FACTORIZATION THEOREM FOR THE TRANSFER FUNCTION OF A 2×2 OPERATOR MATRIX WITH UNBOUNDED COUPLINGS*

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ABSTRACT. We consider the analytic continuation of the transfer function associated with a 2×2 operator matrix having unbounded couplings into unphysical sheets of its Riemann surface. We construct a family of non-selfadjoint operators which factorize the transfer function and reproduce certain parts of its spectrum including the nonreal (resonance) spectrum situated in the unphysical sheets neighboring the physical sheet.

INTRODUCTION

In this work we consider 2×2 operator matrices

$$\mathbf{H}_0 = \begin{pmatrix} A_0 & T_{01} \\ T_{10} & A_1 \end{pmatrix} \tag{0.1}$$

acting in the orthogonal sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of separable Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$. The entry $A_0 : \mathcal{H}_0 \to \mathcal{H}_0$ is assumed to be an unbounded selfadjoint operator with the domain $\mathcal{D}(A_0)$. We suppose that A_0 is semibounded from below, i.e., $A_0 \geq \alpha_0$ for some $\alpha_0 \in \mathbb{R}$ and without loss of generality let $\alpha_0 > 0$. The entry A_1 is assumed to be a bounded selfadjoint operator in $\mathcal{H}_1$. In contrast to [MM1, MM2], in the present paper we consider unbounded coupling operators $T_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, $i, j = 0, 1$, $i \neq j$. Regarding these operators the following conditions are supposed to be fulfilled:

$$T_{01}^* = T_{10} \quad \text{and} \quad \mathcal{D}(T_{10}) \supset \mathcal{D}(A_0^{1/2}). \tag{0.2}$$

These assumptions are similar to those used in the works by V. M. ADAMYAN, H. LANGER, R. MENNICKEN and J. SAURER [ALMSa] and by R. MENNICKEN and A. A. SHKALIKOV [MS]. Under the conditions (0.2) the matrix (0.1) is a symmetric closable operator in $\mathcal{H}$ on the domain $\mathcal{D}(A_0) \oplus \mathcal{H}_1$ and its closure $\mathbf{H} = \overline{\mathbf{H}}_0$ is a

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selfadjoint operator (see [ALMSa, MS]); for an explicit description of $\mathbf{H}$ see [ALMSa, HMM, MS]. Note that in applications arising from physical problems (see e. g., Refs. [G, L, M2] and references cited therein) one typically deals just with the case where $\mathbf{H}_0$ is a selfadjoint operator in a Hilbert space or a symmetric operator admitting a selfadjoint closure.

The second condition in (0.2) yields that the product $B_{10} := T_{10}A_0^{-1/2}$ is a bounded linear operator. It follows that $A_0^{-1/2}T_{01}$ has a bounded extension B_{01} to the whole space $\mathcal{H}_1$. This extension coincides with B_{10}^*. In addition, the hypothesis (0.2) implies that the operator $A_1 - z + T_{10}(z - A_0)^{-1}T_{01}$ for $z \in \varrho(A_0)$ is densely defined and has a bounded extension onto the whole space $\mathcal{H}_1$ which is given by

$$M_1(z) := \widetilde{A}_1 - z + V_1(z) \tag{0.3}$$

where $\widetilde{A}_1 := A_1 - B_{10}B_{01}$ and $V_1(z) := zB_{10}(z - A_0)^{-1}B_{01}$. We call $M_1(z)$ the transfer function associated with the operator matrix $\mathbf{H}$. It is obvious that this function, considered in the resolvent set $\varrho(A_0)$ of the operator A_0, represents a holomorphic operator-valued function. (In the present work we use the standard definition of holomorphy of an operator-valued function with respect to the operator norm topology, see, e. g., [ALMSa]). It is worth noting that the holomorphic operator-valued function $-M_1$ belongs to the class of operator-valued Herglotz functions (see, e. g., [AG, GKMT, KL, N]).

In the present paper like in [MM1, MM2] we study the transfer function $M_1(z)$ under the assumption that it admits analytic continuation through the absolutely continuous spectrum $\sigma_{\rm ac}(A_0)$ of the entry A_0. We are especially interested in the case where the spectrum of $\widetilde{A}_1$ is partly or totally embedded into the absolutely continuous spectrum of A_0. Notice that, since the resolvent of the operator $\mathbf{H}$ can be expressed explicitly in terms of $\left[M_1(z)\right]^{-1}$ (see, e. g., [ALMSa, MS, MM2]), in studying the spectral properties of the transfer function one studies at the same time the spectral properties of the operator matrix $\mathbf{H}$.

Section 1 includes a description of the conditions making analytic continuation of $M_1(z)$ through the spectrum $\sigma_{\rm ac}(A_0)$ possible. Further, a representation of this continuation is given (see (1.3)). In Section 2 we introduce the basic nonlinear equation (2.3) giving a rigorous sense to the formal operator equation $M_1(H_1) = 0$. We explicitly show that eigenvalues and accompanying eigenvectors of a solution H_1 of the equation (2.3) are eigenvalues and eigenvectors of the analytically continued transfer function M_1. The solvability of (2.3) is proved under smallness conditions concerning the operator B_{10}, see (2.8). In Section 3 we first prove a factorization theorem (Theorem 3.1) for the analytically continued transfer function. This theorem implies that there exists certain domains in $\mathbb{C}$ lying partly on the unphysical sheet(s) where the spectrum of the analytically continued transfer function is represented by the spectrum of the corresponding solutions of the basic equation (2.3). Further in Section 3 we describe some relations between different solutions of (2.3) and some relations between their spectra. Finally, in Section 4 we present a simple example.

A detailed exposition of the material presented here including proofs in the case of essentially more general spectral situations will be given in the extended paper [HMM].

1. Analytic continuation of the transfer function M_1

For the sake of simplicity we assume in this paper that the spectrum $\sigma(A_0)$ of the entry A_0 is absolutely continuous consisting of the interval $\Delta_0 := [\alpha_0, +\infty)$ with $\alpha_0 > 0$ while the spectrum of $\widetilde{A}_1$ is totally embedded into the interval Δ_0, i.e., $\sigma(\widetilde{A}_1) \subset \Delta_0$.

Let E_0 be the spectral measure for the entry A_0, $A_0 = \int_{\Delta_0} \lambda\, dE_0(\lambda)$. Then the function $V_1(z)$ can be written

$$V_1(z) \;=\; \int\limits_{\Delta_0} dK_B(\mu) \frac{z}{z-\mu}$$

with

$$K_B(\mu) \;:=\; B_{10} E^0(\mu) B_{01}$$

where $E^0(\mu)$ stands for the spectral function of A_0, $E^0(\mu) = E_0\big([\alpha_0, \mu)\big)$. Thus, it is convenient to introduce the quantities

$$\mathrm{Var}_\theta(B) \;:=\; \sup_{\{\delta_k,\, \mu_k \in \delta_k\}} \sum_k (1+|\mu_k|)^{-\theta} \|B_{10} E_0(\delta_k) B_{01}\|, \tag{1.1}$$

where θ is some real number and $\{\delta_k\}$ stands for a finite or countable complete system of Borel subsets of $\sigma(A_0) = \Delta_0$ such that $\delta_k \cap \delta_l = \emptyset$, if $k \neq l$, and $\bigcup_k \delta_k = \Delta_0$. The points μ_k are arbitrarily chosen points of δ_k. The number $\mathrm{Var}_\theta(B)$ is called weighted variation of the operators B_{ij} with respect to the spectral measure E_0.

Notice that in contrast to [MM1, MM2], where the variation (1.1) was considered in case of $\theta = 0$, we now will mainly consider $\theta = 1$. Of course, introducing the variation $\mathrm{Var}_\theta(B)$ for $\theta \neq 0$ only makes sense when the entry A_0 is an unbounded operator.

We suppose that the function $K_B(\mu)$ is differentiable in $\mu \in \Delta_0$ in the operator norm topology. The derivative $K_B'(\mu)$ is non-negative, $K_B'(\mu) \geq 0$, since $K_B(\mu)$ is a non-decreasing function. Obviously,

$$\mathrm{Var}_\theta(B) = \int\limits_{\Delta_0} d\mu\, (1+|\mu|)^{-\theta} \|K_B'(\mu)\|.$$

Further, we suppose that the function $K_B'(\mu)$ is continuous within the interval Δ_0 and, moreover, that it admits analytic continuation from this interval to a simply connected domain situated, say, in $\mathbb{C}^+$. Let this domain be called D^+. We assume that the boundary of the domain D^+ includes the entire spectral interval Δ_0. Since $K_B'(\mu)$ represents a selfadjoint operator for $\mu \in \Delta_0$ and $\Delta_0 \subset \mathbb{R}$, the function $K_B'(\mu)$ admits an analytic continuation from Δ_0 into the domain D^-, symmetric to D^+ with respect to the real axis, $D^- = \{z : \overline{z} \in D^+\}$. For the continuation into D^- we will use the same notation $K_B'(\mu)$. The selfadjointness of $K_B'(\mu)$ for

$\mu \in \Delta_0$ implies $[K'_B(\mu)]^* = K'_B(\bar{\mu})$, $\mu \in D^\pm$. Also, we shall suppose that the $K'_B(\mu)$ satisfies the following condition at the end point α_0 of the spectral interval Δ_0:

$$\|K'_B(\mu)\| \leq C|\mu - \alpha_0|^\gamma, \quad \mu \in D^\pm,$$

with some $C > 0$ and $\gamma \in (-1, 0]$.

Let Γ_l ($l = \pm 1$) be a rectifiable Jordan curve in D^l resulting from continuous deformation of the interval Δ_0, the finite end point of this interval being fixed. As mentioned above, in the following we deal with the variation $\mathrm{Var}_1(B)$. We extend the definition of this variation also to the curve Γ_l by introducing the modified variation

$$\mathrm{Var}_1(B, \Gamma_l) := \int_{\Gamma_l} |d\mu|\,(1 + |\mu|)^{-1}\|K'_B(\mu)\| \tag{1.2}$$

where $|d\mu|$ denotes the Lebesgue measure on Γ_l. We suppose that the operators B_{ij} are such that there exists a contour (exist contours) Γ_l on which the value $\mathrm{Var}_1(B, \Gamma_l)$ is finite, i.e., $\mathrm{Var}_1(B, \Gamma_l) < \infty$. The contours Γ_l satisfying the condition $\mathrm{Var}_1(B, \Gamma_l) < \infty$ are said to be K_B-bounded contours. Surely, in the case of unbounded A_0 the condition of boundedness of $\mathrm{Var}_1(B, \Gamma_l)$ is much weaker than the condition of boundedness of $\mathrm{Var}_0(B, \Gamma_l)$ used in [MM1, MM2]).

Lemma 1.1. *The analytic continuation of the transfer function $M_1(z)$, $z \in \mathbb{C}\setminus\Delta_0$, through the spectral interval Δ_0 into the subdomain $D(\Gamma_l) \subset D^l$ ($l = \pm 1$) bounded by the set Δ_0 and a K_B-bounded contour Γ_l is given by*

$$M_1(z, \Gamma_l) := \widetilde{A}_1 - z + V_1(z, \Gamma_l) \tag{1.3}$$

where

$$V_1(z, \Gamma_l) := \int_{\Gamma_l} d\mu\, K'_B(\mu)\, \frac{z}{z - \mu}. \tag{1.4}$$

For $z \in D^l \cap D(\Gamma_l)$ the function $M_1(z, \Gamma_l)$ may be written as

$$M_1(z, \Gamma_l) = M_1(z) + 2\pi \mathrm{i}\, l z K'_B(z). \tag{1.5}$$

Proof. Obviously, the function (1.4) is well defined due to the K_B–boundedness of the contour Γ_l and since for all $z \in \mathbb{C} \setminus \Gamma_l$ there exist a $c(z) > 0$ such that the estimate $\left|(z - \mu)^{-1}\right| < c(z)\left(1 + |\mu|\right)^{-1}$ ($\mu \in \Gamma_l$) holds. Thus, the proof of this lemma is reduced to the observation that the function $M_1(z, \Gamma_l)$ is holomorphic for $z \in \mathbb{C} \setminus \Gamma_l$ and coincides with $M_1(z)$ for $z \in \mathbb{C} \setminus \overline{D(\Gamma_l)}$. The equation (1.5) is obtained from (1.4) using the Residue Theorem. □

The formula (1.5) shows that in general the transfer function $M_1(z)$ has a Riemann surface with at least two sheets. The sheet of the complex plane where the transfer function $M_1(z)$ together with the resolvent $\mathbf{R}(z) = (\mathbf{H} - z)^{-1}$ is initially considered is said to be the physical sheet. The remaining sheets of the Riemann surface of $M_1(z)$ are said to be unphysical sheets (see, e.g., [RS]). In the present work we deal with the unphysical sheets neighboring the physical one, i.e., with the sheets connected through the interval Δ_0 immediately to the physical sheet.

Remark 1.1. For $z \in \mathbb{C} \setminus \Gamma_l$, the equation (1.4) defines values of the function $V_1(\cdot, \Gamma_l)$ in the space of bounded operators in $\mathcal{H}_1$. The inverse transfer function $\left[M_1(z)\right]^{-1}$ coincides with the right lower block component $\mathbf{R}_{11}(z)$ of the resolvent $\mathbf{R}(z) = (\mathbf{H} - z)^{-1}$ and, thus, it is holomorphic in $\mathbb{C} \setminus \sigma(\mathbf{H}) \supset \mathbb{C} \setminus \mathbb{R}$. Since $M_1(z, \Gamma_l)$ coincides with $M_1(z)$ for all $z \in \mathbb{C} \setminus \overline{D(\Gamma_l)}$, one concludes that $[M_1(z, \Gamma_l)]^{-1}$ exists as a bounded operator and is holomorphic in z at least for $z \in \mathbb{C} \setminus \left(\sigma(\mathbf{H}) \cup \overline{D(\Gamma_l)}\right)$.

2. The basic equation

Let Γ be a K_B-bounded contour. If Y stands for an arbitrary bounded operator in $\mathcal{H}_1$ such that the spectrum of Y is separated from the set Γ then, following to [MM2, M1, M2], one can define the operator

$$V_1(Y, \Gamma) \ := \ \int\limits_{\Gamma} d\mu\, K_B'(\mu)\, Y (Y - \mu)^{-1}. \tag{2.1}$$

This operator is bounded, $V_1(Y, \Gamma) \in \mathbf{B}(\mathcal{H}_1, \mathcal{H}_1)$, and its norm admits the estimate

$$\|V_1(Y, \Gamma)\| \ \le \ \mathrm{Var}_1(B, \Gamma)\ \|Y\| \ \sup_{\mu \in \Gamma} (1 + |\mu|) \|(Y - \mu)^{-1}\|. \tag{2.2}$$

In what follows we consider the equation (cf. [MM2, M1, M2])

$$Y \ = \ \widetilde{A}_1 + V_1(Y, \Gamma). \tag{2.3}$$

This equation possesses the following characteristic property: If an operator H_1 is a solution of (2.3) and u_1 is an eigenvector of H_1, $H_1 u_1 = z u_1$, then

$$\begin{aligned} z u_1 \ &= \ \widetilde{A}_1 u_1 + V_1(H_1, \Gamma) u_1 \ = \ \widetilde{A}_1 u_1 + \int\limits_{\Gamma} d\mu\, K_B'(\mu) H_1 (H_1 - \mu)^{-1} u_1 \\ &= \ \widetilde{A}_1 u_1 + \int\limits_{\Gamma} d\mu\, K_B'(\mu) \frac{z}{z - \mu} u_1 \ = \ \widetilde{A}_1 u_1 + V_1(z, \Gamma) u_1. \end{aligned}$$

This means that any eigenvalue z of such an operator H_1 is automatically an eigenvalue for the analytically continued transfer function $M_1(z, \Gamma)$ and u_1 is a corresponding eigenvector. Thus, having found the solution(s) of the equation (2.3) one obtains an effective means of studying the spectral properties of the transfer function $M_1(z, \Gamma)$, referring to well-known facts of operator theory [GK, K]. It is convenient to rewrite the equation (2.3) in the form

$$X \ = \ V_1(\widetilde{A}_1 + X, \Gamma) \tag{2.4}$$

where $X := Y - \widetilde{A}_1$.

Let the spectrum of the operator $\widetilde{A}_1$ be separated from Γ, i.e.,

$$d_0(\Gamma) \ := \ \mathrm{dist}\{\sigma(\widetilde{A}_1), \Gamma\} > 0\,. \tag{2.5}$$

Then, since $\widetilde{A}_1$ is selfadjoint and bounded, it is obvious that the following quantity

$$\mathrm{Var}_{\widetilde{A}_1}(B, \Gamma) \ := \ \int\limits_{\Gamma} |d\mu| \frac{\|K_B'(\mu)\|}{\mathrm{dist}\{\mu, \sigma(\widetilde{A}_1)\}} \tag{2.6}$$

is finite,

$$\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma) \;\le\; \mathrm{Var}_1(B,\Gamma)\,\sup_{\mu\in\Gamma}(1+|\mu|)\big[\mathrm{dist}\{\mu,\sigma(\widetilde{A}_1)\}\big]^{-1} \;<\; \infty\,. \tag{2.7}$$

It is more convenient to make the subsequent estimations in terms of the variation $\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma)$ rather then in terms of the variation $\mathrm{Var}_1(B,\Gamma)$.

Theorem 2.1. *Let $\widetilde{A}_1$ be a bounded operator, the contour Γ be K_B-bounded and*

$$\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma)<1\,, \qquad \mathrm{Var}_{\widetilde{A}_1}(B,\Gamma)\|\widetilde{A}_1\| < \frac{1}{4}\,d_0(\Gamma)\,[1-\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma)]^2\,. \tag{2.8}$$

Let

$$\begin{aligned} r_{\min}(\Gamma) \;&:=\; \frac{1}{2}\,d_0(\Gamma)\,[1-\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma)] \\ &\quad -\sqrt{\frac{1}{4}\,d_0^2(\Gamma)\,[1-\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma)]^2 - d_0(\Gamma)\,\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma)\,\|\widetilde{A}_1\|} \end{aligned} \tag{2.9}$$

and

$$r_{\max}(\Gamma) \;:=\; d_0(\Gamma) - \sqrt{\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma)\,d_0(\Gamma)\,[d_0(\Gamma)+\|\widetilde{A}_1\|]}. \tag{2.10}$$

Then the equation (2.4) *is uniquely solvable in any closed ball*

$$\mathcal{S}_1(r) \;:=\; \big\{X\in\mathbf{B}(\mathcal{H}_1,\mathcal{H}_1)\;:\;\|X\|\le r\big\}$$

where

$$r_{\min}(\Gamma) \;\le\; r \;<\; r_{\max}(\Gamma). \tag{2.11}$$

The solution X of the equation (2.4) *is the same for any r satisfying* (2.11) *and in fact it belongs to the smallest ball $\mathcal{S}_1(r_{\min})$, $\|X\|\le r_{\min}(\Gamma)$.*

Proof. One can prove this theorem making use of Banach's Fixed Point Theorem (see [HMM]). □

The following statement is a direct consequence of the conditions (2.8).

Remark 2.1. The values of $r_{\min}(\Gamma)$ and $r_{\max}(\Gamma)$ satisfy the estimates

$$r_{\min}(\Gamma) \;<\; \frac{1}{2}\,d_0(\Gamma)\,[1-\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma)] \;<\; r_{\max}(\Gamma)\,.$$

Theorem 2.2. *Let the conditions of* Theorem 2.1 *be fulfilled for a K_B-bounded contour $\Gamma\subset D^l$ and let X be the solution of the equation* (2.4)*. Then X coincides with the analogous solution $\widetilde{X}$ for any other K_B-bounded contour $\widetilde{\Gamma}\subset D^l$ satisfying the estimates*

$$\mathrm{Var}_{\widetilde{A}_1}(B,\tilde{\Gamma})<1 \quad \textit{and} \quad \mathrm{Var}_{\widetilde{A}_1}(B,\tilde{\Gamma})\|\widetilde{A}_1\| < \frac{1}{4}\,\widetilde{d}_0[1-\mathrm{Var}_{\widetilde{A}_1}(B,\tilde{\Gamma})]^2$$

where $0<\widetilde{d}_0=\mathrm{dist}\{\sigma(\widetilde{A}_1),\sigma'(A_0)\cup\widetilde{\Gamma}\}\le d_0(\Gamma)$. Moreover, this solution satisfies the inequality $\|X\|\le r_0(B)$ where

$$r_0(B) := \inf\big\{r_{\min}(\Gamma_l)\;:\;\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma_l)<1\,,\,\omega(B,\Gamma_l)>0\big\}$$

with $r_{\min}(\Gamma_l)$ *given by* (2.9) *and*

$$\omega(B,\Gamma_l) \;:=\; d_0(\Gamma_l)\,[1-\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma_l)]^2 - 4\|\widetilde{A}_1\|\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma_l).$$

The value of $r_0(B)$ *does not depend on* l*.*

So, for a given holomorphy domain D^l $(l=\pm 1)$ the solutions X and H_1, $H_1=\widetilde{A}_1+X$, do not depend on the K_B-bounded contours $\Gamma_l\subset D^l$ satisfying the conditions (2.8). But when the index l changes, X and H_1 can also change. For this reason we shall supply them in the following, when it is necessary, with the index l writing $X^{(l)}$ and $H_1^{(l)}$, $H_1^{(l)}=\widetilde{A}_1+X^{(l)}$. Surely, the equations (2.3) and (2.4) are nonlinear equations and, outside the balls $\|X\|<r_{\max}(\Gamma_l)$, they may, in principle, have other solutions, different from the $X^{(l)}$ or $H_1^{(l)}$ the existence of which is guaranteed by Theorem 2.1. In the following we only deal with the solutions $X^{(l)}$ or $H_1^{(l)}$ for $l=\pm 1$.

3. Factorization theorem

Now we prove a *factorization theorem* for the transfer function $M_1(z,\Gamma_l)$. Note that this theorem recalls the corresponding statements from [MrMt, VM].

Theorem 3.1. *Let* Γ_l *be a* K_B*-bounded contour satisfying the conditions* (2.8)*. Suppose* $X^{(l)}$ *is the solution of the basic equation* (2.4)*,* $\|X^{(l)}\|\le r_0(B)$*, and* $H_1^{(l)}=\widetilde{A}_1+X^{(l)}$*. Then, for* $z\in\mathbb{C}\setminus\Gamma_l$*, the transfer function* $M_1(z,\Gamma_l)$ *admits the factorization*

$$M_1(z,\Gamma_l) \;=\; W_1(z,\Gamma_l)\,(H_1^{(l)}-z) \tag{3.1}$$

where $W_1(z,\Gamma_l)$ *is a bounded operator in* $\mathcal{H}_1$*,*

$$\begin{aligned} W_1(z,\Gamma_l) \;&=\; I_1-\int_{\Gamma_l} d\mu\, K_B'(\mu)\,(H_1^{(l)}-\mu)^{-1}\\ &\quad +z\int_{\Gamma_l} d\mu\, K_B'(\mu)(z-\mu)^{-1}(H_1^{(l)}-\mu)^{-1}. \end{aligned}\tag{3.2}$$

If $\mathrm{dist}\{z,\sigma(\widetilde{A}_1)\}\le d_0(\Gamma_l)[1-\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma_l)]/2$*, then the operator* $W_1(z,\Gamma_l)$ *is boundedly invertible and*

$$\big\|[W_1(z,\Gamma_l)]^{-1}\big\| \;\le\; \left(1-\frac{4\,\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma_l)\,\big[d_0(\Gamma_l)+\|\widetilde{A}_1\|\big]}{d_0(\Gamma_l)\big[1+\mathrm{Var}_{\widetilde{A}_1}(B,\Gamma_l)\big]^2}\right)^{-1} \;<\;\infty. \tag{3.3}$$

Proof. First we prove the formula (3.1). Note that, according to (2.1) and (2.4),

$$\widetilde{A}_1 \;=\; H_1^{(l)}-V_1(\widetilde{A}_1+X^{(l)},\Gamma_l) \;=\; H_1^{(l)}-\int_{\Gamma_l} d\mu\, K_B'(\mu)H_1^{(l)}(H_1^{(l)}-\mu)^{-1}. \tag{3.4}$$

Thus, in view of the representations (1.3) and (1.4), the function $M_1(z,\Gamma_l)$ can be written as

$$\begin{aligned} M_1(z,\Gamma_l) &= \widetilde{A}_1 - z + \int\limits_{\Gamma_l} d\mu\, K'_B(\mu)\frac{z}{z-\mu} \\ &= H_1^{(l)} - z - \int\limits_{\Gamma_l} d\mu\, K'_B(\mu)(H_1^{(l)}-\mu)^{-1}(H_1^{(l)}-z) \\ &\quad + z\int\limits_{\Gamma_l} d\mu\, K'_B(\mu)\left[\frac{1}{z-\mu} - (H_1^{(l)}-\mu)^{-1}\right] \\ &= (H_1^{(l)} - z) - \int\limits_{\Gamma_l} d\mu\, K'_B(\mu)(H_1^{(l)}-\mu)^{-1}(H_1^{(l)}-z) \\ &\quad + z\int\limits_{\Gamma_l} d\mu\, K'_B(\mu)(z-\mu)^{-1}(H_1^{(l)}-\mu)^{-1}(H_1^{(l)}-z). \end{aligned}$$

which proves the equation (3.1). The boundeness of the operator $W_1(z,\Gamma_l)$ for $z\in\mathbb{C}\setminus\Gamma_l$ is obvious.

Further, we give a sketch of the proof that the factor $W_1(z,\Gamma_l)$ is a boundedly invertible operator if the condition $\operatorname{dist}\{z,\sigma(\widetilde{A}_1)\} \le d_0(\Gamma_l)\,[1-\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)]/2$ holds. The formula

$$\|(\widetilde{A}_1 + X^{(l)} - \mu)^{-1}\| \;\le\; \frac{1}{\operatorname{dist}\{\mu,\sigma(\widetilde{A}_1)\} - \|X\|}\,, \tag{3.5}$$

the definitions of $d_0(\Gamma_l)$ and $r_{\min}(\Gamma_l)$ and Remark 2.1 imply that

$$\left\| \int\limits_{\Gamma_l} d\mu\, K'_B(\mu)\,(H_1^{(l)}-\mu)^{-1} \right\| \;\le\; \frac{2\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)}{1+\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)}\,. \tag{3.6}$$

Using again the inequality (3.5) and Remark 2.1 we find

$$\left\| z\int\limits_{\Gamma_l} d\mu\, K'_B(\mu)\,(H_1^{(l)}-\mu)^{-1}(z-\mu)^{-1} \right\| \;\le\; |z|\,\frac{2\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)}{1+\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)}\,\sup_{\mu\in\Gamma_l}|z-\mu|^{-1}\,.$$

The inequality $\operatorname{dist}\{z,\sigma(\widetilde{A}_1)\} \le d_0(\Gamma_l)[1-\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)]/2$ yields

$$|z| \;\le\; \|\widetilde{A}_1\| + \operatorname{dist}\{z,\sigma(\widetilde{A}_1)\} \le \|\widetilde{A}_1\| + \frac{1}{2}d_0(\Gamma_l)[1-\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)]$$

and one obtains for $\mu\in\Gamma_l$ that

$$\sup_{\mu\in\Gamma_l}|z-\mu|^{-1} \;\le\; \frac{2}{d_0(\Gamma_l)[1+\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l]}.$$

Hence, for $\operatorname{dist}\{z,\sigma(\widetilde{A}_1)\} \le d_0(\Gamma_l)[1-\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)]/2$,

$$\begin{aligned} \|W_1(z,\Gamma_l)-I_1\| &\le \frac{2\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)}{1+\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)} \\ &\quad + \frac{4\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)\left\{\|\widetilde{A}_1\|+\frac{1}{2}\,d_0(\Gamma_l)[1-\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)]\right\}}{d_0(\Gamma_l)\,[1+\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)]^2} \\ &= \frac{4\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)\,[d_0(\Gamma_l)+\|\widetilde{A}_1\|]}{d_0(\Gamma_l)\,[1+\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)]^2} \;<\; 1\,. \end{aligned}$$

The last inequality is a direct consequence of the second assumption in (2.8). We conclude that $W_1(z,\Gamma_l)$ is invertible and that the inequality (3.3) holds. $\square$

The following theorems can be proved in the same way as Theorem 4.4 and Theorem 4.7 in [MM2].

Theorem 3.2. *The spectrum $\sigma(H_1^{(l)})$ of the operator $H_1^{(l)}=\widetilde{A}_1+X^{(l)}$ belongs to the closed $r_0(B)$-neighbourhood*

$$\mathcal{O}_{r_0(B)}(\widetilde{A}_1) \;:=\; \left\{z\in\mathbb{C}:\ \operatorname{dist}\{z,\sigma(\widetilde{A}_1)\}\le r_0(B)\right\}$$

of the spectrum of $\widetilde{A}_1$. If the contour $\Gamma_l\subset D^l$ satisfies (2.8), then the nonreal spectrum of $H_1^{(l)}$ belongs to $D^l\cap\mathcal{O}_{r_0(B)}(\widetilde{A}_1)$. Moreover, the spectrum $\sigma(H_1^{(l)})$ coincides with a subset of the spectrum of the transfer function $M_1(\cdot,\Gamma_l)$. More precisely, the spectrum of $M_1(\cdot,\Gamma_l)$ in the set

$$\mathcal{O}(\widetilde{A}_1,\Gamma_l) \;:=\; \left\{z\in\mathbb{C}\,:\,\operatorname{dist}\{z,\sigma(\widetilde{A}_1)\}\le d_0(\Gamma_l)\,[1-\operatorname{Var}_{\widetilde{A}_1}(B,\Gamma_l)]/2\right\}$$

equals the spectrum of $H_1^{(l)}$, i. e.,

$$\sigma\big(M_1(\cdot,\Gamma_l)\big)\cap\mathcal{O}(\widetilde{A}_1,\Gamma_l) \;=\; \sigma(H_1^{(l)}). \tag{3.7}$$

In fact such a statement separately holds for the point and continuous spectra.

In the following lemma we state a simple but useful relation between $H_1^{(l)}$ and the adjoint operator of $H_1^{(-l)}$. According to our convention $\Gamma_{(-l)}\subset D^{(-l)}$ is the contour which is conjugate to the contour Γ_l.

Lemma 3.1. *Let $\Gamma_l\subset D^l$ be a K_B-bounded contour for which the conditions of* Theorem 2.1 *are fulfilled. Then for any $z\in\mathbb{C}\setminus\Gamma_l$ the following equality holds true:*

$$W_1(z,\Gamma_l)\,\left(H_1^{(l)}-z\right) \;=\; \left(H_1^{(-l)*}-z\right)\,\left[W_1(\overline{z},\Gamma_{(-l)})\right]^*\,. \tag{3.8}$$

Further the spectrum of $H_1^{(-l)*}$ coincides with the spectrum of $H_1^{(l)}$.

Proof. Let $z\in\mathbb{C}\setminus\Gamma_l$. By definition $\overline{z}\in\mathbb{C}\setminus\Gamma_{(-l)}$ and

$$M_1(z,\Gamma_l)^* \;=\; M_1(\overline{z},\Gamma_{(-l)}). \tag{3.9}$$

Therefore, the relation (3.8) follows from the factorizations

$$M_1(z,\Gamma_l) \;=\; W_1(z,\Gamma_l)\,(H_1^{(l)}-z)$$

and

$$M_1(\overline{z},\Gamma_{(-l)}) \;=\; W_1(\overline{z},\Gamma_{(-l)})\,(H_1^{(-l)}-\overline{z}).$$

By the relation (3.7) z belongs to the spectrum of the operator $H_1^{(-l)*}$ if and only if $\overline{z}\in\mathcal{O}(\widetilde{A}_1,\Gamma_{(-l)})$ and $0\in\sigma\bigl([M_1(\overline{z},\Gamma_{(-l)})]^*\bigr)$. From (3.9) we conclude that $0\in\sigma\bigl([M_1(\overline{z},\Gamma_{(-l)})]^*\bigr)$ if and only if $0\in\sigma\bigl(M_1(z,\Gamma_l)\bigr)$. Again by (3.7) the coincidence of the spectra of $H_1^{(l)}$ and $H_1^{(-l)*}$ follows. □

Let

$$\Omega^{(l)} \;:=\; \int\limits_{\Gamma_l} d\mu\,\mu\;(H_1^{(-l)*}-\mu)^{-1}K_B'(\mu)\,(H_1^{(l)}-\mu)^{-1} \tag{3.10}$$

where as previously Γ_l denotes a K_B-bounded contour satisfying the conditions (2.8). The operator $\Omega^{(l)}$ does not depend on the choice of such a Γ_l.

Theorem 3.3. *The operators $\Omega^{(l)}$ $(l=\pm1)$ possess the following properties (cf.* [HMM, MrMt, MM1, MM2, MS, VM]):

$$\|\Omega^{(l)}\|<1, \qquad \Omega^{(-l)} \;=\; \Omega^{(l)*}, \tag{3.11}$$

$$-\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma} dz\,[M_1(z,\Gamma_l)]^{-1} \;=\; (I_1+\Omega^{(l)})^{-1}\,, \tag{3.12}$$

$$-\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma} dz\,z\,[M_1(z,\Gamma_l)]^{-1} \;=\; (I_1+\Omega^{(l)})^{-1}H_1^{(-l)*}=H_1^{(l)}(I_1+\Omega^{(l)})^{-1}\,, \tag{3.13}$$

where γ stands for an arbitrary rectifiable closed contour going around the spectrum of $H_1^{(l)}$ inside the set $\mathcal{O}(\widetilde{A}_1,\Gamma_l)$ in the positive direction. The integration along γ is understood in the strong sense.

Proof. The estimate in (3.11) can be proved by using the relation (3.8) following the proof of the estimate (3.3). This estimate yields that the sum $I_1+\Omega^{(l)}$ is a boundedly invertible operator in $\mathcal{H}_1$.

To prove the formula (3.12) we recall that due to the factorization theorem 3.1 and the formula (3.8) the following factorization holds for for $z\in\mathcal{O}(\widetilde{A}_1,\Gamma_l)\backslash\sigma(H_1^{(l)})$:

$$\begin{aligned}[M_1(z,\Gamma_l)]^{-1} \;&=\; \left(H_1^{(l)}-z\right)^{-1}[W_1(z,\Gamma_l)]^{-1}\\ &=\; [W_1(\overline{z},\Gamma_{(-l)})]^{*-1}\left(H_1^{(-l)*}-z\right)^{-1}\end{aligned} \tag{3.14}$$

where $[W_1(z,\Gamma_l)]^{-1}$ and $[W_1(\overline{z},\Gamma_{(-l)})]^{*-1}$ are holomorphic functions with values in $\mathbf{B}(\mathcal{H}_1,\mathcal{H}_1)$. By the resolvent equation and the definition (3.2) the product $\Omega^{(l)}(H_1^{(l)}-z)^{-1}$ can be written as

$$\Omega^{(l)}(H_1^{(l)}-z)^{-1} \;=\; F_1(z)+F_2(z) \tag{3.15}$$

where

$$F_1(z) \;:=\; \int\limits_{\Gamma_l} d\mu\, \mu\, (H_1^{(-l)*} - \mu)^{-1}\, K_B'(\mu)\, (H_1^{(l)} - \mu)^{-1}\, (\mu - z)^{-1} \tag{3.16}$$

and

$$\begin{aligned} F_2(z) \;&:=\; \left(-\int\limits_{\Gamma_l} d\mu \frac{\mu}{\mu - z}\, (H_1^{(-l)*} - \mu)^{-1}\, K_B'(\mu) \right) (H_1^{(l)} - z)^{-1} \\ &= \; \left([W_1(\overline{z}, \Gamma_{(-l)})]^* - I_1 \right) (H_1^{(l)} - z)^{-1}. \end{aligned} \tag{3.17}$$

Further, the formula (3.14) yields that

$$(I_1 + \Omega^{(l)})\, [M_1(z, \Gamma_l)]^{-1} \;=\; F_1(z)\, [W_1(z, \Gamma_l)]^{-1} + (H_1^{(-l)*} - z)^{-1}.$$

The function $F_1(z)$ is holomorphic inside the contour γ, $\gamma \subset \mathcal{O}(\widetilde{A}_1, \Gamma_l)$, since the argument μ of the integrand in the formula (3.16) belongs to Γ_l and thereby $|z - \mu| \geq [d_0(\Gamma_l) + \mathrm{Var}_{\widetilde{A}_1}(B, \Gamma_l)]/2 > 0$. Thus the term $F_1(z)[W_1(z, \Gamma_l)]^{-1}$ does not contribute to the integral

$$-\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} dz (I_1 + \Omega^{(l)}) [M_1(z, \Gamma_l)]^{-1}$$

while the resolvent $(H_1^{(-l)*} - z)^{-1}$ gives the identity I_1 which proves the equation (3.12).

Regarding the equation (3.13) we obtain

$$\begin{aligned} &-\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} dz (I_1 + \Omega^{(l)})\, z\, [M_1(z, \Gamma_l)]^{-1} = \\ &= \; -\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} dz\, z\, F_1(z)\, [W_1(z, \Gamma_l)]^{-1} - \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} dz\, z\, (H_1^{(-l)*} - z)^{-1}\,. \end{aligned}$$

The first integral vanishes whereas the second integral equals $H_1^{(-l)*}$. The second equation of (3.13) can be checked in the same way. $\square$

Note that the formulae (3.12) and (3.13) allow, in principle, to construct the operators $H_1^{(l)}$, $l = \pm 1$, and, thus, to resolve the equation (2.4) by a contour integration of the inverse of the transfer function $M_1(z, \Gamma_l)$.

Remark 3.1. The formula (3.13) implies that

$$H_1^{(l)*} \;=\; (I_1 + \Omega^{(-l)})\, H_1^{(-l)}\, (I_1 + \Omega^{(-l)})^{-1}\,.$$

Therefore the spectrum of $H_1^{(-l)*}$ coincides with the spectrum of $H_1^{(l)}$.

Theorem 3.4. *Let λ be an isolated eigenvalue of the operator $H_1^{(l)}$ and, consequently, of the operator $H_1^{(-l)*}$ and of the transfer function $M_1(z, \Gamma_l)$ taken for a*

K_B-bounded contour Γ_l satisfying the conditions (2.8). By $\mathsf{P}_\lambda^{(l)}$ and $\mathsf{P}_\lambda^{(-l)*}$ we denote the eigenprojections of the operators $H_1^{(l)}$ and $H_1^{(-l)*}$, respectively, and by $P_\lambda^{(l)}$ the residue of $M_1(z,\Gamma_l)$ at $z=\lambda$,

$$\mathsf{P}_\lambda^{(l)} := -\frac{1}{2\pi\mathrm{i}}\int\limits_\gamma dz\,(H_1^{(l)}-z)^{-1}, \tag{3.18}$$

$$\mathsf{P}_\lambda^{(-l)*} := -\frac{1}{2\pi\mathrm{i}}\int\limits_\gamma dz\,(H_1^{(-l)*}-z)^{-1} \tag{3.19}$$

and

$$P_\lambda^{(l)} := -\frac{1}{2\pi\mathrm{i}}\int\limits_\gamma dz\,[M_1(z,\Gamma_l)]^{-1} \tag{3.20}$$

where γ stands for an arbitrary rectifiable closed contour going around λ in the positive direction in a sufficiently close neighbourhood such that $\gamma\cap\Gamma_l=\emptyset$ and no points of the spectrum of $M_1(\cdot,\Gamma_l)$, except the eigenvalue λ, lie inside γ. Then the following relations hold:

$$P_\lambda^{(l)} = \mathsf{P}_\lambda^{(l)}\,(I_1+\Omega^{(l)})^{-1} = (I_1+\Omega^{(l)})^{-1}\,\mathsf{P}_\lambda^{(-l)*}. \tag{3.21}$$

Proof. The proof is carried out in the same way as the proof of the relation (3.12), only the path of integration is changed. □

4. An example

Let $\mathcal{H}_0=\mathcal{H}_1=L_2(\mathbb{R})$ and $A_0=\mathsf{D}^2+\lambda_0 I_0$ where $\mathsf{D}=\mathrm{i}\dfrac{d}{dx}$ and λ_0 is some positive number. It is assumed that the domain $\mathcal{D}(A_0)$ is the Sobolev space $W_2^2(\mathbb{R})$. The spectrum of A_0 is absolutely continuous and fills the semiaxis $\Delta_0=[\lambda_0,+\infty)$. By the operator A_1 we understand the multiplication by a bounded real-valued function a_1, $A_1f_1=a_1f_1$, $f_1\in\mathcal{H}_1$. The operator T_{01} reads as

$$T_{01}=(\mathsf{D}^2+\lambda_0 I_0)^{1/2}B$$

where B is the multiplication by a bounded real-valued function $b\in W_2^1(\mathbb{R})$, $Bf=bf$, $f\in L_2(\mathbb{R})$. Moreover, we assume that the function b is decreasing at infinity at least exponentially, so that for any $x\in\mathbb{R}$ the estimate

$$|b(x)| \le c\,\exp(-\alpha|x|) \tag{4.1}$$

holds with some $c>0$ and $\alpha>0$. Finally, we assume that the range of the function

$$\widetilde{a}_1(x) = a_1(x)-b^2(x)$$

is embedded in $[\lambda_0+\widetilde{c},\infty)$ with some $\widetilde{c}>0$. The operator $\widetilde{A}_1$ is the multiplication by the function $\widetilde{a}_1$.

It is easy to check that the value $E^0(\mu)$ of the spectral function of the operator $A_0 = \mathsf{D}^2 + \lambda_0 I_0$ is represented by the integral operator whose kernel reads

$$E^0(\mu; x, x') = \begin{cases} 0 & \text{if } \mu < \lambda_0, \\ \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\lambda_0}^{\mu} d\nu \, \dfrac{\cos[(\nu-\lambda_0)^{1/2}(x-x')]}{(\nu-\lambda_0)^{1/2}} & \text{if } \mu \geq \lambda_0. \end{cases}$$

Thus the derivative $K_B'(\mu)$ is also an integral operator in $L_2(\mathbb{R})$. Its kernel $K_B'(\mu; x, x')$ is only nontrivial for $\mu > \lambda_0$ and, moreover, for these μ

$$K_B'(\mu; x, x') = \frac{1}{\sqrt{2\pi}} \frac{\cos[(\mu-\lambda_0)^{1/2}(x-x')]}{(\mu-\lambda_0)^{1/2}} \, b(x) \, b(x').$$

Obviously, this kernel is degenerate for $\mu > \lambda_0$,

$$K_B'(\mu; x, x') = \frac{1}{2\sqrt{2\pi}\,(\mu-\lambda_0)^{1/2}} [\widetilde{b}_+(\mu, x)\widetilde{b}_-(\mu, x') + \widetilde{b}_-(\mu, x)\widetilde{b}_+(\mu, x')] \tag{4.2}$$

where $\widetilde{b}_\pm(\mu, x) = \mathrm{e}^{\pm \mathrm{i}\,(\mu-\lambda_0)^{1/2} x}\, b(x)$. From the assumption (4.1) on b we conclude that in the domain $\pm \operatorname{Im} \sqrt{\mu - \lambda_0} < \alpha$, i.e., inside the parabola

$$\operatorname{Re} \mu > \lambda_0 - \alpha^2 + \frac{1}{4\alpha^2} (\operatorname{Im} \mu)^2, \tag{4.3}$$

the functions $\widetilde{b}_\pm(\mu, \cdot)$ are elements of $L_2(\mathbb{R})$. The function $K_B'(\mu)$ admits an analytic continuation onto this domain (cut along the interval $\lambda_0 - \alpha^2 < \mu \leq \lambda_0$) as a holomorphic function with values in $\mathbf{B}(\mathcal{H}_1, \mathcal{H}_1)$ and the equation (4.2) implies that

$$\|K_B'(\mu)\| \leq \frac{1}{\sqrt{2\pi}} \frac{1}{|(\mu-\lambda_0)^{1/2}|} \|\widetilde{b}_+(\mu, \cdot)\| \, \|\widetilde{b}_-(\mu, \cdot)\|.$$

Obviously, for real μ we have $\|\widetilde{b}_\pm(\mu, \cdot)\| = \|b\|$. Since $\widetilde{A}_1$ is bounded, one can always choose a K_B-bounded contour Γ lying in the domain (4.3). Indeed, for the K_B-boundedness of the contour Γ it is sufficient to have its infinite part presented by an appropriate semi-infinite real interval. Thus, if the function b is sufficiently small in the sense that the conditions (2.8) hold, one can apply all the statements of the Section 2 and 3 to the corresponding transfer function $M_1(z, \Gamma)$.

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