliated to $\mathfrak{C}$ for all $\nu\in(0,1+\varepsilon)$. ■

If $\mathfrak{C}$ is a C^*-subalgebra of $B(\mathcal{H})$ we set

$$\mathcal{M}(\mathfrak{C}) = \{S\in B(\mathcal{H})\,|\,T\in\mathfrak{C}\Rightarrow ST,TS\in\mathfrak{C}\}. \tag{2.2}$$

This is clearly a C^*-algebra. It can be identified with the usual multiplier algebra of $\mathfrak{C}$ if and only if $\mathfrak{C}$ is nondegenerate on $\mathcal{H}$. This is the case for the algebra considered here because the self-adjoint operator H_0 is affiliated to it. Note that if H_0 is $\mathfrak{C}$-nondegenerate then $\varphi(H_0)\in\mathcal{M}(\mathfrak{C})$ for all $\varphi\in C_b(\mathbb{R})$ (see Lemma 2.1). We also mention that $\mathcal{M}(\mathfrak{C})$ is considerably larger than $\mathfrak{C}$ in general. For example, if $\mathfrak{C}=C_0(\mathbb{R}^n;K(\mathcal{H}_0))$ for some Hilbert space $\mathcal{H}_0$ and if $\mathcal{H}=L^2(\mathbb{R}^n;\mathcal{H}_0)$, then $\mathcal{M}(\mathfrak{C})$ is the algebra of bounded strongly continuous functions $\mathbb{R}^n\to B(\mathcal{H}_0)$ (this is relevant for us in view of (3.18)).

If H_0 is affiliated to $\mathfrak{C}$ and $V\in\mathcal{M}(\mathfrak{C})$ is symmetric then it is easy to show that the self-adjoint operator $H=H_0+V$ is affiliated to $\mathfrak{C}$. More generally:

Theorem 2.8 *Let $\mathfrak{C}$ be a C^*-algebra of bounded operators on a Hilbert space $\mathcal{H}$ and H_0 a self-adjoint operator affiliated to $\mathfrak{C}$. Let V be a standard form perturbation of H_0 and $H=H_0+V$. If $U\equiv(|H_0|+1)^{-1/2}V(|H_0|+1)^{-1/2}$ belongs to $\mathcal{M}(\mathfrak{C})$ then H is affiliated to $\mathfrak{C}$. If H_0 is $\mathfrak{C}$-nondegenerate then H is $\mathfrak{C}$-nondegenerate too.*

Proof. We check the conditions of Lemma 2.7. Let $\Lambda=(|H_0|+1)^{-1/2}$ and $S=(|H_0|+1)(H_0-z)^{-1}$, so that $(H_0-z)^{-1}=S\Lambda^2$. Then

$$(H_0-z)^{-1}\,V\,(H_0-z)^{-1}\,\cdots\,V\,(H_0-z)^{-1} = S\,\Lambda\,U\,S\,U\,S\,\cdots\,U\,S\,\Lambda.$$

The operators U and S belong to $\mathcal{M}(\mathfrak{C})$ and $\mathcal{M}(\mathfrak{C})$ is an algebra. Moreover, $\Lambda\in\mathfrak{C}$, so $(H_0-z)^{-1}\,V\,(H_0-z)^{-1}\,\cdots\,V\,(H_0-z)^{-1}\in\mathfrak{C}$.

Now assume that H_0 is $\mathfrak{C}$-nondegenerate. In order to prove that H is $\mathfrak{C}$-nondegenerate it suffices to show that $\lim_{\varepsilon\to0}\|\theta(\varepsilon H)R_0-R_0\|=0$ when $\theta(\varepsilon H)=(1+i\varepsilon H)^{-1}$ and $R_0=(H_0+i)^{-1}$ (by Lemma 2.1(a) and the fact that H_0 is $\mathfrak{C}$-nondegenerate). We have $\|\theta(\varepsilon H_0)R_0-R_0\|\to0$, so it is enough to prove that $\|[\theta(\varepsilon H)-\theta(\varepsilon H_0)]R_0\|\to0$. The identity $\theta(\varepsilon H)-\theta(\varepsilon H_0)=$

$-i\varepsilon\theta(\varepsilon H)V\theta(\varepsilon H_0)$ holds in $B(\mathcal{G}^*, \mathcal{G})$ and implies:

$$\begin{aligned} \|[\theta(\varepsilon H) - \theta(\varepsilon H_0)]R_0\| &\leq \varepsilon \left\| \frac{(1+|\,H_0\,|)^{1/2}}{(1+i\varepsilon H)} \right\| \cdot \|U\| \cdot \left\| \frac{(1+|\,H_0\,|)^{1/2}}{(H_0+i)(1+i\varepsilon H_0)} \right\| \\ &\leq c\,\varepsilon \left\| (1+i\varepsilon H)^{-1}(1+|\,H_0\,|)^{1/2} \right\| \end{aligned}$$

for some finite constant c. If we are in situation (ii) of Definition 2.5 then the operators H_0 and H have the same form domain, hence the operator $(1+|\,H\,|)^{-1/2}(1+|\,H_0\,|)^{1/2}$ is bounded. This clearly implies that the last term above is dominated by $\sqrt{\varepsilon}$, hence tends to zero with ε. In situation (i) of Definition 2.5 the form domains of H_0 and H could be different, but we can argue as follows.

Let $\Lambda_\varepsilon = (1+\varepsilon^2 {H_0}^2)^{-1/4}$ and $S_\varepsilon = (1+i\varepsilon H_0)\Lambda_\varepsilon^2$, so that S_ε is a unitary operator in $\mathcal{H}$. Then

$$1 + i\varepsilon H = 1 + i\varepsilon H_0 + i\varepsilon V = \Lambda_\varepsilon^{-1}[S_\varepsilon + i\varepsilon\Lambda_\varepsilon V \Lambda_\varepsilon]\Lambda_\varepsilon^{-1}.$$

From $\pm\,\varepsilon V \leq \mu\,|\,\varepsilon H_0\,| + \varepsilon\delta$ we get $\pm\,\varepsilon\Lambda_\varepsilon V\Lambda_\varepsilon \leq \mu + \varepsilon\delta \leq \mu_0 < 1$ if ε is small enough, hence $\|\varepsilon\Lambda_\varepsilon V\Lambda_\varepsilon\| \leq \mu_0$. Then by using

$$(1 + i\varepsilon H)^{-1} = \Lambda_\varepsilon[S_\varepsilon + i\varepsilon\Lambda_\varepsilon V\Lambda_\varepsilon]^{-1}\Lambda_\varepsilon$$

we see that there is a finite number c such that for ε small:

$$\left\|(1 + i\varepsilon H)^{-1}(1 + |\,H_0\,|)^{1/2}\right\| \leq c\left\|\Lambda_\varepsilon(1 + |\,H_0\,|)^{1/2}\right\|$$

which is dominated by $\varepsilon^{-1/2}$. ∎

If H_0 is bounded from below one can obviously replace $(|\,H_0\,|+1)^{-1/2}$ in the definition of U from Theorem 2.8 by $(H_0+\lambda)^{-1/2}$ if $H_0+\lambda \geq c > 0$. More generally, the following result is useful in applications for checking the hypotheses of the theorem.

Lemma 2.9 *Assume that H_0 is a self-adjoint operator strictly affiliated to a C^*-algebra $\mathfrak{C}$.*
(a) A symmetric operator $U \in B(\mathcal{H})$ belongs to $\mathcal{M}(\mathfrak{C})$ if and only if one has $\varphi(H_0)U \in \mathfrak{C}$ $\forall\varphi \in C_c(\mathbb{R})$ and also if and only if $\exists\varphi \in C_0(\mathbb{R})$ with $\varphi(x) \neq 0$ $\forall x \in \mathbb{R}$ such that $\varphi(H_0)U \in \mathfrak{C}$.
(b) Let V be a symmetric continuous sesquilinear form on $D(|\,H_0\,|^{1/2})$ and set $U = (|\,H_0\,|+1)^{-1/2}V(|\,H_0\,|+1)^{-1/2}$. Then $U \in \mathcal{M}(\mathfrak{C})$ if and only if there is a continous function $\theta : \mathbb{R} \to \mathbb{C}$ with $c_1(|\,x\,|+1)^{-1/2} \leq |\,\theta(x)\,| \leq c_2(|\,x\,|+1)^{-1/2}$ for some constants $0 < c_1 < c_2$ and all $x \in \mathbb{R}$ such that $\varphi(H_0)V\theta(H_0) \in \mathfrak{C}$ for all $\varphi \in C_c(\mathbb{R})$.

Proof. The first assertion follows from the fact that the elements of the form $T\varphi(H_0)$, with $T \in \mathfrak{C}$ and $\varphi \in C_c(\mathbb{R})$, generate a dense subspace of $\mathfrak{C}$.

For the second one observe that there is $\eta \in C_b(\mathbb{R})$ such that $(|x|+1)^{-1/2} = \theta(x)\eta(x)$. Hence from $\varphi(H_0)V\theta(H_0) \in \mathfrak{C}$ we get $\varphi(H_0)V(|H_0|+1)^{-1/2} \in \mathfrak{C}$ by using (b) of Lemma 2.1. Replacing $\varphi(x)$ by $\varphi(x)(|x|+1)^{-1/2}$ we get then $\varphi(H_0)U \in \mathfrak{C}$ for all $\varphi \in C_c(\mathbb{R})$, so $U \in \mathcal{M}(\mathfrak{C})$ by the first assertion of the lemma. The reciprocal assertion is proved similarly. ∎

2.3. We consider here a class of $\mathcal{L}$-graded C*-algebras discussed in [ABG] when $\mathcal{L}$ is finite and in [DaG] in general. Note that our conventions differ from those in [ABG], the roles of the lower and upper bounds in the definition of the grading being interchanged.

If $\mathfrak{A}, \mathfrak{B}$ are subalgebras of an algebra $\mathfrak{C}$ then we denote by $\mathfrak{A}\cdot\mathfrak{B}$ the linear subspace of $\mathfrak{C}$ generated by the elements of the form AB with $A \in \mathfrak{A}, B \in \mathfrak{B}$. A family $\{\mathfrak{C}_i\}_{i\in I}$ of subalgebras of $\mathfrak{C}$ is *linearly independent* if for each family $\{S_i\}_{i\in I}$ such that $S_i \in \mathfrak{C}_i \ \forall i, S_i \neq 0$ for at most a finite number of i and $\sum_{i\in I} S_i = 0$, one has $S_i = 0$ for all $i \in I$.

Let $\mathcal{L}$ be a *semilattice*, i.e. a partially ordered set in which each pair of elements a, b has a lower bound $a \wedge b$. For $a \in \mathcal{L}$ we set $\mathcal{L}_a = \{b \in \mathcal{L} | b \geq a\}$; this is also a semilattice.

We say that a C^*-algebra $\mathfrak{C}$ is $\mathcal{L}$*-graded* if a linearly independent family $\{\mathfrak{C}(a)\}_{a\in\mathcal{L}}$ of C^*-subalgebras of $\mathfrak{C}$ has been given such that:

(i) $\mathfrak{C}(a)\cdot\mathfrak{C}(b) \subset \mathfrak{C}(a \wedge b)$ for all $a, b \in \mathcal{L}$;

(ii) if $\mathcal{E} \subset \mathcal{L}$ is finite then $\sum_{a\in\mathcal{E}} \mathfrak{C}(a)$ is a closed subspace of $\mathfrak{C}$;

(iii) $\sum_{a\in\mathcal{L}} \mathfrak{C}(a)$ is dense in $\mathfrak{C}$.

For each $\mathcal{E} \subset \mathcal{L}$ we set $\mathfrak{C}(\mathcal{E})^\circ = \sum_{a\in\mathcal{E}} \mathfrak{C}(a)$ and we define $\mathfrak{C}(\mathcal{E})$ as the closure of $\mathfrak{C}(\mathcal{E})^\circ$. Observe that

$$\mathfrak{C} = \bigcup \mathfrak{C}(\mathcal{E}) \tag{2.3}$$

where $\mathcal{E}$ runs over the set of countable subsets of $\mathcal{L}$ which are $\wedge$-stable, so that each $\mathfrak{C}(\mathcal{E})$ is an $\mathcal{E}$-graded C^*-subalgebra of $\mathfrak{C}$. Indeed, if $T \in \mathfrak{C}$ then T is equal to the limit of a sequence of elements $T_n \in \mathfrak{C}(\mathcal{E}_n)$ where $\mathcal{E}_n$ are finite sets which can be assumed $\wedge$-stable and such that $\mathcal{E}_n \subset \mathcal{E}_{n+1}$. Then it suffices to take $\mathcal{E} = \bigcup \mathcal{E}_n$.

If $a \in \mathcal{L}$ we set $\mathfrak{C}_a = \mathfrak{C}(\mathcal{L}_a)$. It is clear that $\mathfrak{C}_a$ is a $\mathcal{L}_a$-graded C^*-subalgebra of $\mathfrak{C}$. There is a natural map $\mathcal{P}_a^\circ : \mathfrak{C}(\mathcal{L})^\circ \to \mathfrak{C}(\mathcal{L}_a)^\circ$ defined by $\mathcal{P}_a^\circ \sum_{b\in\mathcal{L}} T(b) = \sum_{b\in\mathcal{L}_a} T(b)$ if $T(b) \in \mathfrak{C}(b)$ and $T(b) \neq 0$ only for a finite numbers of b. This map is clearly a surjective morphism of $*$-algebras. It is shown in [DaG] that this map is continuous, so it extends to a surjective morphism $\mathcal{P}_a : \mathfrak{C} \to \mathfrak{C}_a$. Moreover, $\mathcal{P}_a$ is a projection (in the sense of linear spaces) and its kernel is $\mathfrak{C}(\mathcal{L}'_a)$, where $\mathcal{L}'_a = \{b \in \mathcal{L} | b \not\geq a\}$.

We need a second abstract result proved in [DaG] for infinite $\mathcal{L}$. Assume that $\mathcal{L}$ has a least element $\min\mathcal{L}$, denote $\mathcal{M}$ the set of atoms of $\mathcal{L}$ (i.e. minimal elements of $\mathcal{L} \setminus \{\min\mathcal{L}\}$) and assume that $\mathcal{L}$ is atomic (i.e. each

$a \neq \min \mathcal{L}$ is minorated by an atom). Observe that $\mathfrak{C}(\min \mathcal{L})$ is a closed self-adjoint ideal in $\mathfrak{C}$, so the "abstract" C^*-algebra $\mathfrak{C}/\mathfrak{C}(\min \mathcal{L})$ is well defined. The result we need says that one can "explicitly" realize this algebra as follows. There is a natural morphism

$$\mathfrak{C} \ni T \longmapsto (\mathcal{P}_a[T])_{a\in\mathcal{M}} \in \bigoplus_{a\in\mathcal{M}} \mathfrak{C}_a, \tag{2.4}$$

where the direct sum is in the C^*-algebra sense, and the kernel of this morphism is equal to $\mathfrak{C}(\min \mathcal{L})$. This gives us a canonical embedding

$$\mathfrak{C}/\mathfrak{C}(\min \mathcal{L}) \subset \bigoplus_{a\in\mathcal{M}} \mathfrak{C}_a. \tag{2.5}$$

We make one more remark concerning the range of the embedding (2.5). If $T \in \mathfrak{C}$ we denote $T_a = \mathcal{P}_a[T]$. Then *for each $T \in \mathfrak{C}$ the set $\{T_a \mid a \in \mathcal{M}\}$ is relatively compact in $\mathfrak{C}$.* Indeed, for each $\varepsilon > 0$ there is $\mathcal{E} \subset \mathcal{L}$ finite and there is $S \in \mathfrak{C}(\mathcal{E})$ such that $\|T - S\| < \varepsilon$. Then $\|T_a - S_a\| < \varepsilon$ for all a, so it suffices to prove that $\{S_a \mid a \in \mathcal{M}\}$ is a finite set. But this is clear because $S_a = \sum_{b\in\mathcal{E}, b\geq a} S(b)$ and the set of subsets of $\mathcal{E}$ is finite.

It is sometimes necessary to show that the set $\{T_a \mid a \in \mathcal{M}\}$ is actually compact (this is especially important for the proof of the Mourre estimate in the case of infinite semilattices). Although this can be done in an abstract setting for a class of finite rank lattices, we shall treat this question in Section 5.2 only for the case of interest for us when $\mathcal{L}$ is the grassmannian of a vector space.

We stress the fact that all the preceding results are elementary and very easy to prove. They have, however, non-trivial consequences in spectral analysis, e.g. they allow one to compute the essential spectrum and to prove the Mourre estimate for large classes of many channel hamiltonians. We consider here only the question of the essential spectrum and assume that $\mathfrak{C}$ is realized as a C^*-algebra of operators on a Hilbert space $\mathcal{H}$ such that $\mathfrak{C}(\min \mathcal{L}) = K(\mathcal{H})$. If H is a self-adjoint operator on $\mathcal{H}$ affiliated to $\mathfrak{C}$ and $a \in \mathcal{L}$ then the map $\varphi \longmapsto \mathcal{P}_a[\varphi(H)]$ is an observable affiliated to $\mathfrak{C}_a$ (see the Remark 2.4) and cannot be realized as a densely defined self-adjoint operator on $\mathcal{H}$ in general. If, however, $\mathfrak{C}_a$ is nondegenerate on $\mathcal{H}$ and H is $\mathfrak{C}$-nondegenerate then by Proposition 2.3 there is a unique self-adjoint operator H_a on $\mathcal{H}$ such that $\mathcal{P}_a[\varphi(H)] = \varphi(H_a)$ for all $\varphi \in C_0(\mathbb{R})$. We shall set $H_a = \mathcal{P}_a[H]$. We shall make below these nondegeneracy hypotheses although they are not necessary in this context (one could proceed as in Chapter 8 from [ABG]).

Theorem 2.10 *Under the preceding conditions one has*

$$\sigma_{\text{ess}}(H) = \overline{\bigcup_{a\in\mathcal{M}} \sigma(H_a)}. \tag{2.6}$$

If for each $T \in \mathfrak{C}$ the subset $\{T_a \mid a \in \mathcal{M}\}$ is compact in $\mathfrak{C}$ then the set $\bigcup_{a \in \mathcal{M}} \sigma(H_a)$ is closed.

Proof. Let λ be a real number. Then $\lambda \notin \sigma_{\text{ess}}(H)$ if and only if there is a function $\varphi \in C_0(\mathbb{R})$ with $\varphi(\lambda) \neq 0$ and $\varphi(H) \in K(\mathcal{H})$. By (2.5) and the hypothesis $\mathfrak{C}(\min \mathcal{L}) = K(\mathcal{H})$ this is equivalent to $\varphi(H_a) = \mathcal{P}_a[\varphi(H)] = 0$ for all $a \in \mathcal{M}$. So $\lambda \notin \sigma_{\text{ess}}(H)$ if and only if there is a neighbourhood V of λ such that $V \cap \sigma(H_a) = \emptyset$ for all $a \in \mathcal{M}$. But this is clearly equivalent to the fact that λ does not belong to the closure of the union of the sets $\sigma(H_a)$.

In order to prove the last assertion of the theorem, let us equip the set $\mathcal{S}$ of all self-adjoint operators on $\mathcal{H}$ with the metric

$$d(S_1, S_2) = \left\| (S_1 - i)^{-1} - (S_2 - i)^{-1} \right\|.$$

Then for each $\varphi \in C_0(\mathbb{R})$ the map $\mathcal{S} \ni S \longmapsto \varphi(S) \in B(\mathcal{H})$ is continuous. Let λ a real number not in $\bigcup_{a \in \mathcal{M}} \sigma(H_a)$. Then for each a there is $\varphi_a \in C_0(\mathbb{R})$ such that $\varphi_a(\mu) = 1$ near λ and $\varphi_a(H_a) = 0$. For each a there is a neighbourhood V_a of H_a in $\mathcal{S}$ such that $\|\varphi_a(S)\| \leq 1/2$ if $S \in V_a$. Since by our hypothesis the set $\{H_a \mid a \in \mathcal{M}\}$ is compact in $\mathcal{S}$, there is a finite subset $\mathcal{N} \subset \mathcal{M}$ such that the V_b, $b \in \mathcal{N}$, cover $\{H_a \mid a \in \mathcal{M}\}$. Let $\varphi \in C_0(\mathbb{R})$ such that $0 \leq \varphi \leq 1$, $\varphi(\mu) = 1$ near λ and $\varphi\varphi_b = \varphi$ for all $b \in \mathcal{N}$. Clearly we shall have $\|\varphi(H_a)\| \leq 1/2$ for all $a \in \mathcal{M}$. Then an open neighbourhood of λ on which $\varphi(\mu) = 1$ has empty intersection with $\sigma(H_a)$, $\forall a \in \mathcal{M}$. ■

If $\mathcal{L}$ has a greatest element $\max \mathcal{L}$ then each $\mathfrak{C}(a)$ is equipped with a natural left and right action of the C*-algebra $\mathfrak{C}(\max \mathcal{L})$: indeed, one has $\mathfrak{C}(\max \mathcal{L}) \cdot \mathfrak{C}(a) \subset \mathfrak{C}(a)$ and $\mathfrak{C}(a) \cdot \mathfrak{C}(\max \mathcal{L}) \subset \mathfrak{C}(a)$. It follows immediately from the definition given in the subsection 2.1 that the action of $\mathfrak{C}(\max \mathcal{L})$ on $\mathfrak{C}$ is nondegenerate if (and only if) its action on each $\mathfrak{C}(a)$ is nondegenerate. In this case for each $S \in \mathfrak{C}$ there are $T \in \mathfrak{C}$ and $A, B \in \mathfrak{C}(\max \mathcal{L})$ such that $S = ATB$.

From now on in this subsection we assume that $\mathcal{L}$ has a greatest element and that $\mathfrak{C}$ is realized as a C*-algebra of bounded operators on a Hilbert space $\mathcal{H}$. Let H_0 be a self adjoint operator on $\mathcal{H}$ strictly affiliated to $\mathfrak{C}(\max \mathcal{L})$. If $\mathfrak{C}(\max \mathcal{L})$ acts nondegenerately on each algebra $\mathfrak{C}(a)$ then H_0 will be strictly affiliated to $\mathfrak{C}$. In this case for each $S \in \mathfrak{C}$ there are $T \in \mathfrak{C}$ and $\varphi, \psi \in C_0(\mathbb{R})$ such that $S = \varphi(H_0)T\psi(H_0)$; if $S \in \mathfrak{C}(a)$ then one may choose $T \in \mathfrak{C}(a)$.

For the next theorem we assume that a bounded from below self-adjoint operator H_0 on $\mathcal{H}$ and a family $\{H(a)\}_{a \in \mathcal{L}}$ of symmetric continuous s-esquilinear forms on $\mathcal{G} = D(|H_0|^{1/2})$ are given such that $H(a) \geq -\mu_a H_0 - \delta_a$ for some positive numbers μ_a, δ_a with $\sum_{a \in \mathcal{L}} \mu_a < 1$ and $\sum_{a \in \mathcal{L}} \delta_a < \infty$. Moreover, we assume that the family $\{H(a)\}_{a \in \mathcal{L}}$ is (norm) summable in $B(\mathcal{G}; \mathcal{G}^*)$, in particular $H(a) \neq 0$ for a set of a which is at most countable. We shall take $H(\max \mathcal{L}) = 0$ (this amounts to a redefinition of H_0).

If we denote $V = \sum_{b \in \mathcal{L}} H(b)$ and $V_a = \sum_{b \geq a} H(b)$ then V and V_a are standard form perturbations of H_0. Hence $H = H_0 + V$ and $H_a = H_0 + V_a$ are bounded from below self-adjoint operators on $\mathcal{H}$ with form domain equal to $\mathcal{G}$ and $H_{\max \mathcal{L}} = H_0$.

Theorem 2.11 *Assume that H_0 is strictly affiliated to $\mathfrak{C}(\max \mathcal{L})$ and that $\mathfrak{C}(\max \mathcal{L})$ acts nondegenerately on each $\mathfrak{C}(a)$. Moreover, assume that for some numbers $\alpha > 1/2$ and $\lambda > -\inf H_0$ and for each $a \in \mathcal{L}$ one has:*

$$(\lambda + H_0)^{-\alpha} H(a) (\lambda + H_0)^{-1/2} \in \mathfrak{C}(a) \tag{2.7}$$

Then H is strictly affiliated to $\mathfrak{C}$ and $\mathcal{P}_a[H] = H_a$ for all $a \in \mathcal{L}$.

Remark: If (2.7) holds for some $\alpha > 1/2$ and $\lambda > -\inf H_0$ then it holds for all such α, λ. Moreover, let $\theta \in C_0(\mathbb{R})$ such that $\theta(x) \sim x^{-1/2}$ as $x \to +\infty$. Then (2.7) is a consequence of

$$\varphi(H_0) H(a) \theta(H_0) \in \mathfrak{C}(a) \quad \forall \varphi \in C_c(\mathbb{R}) \tag{2.8}$$

and it implies

$$\varphi(H_0) \theta(H_0) H(a) \theta(H_0) \in \mathfrak{C}(a) \quad \forall \varphi \in C_0(\mathbb{R}). \tag{2.9}$$

Proof. We apply Theorem 2.8 with $U = (\lambda + H_0)^{-1/2} V (\lambda + H_0)^{-1/2}$ (see the comment after Theorem 2.8). By the preceding remark we have $\varphi(H_0) U \in \mathfrak{C}$ for all $\varphi \in C_0(\mathbb{R})$. We have seen that H_0 is strictly affiliated to $\mathfrak{C}$ so each $S \in \mathfrak{C}$ is of the form $S = T\varphi(H_0)$ with $T \in \mathfrak{C}$ and $\varphi \in C_0(\mathbb{R})$. We get $SU \in \mathfrak{C}$ for all $S \in \mathfrak{C}$, hence $U \in \mathcal{M}(\mathfrak{C})$. So, by Theorem 2.8, H is strictly affiliated to $\mathfrak{C}$. It remains to prove that $\mathcal{P}_a[H] = H_a$ for all $a \in \mathcal{L}$.

Let us consider the operators $H^\nu = H_0 + \nu V$ and $H_a^\nu = H_0 + \nu V_a$ for $0 \leq \nu \leq 1 + \varepsilon$. The numbers λ, ε are chosen exactly as in the proof of the second part of Lemma 2.7. Then H^ν is affiliated to $\mathfrak{C}$ and $\mathfrak{C}$-nondegenerate and the operators $(\lambda + H^\nu)^{-1}$ and $(\lambda + H_a^\nu)^{-1}$ depend analytically on $\nu \in (0, 1 + \varepsilon)$. Clearly it suffices to prove that the equality $\mathcal{P}_a[(\lambda + H^\nu)^{-1}] = (\lambda + H_a^\nu)^{-1}$ holds for small values of $\nu > 0$.

Set $\Lambda = (H_0 + \lambda)^{-1/2}, U_a = \Lambda V_a \Lambda$ and $U = \Lambda V \Lambda$. Let us assume that $0 < \nu < \min(\|U\|, \|U_a\|)$. Then (use the proof of Lemma 2.7) we have a norm convergent expansion

$$(\lambda + H^\nu)^{-1} = \sum_{k=0}^{\infty} (-\nu)^k \Lambda \, U^k \Lambda.$$

This remains valid if we replace H^ν by H_a^ν and U bu U_a. So it suffices to prove that $\Lambda U^k \Lambda$ belongs to $\mathfrak{C}$ and that its projection onto $\mathfrak{C}_a$ is $\Lambda U_a^k \Lambda$.

Since $U = \sum_b \Lambda H(b)\Lambda$ one can develop the power U^k and see that $\Lambda U^k \Lambda$ is a linear combination of terms of the form $\mathscr{J} = \Lambda U(b_1)...U(b_k)\Lambda$ with $U(b) = \Lambda H(b)\Lambda$. If in this linear combination one replaces by zero the terms such that b_j is not $\leq a$ for some j then one clearly gets $\Lambda U_a^k \Lambda$. So it suffices to prove that $\mathscr{J} \in \mathfrak{C}(b_1 \wedge ... \wedge b_k)$.

We shall show that $\Lambda U(b_1) \cdots U(b_k) \in \mathfrak{C}(b_1 \wedge \cdots \wedge b_k)$ by induction over k. If $k = 1$ this is clear by the remark after the statement of the theorem. If the property holds for some k then by the Cohen-Hewitt theorem one can write $\Lambda U(b_1) \cdots U(b_k) = T\varphi(H_0)$ for some $T \in \mathfrak{C}(b_1 \wedge \cdots \wedge b_k)$ and $\varphi \in C_0(\mathbb{R})$ (cf. the discussion before the theorem). Then

$$\Lambda U(b_1) \cdots U(b_k) U(b_{k+1}) \;=\; T\varphi(H_0) U(b_{k+1}) \;\in\; \mathfrak{C}(b_1 \wedge \cdots \wedge b_{k+1})$$

because $\varphi(H_0) U(b_{k+1}) \in \mathfrak{C}(b_{k+1})$. ■

It is clear that Theorem 2.8 can be used to get better results than Theorem 2.11. Indeed, the decomposition into a sum $\sum H(a)$ of the perturbation V is not necessary. For example, a symmetric element $V \in \mathfrak{C}$ could not be of this type and $H_0 + V$ is affiliated to $\mathfrak{C}$. On the other hand, observe that a self-adjoint operator affiliated to $\mathfrak{C}$ is necessarily affiliated to a C^*-subalgebra of the form $\mathfrak{C}(\mathcal{E})$, where $\mathcal{E}$ is a *countable* $\wedge$-stable subset of $\mathcal{L}$ (see (2.3)).

3 C^*-Algebras Associated to Vector Spaces

3.1. Let X be a finite dimensional real vector space and $\mathcal{H}(X)$ the space of (equivalence classes of) square integrable functions with respect to a Haar measure on X. A Haar measure (i.e. a translation invariant positive Radon measure) being defined only modulo a constant factor, there is no canonical norm on $\mathcal{H}(X)$. However, the topology on $\mathcal{H}(X)$ and the norm on $B(\mathcal{H}(X))$ are uniquely defined. So $\mathfrak{B}(X) = B(\mathcal{H}(X))$ and $\mathfrak{K}(X) = K(\mathcal{H}(X))$ are C^*-algebras independent of the choice of the measure. Note also that the notion of self-adjoint operator on $\mathcal{H}(X)$ is meaningful, the scalar products associated to different Haar measures being proportional.

Let X^* be the dual space of X, for $x \in X$, $k \in X^*$ set $\langle x, k\rangle = k(x)$, and if $Y \subset X$ is a subspace denote $Y^\perp = \{k \in X^* \mid \langle y, k\rangle = 0 \, \forall y \in Y\}$. For $x \in X$ and $k \in X^*$ we define unitary operators $\mathrm{e}_x(P) \equiv \mathrm{e}^{i\langle x, P\rangle}$, $\mathrm{e}_k(Q) \equiv \mathrm{e}^{i\langle Q, k\rangle}$ in $\mathcal{H}(X)$ by $(\mathrm{e}_x(P)f)(y) = f(x+y)$ and $(\mathrm{e}_k(Q)f)(y) = \mathrm{e}^{i\langle y,k\rangle} f(y)$. We make the following notational convention: if a symbol like $T^{(*)}$ appears in a relation, then the relation must hold for the operator T and for its adjoint T^*. The operator limits must be interpreted in the norm topology if not otherwise stated.

Definition 3.1 *If Y is a linear subspace of X then $\mathfrak{C}^X(Y)$ is the set of operators $T \in \mathfrak{B}(X)$ satisfying the following conditions:*
(i) $[T, \mathrm{e}_y(P)] = 0 \quad \forall\ y \in Y$;
(ii) $[T, \mathrm{e}_k(Q)] \to 0$ *if* $k \to 0$ *in* X^*;
(iii) $[\mathrm{e}_x(P) - 1]T^{(*)} \to 0$ *if* $x \to 0$ *in* X;
(iv) $[\mathrm{e}_k(Q) - 1]T^{(*)} \to 0$ *if* $k \to 0$ *in* $Y^\perp$.

Note that (iii) and

$$\|[T, \mathrm{e}_x(P)]\| \le \|T[\mathrm{e}_x(P) - 1]\| + \|[\mathrm{e}_x(P) - 1]T\| \tag{3.1}$$

imply that $[T, \mathrm{e}_x(P)] \to 0$ if $x \to 0$ in X. Similarly, (iv) gives $[T, \mathrm{e}_k(Q)] \to 0$ if $k \to 0$ in $Y^\perp$. In particular, it suffices in (ii) to consider $k \to 0$ in Y', where Y' is a subspace supplementary to $Y^\perp$ in X^*. It is less obvious, but we shall prove it in Proposition 3.10, that it suffices to require (iii) and (iv) to hold only for T (then they will hold for T^* too).

Obviously $\mathfrak{C}^X(Y)$ is a C^*-algebra on $\mathcal{H}(X)$. In Theorem 3.9 we shall describe it as a C^*-algebra of pseudo-differential operators.

We denote by Q and P the position and momentum observables in $\mathcal{H}(X)$, more precisely, if $\varphi : X \to \mathbb{C}$ and $\psi : X^* \to \mathbb{C}$ are Borel functions, then $\varphi(Q)$ is the operator of multiplication by φ in $\mathcal{H}(X)$ and $\psi(P)$ is such that $\mathcal{F}\psi(P)\mathcal{F}^{-1}$ is the operator of multiplication by ψ in $\mathcal{H}(X^*)$. The Fourier transformation $\mathcal{F}$ is defined by $(\mathcal{F}f)(k) = \int_X e^{-i\langle x,k\rangle} f(k)dk$; $\mathcal{F}$ induces a map $\mathcal{H}(X) \to \mathcal{H}(X^*)$ which is unitary if norms on these spaces are chosen conveniently. When it will be necessary to specify the space X in the notations, we shall use it as an upper index, e.g. Q^X, P^X, or $\mathcal{F}^X$. For example, we have $\psi(P^X) = \mathcal{F}^{-1}\psi(Q^{X^*})\mathcal{F}$.

The map $\psi \mapsto \psi(P)$ is an injective morphism of $C_0(X^*)$ into $\mathfrak{B}(X)$, and we shall often identify $C_0(X^*)$ with its range through this map. So, for example, a subspace $\mathfrak{C} \subset \mathfrak{B}(X)$ is a $C_0(X^*)$-bimodule if $\psi(P)T$ and $T\psi(P)$ belong to $\mathfrak{C}$ for each $\psi \in C_0(X^*)$ and $T \in \mathfrak{C}$. Similarly if C_0 is replaced by C_b. If Y is a subspace of X then $\pi_Y = \pi_Y^X$ is the canonical surjection of X onto the quotient space X/Y. Then for $\varphi : X/Y \to \mathbb{C}$ Borel we set $\varphi(Q_Y^X) \equiv \varphi(Q_Y) = (\varphi \circ \pi_Y)(Q)$. Note that $\varphi \mapsto \varphi(Q_Y)$ gives an embedding of $C_0(X/Y)$ into $\mathfrak{B}(X)$. The next two relations will often be used without comment:

$$\mathrm{e}_x(P)\, \varphi(Q_Y)\, \mathrm{e}_x(P)^* = \varphi(Q_Y + \pi_Y(x)) \tag{3.2}$$

$$\mathrm{e}_k(Q)\, \psi(P)\, \mathrm{e}_k(Q)^* = \psi(P - k). \tag{3.3}$$

It will be convenient to express conditions (iii), (iv) in a different form. The next lemma is a particular case of a result from [Geo Ift], where representations of general abelian groups are considered.

Lemma 3.2 *Let $T \in \mathfrak{B}(X)$ and $Y \subset X$ a subspace.*
(a) *One has $[\mathrm{e}_x(P) - 1]T \to 0$ as $x \to 0$ in X if and only if there is $\psi \in C_0(X^*)$ with $\psi(0) = 1$ such that $\lim_{\varepsilon\to 0}[\psi(\varepsilon P) - 1]T = 0$. And then the last property holds for all $\psi \in C_b(X^*)$ with $\psi(0) = 1$.*
(b) *One has $[\mathrm{e}_k(Q) - 1]T \to 0$ as $k \to 0$ in $Y^\perp$ if and only if there is $\varphi \in C_0(X/Y)$ with $\varphi(0) = 1$ such that $\lim_{\varepsilon\to 0}[\varphi(\varepsilon Q_Y) - 1]T = 0$. And then the last relation holds for all $\varphi \in C_b(X/Y)$ with $\varphi(0) = 1$.*

Proof. We prove (a), the proof of (b) being similar. Denote $\mathrm{e}_x(P) = U_x$, then

$$\begin{aligned} \|(U_x - 1)T\| &\leq 2\,\|[1 - \psi(\varepsilon P)]T\| + \|(U_x - 1)\psi(\varepsilon P)\| \cdot \|T\| \\ &\leq 2\,\|[1 - \psi(\varepsilon P)]T\| + \sup_{k\in X^*} \left| (e^{i\langle x,k\rangle} - 1)\psi(\varepsilon k) \right| \cdot \|T\|\,. \end{aligned}$$

So if $\|[\psi(\varepsilon P) - 1]T\| \to 0$ for $\varepsilon \to 0$ and some $\psi \in C_0(X^*)$, then we shall have $\|(U_x - 1)T\| \to 0$ if $x \to 0$ in X. Reciprocally, assume that $\psi(k) = \int_X e^{i\langle x,k\rangle}\mu(dx)$ for an integrable measure on X and $\psi(0) = 1$, then $\psi(\varepsilon P) - 1 = \int_X (U_{\varepsilon x} - 1)\mu(dx)$, hence for each compact set $K \subset X$:

$$\|[\psi(\varepsilon P) - 1]T\| \leq \int_K \|(U_{\varepsilon x} - 1)T\|\, |\,\mu\,|\,(dx) + 2\,\|T\| \int_{X\setminus K} |\,\mu\,|\,(dx).$$

Given $\delta > 0$ we choose K such that the last term be $< \delta/2$. Then there is $\varepsilon_0 > 0$ such that for $|\,\varepsilon\,| < \varepsilon_0$ and $x \in K$ one has $\|(U_{\varepsilon x} - 1)T\| < \delta[2\,|\,\mu\,|\,(K)]^{-1}$; hence $\|[\psi(\varepsilon P) - 1]T\| < \delta$ if $|\,\varepsilon\,| < \varepsilon_0$. The fact that we shall also have $\|[\psi(\varepsilon P) - 1]T\| \to 0$ if $\psi \in C_b(X^*)$ is shown as in the proof of Lemma 2.1. ■

Proposition 3.3 *The C^*-algebra $\mathfrak{C}^X(Y)$ has the following properties:*
(i) *it acts nondegenerately on $\mathcal{H}(X)$;*
(ii) *it is a nondegenerate $C_0(X^*)$-bimodule and a $C_b(X^*)$-bimodule;*
(iii) *it is a nondegenerate $C_0(X/Y)$-bimodule and a $C_b(X/Y)$-bimodule.*

Proof. It is easy to check, by using (3.2), (3.3) that $\varphi(Q_Y)\psi(P) \in \mathfrak{C}^X(Y)$ if $\varphi \in C_0(X/Y)$ and $\psi \in C_0(X^*)$. Since $s-\lim_{\varepsilon\to 0} \varphi(\varepsilon Q_Y)\psi(\varepsilon P) = \varphi(0)\psi(0)$, it is clear that $\mathfrak{C}^X(Y)$ acts nondegenerately on $\mathcal{H}(X)$. It is also clear that $\varphi(Q_Y)T$, $T\varphi(Q_Y)$, $\psi(P)T$ and $T\psi(P)$ belong to $\mathfrak{C}^X(Y)$ if $T \in \mathfrak{C}^X(Y)$, i.e. $\mathfrak{C}^X(Y)$ is a $C_0(X/Y)$-bimodule and a $C_0(X^*)$-bimodule. Lemma 3.2 implies that these bimodules are nondegenerate. Finally, the operators $\varphi(Q_Y)T$, $T\varphi(Q_Y)$, $\psi(P)T$ and $T\psi(P)$ belong to $\mathfrak{C}^X(Y)$ if $\varphi \in C_b(X/Y)$, $\psi \in C_b(X^*)$ as a consequence of the nondegeneracy (see Theorem 2.2 and the Remark 2.4). ■

Next we present the main properties of the family of C^*-algebras $\mathfrak{C}^X(Y)$ when Y runs over the set of subspaces of X. We set $O = \{0\}$ for the zero dimensional subspace.

Theorem 3.4 (a) $\mathfrak{C}^X(O) = \mathfrak{K}(X)$ *and* $\mathfrak{C}^X(X) = C_0(X^*)$.
(b) $\mathfrak{C}^X(Y) \cdot \mathfrak{C}^X(Z) \subset \mathfrak{C}^X(Y \cap Z)$ *if* Y, Z *are subspaces of* X.
(c) *The family* $\{\mathfrak{C}^X(Y) \mid Y$ *subspace of* $X\}$ *of* C^**-subalgebras of* $\mathfrak{B}(X)$ *is linearly independent.*
(d) *If* $\mathcal{E}$ *is a finite family of subspaces of* X, *then* $\mathfrak{C}^X(\mathcal{E}) := \sum_{Y\in\mathcal{E}} \mathfrak{C}^X(Y)$ *(linear subspace of* $\mathfrak{B}(X)$ *generated by* $\bigcup_{Y\in\mathcal{E}} \mathfrak{C}^X(Y)$*) is a norm closed subspace of* $\mathfrak{B}(X)$.

Proof. 1) If $Y = O$ then only conditions (iii) and (iv) of Definition 3.1 have to be taken into account (clearly (iv) $\Rightarrow$ (ii) in this case). From the strong continuity of the representations $\mathrm{e}_x(P)$ of X and $\mathrm{e}_k(Q)$ of X^* it follows that each compact operator T verifies (iii) and (iv). Reciprocally, we prove the following slightly stronger fact: if $T \in \mathfrak{B}(X)$ and

$$\lim_{x\to 0} \|[\mathrm{e}_x(P) - 1]T\| = \lim_{k\to 0} \|[\mathrm{e}_k(Q) - 1]T\| = 0 \tag{3.4}$$

then T is compact (so the conditions on $T^{(*)}$ from (iii) and (iv) are not needed). Let $\varphi \in C_0(X)$ and $\psi \in C_0(X^*)$ such that $\varphi(0) = \psi(0) = 1$. Then by Lemma 3.2(a) we have $\psi(\varepsilon P)T - T \to 0$ if $\varepsilon \to 0$. So for each $\delta > 0$ there is $\varepsilon_1 > 0$ such that $\|\psi(\varepsilon_1 P)T - T\| < \delta/2$. Since ψ is a uniformly continuous function we have $\|[\mathrm{e}_k(Q) - 1]\psi(\varepsilon_1 P)T\| \to 0$ if $k \to 0$ in X^* (use (3.3)) hence from Lemma 3.2(b) we see that there is $\varepsilon_2 > 0$ such that $\|[\varphi(\varepsilon_2 Q) - 1]\psi(\varepsilon_1 P)T\| < \delta/2$. Thus we have $\|\varphi(\varepsilon_2 Q)\psi(\varepsilon_1 P)T - T\| < \delta$. On the other hand $\varphi(\varepsilon_2 Q)\psi(\varepsilon_1 P)$ is a compact operator. Hence T is compact. So $\mathfrak{C}^X(O) = \mathfrak{K}(X)$.

Clearly $C_0(X^*) \subset \mathfrak{C}^X(X)$. To prove the oposite inclusion note that (i) with $Y = X$ implies $T = \psi(P)$ for some bounded Borel function ψ on X^*. Then (3.3) and (ii) imply that ψ is uniformly continuous and (iii) that ψ tends to zero at infinity (use Lemma 3.2(a)), i.e. $\psi \in C_0(X^*)$.

2) Now we prove (b). Let $S \in \mathfrak{C}^X(Y)$ and $T \in \mathfrak{C}^X(Z)$. Clearly ST satisfies (i) if Y is replaced by $Y \cap Z$, and also (ii), (iii). Since $(Y \cap Z)^\perp = Y^\perp + Z^\perp$, if $\{k_n\}$ is a sequence in $(Y \cap Z)^\perp$ which tend to zero we can write $k_n = u_n + v_n$ with $u_n \in Y^\perp$, $u_n \to 0$, $v_n \in Z^\perp$, $v_n \to 0$. We easily deduce then $[\mathrm{e}_{k_n}(Q) - 1](ST)^{(*)} \to 0$. Thus (b) is proved.

3) For the proof of (c) and (d) we need a simple preliminary fact:

$$\lim S\,\mathrm{e}_x(P)\,T = 0 \;\text{ if }\; S, T \in \mathfrak{C}^X(Y) \;\text{ and }\; \pi_Y(x) \to \infty. \tag{3.5}$$

Indeed, $\mathfrak{C}^X(Y)$ is a nondegenerate $C_0(X/Y)$-bimodule, so we may assume $S = S_0\varphi_1(Q_Y)$, $T = \varphi_2(Q_Y)T_0$ for some $\varphi_1, \varphi_2 \in C_0(X/Y)$. So (3.5) follows from (3.2).

4) We can now prove (c) and (d). Let $\mathcal{E}$ be a finite family of subspaces of X, for each $Y \in \mathcal{E}$ let $T_Y \in \mathfrak{C}^X(Y)$, and let us set $T = \sum_{Y\in\mathcal{E}} T_Y$. For $\omega \in X$ we set $\mathcal{E}(\omega) = \{Y \in \mathcal{E} \mid \omega \in Y\}$. If $\lambda \in \mathbb{R}$ and $Y \in \mathcal{E}(\omega)$ we

have $\mathrm{e}_x(P)^*T_Y\mathrm{e}_x(P) = T_Y$ for $x = \lambda\omega$ (because of (i)). Thus (3.5) and the nondegeneracy of $\mathfrak{C}^X(Y)$ imply

$$s - \lim_{\lambda\to\infty} \mathrm{e}_{\lambda\omega}(P)^* T \mathrm{e}_{\lambda\omega}(P) = \sum_{Y\in\mathcal{E}(\omega)} T_Y. \tag{3.6}$$

In particular

$$\left\| \sum_{Y\in\mathcal{E}(\omega)} T_Y \right\| \le \left\| \sum_{Y\in\mathcal{E}} T_Y \right\|. \tag{3.7}$$

We shall prove that there is a constant C such that for all $Z \in \mathcal{E}$ and all families $\{T_Y\}_{Y\in\mathcal{E}}$ with $T_Y \in \mathfrak{C}^X(Y)$

$$\|T_Z\| \le C \left\| \sum_{Y\in\mathcal{E}} T_Y \right\|. \tag{3.8}$$

This clearly finishes the proof of (c), (d). If the set

$$Z_0 = Z \setminus \bigcup_{Y\neq Z} Y = \bigcap_{Y\neq Z} [Z \setminus (Y \bigcap Z)]$$

is not empty, then (3.7) with a choice $\omega \in Z_0$ gives (3.8) with $C = 1$. Since $Y \cap Z$ are linear subspaces of Z, one has $Z_0 = \emptyset$ if and only if there is $Y \in \mathcal{E}$ such that $Z \subset Y$ strictly. This cannot happen if Z is a maximal element in $\mathcal{E}$ (for the inclusion relation), hence (3.8) holds for such elements. Let $\mathcal{E}_1$ be the set of $Y \in \mathcal{E}$ which are not maximal elements in $\mathcal{E}$. Then we get $\left\|\sum_{Y\in\mathcal{E}_1} T_Y\right\| \le C_1 \left\|\sum_{Y\in\mathcal{E}} T_Y\right\|$ for some constant C_1. By what we already proved we see then that (3.8) holds for the maximal elements of $\mathcal{E}_1$, etc. ■

The object of main interest for us will be the C^*-subalgebra $\mathfrak{C}^X$ of $\mathfrak{B}(X)$ generated by the family $\{\mathfrak{C}^X(Y)\}$ when Y runs over the family of all subspaces of X. Theorem 3.4(b) allows us to define

$$\mathfrak{C}^X = \text{norm closure in } \mathfrak{B}(X) \text{ of } \sum_{Y\subset X} \mathfrak{C}^X(Y). \tag{3.9}$$

Let $\mathbb{G}(X)$, the *grassmanian* of X, be the set of all vector subspaces of X equipped with the order relation defined by inclusion. Then $\mathbb{G}(X)$ is a complete lattice and $Y \wedge Z = Y \cap Z, \min \mathbb{G}(X) = O = \{0\}, \max \mathbb{G}(X) = X$. From Section 2.3 and Theorem 3.4 we see that $\mathfrak{C}^X$ *is a* $\mathbb{G}(X)$*-graded* C^**-algebra.*

We shall use in this context the abstract theory presented in Section 2.3. In particular, for each subspace Y of X we define

$$\mathfrak{C}^X_Y = \text{norm closure in } \mathfrak{B}(X) \text{ of } \sum_{Z\supset Y} \mathfrak{C}^X(Z).$$

This is a C^*-subalgebra of $\mathfrak{C}^X$ and there is a canonical morphism $\mathcal{P}_Y : \mathfrak{C}^X \to \mathfrak{C}^X_Y$ which is at the same time a projection in the sense of linear spaces. $\mathcal{P}_Y$ is uniquely determined by the property $\mathcal{P}_Y \sum_Z T(Z) = \sum_{Z\supset Y} T(Z)$ if $T(Z) \in \mathfrak{C}^X(Z)$ for each $Z \in \mathbb{G}(X)$ and $T(Z) \neq 0$ only for a finite number of Z. Moreover, $\mathfrak{J}^X_Y := \ker \mathcal{P}_Y$ (a closed self-adjoint ideal) is equal to the closure of $\sum_{Z\not\supset Y} \mathfrak{C}^X_Z$ and we have $\mathfrak{C}^X = \mathfrak{C}^X_Y + \mathfrak{J}^X_Y$, a topological direct sum. If T belongs to $\mathfrak{C}^X$ we shall often denote by $T_Y = \mathcal{P}_Y[T]$ its component in $\mathfrak{C}^X_Y$.

3.2. What we have done until now suffices for an elementary treatment of the N-body hamiltonians. It is, however, useful and interesting to study a class of larger algebras, which appear quite naturally in our context. This will give us a better understanding of the structure of the algebras $\mathfrak{C}^X(Y)$.

Theorem 3.5 *Let $\mathfrak{U}^X$ be the set of operators $T \in \mathfrak{B}(X)$ such that:*
(i) $[T, \mathrm{e}_k(Q)] \to 0$ *if* $k \to 0$ *in* X^*, *and* (ii) $[\mathrm{e}_x(P) - 1]T^{(*)} \to 0$ *if* $x \to 0$ *in* X. *Then $\mathfrak{U}^X$ coincides with the norm closed subspace of $\mathfrak{B}(X)$ generated by the operators of the form $\varphi(Q)\psi(P)$, with $\varphi \in C_{bu}(X)$, $\psi \in C_0(X^*)$.*

Proof. The method of proof is based on an idea from [Geo Ift]. Let $\mathcal{A}$ be the closed subspace of $\mathfrak{B}(X)$ generated by the operators $\varphi(Q)\psi(P)$ with φ, ψ as above. Clearly $\mathfrak{U}^X$ is a norm closed subspace of $\mathfrak{B}(X)$ and $\varphi(Q)\psi(P) \in \mathfrak{U}^X$ if $\varphi \in C_{bu}(X), \psi \in C_0(X^*)$. So we have $\mathcal{A} \subset \mathfrak{U}^X$ and it remains to prove the oposite inclusion. From now we assume $X = \mathbb{R}^n = X^*$ and choose a function $v \in C_c^\infty(\mathbb{R}^n)$ with $\int v dx = 1$ and such that the support of its Fourier transform $\widehat{v}$ is included in the unit ball. For $T \in \mathfrak{B}(X)$ and $\varepsilon > 0$ we set

$$T_\varepsilon = \int_{\mathbb{R}^n} \mathrm{e}_{\varepsilon k}(Q)\, T\, \mathrm{e}_{\varepsilon k}(Q)^*\, v(k) dk.$$

If $T \in \mathfrak{U}^X$ then $\|T_\varepsilon - T\| \to 0$ (because of (i)) and clearly $T_\varepsilon \in \mathfrak{U}^X$. The property of T_ε which is important for us is the following: if θ_1, θ_2 are bounded functions and $\mathrm{dist}(\mathrm{supp}\,\theta_1, \mathrm{supp}\,\theta_2) > \varepsilon^{-1}$, then $\theta_1(Q)T_\varepsilon\theta_2(Q) = 0$. This is proven, in a more general context, in [Geo Ift] and the computations will not be repeated here (it suffices to assume that T is a rank one operator and then the computation is straightforward).

Now we are clearly reduced to proving the following: if $T \in \mathfrak{U}^X$ and if there is a number $r > 0$ such that $\theta_1(Q)T\theta_2(Q) = 0$ if $\theta_1, \theta_2 \in C_c(X)$ and $\mathrm{dist}(\mathrm{supp}\,\theta_1, \mathrm{supp}\,\theta_2) > r$, then T is a norm limit of linear combinations of operators of the form $\varphi(Q)\psi(P)$ with $\varphi \in C_{bu}(X), \psi \in C_0(X^*)$.

From Theorem 1.4.6 in [H] it follows that there is $\theta \in C_c^\infty(\mathbb{R}^n)$ with support in the unit ball such that $\sum_{a\in L} \theta(x - a) = 1$, where $L = \mathbb{Z}^n$. We set $\theta^a = \theta(Q - a) = \mathrm{e}_a(P)^*\theta(Q)\mathrm{e}_a(P)$ and $\theta \equiv \theta(Q)$. Then we get:

$$T \;=\; \sum_{a,b} \theta^a\, T\, \theta^b = \sum_a \theta^a\, T \sum_{|\,b-a\,|<r+2} \theta^b$$

$$= \sum_a \mathrm{e}_a(P)^* \, \theta \, \mathrm{e}_a(P) \, T \, \mathrm{e}_a(P)^* \sum_{|\, b-a \,|<r+2} \theta(Q-(b-a))\mathrm{e}_a(P).$$

Denote $\widetilde{\theta}(x) = \sum_{|\, c\,|<r+2} \theta(x-c)$ and $T_a = \mathrm{e}_a(P)T\mathrm{e}_a(P)^*$. Then we get:

$$T = \sum_a \mathrm{e}_a(P)^* \, \theta \, T_a \widetilde{\theta} \, \mathrm{e}_a(P).$$

Set $K_a = \theta T_a \widetilde{\theta}$. Then there is a compact set $\Lambda \subset X$ such that $\chi K_a \chi = K_a$ for all a; here χ is the operator of multiplication by the characteristic function of the set Λ. Moreover, from $\left\|[\mathrm{e}_x(P)-1]T^{(*)}\right\| \to 0$ if $x \to 0$ we easily get $\left\|[\mathrm{e}_x(P)-1]K_a^{(*)}\right\| \to 0$ as $x \to 0$, uniformly in a. From the Riesz-Kolmogorov compacity criterion it follows now that $\{K_a\}_{a\in\mathbb{Z}^n}$ is a relatively compact family of compact operators. Hence for each $\varepsilon > 0$ there is a finite set $M \subset L$ such that L can be written as a disjoint union of subsets L_m, $m \in M$, with the property: $a \in L_m \Rightarrow \|K_a - K_m\| < \varepsilon$. Let $S_m = \sum_{a\in L_m} \mathrm{e}_a(P)^* K_m \mathrm{e}_a(P)$ and $S = \sum_{m\in M} S_m$. Then we have

$$\begin{aligned} T - S &= \sum_{m\in M} \sum_{a\in L_m} \mathrm{e}_a(P)^*[K_a - K_m]\mathrm{e}_a(P) \\ &= \sum_{m\in M} \sum_{a\in L_m} \chi^a \mathrm{e}_a(P)^*[K_a - K_m]\mathrm{e}_a(P)\chi^a \end{aligned}$$

where χ^a is the characteristic function of the set $a+\Lambda$. Since Λ is a compact set and a belongs to $\mathbb{Z}^n$, it is elementary to prove that there is $N < \infty$, depending only on a and Λ, such that the norm of the last term above is smaller than $N \sup\{\|K_a - K_m\| \mid m \in M, a \in L_m\}$, which in turn is $\leq N\varepsilon$ (this is also a consequence of a simple version of the Cotlar-Stein lemma).

Finally, we see that it suffices to show that an operator of the form S_m belongs to $\mathcal{A}$. K_m being compact and supported by a compact set, for each $\lambda \geq 0$ and each $\varepsilon > 0$ there are $\varphi_1, \cdots, \varphi_j \in C_c(X)$ and $\psi_1, \cdots, \psi_j \in C_c(X^*)$ such that $\left\|[K_m - \sum_{i=1}^j \varphi_i(Q)\psi_i(P)]\langle Q\rangle^\lambda\right\| < \varepsilon$. Let

$$\begin{aligned} \widetilde{S}_m &= \sum_{a\in L_m} \mathrm{e}_a(P)^* \sum_{i=1}^j \varphi_i(Q)\psi_i(P)\mathrm{e}_a(P) \\ &= \sum_{i=1}^j \left[\sum_{a\in L_m} \varphi_i(Q-a)\right] \psi_i(P) \equiv \sum_{i=1}^j \Phi_i(Q)\psi_i(P). \end{aligned}$$

We have $\Phi_i \in C_{bu}(X)$, so $\widetilde{S}_m$ belongs to $\mathcal{A}$. Hence it suffices to prove that $\left\|S_m - \widetilde{S}_m\right\| \leq \text{const.}\,\varepsilon$. Such an estimate follows easily from the following more general fact: if $J \subset \mathbb{R}^n$ is such that $|j-k| \geq \text{const.} > 0$ for $j, k \in J$, $j \neq k$, then for each $\lambda > 2n$ there is $C(n) < \infty$ such that

$\left\|\sum_{a\in J}\mathrm{e}_a(P)^* B\mathrm{e}_a(P)\right\| \le C(n)\left[\left\|\langle Q\rangle^\lambda B\right\| + \left\|B\langle Q\rangle^\lambda\right\|\right]$ for all bounded operators B. This is is a straightforward consequence of the Cotlar-Stein lemma, see [GI] for details. ■

The space $\mathfrak{U}^X$ is clearly a C^*-subalgebra of $\mathfrak{B}(X)$. It can also be proved directly that $\mathcal{A}$ (notation as in the preceding proof) is a C^*-algebra (show that $\psi(P)\varphi(Q) \in \mathcal{A}$ if $\psi \in C_c^\infty(X^*)$, $\varphi \in C_{bu}(X)$, cf. [DaG]). In fact, $\mathcal{A}$ and the C^*-crossed product $C_{bu}(X) \rtimes X$ are canonically isomorphic.

To each subspace Y of X we associate the C^*-subalgebra of $\mathfrak{B}(X)$ defined by

$$\mathfrak{U}^X_Y = \{T \in \mathfrak{U}^X \mid [T, \mathrm{e}_y(P)] = 0\,\forall\, y \in Y\}.$$

Then $\mathfrak{U}^X_Z \subset \mathfrak{U}^X_Y$ if $Y \subset Z$ and it is easy to prove that $\mathfrak{U}^X_X = C_0(X^*)$. More generally:

Lemma 3.6 *Each supplementary subspace of Y in X determines an isomorphism*

$$\mathfrak{U}^X_Y \simeq C_0(Y^*) \otimes \mathfrak{U}^{X/Y} = C_0(Y^*; \mathfrak{U}^{X/Y}). \tag{3.10}$$

Proof. Let $Y_1 \subset X$ be a subspace such that $X = Y + Y_1$ and $Y \cap Y_1 = \{0\}$. Then we have a natural identification $X/Y \simeq Y_1$ and

$$\mathcal{H}(X) = \mathcal{H}(Y) \otimes \mathcal{H}(Y_1) = \mathcal{H}(Y; \mathcal{H}(Y_1)). \tag{3.11}$$

Note that we make the natural conventions: if $Y = O = \{0\}$ then $\mathcal{H}(Y) = \mathbb{C}$, $\mathfrak{B}(Y) = \mathfrak{K}(Y) = \mathbb{C}$. Denote $\mathcal{F}_Y$ the operator $\mathcal{H}(Y; \mathcal{H}(Y_1)) \to \mathcal{H}(Y^*; \mathcal{H}(Y_1))$ defined by the Fourier transformation in the Y-variables (so $\mathcal{F}_Y = \mathcal{F}^Y \otimes 1$ in the factorization (3.11)). Then to each bounded continuous map $F : Y^* \to \mathfrak{B}(X/Y)$ we associate an operator $F(P^Y) \in \mathfrak{B}(X)$ by the following rule: $\mathcal{F}_Y F(P^Y) {\mathcal{F}_Y}^{-1}$ is the operator of multiplication by the operator valued function F in $\mathcal{H}(Y^*; \mathcal{H}(Y_1))$. We shall prove:

(3.12) $F \mapsto F(P^Y)$ *is an isomorphism of* $C_0(Y^*; \mathfrak{U}^{X/Y})$ *onto* $\mathfrak{U}^X_Y$.

For each $T \in \mathfrak{B}(X)$ let us set $\widetilde{T} = \mathcal{F}_Y T {\mathcal{F}_Y}^{-1}$, a bounded operator in $\mathcal{H}(Y^*; \mathcal{H}(Y_1))$. It is well known that the condition $[\mathrm{e}_y(P), T] = 0\ \forall y \in Y$ is equivalent to the existence of a bounded weakly measurable function $F : Y^* \to \mathfrak{B}(Y_1)$ such that $\widetilde{T}$ is the operator of multiplication by F. Let $Y' = Y_1^\perp$, then we have a direct sum decomposition $X^* = Y^\perp + Y'$ which allows us to identify $Y^\perp = Y_1^*$ and $Y' = Y^*$. Then the condition $[T, \mathrm{e}_k(Q)] \to 0$ if $k \to 0$ in Y' is equivalent to

$$\mathrm{ess-}\sup_{p\in Y^*} \|F(p+k) - F(p)\| \to 0 \text{ if } k \to 0 \text{ in } Y^*.$$

This means that F is uniformly norm continuous. If we also require that $[\mathrm{e}_y(P) - 1]T \to 0$ if $y \to 0$ in Y, then

$$\mathrm{ess-}\sup_{p\in Y^*} \left\|[\mathrm{e}^{i\langle y,p\rangle} - 1]F(p)\right\| \to 0 \text{ if } y \to 0 \text{ in } Y.$$

By using Lemma 3.2 we thus get $F \in C_0(Y^*; \mathfrak{B}(Y_1))$. The conditions $[T, \mathrm{e}_k(Q)] \to 0$ if $k \to 0$ in $Y^\perp$ and $[\mathrm{e}_x(P) - 1]T^{(*)} \to 0$ if $x \to 0$ in Y_1 are equivalent to

$$\sup_{p \in Y^*} \left\| [\mathrm{e}_k(Q^{Y_1}), F(p)] \right\| \to 0 \text{ if } k \to 0 \text{ in } Y_1^*, \tag{3.13}$$

$$\sup_{p \in Y^*} \left\| [\mathrm{e}_y(P^{Y_1}) - 1] F(p)^{(*)} \right\| \to 0 \text{ if } y \to 0 \text{ in } Y_1. \tag{3.14}$$

Thus we have $F(p) \in \mathfrak{U}^{Y_1}$ for all $p \in Y^*$. ■

Theorem 3.7 $\mathfrak{U}_Y^X$ *coincides with the closed linear subspace of* $\mathfrak{B}(X)$ *generated by the operators* $\varphi(Q_Y)\psi(P)$, *with* $\varphi \in C_{bu}(X/Y)$ *and* $\psi \in C_0(X^*)$.

Proof. It is clear that $\varphi(Q_Y)\psi(P) \in \mathfrak{U}_Y^X$ if φ, ψ are as above, so one inclusion is obvious. In order to prove the opposite inclusion we use the isomorphism (3.10) determined by a supplementary subspace Y_1. The linear space generated by the functions of the form $F(p) = \theta(p)S$, with $\theta \in C_0(Y^*)$ and $S \in \mathfrak{U}^{X/Y}$, is dense in $C_0(Y^*; \mathfrak{U}^{X/Y})$, so it suffices to consider operators T such that $\widetilde{T}$ is of this form. Moreover, by Theorem 3.5 (with X replaced by X/Y), it suffices to assume that $S = \varphi(Q^{X/Y})\xi(P^{X/Y})$ for some $\varphi \in C_{bu}(X/Y)$ and $\xi \in C_0((X/Y)^*)$. Recall that $(X/Y)^* = Y^\perp$, $Y^* \simeq {Y_1}^\perp$, and $X^* = {Y_1}^\perp + Y^\perp$ direct sum. Thus we can define $\psi \in C_0(X^*)$ by $\psi(p_1 + p_2) = \theta(p_1)\xi(p_2)$ if $p_1 \in {Y_1}^\perp$, $p_2 \in Y^\perp$. Then $T = \varphi(Q_Y)\psi(P)$. ■

3.3. The isomorphism of Lemma 3.6 depends on the choice of a supplementary space for Y, and this is esthetically and practically annoying. We present below an intrinsic version of this construction.

We define $C_0^X(Y^*; \mathfrak{U}^{X/Y})$ as the C^*-algebra of norm continuous maps $F : X^* \to \mathfrak{U}^{X/Y}$ such that:
(a) $F(k+p) = \mathrm{e}_p(Q^{X/Y})^* \, F(k) \, \mathrm{e}_p(Q^{X/Y})$ for all $k \in X^*, p \in Y^\perp$;
(b) $\|F(k)\| \to 0$ if $\pi_{Y^\perp}^{X^*}(k) \to \infty$.
Note that $Y^\perp = (X/Y)^*$ and $\pi_{Y^\perp}^{X^*}$ is the projection map $X^* \to X^*/Y^\perp = Y^*$. Such functions F are sections of a (trivialisable but not trivial) fiber bundle of C^*-algebras over Y^*. One can similarly define $C_0^X(Y^*; \mathfrak{C})$ for any C^*-subalgebra $\mathfrak{C} \subset \mathfrak{B}(X/Y)$ which has the property: if $S \in \mathfrak{C}$ then for each $p \in Y^\perp$ the operator $\mathrm{e}_p(Q^{X/Y})^* S \mathrm{e}_p(Q^{X/Y})$ belongs to $\mathfrak{C}$ and depends norm-continuously on p. A map $F : X^* \to \mathfrak{B}(X/Y)$ satisfying (a) will be called Y*-covariant.*

The choice of a supplementary subspace Y' for $Y^\perp$ in X^* determines an isomorphism

$$C_0^X(Y^*; \mathfrak{U}^{X/Y}) \simeq C_0(Y^*; \mathfrak{U}^{X/Y}). \tag{3.15}$$

Indeed, we have a direct sum decomposition $X^* = Y' + Y^\perp$ and the identification $Y^* \simeq Y'$, so we may write $X^* = Y^* \oplus Y^\perp$. We get the isomorphism

(3.15) by associating to each $F : X^* \to \mathfrak{U}^{X/Y}$ the map $F_0 : Y^* \to \mathfrak{U}^{X/Y}$ obtained by restriction: $F_0 = F|Y'$. Then F_0 uniquely determines F if (a) has to be satisfied: if $k \in Y'$ and $p \in Y^\perp$ we must take $F(k+p) = \mathrm{e}_p(Q^{X/Y})^* F_0(k)\, \mathrm{e}_p(Q^{X/Y})$.

If $\psi \in C_0(X^*)$ and $k \in X^*$ we denote ψ_Y^k the map $Y^\perp \ni p \mapsto \psi(k+p)$. Since $Y^\perp = (X/Y)^*$ we may define $\psi(k + P^{X/Y}) := \psi_Y^k(P^{X/Y}) \in \mathfrak{U}^{X/Y}$. The map $k \mapsto \psi(k + P^{X/Y})$ is norm continuous and satisfies (a), (b) as a consequence of the corresponding properties of the map $k \mapsto \psi_Y^k \in C_0(Y^\perp)$.

Theorem 3.8 *There is a unique isomorphism $T \mapsto \widetilde{T}$ of $\mathfrak{U}_Y^X$ onto the algebra $C_0^X(Y^*; \mathfrak{U}^{X/Y})$ such that, if $T = \varphi(Q_Y^X)\psi(P^X)$ for some functions $\varphi \in C_{bu}(X/Y)$, $\psi \in C_0(X^*)$, then $\widetilde{T}(k) = \varphi(Q^{X/Y})\psi(k + P^{X/Y})$.*

Proof. If such a morphism exists, then it is unique by Theorem 3.5. So it suffices to show that there is an isomorphism with the required property, and this can be donne with the help of a supplementary subspace Y_1 for Y and the isomorphisms (3.10), (3.15) determined by it ($Y' = Y_1^\perp$). The proof of Theorem 3.7 shows that $\varphi(Q_Y^X)\psi(P^X)$ is sent into $k \mapsto \varphi(Q^{X/Y})\psi(k+P^{X/Y})$ by this isomorphism. ■

The function $\widetilde{T} \in C_0^X(Y^*; \mathfrak{U}^{X/Y})$ will be called *covariant symbol of T*. It will be convenient to set $T(k) \equiv \widetilde{T}(k)$ for $k \in X^*$, hence to denote $T(\cdot) \equiv \widetilde{T}$.

It is possible to reformulate the definition of the covariant symbol as follows. First note that *there is a unique surjective morphism*

$$\mathfrak{U}_Y^X \ni T \longmapsto T^{X/Y} \in \mathfrak{U}^{X/Y} \tag{3.16}$$

such that $T^{X/Y} = \varphi(Q^{X/Y})\psi(P^{X/Y})$ if $T = \varphi(Q_Y^X)\psi(P^X)$ (in the computation of the operator $\psi(P^{X/Y})$ the restriction of ψ to the subspace $(X/Y)^* = Y^\perp$ of X^* is involved). This is clear from Theorem 3.8. Then one has

$$T(k) \equiv \widetilde{T}(k) = [\mathrm{e}_k(Q^X)^* T \mathrm{e}_k(Q^X)]^{X/Y} \tag{3.17}$$

for all $k \in X^*$.

3.4. Now we return to the study of the C^*-algebras $\mathfrak{C}^X(Y)$. We have

$$\mathfrak{C}^X(Y) = \{T \in \mathfrak{U}_Y^X \mid [\mathrm{e}_k(Q) - 1]T^{(*)} \to 0 \text{ if } k \to 0 \text{ in } Y^\perp\}.$$

It will be clear from the proof of the next theorem that the condition on T^* is not needed.

Theorem 3.9 *$\mathfrak{C}^X(Y)$ coincides with the norm closed linear subspace of $\mathfrak{B}(X)$ generated by the operators of the form $\varphi(Q_Y)\psi(P)$, with $\varphi \in C_0(X/Y)$ and $\psi \in C_0(X^*)$.*

Proof. We have seen in the proof of Proposition 3.3 that the operators

$\varphi(Q_Y)\psi(P)$ belong to $\mathfrak{C}^X(Y)$. Reciprocally, let $T \in \mathfrak{U}^X_Y$ such that $[\mathrm{e}_k(Q) - 1]T \to 0$ if $k \to 0$ in $Y^\perp$. Let $\theta \in C_0(X/Y)$ with $\theta(0) = 1$. Then $[\theta(\varepsilon Q_Y) - 1]T \to 0$ as $\varepsilon \to 0$ (see Lemma 3.2). On the other hand, T is a norm limit of operators of the form $\sum \varphi_i(Q_Y)\psi_i(P)$ with $\varphi_i \in C_{bu}(X/Y)$, $\psi_i \in C_0(X^*)$ (see Theorem 3.5). Thus T is a norm limit of operators of the form $\sum \theta(\varepsilon Q_Y)\varphi_i(Q_Y)\psi_i(P)$. ■

Proposition 3.10 *An operator $T \in \mathfrak{B}(X)$ belongs to $\mathfrak{C}^X(Y)$ if and only if there is a subspace Y' supplementary to $Y^\perp$ in X^* such that the next relations hold:*
$(\mathrm{i})_0$ $[\mathrm{T}, \mathrm{e_y(P)}] = 0 \quad \forall\, \mathrm{y} \in \mathrm{Y}$;
$(\mathrm{ii})_0$ $[\mathrm{T}, \mathrm{e_k(Q)}] \to 0$ *if* $k \to 0$ *in* Y';
$(\mathrm{iii})_0$ $[\mathrm{e_x(P)} - 1]\mathrm{T} \to 0$ *if* $x \to 0$ *in* X;
$(\mathrm{iv})_0$ $[\mathrm{e_k(Q)} - 1]\mathrm{T} \to 0$ *if* $k \to 0$ *in* $Y^\perp$.

Proof. It will be convenient to decompose the condition the condition $(\mathrm{iii})_0$ in two parts: $(\mathrm{iii})_\mathrm{Y}$, obtained by restricting x to Y in $(\mathrm{iii})_0$, and $(\mathrm{iii})_{\mathrm{Y'}}$, obtained by restricting x to $Y_1 := Y'^\perp$ in $(\mathrm{iii})_0$. We use the same conventions and notations as in the proof of Lemma 3.6. We have seen there that conditions $(\mathrm{i})_0$, $(\mathrm{ii})_0$ and $(\mathrm{iii})_\mathrm{Y}$ are equivalent to the existence of $F \in C_0(Y^*; \mathfrak{B}(Y_1))$ such that $T = F(P^Y)$. Then $(\mathrm{iv})_0$ is equivalent to (3.13) while $(\mathrm{iii})_{\mathrm{Y'}}$ can be written as (3.14) but with $F(p)^{(*)}$ replaced by $F(p)$. Now we can use the Riesz-Kolmogorov compacity criterion to get that $F(p)$ is a compact operator for each p. Clearly, for each $\varepsilon > 0$ we can find positive functions $\theta_1, \ldots, \theta_n \in C_c(Y^*)$ and points $p_1, \ldots, p_n \in Y^*$ such that $\sum_i \theta_i(p) \le 1$ on Y^* and $\|F(p) - \sum_i F(p_i)\theta_i(p)\| \le \varepsilon$. Moreover, there are operators of the form $K_i = \sum_j \varphi_{ij}(Q^{Y_1})\psi_{ij}(P^{Y_1})$, with $\varphi_{ij} \in C_c(Y_1)$, $\psi_{ij} \in C_c(Y_1^*)$, such that $\|F(p_i) - K_i\| \le \varepsilon$. Thus

$$\left\| F(p) - \sum_{ij} \varphi_{ij}(Q^{Y_1})\psi_{ij}(P^{Y_1})\theta(p) \right\| \le 2\varepsilon \quad \forall p \in Y^*.$$

This means that there are functions $\varphi_i \in C_c(X/Y)$ and $\psi_i \in C_c(X^*)$ such that $\|T - \sum \varphi_i(Q_Y)\psi(P)\| \le 2\varepsilon$, and this finishes the proof. ■

Observe that the elements of $\mathfrak{C}^X(Y)$ can be characterized in terms of their covariant symbols as follows: the map $T \mapsto T(\cdot)$ is an isomorphism

$$\mathfrak{C}^X(Y) \to C_0^X(Y^*; \mathfrak{K}(X/Y)). \tag{3.18}$$

This map is uniquely determined by the property $T(k) = \varphi(Q^{X/Y})\psi(k + P^{X/Y})$ if $T = \varphi(Q_Y^X)\psi(P^X)$ with $\varphi \in C_0(X/Y)$ and $\psi \in C_0(X^*)$.

More generally, let Y, Z be subspaces of X such that $Y \subset Z$. Then $T \mapsto T(\cdot)$ induces an isomorphism

$$\mathfrak{C}^X(Z) \to C_0^X(Y^*; \mathfrak{C}^{X/Y}(Z/Y))$$

uniquely determined by the property $T(k) = \varphi(Q^{X/Y}_{Z/Y})\psi(k + P^{X/Y})$ if $T = \varphi(Q^X_Z)\psi(P^X)$ and $\varphi \in C_0(X/Z)$, $\psi \in C_0(X^*)$. Note that one has $Z/Y \subset X/Y$ and $(X/Y)/(Z/Y) = X/Z$, $\pi^X_Z = \pi^{X/Y}_{Z/Y} \circ \pi^X_Y$.

Clearly $\mathfrak{C}^X \subset \mathfrak{U}^X$ and $\mathfrak{C}^X_Y \subset \mathfrak{U}^X_Y$. By taking into account the remarks made above in connection with $\mathfrak{C}^X(Z)$, we have the following description of $\mathfrak{C}^X_Y$ in terms of covariant symbols:

$$\mathfrak{C}^X_Y = \{T \in \mathfrak{U}^X_Y \mid T(k) \in \mathfrak{C}^{X/Y}\ \forall k \in X^*\}. \tag{3.19}$$

In other terms, $T \mapsto T(\cdot)$ gives a canonical isomorphism

$$\mathfrak{C}^X_Y \cong C^X_0(Y^*; \mathfrak{C}^{X/Y}) \tag{3.20}$$

and a choice of a supplementary subspace Y' for $Y^\perp$ in X^* determines a non-canonical isomorphism (cf. (3.15)):

$$\mathfrak{C}^X_Y \simeq C_0(Y^*; \mathfrak{C}^{X/Y}) = C_0(Y^*) \otimes \mathfrak{C}^{X/Y}. \tag{3.21}$$

We discuss now a property of the covariant symbols which will be needed for the proof of the Mourre estimate. Let $Y \subsetneq Z \subsetneq X$ and $T \in \mathfrak{C}^X$. Then $T_Y \in \mathfrak{C}^X_Y$ and its covariant symbol is a map $T_Y(\cdot) : X^* \to \mathfrak{C}^{X/Y}$, hence for each $k \in X^*$ one gets an operator $T_Y(k) \in \mathfrak{C}^{X/Y}$. But Z/Y is a subspace of X/Y so we can consider the operator $[T_Y(k)]_{Z/Y} \in \mathfrak{C}^{X/Y}_{Z/Y}$ whose covariant symbol is a map $[T_Y(k)]_{Z/Y}(\cdot) : (X/Y)^* = Y^\perp \to \mathfrak{C}^{X/Z}$, where we used the identification $(X/Y)/(Z/Y) = X/Z$. Thus for each $p \in Y^\perp$ we get an operator $[T_Y(k)]_{Z/Y}(p) \in \mathfrak{C}^{X/Z}$. On the other hand, we may also consider the operator $T_Z \in \mathfrak{C}^X_Z$ whose covariant symbol is a map $T_Z(\cdot) : X^* \to \mathfrak{C}^{X/Z}$. We shall prove that

$$[T_Y(k)]_{Z/Y}(p) = T_Z(k + p). \tag{3.22}$$

Indeed, it suffices to show this for $T = \varphi(Q^X_E)\psi(P^X)$, where E is a subspace of X and $\varphi \in C_0(X/E), \psi \in C_0(X^*)$. Moreover, we may assume $E \supset Z$, otherwise both members above are zero. Then $T_Y = T$ and from $\pi^X_E = \pi^{X/Y}_{E/Y} \circ \pi^X_Y$ we get $T_Y(k) = \varphi \circ \pi^{X/Y}_{E/Y}(Q^{X/Y})\psi^k_Y(P^{X/Y})$ for all $k \in X^*$. Since $E/Y \supset Z/Y$ we shall also have $[T_Y(k)]_{Z/Y} = T_Y(k)$. From $\pi^{X/Y}_{E/Y} = \pi^{X/Z}_{E/Z} \circ \pi^{X/Y}_{Z/Y}$ we obtain $[T_Y(k)]_{Z/Y}(p) = \varphi \circ \pi^{X/Z}_{E/Z}(Q^{X/Z})\psi^k_Y(p + P^{X/Z})$ which is the same as $\varphi(Q^{X/Z}_{E/Z})\psi(k + p + P^{X/Z})$, so (3.22) is proved.

We also observe that by taking $k = 0$ in (3.22) we get an expression for the covariant symbol of the projection onto $\mathfrak{C}^{X/Y}_{Z/Y}$ of the operator $T^{X/Y}$:

$$T^{X/Y}_{Z/Y}(p) = T_Z(p) \tag{3.23}$$

for all $p \in (X/Y)^* = Y^\perp$. And then by taking $p = 0$ we get

$$[T^{X/Y}]^{Z/Y} = T^{X/Z}. \tag{3.24}$$

4 The Hamiltonian and its Essential Spectrum

4.1. We are interested in the spectral analysis of the self-adjoint operators affiliated to the $\mathbb{G}(X)$-graded C^*-algebra $\mathfrak{C}^X$. Note that according to Proposition 3.3 all the nondegeneracy requirements from Section 2.3 are satisfied. Observe also that $\mathfrak{C}^X(\min \mathbb{G}(X)) = \mathfrak{C}^X(O) = \mathfrak{K}(X)$, $\mathfrak{C}^X(\max \mathbb{G}(X)) = \mathfrak{C}^X(X) = C_0(X^*)$.

Let H be a self-adjoint operator on $\mathcal{H}(X)$ strictly affiliated to $\mathfrak{C}^X$. Then for each subspace Y of X there is a unique self-adjoint operator H_Y on $\mathcal{H}(X)$, called *sub-hamiltonian* of H, such that $\mathcal{P}_Y[\varphi(H)] = \varphi(H_Y)$ for all $\varphi \in C_0(\mathbb{R})$, and this operator is strictly affiliated to $\mathfrak{C}^X_Y$ (use Proposition 2.3). According to standard conventions we should set $H_0 = H_X$, but note that $H = H_O$, where O is the zero dimensional subspace. We may introduce the covariant symbol $H_Y(\cdot)$; this is the map $k \mapsto H_Y(k)$ from X^* to the space of self-adjoint operators on $\mathcal{H}(X/Y)$ defined by the rule:

$$\theta(H)_Y(k) = \theta(H_Y(k)) \text{ for all } \theta \in C_0(\mathbb{R}) \text{ and } k \in X^*.$$

Then we have

(i) $H_Y(k+p) = \mathrm{e}_p(Q^{X/Y})^* H_Y(k) \mathrm{e}_p(Q^{X/Y})$ if $k \in X^*$, $p \in (X/Y)^* = Y^\perp$;

(ii) $k \mapsto H_Y(k)$ is continuous in norm resolvent sense;

(iii) $H_Y(k) \to \infty$ if $\mathrm{dist}(k, Y^\perp) \to \infty$, in the sense that: for each compact $\Lambda \subset \mathbb{R}$ there is $r < \infty$ such that $\sigma(H_Y(k)) \cap \Lambda = \emptyset$ if $\mathrm{dist}(k, Y^\perp) > r$.

It is clear from (i) that $\sigma(H_Y(k))$ depends only on the class of k in $X^*/Y^\perp = Y^*$. For $p \in Y^*$ let $\sigma(p) := \sigma(H_Y(k))$ if $k \in X^*$ is in the class p. Then we have $\sigma(H_Y) = \cup_{p \in Y^*} \sigma(p)$.

We also define $H^{X/Y} = H_Y(0)$; then $H^{X/Y}$ is a self-adjoint operator strictly affiliated to $\mathfrak{C}^{X/Y}$. These operators will play an important rôle in the proof of the Mourre estimate.

Our first theorem shows that the essential spectrum $\sigma_{\mathrm{ess}}(H)$ of H can be computed in terms of the spectra of its sub-hamiltonians. The next lemma will be needed in the proof.

Lemma 4.1 *If $Y \neq O$ and $S \in \mathfrak{C}^X_Y$ is a self-adjoint operator then $\sigma(S)$ is an interval.*

Proof. By using $\mathfrak{C}^X_Y \subset \mathfrak{U}^X_Y$ and Lemma 3.6 we see that we may suppose that there is a Hilbert space $\mathcal{H}_0$ such that S is the operator of multiplication by a continuous function $S(\cdot) : Y^* \to B(\mathcal{H}_0)$ with the property $\|S(k)\| \to 0$ if $k \to \infty$. Clearly then $\sigma(S)$ is equal to the closure of the union of the spectra of the operators $S(k)$. If $\alpha(k), \beta(k)$ are the lower and upper bounds respectively of $S(k)$ then $\sigma(S(k)) \subset [\alpha(k), \beta(k)]$. Moreover, we have

$\alpha(k), \beta(k) \in \sigma(S(k))$ for all $k \in Y^*$ and $\alpha, \beta \in C_0(Y^*)$. This clearly implies $\sigma(S) = [\min \alpha, \max \beta]$. ■

The set of atoms of the lattice $\mathbb{G}(X)$ is the projective space $\mathbb{P}(X)$ of X, i.e. the set of one dimensional subspaces of X.We mention a consequence of relation (3.6): for each $Y \in \mathbb{P}(X)$ and $\omega \in Y$, $\omega \neq 0$:

$$H_Y = s - \lim_{\lambda \to \infty} \mathrm{e}_{\lambda\omega}(P)^* H \mathrm{e}_{\lambda\omega}(P) \tag{4.1}$$

in the strong resolvent sense.

Theorem 4.2 *Let H be a self-adjoint bounded from below operator on $\mathcal{H}(X)$ and let us assume that H is strictly affiliated to the C^*-algebra $\mathfrak{C}^X$. Set $\tau_Y = \inf H_Y$ and let $\tau = \min\{\tau_Y \mid Y \in \mathbb{P}(X)\}$. Then for $Y \neq O$ one has $\sigma(H_Y) = [\tau_Y, \infty)$. Moreover, $\sigma_{\mathrm{ess}}(H) = [\tau, \infty)$.*

Proof. We first prove that the spectrum of H_Y is an interval if $Y \neq O$. Let c be a real number such that $H_Y + c \geq \text{const} > 0$ and set $S = (H_Y + c)^{-1}$. Then it suffices to show that the spectrum of S is an interval. Since $S \in \mathfrak{C}_Y^X$ and is a self-adjoint operator, this follows from Lemma 4.1. Now according to Theorem 2.10 and Proposition 5.4 below we have

$$\sigma_{\mathrm{ess}}(H) = \bigcup_{Y \in \mathbb{P}(X)} \sigma(H_Y) \tag{4.2}$$

which clearly proves the theorem. ■

If H is affiliated to $\mathfrak{C}^X(\mathcal{E})$ for some stable under intersections set $\mathcal{E} \subset \mathbb{G}(X)$ and if Z is the smallest element of $\mathcal{E}$ such that $Z \supset Y$ for a given $Y \in \mathbb{P}(X)$, then clearly $H_Y = H_Z$. This gives a version of Theorem 4.2 which looks like the usual HVZ theorem.

There is a second version of Theorem 4.2 which is worth to be mentioned. This is an immediate consequence of (4.1) and (4.2).

Theorem 4.3 *Let H be a self-adjoint operator on $\mathcal{H}(X)$ strictly affiliated to the $\mathfrak{C}^X$. Then for each $\omega \in X$ the limit $H_\omega = s - \lim_{\lambda\to\infty} e_{\lambda\omega}(P)^* H e_{\lambda\omega}(P)$ exists in the strong resolvent sense and $\sigma_{\mathrm{ess}}(H) = \bigcup_{\omega\neq 0} \sigma(H_\omega)$.*

4.2. In this section we apply the results of Section 2.3 and get an explicit class of operators strictly affiliated to the algebra $\mathfrak{C}^X$. Let $H_0 \equiv H_X$ be a self-adjoint, bounded from below operator on $\mathcal{H}(X)$, strictly affiliated to $\mathfrak{C}^X(X) = C_0(X^*)$. Equivalently, let $H_0 = h(P)$ for some continuous divergent function $h : X^* \to \mathbb{R}$ (we say that h is divergent if $h(k) \to \infty$ if $k \to \infty$ in X^*). We keep the notations $\mathcal{G}$, $\mathcal{G}^*$ from Section 2.2.

We recall that the Sobolev scale $\{\mathcal{H}^s(X)\}_{s\in\mathbb{R}}$ associated to the translation group is defined by the norms $\|f\|_s = \|\langle P\rangle^s f\|$. Here (and later) we use norms $|\cdot|$ on X^* and $\|\cdot\|$ on $\mathcal{H}(X)$ and set $\langle k\rangle = (1+|k|^2)^{1/2}$. One has $\mathcal{H}^0(X) = \mathcal{H}(X)$ and $\mathcal{H}^s(X) \subset \mathcal{H}^t(X)$ as topological vector spaces if $t \le s$. We denote $\|\cdot\|_{s,t}$ the norm in the space $B(\mathcal{H}^s(X), \mathcal{H}^t(X))$. Of course, the choice of norms is not important.

In this section we shall consider only the case when $\mathcal{G} = \mathcal{H}^s(X)$ for some $s > 0$. Since one may take as norm on $\mathcal{G}$ the expression $\langle f, [\lambda + h(P)]f\rangle^{1/2}$ for λ large enough, one has $\mathcal{G} = \mathcal{H}^s(X)$ for some $s > 0$ if and only if there are constants $C, R > 0$ such that

$$C^{-1}\langle k\rangle^{2s} \;\le\; h(k) \;\le\; C\langle k\rangle^{2s} \quad \text{if } |k| > R. \tag{4.3}$$

In order to state conditions on the perturbations $H(Y)$ (cf. Theorem 2.11) it is convenient to introduce analogues of the spaces $\mathfrak{C}^X(Y)$ consisting of unbounded operators.

Definition 4.4 *Let $s > 0$ real and Y a subspace of X. Then $\mathfrak{C}^X_s(Y)$ is the set of continuous operators $S : \mathcal{H}^s(X) \to \mathcal{H}^{-s}(X)$ such that the following properties hold for some real $t > s$:*

(i) $[S, \mathrm{e}_y(P)] = 0 \quad \forall y \in Y$,

(ii) $\|[S, \mathrm{e}_k(Q)]\|_{s,-t} \to 0$ if $k \to 0$ in X^*,

(iii) $\|[\mathrm{e}_k(Q) - 1]S\|_{s,-t} \to 0$ if $k \to 0$ in $Y^\perp$.

It could seem more natural to consider the more restricted class of operators obtained by requiring $t = s$. However, the little gain of generality afforded by the preceding definition is useful in applications. We shall prove below that one gets the same class of operators if in (ii) k is restricted to a subspace Y' supplementary to $Y^\perp$ in X^* (as in (ii)_0 of Proposition 3.10). An easy convexity argument shows that if (ii) (or (iii)) holds for one $t > s$, then it holds for all $t > s$. Moreover, by an argument similar to that in the proof of Lemma 3.2, the condition (iii) is equivalent with the existence of a function $\theta \in C^\infty(X/Y)$, with $\theta(z) = 0$ if z is in a neighbourhood of zero and $\theta(z) = 1$ if z is outside a compact set, such that $\|\theta(\varepsilon Q_Y)S\|_{s,-t} \to 0$ if $\varepsilon \to 0$.

Lemma 4.5 *An operator $S \in B(\mathcal{H}^s(X), \mathcal{H}^{-s}(X))$ belongs to $\mathfrak{C}^X_s(Y)$ if and only if one has $\langle P\rangle^{-t}S\langle P\rangle^{-s} \in \mathfrak{C}^X(Y)$ for some $t > s$. And then the last property holds for all $t > s$.*

Proof. Let $T = \langle P\rangle^{-t}S\langle P\rangle^{-s}$. It is clear that T satisfies (i)_0 of Proposition 3.10 if and only if S satisfies (i) of Definition 4.4. Let $\omega(k) = \langle k\rangle^s$ and $u = t/s$, $v = u - 1 > 0$. Then

$$\|[\mathrm{e}_x(P) - 1]T\| \le \left\|[\mathrm{e}_x(P) - 1]\omega(P)^{-v}\right\| \cdot \left\|\omega(P)^{-1}S\omega(P)^{-1}\right\|$$

hence (iii)_0 of Proposition 3.10 follows from $\omega(k)^{-v} \to 0$ as $k \to \infty$.

For the rest of the proof we need the following observation: if r is an arbitrary real number then

$$\left\| [\mathrm{e}_k(Q), \omega(P)^r]\omega(P)^{-r} \right\| \to 0 \text{ if } k \to 0 \text{ in } X^*. \tag{4.4}$$

To prove, this, note that the norm above is equal to

$$\sup_{p \in X^*} \left| 1 - \frac{w(p+k)}{w(p)} \right|$$

where $w(k) = \omega(k)^r$, so it suffices to show that this quantity tends to zero when $k \to 0$. Clearly $\langle p+k \rangle \leq (1+|\,k\,|)\langle p \rangle$ for all $k, p \in X^*$. Replacing p by $p-k$ and then changing k in $-k$ we also get $\langle p+k \rangle \geq (1+|\,k\,|)^{-1}\langle p \rangle$. Thus for $r \in \mathbb{R}$ we have $w(k+p) \leq (1+C\,|\,k\,|)^N w(p)$ with $C=1$ and $N = sr$ (we write this estimate in a form more general than needed here in view of later applications). Then clearly

$$(1+C\,|\,k\,|)^{-N} \leq \frac{w(p+k)}{w(p)} \leq (1+C\,|\,k\,|)^N$$

from which the result follows.

Now we can check $(\mathrm{ii})_0$ of Proposition 3.10 by using

$$\begin{aligned} [\mathrm{e}_k(Q), T] \quad = \quad & [\mathrm{e}_k(Q), \omega(P)^{-u}]\omega(P)^u \cdot T \\ & + \omega(P)^{-u}[\mathrm{e}_k(Q), S]\omega(P)^{-1} + T \cdot \omega(P)[\mathrm{e}_k(Q), \omega(P)^{-1}]. \end{aligned}$$

For $(\mathrm{iv})_0$ we use

$$[\mathrm{e}_k(Q) - 1]T = [\mathrm{e}_k(Q), \omega(P)^{-u}]\omega(P)^u \cdot T + \omega(P)^{-u}[\mathrm{e}_k(Q) - 1]S\omega(P)^{-1}.$$

Thus we see that T verifies all the conditions of Proposition 3.10, so $T \in \mathfrak{C}^X(Y)$. Reciprocally, if T has this property then we check (ii) of Definition 4.4 by writing

$$\begin{aligned} \|[\mathrm{e}_k(Q), S]\|_{s,-t} &= \left\| \langle P \rangle^{-t} [\mathrm{e}_k(Q), S] \langle P \rangle^{-t} \right\| \\ &= \left\| [\mathrm{e}_k(Q), T] - [\mathrm{e}_k(Q), \langle P \rangle^{-t}]\langle P \rangle^t \cdot T - T \cdot \langle P \rangle^s [\mathrm{e}_k(Q), \langle P \rangle^{-s}] \right\| \end{aligned}$$

and using (4.4). Then (iii) is proved similarly, hence $S \in \mathfrak{C}^X_s(Y)$. Observe that this also proves $\|S[\mathrm{e}_k(Q) - 1]\|_{s,-t} \to 0$ if $k \to 0$ in $Y^\perp$.

The last assertion of the lemma is a consequence of the analyticity of the map $t \mapsto \langle P \rangle^{-t} S \langle P \rangle^{-s} \in B(\mathcal{H})$ on the interval (s, ∞) (see the proof of Lemma 2.6). ■

The preceding lemma and Proposition 3.10 imply an assertion we made before: in condition (ii) of Definition 4.4 it suffices to assume $k \in Y'$, where Y' is a supplementary subspace for $Y^\perp$ in X^*.

Theorem 4.6 *Let $H_0 = h(P)$ for some continuous function h on X^* satisfying (4.3) for some $s > 0$. Set $H(X) = 0$ and for each $Y \in \mathbb{G}(X) \setminus \{X\}$ let $H(Y) \in \mathfrak{C}^X_s(Y)$ be symmetric and such that $H(Y) \geq -\mu_Y H_0 - \delta_Y$ for some positive numbers μ_Y, δ_Y with $\sum \mu_Y < 1$, $\sum \delta_Y < \infty$. Assume that the family $\{H(Y)\}_{Y \in \mathbb{G}(X)}$ is summable in $B(\mathcal{H}^s; \mathcal{H}^{-s})$. Let $H = H_0 + \sum H(Y)$, the sum being taken in the sense of forms. Then H is a self-adjoint operator strictly affiliated to $\mathfrak{C}^X$ and $\mathcal{P}_Y[H] = H_0 + \sum_{Z \supset Y} H(Z)$ for all $Y \in \mathbb{G}(X)$.*

Proof. The only condition of Theorem 2.11 which has not been already checked is (2.7). Let $\omega(k) = \langle k \rangle^s$, then we write

$$(\lambda + H_0)^{-\alpha} H(Y)(\lambda + H_0)^{-1/2} = (\lambda + H_0)^{-\alpha}\omega(P)^{2\alpha} \cdot T \cdot \omega(P)(\lambda + H_0)^{-1/2}$$

where $T = \omega(P)^{-2\alpha} H(Y)\omega(P)^{-1}$. We have $H_0 = h(P)$ for some $h : X^* \to \mathbb{R}$ continuous and verifying (4.3). So the operators $(\lambda + H_0)^{-\alpha}\omega(P)^{2\alpha}$ and $\omega(P)(\lambda + H_0)^{-1/2}$ are of the form $\psi(P)$ for some $\psi \in C_b(X^*)$. By taking into account Proposition 3.3(ii) we see that (2.7) is a consequence of $T \in \mathfrak{C}^X(Y)$. But this holds by Lemma 4.5. ■

We recall that an operator affiliated to $\mathfrak{C}^X$ is necessarily affiliated to the closure of $\sum_{Y \in \mathcal{E}} \mathfrak{C}^X(Y)$ for some countable, stable under intersections subset $\mathcal{E}$ of $\mathbb{G}(X)$. However, the requirement of convergence in $B(\mathcal{H}^s; \mathcal{H}^{-s})$ of the series $\sum H(Y)$ can be relaxed, e.g. one can take $H = H_0 + V$ with V in the closure of $\sum_{Y \in \mathbb{G}(X)} \mathfrak{C}^X_s(Y)$ in $B(\mathcal{H}^s; \mathcal{H}^{-s})$.

It is possible to give a more explicit description of the class $\mathfrak{C}^X_s(Y)$ by using the covariant symbols of its elements but this not useful for our purposes. We shall give only an explicit construction of a large class of operators belonging to this space. For this it is convenient to use the following notion (see Definition 9.4.7 in [ABG]). Let Z be a real finite dimensional vector space and let $L : \mathcal{H}^s(Z) \to \mathcal{H}^t(Z)$ be a continuous linear operator for some $s, t \in \mathbb{R}$. One says that L *is small at infinity* if for some (and hence for all) $u < t$ one of the following equivalent conditions is satisfied:

(1) $L : \mathcal{H}^s(Z) \to \mathcal{H}^u(Z)$ is compact.
(2) $\left\|[\mathrm{e}_k(Q^Z) - 1]L\right\|_{s,u} \to 0$ if $k \to 0$ in Z^*.
(3) There is $\theta \in C^\infty(Z)$, with $\theta(z) = 0$ on a neighbourhood of zero and $\theta(z) = 1$ for large z, such that $\left\|\theta(\varepsilon Q^Z)L\right\|_{s,u} \to 0$ as $\varepsilon \to 0$.

In the next proposition we use the same notations as in the proof of Lemma 3.6.

Proposition 4.7 *Let $F : Y^* \to B(\mathcal{H}^s(X/Y); \mathcal{H}^{-s}(X/Y))$ be a norm continuous map such that the operators $F(k)$ are symmetric and satisfy:*
(i) *$\pm F(k) \leq C[\langle P^{X/Y}\rangle^{2s} + \langle k \rangle^{2s}]$ for some constant C and all $k \in Y^*$,*
(ii) *$F(k) : \mathcal{H}^s(X/Y) \to \mathcal{H}^{-s}(X/Y)$ is small at infinity for each $k \in Y^*$.*

Let Y_1 be a supplementary subspace for Y in X, identify $X/Y = Y_1$, and define $F(P^Y) : \mathcal{H}^s(X) \to \mathcal{H}^{-s}(X)$ by

$$\langle f, F(P^Y)f\rangle = \int_{Y^*} \langle (\mathcal{F}_Y f)(k), F(k)\,(\mathcal{F}_Y f)(k)\rangle\, dk.$$

Then $F(P^Y) \in \mathfrak{C}^X_s(Y)$.

Proof. Clearly $\mathcal{F}_Y\langle P\rangle^{-t}F(P^Y)\langle P\rangle^{-s}\mathcal{F}^*_Y$ is the operator of multiplication by the operator valued map

$$k \mapsto (1 + |P^{X/Y}|^2 + |\,k\,|^2)^{-t/2}F(k)(1 + |P^{X/Y}|^2 + |\,k\,|^2)^{-s/2}$$

and the result follows from the lemmas 4.5 and 3.6 and from (3.18). ■

Not all the symmetric operators in $\mathfrak{C}^X_s(Y)$ are of the form $F(P^Y)$ with F as above: indeed, we could require that $F : Y^* \to B(\mathcal{H}^s(Y_1); \mathcal{H}^{-t}(Y_1))$ be norm continuous for some $t > s$.

Finally, we give an explicit example in the case $s = t = 0$ which shows that the notion of smallness at infinity used above goes beyond the classical setting. Let $m : Z \to \mathbb{C}$ be a bounded Borel function. If $m(z) \to 0$ as $|\,z\,| \to \infty$ then clearly the linear continuous operator $m(Q) : \mathcal{H}(Z) \to \mathcal{H}(Z)$ of multiplication by m is small at infinity. But such a condition is by no means necessary:

Proposition 4.8 *Let $m : Z \to \mathbb{C}$ be a bounded Borel function such that the measure of the set $\{z \in Z \,|\, |\,m(z)\,| > \varepsilon\}$ is finite for each $\varepsilon > 0$. Then the linear operator $m(Q) : \mathcal{H}(Z) \to \mathcal{H}(Z)$ is small at infinity.*

Remark The preceding hypotheses are satisfied if $m \in L^p(Z) \cap L^\infty(Z)$ for some $p < \infty$. Note that such a function could not tend to zero at infinity.

Proof. (i) Let $A \subset Z$ be a Borel set of finite measure. We first prove that for each $s > 0$ the operator χ_A of multiplication by the characteristic measure of A is compact operator $\mathcal{H}^s(Z) \to \mathcal{H}(Z)$. By interpolation it suffices to prove this for large s; we shall take $s > \dim Z/2$. Then $\chi_A = \chi_A\langle P\rangle^{-s}.\langle P\rangle^s$, $\langle P\rangle^s : \mathcal{H}^s(Z) \to \mathcal{H}(Z)$ is an isomorphism, and $\chi_A\langle P\rangle^{-s}$ is a Hilbert-Schmidt operator on $\mathcal{H}(Z)$.
(ii) Let $\theta \in C^\infty(Z)$ with $\theta(z) = 0$ on a neighbourhood of the origin and $\theta(z) = 1$ on a neighbourhood of infinity and set $\theta_r = \theta(Qr^{-1})$. We show that $\|\theta_r m(Q)\|_{s,0} \to 0$ as $r \to \infty$; this suffices to prove the proposition, as is clear by a transposition argument. Let $\varepsilon > 0$ and $A = \{z \in Z \,|\, |\,m(z)\,| > \varepsilon\}$. Denote $c = \sup|\,m(z)\,|$. Then for each $f \in \mathcal{H}(Z)$ one has:

$$\begin{aligned}\|\theta_r m f\|^2 &= \int_A |\,\theta_r\, m f\,|^2\, dz + \int_{Z\setminus A} |\,\theta_r\, m f\,|^2\, dz \\ &\le c^2\, \|\theta_r\, \chi_A f\|^2 + \varepsilon^2 \|f\|^2\end{aligned}$$

Since $\theta_r \to 0$ strongly on $\mathcal{H}(Z)$ and $\chi_A : \mathcal{H}^s \to \mathcal{H}$ is compact one has $\|\theta_r \chi_A\|_{s,0} \to 0$ as $r \to \infty$. So there is $R > 0$ such that $\|\theta_r\, \chi_A f\| \le \frac{\varepsilon}{c}\, \|f\|_{\mathcal{H}^s}$ for all $f \in \mathcal{H}^s$ and $r \ge R$. Hence for $r \ge R$ we get $\|\theta_r\, m(Q)\|_{s,0} \le \sqrt{2}\varepsilon$. ■

4.3. We end this section with a result concerning the algebra $\mathfrak{U}^X$ and some remarks on a more general class of kinetic energies.

Theorem 4.9 *Let $H_0 = h(P)$ for some continuous function h on X^* satisfying (4.3) for some $s > 0$ and such that $|h(p+k) - h(p)|\langle p\rangle^{-2s} \to 0$ when $k \to 0$, uniformly in p. Let V be a standard form perturbation of H_0 such that $\|[V, \mathrm{e}_k(Q)]\|_{s,-s} \to 0$ if $k \to 0$ in X^*. Then the operator $H = H_0 + V$ is strictly affiliated to $\mathfrak{U}^X$.*

Proof. We check the conditions of Lemma 2.7 by using directly the definition of $\mathfrak{U}^X$. Let $\lambda > 0$ be a large positive number, $R = (H+\lambda)^{-1}$, $\Omega = \langle P\rangle^s$ and $U = \Omega^{-1} V \Omega^{-1}$, so that

$$RVR \ldots VR = R\Omega \cdot U \cdot \Omega^2 R \cdot U \cdot \ldots \cdot \Omega^2 R \cdot U \cdot \Omega R.$$

The right hand side above is written as a product of bounded operators and $R\Omega = \psi(P)$ with $\psi \in C_0(X^*)$, hence condition (ii) of Theorem 3.5 is satisfied. From (4.4) and our hypothesis on V it follows that $T = U$ fulfills (i) of Theorem 3.5. The fact that $\Omega^2 R$ also has this property follows immediately from our hpotheses on h. Thus $RVR \ldots VR \in \mathfrak{U}^X$. ■

We shall explain now how one can treat operators whose form domain is not a Sobolev space (e.g. hypoelliptic operators). The proof of Lemma 4.5 shows that if $w : X^* \to (0, \infty)$ satisfies $w(k+p) \le (1 + C|k|)^N w(p)$ for some $C > 0, N \in \mathbb{R}$ and a norm $|\cdot|$ on X^*, then $\left\|[\mathrm{e}_k(Q), w(P)]\, w(P)^{-1}\right\| \to 0$ if $k \to 0$ in X^*. We call such a function w a strict $N-$weight (see Section 10.1 in [H]). If $w : X^* \to (0, \infty)$ is continuous and satisfies $w(k+p) \le C\langle k\rangle^N w(p)$, we say that w is a $N-$weight. We have:

Lemma 4.10 *Let $H_0 = h(P)$ for some continuous divergent function h on X^*. Denote $\mathcal{G}$ the space $D(|H_0|^{1/2})$ equipped with the graph topology. Then for each $N > 0$ the following assertions are equivalent:*
(i) *$\mathrm{e}_k(Q)\mathcal{G} \subset \mathcal{G}$ and $\|\mathrm{e}_k(Q)\|_{B(\mathcal{G})} \le C\langle k\rangle^N$ for a constant C and all $k \in X^*$;*
(ii) *there is a divergent $N-$weight w such that $\mathcal{G}$ is the domain of $w(P)$;*
(iii) *there is a divergent strict $N-$weight such that* (ii) *holds.*

Proof. We may assume $h(k) > 0$ for all k and then we may take as norm on $\mathcal{G}$ the expression $\langle f, h(P)f\rangle^{1/2}$. From $\mathrm{e}_{-k}(Q)h(P)\mathrm{e}_k(P) = h(P+k)$ we see that condition (i) is equivalent to $\langle f, h(P+k)f\rangle \le C^2\langle k\rangle^{2N}\langle f, h(P)f\rangle$ for all f which, in turn, is equivalent to $h(k+p) \le C^2\langle k\rangle^{2N} h(p)\ \forall k, p \in X^*$. If we set $\omega(k) = h(k)^{1/2}$ then we get a continuous divergent function on X^* such that $\omega(k+p) \le C\langle k\rangle^N \omega(p)$. Then by using the Remark after Theorem

10.1.5 from [H] we obtain a strict $N-$weight w such that $w \leq \omega \leq C_1 w$ for a constant C_1. ■

Observe that (ii) above is equivalent to $C^{-1}w(k)^2 \leq h(k) \leq Cw(k)^2$ for all $|k| \geq R$ and some constants C, R. It is rather straightforward now to develop the theory replacing above the Sobolev space $\mathcal{H}^s(X)$ by $\mathcal{H}^w(X) = \{f \in \mathcal{H}(X) \mid w(P)f \in \mathcal{H}(X)\}$, see Sections 2.7 and 2.9 in [GM].

5 The Mourre Estimate

5.1. In order to get finer spectral properties of an operator affiliated to $\mathfrak{C}^X$ we shall use the Mourre method with the generator of dilations as conjugate operator. In this section we discuss some general facts concerning the Mourre estimate, which is the main technical point of the theory.

Let A, H be self-adjoint operators on a Hilbert space $\mathcal{H}$. We say that H *is of class* $C^1(A)$ if the map $\tau \longmapsto e^{-iA\tau}(H+i)^{-1}e^{iA\tau}$ is strongly differentiable and that H *is of class* $C^1_u(A)$ if this map is norm differentiable. One can show (see Theorem 6.2.10 from [ABG]) that if H is of class $C^1(A)$ then $D(A) \cap D(H)$ is a core for H and there is a sesquilinear form $[H, iA]$ on $D(H)$ continuous for the graph topology and such that $\langle f, [H, iA]f\rangle = 2Re\langle Hf, iAf\rangle$ for all $f \in D(A) \cap D(H)$. Another characterization of the $C^1(A)$ property is given by the next result.

Proposition 5.1 *Let A, H be self-adjoint operators in a Hilbert space $\mathcal{H}$. Then $H \in C^1(A)$ if and only if the next limit exists for each $f \in D(H)$:*

$$\lim_{t\to 0} \frac{1}{t}[\langle Hf, e^{itA}f\rangle - \langle f, e^{itA}Hf\rangle]. \tag{5.1}$$

Under this condition, there is a unique operator $[H, iA] \in B(D(H); D(H)^*)$ such that the preceding limit be equal to $\langle f, [H, iA]f\rangle$, and we shall have for each $z \notin \sigma(H)$:

$$[iA, (H-z)^{-1}] = (H-z)^{-1}[H, iA](H-z)^{-1}. \tag{5.2}$$

Proof. Let $z \in \mathbb{C}\backslash\sigma(H)$ and denote $R = (H-z)^{-1}$. Then for each $g \in \mathcal{H}$ and $t \neq 0$ we have:

$$\langle g, \frac{1}{t}[\mathrm{e}^{itA}, R]g\rangle = \frac{1}{t}\langle (H-\overline{z})R^*g, \mathrm{e}^{itA}Rg\rangle - \frac{1}{t}\langle R^*g, \mathrm{e}^{itA}(H-z)Rg\rangle$$

$$= \frac{1}{t}[\langle HR^*g, \mathrm{e}^{itA}Rg\rangle - \langle R^*g, \mathrm{e}^{itA}HRg\rangle]. \tag{5.3}$$

If (5.1) holds then by using the polarization identity we see that the limit of the right-hand side above exists. An easy argument shows then that R is of class $C^1(A)$ (Lemma 6.2.3(a) from [ABG] would give a stronger result). The fact that the limit in (5.1) is of the form $\langle f, [H, iA]f\rangle$ for a unique continuous operator $[H, iA] : D(H) \to D(H)^*$ (or a continuous quadratic form on $D(H)$) is clear by the uniform boundedness principle. Then (5.2) implies

$$\langle g, [iA, R]g\rangle = \langle R^* g, [H, iA]Rg\rangle$$

so (5.2) is proved. The reciprocal assertion, i.e. that (5.1) exists for each $f \in D(H)$ if R is of class $C^1(A)$, is also a consequence of (5.3). ■

A bounded operator S is called of class $C^1(A)$ if $e^{-iA\tau}Se^{iA\tau}$ is strongly differentiable as a function of τ. This is equivalent to the fact that the sesquilinear form $\langle f, SAf\rangle - \langle Af, Sf\rangle$ with domain $D(A)$ extends to a continuous form, denoted $[S, A]$, on $\mathcal{H}$. From the Helfer-Sjöstrand formula (see (8.1.1) in [ABG]) it follows that if H is of class $C^1(A)$ and $\psi \in C_c^2(\mathbb{R})$ then $\psi(H)$ is also of class $C^1(A)$.

One says that an operator H of class $C^1(A)$ satisfies a Mourre estimate (with respect to A) at some point $\lambda \in \mathbb{R}$ if there are a strictly positive real number c, a real function $\varphi \in C_c(\mathbb{R})$ with $\varphi(\lambda) \neq 0$, and a compact operator K such that

$$\varphi(H)[H, iA]\varphi(H) \geq c\varphi(H)^2 + K. \tag{5.4}$$

Note that the sesquilinear form from the l.h.s. above can be identified with a bounded symmetric operator in $\mathcal{H}$. Indeed, (5.1) implies

$$\varphi(H)[H, iA]\varphi(H) = \varphi(H)[\psi(H), iA]\varphi(H) \tag{5.5}$$

if $\psi \in C_c^2(\mathbb{R})$ and $\psi(x)\varphi(x) = x\varphi(x)\,\forall x$. The closed real set

$$\tau_A(H) = \{\lambda \in \mathbb{R} \mid H \text{ does not satisfy a Mourre estimate at } \lambda\,\}$$

will be called the set of *A-thresholds of H*. If (5.4) holds with $K = 0$ we say that H satisfies a strict Mourre estimate at λ and we define the (closed) set of *A-critical points of H* by

$$\kappa_A(H) = \{\lambda \in \mathbb{R} \mid H \text{ does not satisfy a strict Mourre estimate at } \lambda\,\}.$$

For the computation of the sets $\tau_A(H)$ and $\kappa_A(H)$ it is convenient to introduce the lower semicontinuous functions $\widetilde{\rho}_H : \mathbb{R} \to (-\infty, +\infty]$ and $\rho_H : \mathbb{R} \to (-\infty, +\infty]$ defined as follows: $\widetilde{\rho}_H(\lambda)$ is the upper bound of the numbers c such that the inequality (5.4) holds for some φ, K; similarly, $\rho_H(\lambda)$ is the upper bound of the numbers c such that the inequality (5.4) holds for some φ and $K = 0$. Clearly $\tau_A(H)$ is just the set of $\lambda \in \mathbb{R}$ such that $\widetilde{\rho}_H(\lambda) \leq 0$ and $\kappa_A(H)$ is the set of λ such that $\rho_H(\lambda) \leq 0$.

An important property of the functions ρ_H and $\widetilde{\rho}_H$ is the following (see Theorem 7.2.13 from [ABG]): $\rho_H(\lambda) = \widetilde{\rho}_H(\lambda)$ with the exception of the points λ which are eigenvalues of H and where $\widetilde{\rho}_H(\lambda) > 0$; at these points one has $\rho_H(\lambda) = 0$. In particular, $\rho_H(\lambda) > 0$ if and only if $\widetilde{\rho}_H(\lambda) > 0$ and $\lambda \notin \sigma_p(H)$. In other terms

$$\kappa_A(H) = \tau_A(H) \cup \sigma_p(H). \tag{5.6}$$

We shall need some information concerning the dependence on H of the function ϱ_H. Let $\mathcal{S}^1_A$ be the set of self-adjoint operators of class $C^1(A)$ on $\mathcal{H}$ equipped with the following metric:

$$d(H_1, H_2) = \left\|(H_1 - i)^{-1} - (H_2 - i)^{-1}\right\| + \left\|[(H_1 - i)^{-1}, A] - [(H_2 - i)^{-1}, A]\right\|.$$

Then for each $\varphi \in C_0(\mathbb{R})$ and $\psi \in C^2_c(\mathbb{R})$ the maps

$$\mathcal{S}^1_A \ni H \longmapsto \varphi(H) \in B(\mathcal{H}), \quad \mathcal{S}^1_A \ni H \longmapsto [\psi(H), A] \in B(\mathcal{H})$$

are continuous. Indeed, the set of φ for which this holds is a self-adjoint closed subalgebra of $C_0(\mathbb{R})$ which contains $\varphi(\lambda) = (\lambda - i)^{-1}$, hence is equal to $C_0(\mathbb{R})$. The second assertion is a consequence of the relation (8.3.1) from [ABG] and of the Helfer-Sjöstrand formula. By using (5.5) we see that if $\varphi \in C_c(\mathbb{R})$ then the map

$$\mathcal{S}^1_A \ni H \longmapsto \varphi(H)[H, iA]\varphi(H) \in B(\mathcal{H})$$

is also continuous.

Proposition 5.2 *Let λ be a real number.*
(a) *The map $\mathcal{S}^1_A \ni H \longmapsto \rho_H(\lambda)$ is lower semicontinuous.*
(b) *Let $\mathcal{U}$ be a compact subset of $\mathcal{S}^1_A$ and c a number such that $c < \rho_H(\lambda)$ for $H \in \mathcal{U}$. Then there is a real $\varphi \in C_c(\mathbb{R})$ with $\varphi(\lambda) \neq 0$ such that $\varphi(H)[H, iA]\varphi(H) \geq c\,\varphi(H)^2$ for all $H \in \mathcal{U}$.*

Proof. Let $c_1 < c_0 < \rho_{H_0}(\lambda)$ for some $H_0 \in \mathcal{S}^1_A$. Then there is a real function $\varphi_0 \in C_c(\mathbb{R})$ with $\varphi_0(\mu) = 1$ on a neighbourhood of λ such that $\varphi_0(H_0)[H_0, iA]\varphi_0(H_0) \geq c_0\varphi_0(H_0)^2$. The continuity properties discussed before the proposition imply that there is a neighbourhood $V(H_0)$ of H_0 in $\mathcal{S}^1_A$ such that $\varphi_0(H)[H, iA]\varphi_0(H) \geq c_0\varphi_0(H)^2 - (c_0 - c_1)$ if H belongs to $V(H_0)$. Let $\varphi \in C_c(\mathbb{R})$ with $\varphi(\mu) = 1$ on a neighbourhood of λ and $\varphi\varphi_0 = \varphi$. By left and right multiplying the preceding inequality with $\varphi(H)$ we get $\varphi(H)[H, iA]\varphi(H) \geq c_1\varphi(H)^2$ hence $\rho_H(\lambda) \geq c_1$. This proves assertion (a) of the proposition. Under the conditions of (b) we take $c_1 = c$ and observe that, due to the fact that a lower semicontinuous function on a compact set reaches its lower bound, there is $c_0 > c_1$ such that $\varrho_{H_0}(\lambda) > c_0$ for all $H_0 \in \mathcal{U}$. We cover $\mathcal{U}$ with neighbourhoods $V(H_0)$ as above and then we can extract a finite covering $V(H_1), \ldots, V(H_n)$. Moreover, by what we have shown, we can find $\varphi_1, \ldots, \varphi_n$ such that $\varphi_k(\mu) = 1$ on a neighbourhood of λ and $\varphi_k(H)[H, iA]\varphi_k(H) \geq c_1\varphi_k(H)^2$ if $H \in V(H_k)$. Now it suffices to choose φ such that $\varphi\varphi_k = \varphi$ for all k and $\varphi(\lambda) \neq 0$. ■

Now let us assume that a $\mathcal{L}$-graded C^*-subalgebra $\mathfrak{C}$ of $B(\mathcal{H})$ is given such that
(i) each $\mathfrak{C}_a$ is nondegenerate on $\mathcal{H}$;
(ii) $\mathcal{L}$ has a least element and $\mathfrak{C}(\min \mathcal{L}) = K(\mathcal{H})$;
(iii) $\mathcal{L}$ is atomic (denote $\mathcal{M}$ the set of its atoms);
(iv) if $T \in \mathfrak{C}$ then the set $\{T_a \mid a \in \mathcal{M}\}$ is compact in $\mathfrak{C}$;
(v) for all $a \in \mathcal{L}$ and $\tau \in \mathbb{R}$

$$e^{-iA\tau}\mathfrak{C}(a)e^{iA\tau} \subset \mathfrak{C}(a). \tag{5.7}$$

We recall that if "compact" is replaced by "relatively compact" then (iii) is always satisfied, see Section 2.3. But we shall need the stronger form (iii) and we shall prove it for the algebra $\mathfrak{C}^X$.

Then, if H is strictly affiliated to $\mathfrak{C}$ and of class $C^1_u(A)$, each operator H_a will also be of class $C^1_u(A)$: indeed, this follows immediately from the norm continuity of $\mathcal{P}_a$ and from the relation

$$e^{-iA\tau}\mathcal{P}_a[T]e^{iA\tau} = \mathcal{P}_a[e^{-iA\tau}Te^{iA\tau}] \tag{5.8}$$

valid for all $T \in \mathfrak{C}$ and $\tau \in \mathbb{R}$. By using (5.5) and (5.8) we also get $\varphi(H)[H, iA]\varphi(H) \in \mathfrak{C}$ if $\varphi \in C_c(\mathbb{R})$ and

$$\mathcal{P}_a[\varphi(H)[H, iA]\varphi(H)] = \varphi(H_a)[H_a, iA]\varphi(H_a). \tag{5.9}$$

Theorem 5.3 *Let H be of class $C^1_u(A)$ and strictly affiliated to $\mathfrak{C}$. Then $\widetilde{\rho}_H = \inf_{a\in\mathcal{M}} \rho_{H_a}$ and for each real number λ:*

$$\widetilde{\rho}_H(\lambda) = \min_{a\in\mathcal{M}} \rho_{H_a}(\lambda). \tag{5.10}$$

Moreover, one has

$$\tau_A(H) = \bigcup_{a\in\mathcal{M}} \kappa_A(H_a), \tag{5.11}$$

in particular the right hand side above is a closed set.

Proof. We first show that the set of operators H_a, $a \in \mathcal{M}$, is compact in $\mathcal{S}^1_A$. Let $R = (H-i)^{-1}$ and $R_a = \mathcal{P}_a[R_a] = (H_a - i)^{-1}$. If $\{H_{a_n}\}$ is a Cauchy sequence in $\mathcal{S}^1_A$ for some sequence $\{a_n\}$ in $\mathcal{M}$ then $\{R_{a_n}\}$ and $\{[R_{a_n}, A]\}$ are Cauchy sequences in $B(\mathcal{H})$, hence there are bounded operators S, T such that $\|R_{a_n} - S\| \to 0$ and $\|[R_{a_n}, A] - T\| \to 0$. By hypothesis (iv) the set of operators R_a, $a \in \mathcal{M}$, is compact in $\mathfrak{C}$, so there is $a \in \mathcal{M}$ such that $S = R_a$. We have $[R_{a_n}, A] \to [R_a, A]$ as forms on the domain of A, so $[R_a, A] = T$. Thus $H_{a_n} \to H_a$ in $\mathcal{S}^1_A$.

Now we prove (5.10). One has $c < \widetilde{\rho}_H(\lambda)$ if and only if (5.4) holds for some φ, K. From (2.5), (5.9) and (ii) we then get $\varphi(H_a)[H_a, iA]\varphi(H_a) \geq c\varphi(H_a)^2$ for all $a \in \mathcal{M}$, hence $c \leq \inf \rho_{H_a}(\lambda)$. Reciprocally, let $c < \min \rho_{H_a}(\lambda)$. Then by Proposition 5.2(b) one has $\varphi(H_a)[H_a, iA]\varphi(H_a) \geq c\varphi(H_a)^2$ for some φ with $\varphi(\lambda) \neq 0$ and all $a \in \mathcal{M}$. By using again (2.5) we see that (5.4) holds for some compact K, so $c \leq \widetilde{\rho}_H(\lambda)$.

The equality (5.11) is a consequence of (5.6) (with H replaced by H_a) because one has $\rho_{H_a}(\lambda) > 0$ for all $a \in \mathcal{M}$ if and only if $\min \rho_{H_a}(\lambda) > 0$. ■

We stress the fact that in (5.10) one takes the minimum and not the infimum of the family of numbers $\rho_{H_a}(\lambda)$. One can also see this as follows: by taking into account Proposition 5.2(a), the compacity of the set $\{H_a \mid a \in \mathcal{M}\}$ and the fact that a lower semicontinuous function on a compact set reaches its lower bound, we get $\min_{a\in\mathcal{M}} \rho_{H_a}(\lambda) = \inf_{a\in\mathcal{M}} \rho_{H_a}(\lambda)$.

5.2. We shall apply the preceding general results to the self-adjoint operators on the Hilbert space $\mathcal{H} = \mathcal{H}(X)$ strictly affiliated to the C^*-algebra $\mathfrak{C}^X$. As conjugate operator A we take the generator of dilations D, more precisely the operator D defined by the property

$$(\mathrm{e}^{i\tau D} f)(x) = \mathrm{e}^{\tau \dim X/2} f(\mathrm{e}^{\tau} x).$$

With slightly formal notations $D = \frac{1}{2}(\langle Q, P\rangle + \langle P, Q\rangle)$. If needed, we specify the space X as an upper index, so we set $D = D^X$. We also use the notation $W_\tau = W_\tau^X = \mathrm{e}^{i\tau D}$.

We mention one fact concerning the regularity condition with respect to D. *Let H be a self-adjoint operator strictly affiliated to $\mathfrak{C}^X$; then H is of class $C^1_u(D)$ if and only if there is $z \in \mathbb{C} \setminus \sigma(H)$ such that the sesquilinear form $[(H-z)^{-1}, D]$ with domain $C_c^\infty(X)$ extends to a bounded operator belonging to $\mathfrak{C}^X$.* This is an immediate consequence of the fact that W_τ induces a group of automorphisms of $\mathfrak{C}^X$ which is norm continuous (the group induced on $\mathfrak{U}^X$ being only strongly continuous).

It is clear that the conditions (i)-(iii) from Section 5.1 are satisfied by the algebra $\mathfrak{C}^X$. We shall prove in Proposition 5.4 that (iv) is fulfilled too. That (v) is verified is easy to check either directly on Definition 3.1, or by using the relations

$$(5.12) \qquad W_\tau^* \varphi(Q) W_\tau = \varphi(\mathrm{e}^{-\tau} Q), \quad W_\tau^* \psi(P) W_\tau = \psi(\mathrm{e}^{\tau} P)$$

and Theorem 3.9.

Proposition 5.4 *If $T \in \mathfrak{C}^X$ then $\{\mathcal{P}_Y[T] \mid Y \in \mathbb{P}(X)\}$ is compact in $\mathfrak{C}^X$.*

Proof. We first show that for each subspace $Z \neq O$ there is $Y \in \mathbb{P}(X)$ such that $T_Y = T_Z$. There is a countable $\cap$-stable set $\mathcal{E} \subset \mathbb{G}(X)$ such that T is affiliated to $\mathfrak{C}^X(\mathcal{E})$, cf. (2.3). Then the set of of subspaces $E \in \mathcal{E}$ which do not include Z is countable, so there is $\omega \in Z$ such that $\omega \notin E$ for all such E. Let $Y = \mathbb{R}\omega$, then for $E \in \mathcal{E}$ with $Y \subset E$ we have $Z \subset E$. Let $S \in \mathfrak{C}^X(\mathcal{E})^\circ$, then

$$\mathcal{P}_Y(S) = \sum_{Y \subset E} S(E) = \sum_{Y \subset E, E \in \mathcal{E}} S(E) = \sum_{Z \subset E, E \in \mathcal{E}} S(E) = \mathcal{P}_Z(S).$$

Since $\mathfrak{C}^X(\mathcal{E})^\circ$ is dense in $\mathfrak{C}^X(\mathcal{E})$, we see that the $\mathcal{P}_Y|\mathfrak{C}^X(\mathcal{E}) = \mathcal{P}_Z|\mathfrak{C}^X(\mathcal{E})$, which proves the assertion.

We make two other remarks that we will need below. Let E, F be subspaces of X, then

$$\mathcal{P}_E \mathcal{P}_F = \mathcal{P}_F \mathcal{P}_E = \mathcal{P}_{E+F}.$$

Indeed, this is obvious on $\mathfrak{C}^X(\mathbb{G}(X))^\circ$ and remains true by continuity on $\mathfrak{C}^X$. Note that $E + F = E \vee F$ in the lattice $\mathbb{G}(X)$. Let $\mathcal{P}'_E = 1 - \mathcal{P}_E$, then

$$\mathcal{P}_E - \mathcal{P}_F = \mathcal{P}_E \mathcal{P}'_F - \mathcal{P}'_E \mathcal{P}_F$$

hence $\mathcal{P}_E(\mathcal{P}_E - \mathcal{P}_F) = \mathcal{P}_E \mathcal{P}'_F$, which gives for all $T \in \mathfrak{C}^X$

$$\left\| \mathcal{P}_E \mathcal{P}'_F T \right\| \leq \left\| \mathcal{P}_E T - \mathcal{P}_F T \right\|.$$

Let S be an accumulation point of the set $\{T_Y \mid Y \in \mathbb{P}(X)\}$. Then there is an injective sequence Y_n such that $\|T_{Y_n} - S\| \to 0$. We define $Z = \bigcap_{n \geq 0} \sum_{m \geq n} Y_m$ and let $k = \dim Z$. A monotonic sequence of subspaces being stationary, for each large enough n one can find $n_1 = n < n_2 < \ldots < n_k$ such that $Z = Y_{n_1} + \ldots + Y_{n_k}$. For each $\varepsilon > 0$ there is N such that $\|T_{Y_n} - T_{Y_m}\| \leq \varepsilon$ if $n, m > N$. We choose a sequence $n_1 = n < n_2 < \ldots < n_k$ as above and denote $Z_i = Y_{n_1} + \ldots + Y_{n_{i-1}}$. Then we have

$$\varepsilon \geq \left\| \mathcal{P}_{Y_n} T - \mathcal{P}_{Y_{n_i}} T \right\| \geq \left\| \mathcal{P}_{Y_n} \mathcal{P}'_{Y_{n_i}} T \right\| \geq \left\| \mathcal{P}_{Z_i} \mathcal{P}_{Y_n} \mathcal{P}'_{Y_{n_i}} T \right\| = \left\| \mathcal{P}_{Z_i} \mathcal{P}'_{Y_{n_i}} T \right\|.$$

On the other hand, we can write

$$\mathcal{P}_{Y_n} = \mathcal{P}_{Y_{n_1}} \mathcal{P}'_{Y_{n_2}} + \mathcal{P}_{Y_{n_1}} \mathcal{P}_{Y_{n_2}} = \mathcal{P}_{Y_{n_1}} \mathcal{P}'_{Y_{n_2}} + \mathcal{P}_{Y_{n_1} + Y_{n_2}}.$$

Repeating this procedure we get

$$\mathcal{P}_{Y_n} = \sum_{i=2}^{k} \mathcal{P}_{Z_i} \mathcal{P}'_{Y_{n_i}} + \mathcal{P}_Z.$$

Thus we get $\|T_{Y_n} - T_Z\| \leq (k-1)\varepsilon$. This shows that $S = T_Z$ and now it suffices to use the assertion established at the beginning of the proof. ∎

Proposition 5.4 allows us to apply Theorem 5.3 to each hamiltonian H of class $C^1_u(D)$ strictly affiliated to $\mathfrak{C}^X$. Since from now on the conjugate operator $A = D$ is fixed, we shall omit it in the notations, so we set

$$\tau(H) = \tau_D(H), \ \kappa(H) = \kappa_D(H) = \tau(H) \cup \sigma_p(H).$$

Thus we have

$$\tau(H) = \bigcup_{Y\in\mathbb{P}(X)} \kappa(H_Y) \tag{5.13}$$

for each h as above. If H_Y has no eigenvalues for $Y \neq O$, as it happens in most cases, we get

$$\tau(H) = \bigcup_{Y\in\mathbb{P}(X)} \tau(H_Y). \tag{5.14}$$

Let $H_Y(\cdot)$ be the covariant symbol of the operator H_Y affiliated to $\mathfrak{C}_Y^X$ (see the beginning of the section 4.1) and $H^{X/Y} = H_Y(0)$, a self-adjoint operator on $\mathcal{H}(X/Y)$ affiliated to the algebra $\mathfrak{C}^{X/Y}$.

Proposition 5.5 *If H is a self-adjoint operator on $\mathcal{H}(X)$ of class $C_u^1(D)$ and strictly affiliated to $\mathfrak{C}^X$, then*

$$\tau(H) \supset \bigcup_{Y\in\mathbb{G}^0(X)} \sigma_p(H^{X/Y}), \tag{5.15}$$

where $\mathbb{G}^0(X)$ is the set of linear subspaces $Y \subset X$ such that $Y \neq O$.

Proof. We have introduced before (see (3.16), (3.19)) a canonical morphism $T \mapsto T^{X/Y}$ of $\mathfrak{C}_Y^X$ onto $\mathfrak{C}^{X/Y}$. By using (5.12) and Theorem 3.9 we obtain for all $\tau \in \mathbb{R}$ and $T \in \mathfrak{C}_Y^X$

$$((W_\tau^X)^* T W_\tau^X)(k) = (W_\tau^{X/Y})^* T(\mathrm{e}^\tau k) W_\tau^{X/Y} \quad \forall k \in X^*. \tag{5.16}$$

Then by taking $k = 0$ we get

$$[(W_\tau^X)^* T W_\tau^X]^{X/Y} = (W_\tau^{X/Y})^* T^{/Y} W_\tau^{X/Y}, \tag{5.17}$$

i.e. the morphism $T \mapsto T^{X/Y}$ intertwines the natural actions of the dilation groups on $\mathfrak{C}_Y^X$ and $\mathfrak{C}^{X/Y}$. It is clear that $H^{X/Y}$, seen as observable affiliated to $\mathfrak{C}^{X/Y}$, is just the image of the observable H_Y through this morphism. Now by an argument that we already used before (cf. (5.8), (5.9)) we see that $H^{X/Y}$ is of class $C_u^1(D^{X/Y})$ and that we have $\kappa(H^{X/Y}) \subset \kappa(H_Y)$. From (5.6) and (5.13) we thus get

$$\tau(H) \supset \bigcup_{Y\in\mathbb{P}(X)} \left[\sigma_p(H_Y) \cup \sigma_p(H^{X/Y}) \cup \tau(H^{X/Y})\right]. \tag{5.18}$$

An operator like $H^{X/Y}$ has all the properties of H except that the space X is replaced by X/Y. Moreover, it is clear that for $Y \subset Z \subset X$ one has $(H^{X/Y})^{Z/Y} = H^{X/Z}$ (use (3.24)). Hence relation (5.18) with H replaced by $H^{X/Y}$ will give

$$\tau(H^{X/Y}) \supset \bigcup_Z \left[\sigma_p(H^{X/Z}) \cup \tau(H^{X/Z})\right]$$

where Z runs over the set of subspaces of dimension 2 of X which contain Y. By continuing in this way and by taking into account that $H^{X/X}$ is an operator on $\mathcal{H}(X/X) = \mathbb{C}$, hence $\sigma_p(H^{X/X}) = \tau(H^{X/X})$, we get (5.15). ■

Under more restrictive assumptions we shall now obtain the inclusion converse to that in (5.15). The positivity conditions (i) and (ii) from the next theorem have been isolated by Christian Gerard in his study of dispersive $N-$body systems [G].

In order to avoid conditions on the domain of H we shall use the following notion. Let $S(\cdot)$ be a map defined on a finite dimensional vector space with values in the set of self-adjoint operators on a Hilbert space $\mathcal{H}$ such that $S(k) \geq a$ for some real a and all k. We say that $S(\cdot)$ is norm C^1 if there is $\lambda < a$ such that the map $k \mapsto R(k) = (\lambda - S(k))^{-1} \in B(\mathcal{H})$ is of class C^1 in norm. We write $k\nabla S(k) \geq b(k)$, where $b(k)$ is a real number, if $k\nabla R(k) = \frac{d}{dt}R(tk)|_{t=1} \geq b(k)R(k)^2$. An inequality like $p\nabla_p S(k+p) \geq 0$ is interpreted similarly; note that $p\nabla_p S(k+p) \geq 0 = \frac{d}{dt}R(k+tp)|_{t=1}$. These conditions are independent of λ because if $\lambda_1, \lambda_2 < a$, $R_i(k) = (\lambda_i - S(k))^{-1}$, $\mu = \lambda_1 - \lambda_2$, then $R_2(k) = R_1(k) + \mu R_1(k)R_2(k)$ and

$$k\nabla R_2(k) = [1 + \mu R_2(k)]\, k\nabla R_1(k)\, [1 + \mu R_2(k)] \, .$$

If the operators $S(k)$ have the same form domain $\mathcal{G}$ and we identify $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^*$ then it suffices to ask that the map $k \mapsto S(k) \in B(\mathcal{G}, \mathcal{G}^*)$ be of class C^1 in norm and that $k\nabla S(k) \geq b(k)$ be satisfied in the sense of forms on $\mathcal{G}$.

Theorem 5.6 *Let H be a self-adjoint bounded from below operator on $\mathcal{H}(X)$, of class $C^1_u(D)$ and strictly affiliated to $\mathfrak{C}^X$. Assume that for each $Y \subset X$ a subspace $Y' \subset X^*$ supplementary to $Y^\perp$ is given such that $Y' \subset Z'$ if $Y \subset Z$, the restriction to Y' of the covariant symbol $H_Y(\cdot)$ is norm C^1, and:*
(i) $k\nabla H_Y(k) \geq c(k)$ for some real $c(k) > 0$ if $k \in Y'$, $k \neq 0$;
(ii) $p\nabla_p H_Z(k+p) \geq 0$ if $Y \subset Z$, $k \in Y'$, $p \in Z' \cap Y^\perp$.
Then $\tau(H)$ is a closed countable real set and we have

$$\tau(H) = \bigcup_{Y \in \mathbb{G}^0(X)} \sigma_p(H^{X/Y}), \tag{5.19}$$

where $\mathbb{G}^0(X)$ is the set of linear subspaces $Y \subset X$ such that $Y \neq O$. If $Y \neq O$ then the operator H_Y has no eigenvalues.

Proof. (a) We begin with some preliminary notations and remarks. We already now that $\tau(H)$ is a closed set and that the right-hand side of (5.19) is included in $\tau(H)$. If we prove the reverse inclusion, then the countability of $\tau(H)$ follows, because only a countable number of operators $H^{X/Y}$ are distinct (see the comment after Theorem 2.11).

We fix a real number λ_0 such that $H \geq 1 + \lambda_0$. Then for each Y one has $H_Y \geq 1 + \lambda_0$, hence the operator $R_Y = (\lambda_0 - H_Y)^{-1} \in \mathfrak{C}^X_Y$ verifies

$-1 \leq R_Y < 0$ and is of class $C^1_u(D)$. Moreover, Proposition 7.2.5 in [ABG] gives

$$\varrho_{H_Y}(\lambda) = (\lambda_0 - \lambda)^2 \varrho_{R_Y}((\lambda_0 - \lambda)^{-1}) \tag{5.20}$$

if $\lambda \in \mathbb{R} \setminus \{\lambda_0\}$. Thus, if we set $\mu = (\lambda_0 - \lambda)^{-1}$, then

$$\varrho_{H_Y}(\lambda) \geq 0 \iff \varrho_{R_Y}(\mu) \geq 0, \tag{5.21}$$

$$\varrho_{H_Y}(\lambda) > 0 \iff \varrho_{R_Y}(\mu) > 0. \tag{5.22}$$

Note that only values $\lambda \geq 1 + \lambda_0$ and $\mu \in [-1, 0)$ are of interest because of the position of the spectra of H_Y and R_Y. Let $J \subset (-\infty, 0)$ be a compact neighbourhood of μ. Since $\|R_Y(k)\| \to 0$ when $k \to \infty$ in Y', there is a compact set $U \subset Y'$ such that $\varphi(R_Y(k)) = 0$ if $\varphi \in C_c(J)$ and $k \in Y' \setminus U$. From now on the functions φ (with various indices) are assumed in $C_c(J)$.

We shall also need a formula for the covariant symbol of an operator of the form $[T, iD]$ with $T \in \mathfrak{C}^X_Y \cap C^1_u(D)$. Note that in this case $[T, iD] \in \mathfrak{C}^X_Y$. From (5.16) we easily get

$$\left[T, iD^X\right](k) = k\nabla T(k) + \left[T(k), iD^{X/Y}\right] \tag{5.23}$$

if $T(\cdot)$ is norm derivable at the point $k \in X$. This formula can be used for $T = R_Y$ and $k \in Y'$ and has the follwing consequence (which has to be used when we apply Proposition 5.2(b)): the maps $k \mapsto [R_Y, iD](k)$ and $k \mapsto k\nabla R_Y(k)$ being norm continuous on Y', the map $k \mapsto [R_Y(k), iD^{X/Y}]$ is norm continuous too.

We have $c < \varrho_{R_Y}(\mu)$ if and only if there is $\varphi \in C_c(\mathbb{R})$ real, with $\varphi(\mu) \neq 0$, such that $\varphi(R_{R_Y})\,[R_Y, iD]\,\varphi(R_{R_Y}) \geq c\varphi(R_{R_Y})^2$, which is equivalent to

$$\varphi(R_Y(k))\left[R_Y, iD^X\right](k)\varphi(R_Y(k)) \geq c\varphi(R_Y(k))^2 \quad \forall k \in X^*. \tag{5.24}$$

By Y-covariance, norm differentiability of R_Y on Y' and (5.23) we see that this relation holds if and only if for each $k \in Y'$

$$\varphi(R_Y(k))\left\{k\nabla R_Y(k) + \left[R_Y(k), iD^{X/Y}\right]\right\}\varphi(R_Y(k)) \geq c\varphi(R_Y(k))^2. \tag{5.25}$$

From hypothesis (ii) of the theorem (with $Y = O$) we get $k\nabla R_Y(k) \geq 0$ for all $k \in Y'$, hence the preceding relation is a consequence of

$$\varphi(R_Y(k))\left[R_Y(k), iD^{X/Y}\right]\varphi(R_Y(k)) \geq c\varphi(R_Y(k))^2 \quad \forall k \in Y'. \tag{5.26}$$

(b) At this step we show that $\varrho_H \geq 0$. The property (i) will not be used here. We make the induction hypothesis that $\varrho_H \geq 0$ if the dimension of X is strictly smaller than a number N (this is trivially satisfied if $N = 1$ and easy to check if $N = 2$). Let X be a vector space of dimension N and $Y \in \mathbb{G}^0(X)$. If $k \in Y'$ then the self-adjoint operator $H_Y(k)$ in the space $\mathcal{H}(X/Y)$, with resolvent $(\lambda_0 - H_Y(k))^{-1} = R_Y(k)$, has properties similar to H in $\mathcal{H}(X)$: it is bounded from below, affiliated to $\mathfrak{C}^{X/Y}$, of class $C^1_u(D^{X/Y})$, and verifies the analogue of the condition (ii). To check this last fact it suffices to use the following consequence of (3.22): if $Y \subset Z$ then

$$[H_Y(k)]_{Z/Y}(p) = H_Z(k+p). \tag{5.27}$$

Then by the induction hypothesis $\varrho_{H_Y(k)} \geq 0$ if $k \in Y'$. Relation (5.26) is satisfied if $k \in Y' \setminus U$ for all c. If $c < 0$ we can apply Proposition 5.2(b) to the family $\mathcal{U} = \{H_Y(k) \mid k \in U\}$ in order to obtain a function $\varphi \in C_c(J)$ with $\varphi(\mu) \neq 0$ such that (5.26) be satisfied. Thus $\varrho_{R_Y}(\mu) \geq 0$ hence, by (5.21), $\varrho_{H_Y}(\lambda) \geq 0$. This holds for all $Y \in \mathbb{P}(X)$, so from (5.10) we get $\widetilde{\varrho}_H(\lambda) \geq 0$. Since $\widetilde{\rho}_H \geq 0 \Leftrightarrow \rho_H \geq 0$ (see the comments before (5.6)) we obtain $\varrho_H(\lambda) \geq 0$ for all λ.

(c) Now we assume that λ does not belong to the right hand side of (5.19) and prove that $\lambda \notin \tau(H)$. Because of (5.13) it suffices to show that $\lambda \notin \kappa(H_Y)$ if $Y \in \mathbb{P}(X)$. Equivalently, we have to show that $\varrho_{R_Y}(\mu) > 0$, which means that there are $c > 0$ and φ such that (5.25) is satisfied for all $k \in Y'$. J and U being as above, it suffices to consider $k \in U$. We have $\lambda \notin \kappa(H^{X/Y})$ because $\lambda \notin \sigma_p(H^{X/Y})$ and

$$\tau(H^{X/Y}) = \bigcup_{E \in \mathbb{G}^0(X/Y)} \sigma_p\left([H^{X/Y}]^{(X/Y)/E}\right) = \bigcup_{Z \in \mathbb{G}(X), Y \subsetneq Z} \sigma_p(H^{X/Z}).$$

The last equality follows from (3.24) or from (5.27) with $k = p = 0$. Since $H^{X/Y} = H_Y(0)$ we see that $\varrho_{R_Y(0)}(\mu) > 0$, hence by using Proposition 5.2(a) we get a compact neighbourhood V of 0 in Y' and a number $c_1 > 0$ such that $\varrho_{R_Y(k)}(\mu) > c_1$ if $k \in V$. Now part (b) of Proposition 5.2 gives $\varphi_1 \in C_c(J)$ with $\varphi_1(\nu) = 1$ on a neighbourhood of μ such that

$$\varphi_1(R_Y(k))\left[R_Y(k), iD^{X/Y}\right]\varphi_1(R_Y(k)) \geq c_1 \varphi_1(R_Y(k))^2 \quad \forall k \in V. \tag{5.28}$$

Thus (5.25) holds if $k \in V$ with $\varphi = \varphi_1$ and $c = c_1$, the term $k\nabla R_Y(k)$ being positive. Let $\alpha = \sup J$. From hypothesis (i) of the theorem we get for $\psi \in C_c(J)$:

$$\begin{aligned} \psi(R_Y(k))k\nabla R_Y(k)\psi(R_Y(k)) &\geq c(k)\psi(R_Y(k))R_Y(k)^2\psi(R_Y(k)) \\ &\geq c(k)\alpha^2\psi(R_Y(k))^2. \end{aligned}$$

Let W be the set of points of U which do not belong to the interior of V, so W is a compact subset of Y'. By the norm continuity of $k\nabla R_Y(k)$ we see that there is $c_2 > 0$ such that $\psi(R_Y(k))k\nabla R_Y(k)\psi(R_Y(k)) \geq 2c_2\psi(R_Y(k))^2$ if $k \in W$. By using once again Proposition 5.2(b) in relation with $\varrho_{R_Y(k)} \geq 0$ we also see that there is $\varphi_2 \in C_c(J)$ with $\varphi_2(\nu) = 1$ on a neighbourhood of μ such that

$$\varphi_2(R_Y(k))\left[R_Y(k), iD^{X/Y}\right]\varphi_2(R_Y(k)) \geq -c_2\varphi_1(R_Y(k))^2 \quad \forall k \in W. \tag{5.29}$$

Thus (5.25) holds if $k \in W$ with $\varphi = \varphi_2$ and $c = c_2$. Finally, let $\varphi \in C_c(J)$ with $\varphi(\mu) \neq 0$ such that $\varphi\varphi_1 = \varphi\varphi_2 = \varphi$ and $c = \min(c_1, c_2)$. Then (5.25) holds for all $k \in Y'$.

(d) We now prove that $\sigma_p(H_Y) = \emptyset$ if $Y \neq O$. It suffices to show that if $u \in \mathcal{H}(X)$ satisfies $R_Y u = \mu u$ for some real μ then $u = 0$. By the virial theorem we have $\langle u, [R_Y, iD]u\rangle = 0$. We choose a Haar measure on Y' and make a Fourier transformation in the Y'-variable. It is possible to identify $\mathcal{H}(X) \cong L^2(Y'; \mathcal{H}(X/Y))$ such that R_Y becomes the operator of multiplication by the operator valued map $R_Y|Y'$ and u becomes a square integrable map $u : Y' \to \mathcal{H}(X/Y)$ such that $R_Y(k)u(k) = \mu u(k)$ for a.e. $k \in Y'$. Since $\|R_Y(k)\| \to 0$ when $k \to \infty$ in Y', μ cannot be in the spectrum of $R_Y(k)$ for large k. Thus there is a compact set $U \subset Y'$ such that $u(k) = 0$ if $k \in Y' \setminus U$. From $\langle u, [R_Y, iD]u\rangle = 0$ we get

$$\int_U \langle u(k), \left\{ k\nabla R_Y(k) + \left[R_Y(k), iD^{X/Y} \right] \right\} u(k)\rangle dk = 0. \tag{5.30}$$

We have proved above that $\varrho_{R_Y} \geq 0$. Proposition 5.2(b) shows that for each $c > 0$ we can find $\varphi \in C_c(\mathbb{R})$ real with $\varphi(\mu) = 1$ such that

$$\varphi(R_Y(k)) \left[R_Y(k), iD^{X/Y} \right] \varphi(R_Y(k)) \geq -c\varphi(R_Y(k))^2$$

for all $k \in U$. But $\varphi(R_Y(k))u(k) = u(k)$, so (5.30) implies

$$\int_U \langle u(k), k\nabla R_Y(k)u(k)\rangle dk \leq c\,\|u\|^2 .$$

Since $c > 0$ is arbitrary, we get $\int_U \langle u(k), k\nabla R_Y(k)u(k)\rangle dk \leq 0$. From hypothesis (i) it follows that $u = 0$. ■

6 Dispersive Systems and Pluristratified Media

6.1. It is clear, by Theorem 4.6 and Proposition 4.7, that the class of operators affiliated to $\mathfrak{C}^X$ contains the *dispersive N-body hamiltonians* first studied by J. Derezinski [De] and C. Gerard [G]. The fact that the dispersive hamiltonians are affiliated to $\mathfrak{C}$ was already pointed out in [ABG] (see also references there) but a detailed presentation of the subject from our point of view appears only in [Da]. However, the class of dispersive hamiltonians considered by these authors is significantly smaller than that covered by Theorem 4.6 (even if we consider only operators affiliated to finite parts of $\mathfrak{C}^X$). Indeed, in Theorem 3.1 from [Da] (which contains the best results on dispersive hamiltonians among the quoted references) the operators V^a are independent of x_a and have to satisfy two other restrictive requirements: 1) their form-bound with respect to $h(P^a)$ has to be less than 1 (or our Theorem

4.6 allows an arbitrary positive part); 2) there must exist $t > s$ such that $V^a \mathcal{H}^t(X^a) \subset \mathcal{H}^{t-2s}(X^a)$. This last condition is especially annoying since it is a rather strong local regularity condition on the coefficients of the principal part of H.

Our purpose now is to show that the class of operators introduced in Theorem 4.6 contains the hamiltonians, describing the propagation of waves in *perturbed pluristratified media*, considered by Dermenjian and Iftimie in [DI]. These hamiltonians are generalizations in the spirit of the N-body theory of a class of operators describing *stratified media* first studied by the conjugate operator method by De Bièvre and Pravica [DBP]. More precisely, they consider operators of the form

$$H = \nabla^* M \nabla + W$$

where ∇ and ∇^* are the gradient and divergence operators associated to an euclidean structure on X. So if ∂_i $(i = 1, ..., \dim X)$ denotes the derivative in the direction of the i-th vector of an orthonormal basis in X then

$$H = -\sum_{i=1}^{\dim X} \partial_i \, M \, \partial_i + W.$$

The operators M and W have the form

$$M = 1 + m = 1 + \sum_{Y \in \mathbb{G}(X)} m(Y) \ , \ W = \sum_{Y \in \mathbb{G}(X)} W(Y).$$

We shall identify $X^* = X$ with the help of the euclidean structure given on X. If Y is a subspace of X then we identify $X/Y = Y^\perp$ and we denote by $\otimes_Y$ the tensor product determined by the direct sum decomposition $X = Y^\perp \oplus Y$, which gives a canonical isomorphism $\mathcal{H}(X) \cong \mathcal{H}(Y^\perp) \bigotimes \mathcal{H}(Y)$. Then the operators $m(Y), W(Y)$ are assumed to have the form

$$m(Y) = m^{Y^\perp} \otimes_Y 1 \ , \ W(Y) = W^{Y^\perp} \otimes_Y 1.$$

If $Y = X$ then $Y^\perp = O$ and we take $m^O = W^O = 0$. For the other $m^{Y^\perp}$, $W^{Y^\perp}$ we make the following hypotheses, where $\Delta = \nabla^* \nabla$ is the Laplace operator associated to the euclidean structure on X:

(i) $m^{Y^\perp} : \mathcal{H}(Y^\perp) \to \mathcal{H}(Y^\perp)$ *is a symmetric and small at infinity operator;*

(ii) $W^{Y^\perp} : \mathcal{H}^1(Y^\perp) \to \mathcal{H}^{-1}(Y^\perp)$ *is symmetric and small at infinity;*

(iii) *one has*

$$-\mu_Y \Delta - \delta_Y \le \nabla^* m(Y) \nabla + W(Y) \le \delta_Y \Delta + \delta_Y$$

for some constants μ_Y, $\delta_Y \ge 0$ *such that* $\sum \mu_Y < 1$ *and* $\sum \delta_Y < \infty$.

This class of operators contains both non-relativistic N-body hamiltonians and hamiltonians of perturbed pluristratified media, see [DI].

We check now the assumptions of Theorem 4.6. We decompose H as follows:

$$H = \nabla^* \nabla + (\nabla^* m\ \nabla + W) \equiv \Delta + V.$$

We set $H_0 = \Delta = h(P)$ where $h(x) = |\,x\,|^2$ and

$$V = \nabla^* m\ \nabla + W = \sum [\nabla^* m(Y) \nabla + W(Y)]$$

By taking an orthonormal basis in X compatible with the direct sum decomposition $X = Y^\perp \oplus Y$ we easily see that

$$\nabla^* m(Y) \nabla = (\nabla_{Y^\perp}{}^* m^{Y^\perp} \nabla_{Y^\perp}) \otimes_Y 1 + m^{Y^\perp} \otimes_Y \Delta_Y.$$

Here $\nabla_{Y^\perp}$ is the gradient operator associated to the euclidean space $Y^\perp$ and Δ_Y is the Laplace operator associated to the euclidean space Y. With the notations of Theorem 4.6 and Proposition 4.7, we take $H(Y) = F(P^Y)$, where

$$F(y) = |\,y\,|^2\, m^{Y^\perp} + \nabla_{Y^\perp}{}^* m^{Y^\perp} \nabla_{Y^\perp} + W^{Y^\perp}$$

and obviously all the conditions of Theorem 4.6 are fulfilled (with $s = 1$).

It follows that the operator H is affiliated to the C^*-algebra $\mathfrak{C}^X$ and for each subspace Y one has

$$H_Y = \nabla^* M_Y \nabla + W_Y,$$
$$M_Y = 1 + \sum_{Z \supset Y} m(Z), \quad W_Z = \sum_{Z \supset Y} W(Z).$$

We notice that as in the N-body non-relativistic situation the subhamiltonians H_Y have a quite special form. For each Y (here the notations Y and $Y^\perp$ are interchanged) let us denote by M^Y the operator in $B(\mathcal{H}(Y))$ defined as follows:

$$M^Y = 1 + \sum_{Z \subset Y} m^Z \otimes_Z^Y 1 \equiv 1 + \sum_{Z \subset Y} m^Y (Z^\perp \cap Y)$$

where $\otimes_Z^Y$ corresponds to the tensor product decomposition

$$\mathcal{H}(Y) \cong \mathcal{H}(Z) \bigotimes \mathcal{H}(Z^\perp \cap Y).$$

Then let H^Y be the self-adjoint operator in $\mathcal{H}(Y)$ given by

$$H^Y = \nabla_Y^* M^Y \nabla_Y + \sum_{Z \subset Y} W^Z \otimes_Z^Y 1 \equiv \nabla_Y^* M^Y \nabla_Y + \sum_{Z \subset Y} W^Y (Z^\perp \cap Y).$$

Note that the operator H^Y in $\mathcal{H}(Y)$ has a structure similar to that of H in $\mathcal{H}(X)$. We clearly have $M_Y = M^{Y^\perp} \otimes_Y 1$ and so

$$H_Y = H^{Y^\perp} \otimes_Y 1 + M^{Y^\perp} \otimes_Y \Delta_Y. \tag{6.1}$$

Now, as in the non-relativistic N-body case, we have:

Theorem 6.1 *For each $Y \neq O$ one has $\inf H_Y = \inf H^Y \equiv \tau_Y$ and $\sigma(H_Y) = [\tau_Y, \infty)$; if $M^{Y^\perp} \geq \text{const} > 0$ then the operator H_Y has no eigenvalues. The set $\{\tau_Y \mid Y \in \mathbb{P}(X)\}$ has a smallest element τ and $\sigma_{ess}(H) = [\tau, \infty)$.*

Proof. We shall use Theorem 4.2. Note that $\mathcal{F}_Y H_Y \mathcal{F}_Y^*$ is the operator of multiplication by the operator valued function $H_Y(k) = H^{Y^\perp} + M^{Y^\perp}|k|^2$ in $L^2(Y; \mathcal{H}(Y^\perp))$. Since the spectrum of H_Y is an interval and is the union of the spectra of the operators $H_Y(k)$, and since $H_Y(0) = H^{Y^\perp}$, we get $\inf H_Y = \inf H^{Y^\perp}$. The fact that H_Y has no eigenvalues also folows from the representation (6.1). We prove it in a more general context, without using the virial theorem, in the next lemma. ■

Lemma 6.2 *Let H^a and Δ_a be self-adjoint operators in the Hilbert spaces $\mathcal{H}^a$ and $\mathcal{H}_a$ respectively. Let $M^a \in B(\mathcal{H}^a)$ be positive and invertible. Then the operator $S_0 = H^a \otimes 1 + M^a \otimes \Delta_a$ in $\mathcal{H} = \mathcal{H}^a \bigotimes \mathcal{H}_a$ is essentially self-adjoint on the algebraic tensor product of $D(H^a)$ and $D(\Delta_a)$. If Δ_a has a purely continuous spectrum then the closure S of S_0 has a purely continuous spectrum too.*

Proof. (i) We shall use the following elementary fact. Let T be a (essentially) self-adjoint operator on a Hilbert space $\mathcal{K}$ and let $B : \mathcal{K} \to \mathcal{K}$ be a self-adjoint isomorphism. Then BTB (with domain $B^{-1}D(T)$) is (essentially) self-adjoint.
(ii) Let $N : \mathcal{H}^a \to \mathcal{H}^a$ be a self-adjoint isomorphism such that $N^2 = M^a$ and set $L = N \otimes 1$. Then on the algebraic tensor product of $D(H^a)$ and $D(\Delta_a)$ one clearly has:

$$S_0 = L\{[N^{-1}H^a N^{-1}] \otimes 1 + 1 \otimes \Delta_a\}L$$

By using (i) two times and a well-known fact concerning operators of the form $H_1 \otimes 1 + 1 \otimes H_2$ one gets the essential self-adjointness of S_0.
(iii) Now assume that Δ_a has no eigenvalues. Assume that $f \in D(S)$ and $Sf = \lambda f$; we show that $f = 0$. Replacing H^a by $H^a - \lambda$ we see that it suffices to assume $\lambda = 0$. It is known that the closure T of the operator $[N^{-1}H^a N^{-1}] \otimes 1 + 1 \otimes \Delta_a$ has a purely continuous spectrum, in particular T is injective. So from $0 = Sf = LTLf = 0$ we get $Lf = 0$, hence $f = 0$. ■

The covariant symbol of H_Y can be computed with the help of relation (3.17) from its restriction to Y. On the other hand, it is easy to see that for $k \in Y$ one has

$$H_Y(k) = H^{Y^\perp} + M^{Y^\perp}|k|^2. \tag{6.2}$$

We want to apply Theorem 5.6 with $Z' = Z \; \forall Z$. If $k \in Y$ then $k\nabla H_Y(k) = 2M^{Y^\perp}|k|^2$, so assumption (i) of Theorem 5.6 is fulfilled if $M^{Y^\perp} \geq \text{const} > 0$. If $Y \subset Z$, $k \in Y$, $p \in Z \cap Y^\perp$, then $p\nabla_p H_Z(k+p) = 2M^{Z^\perp}|p|^2$, hence assumption (ii) is satisfied if $M^{Z^\perp} \geq \text{const} > 0$. Thus the theorem can be applied and the results of [DI] can be recovered.

We do not treat here the case, considered in detail by Dermenjian and Iftimie, when the operators m^Y, W^Y are non zero only for a special class of subspaces Y. Then a conjugate operator different from the generator of the dilation group can be chosen such that condition 5.7 still be satisfied, and the rest of the theory works with minor changes (e.g. the relation (5.23) will have a different form).

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