On the Density of States for the Periodic Schrödinger Operator

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ABSTRACT Asymptotic formula for the density of states of the polyharmonic periodic operator $(-\Delta)^l + V$ in R^n, $n \geq 2$, $l \geq 1$ is obtained. Special consideration is given to the case of the Schrödinger equation $n = 3$, $l = 1$, where the second term of the asymptotic is found.

1 Introduction.

We consider the operator

$$H = (-\Delta)^l + V \tag{1}$$

in $L_2(R^n)$, where V is the operator of multiplication by a real periodic potential, $n \geq 2$, $l \geq 1$. Particular attention is paid to the case of the Schrödinger operator $l = 1, n = 3$. For the sake of simplicity, we assume that the potential has orthogonal periods $a_1, ..., a_n$, however all the results are also valid for nonorthogonal periods. Without the loss of generality we consider that $\int_Q V(x)dx = 0$, $Q = [0, a_1] \times ... \times [0, a_n]$. We use the following representation of the potential:

$$V(x) = \sum_{m \in Z^3, m \neq 0} v_m \exp i(\vec{p}_m(0), x), \tag{2}$$

where $(\cdot, \cdot)$ is the inner product in R^n and $\vec{p}_m(0)$ is a vector of the dual lattice:

$$\vec{p}_m(0) = 2\pi(m_1 a_1^{-1}, m_2 a_2^{-1}, ..., m_m a_n^{-1}). \tag{3}$$

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The condition on the smoothness of the potential is:

$$\|V\|_* = \sum_{m\in Z^n, m\neq 0} |v_m||m|^\nu < \infty, \tag{4}$$

$$\nu > (n-1)/2, \text{ when } n \geq 3, \ \nu > 2(l+1)/3, \text{ when } n = 2.$$

It is well known (see f.e. [RS]) that the spectral analysis of H can be reduced to the study of a family of operators $H(t)$, $t \in K$, where K is the elementary cell of the dual lattice,

$$K = [0, 2\pi a_1^{-1}) \times [0, 2\pi a_2^{-1}) \times ... \times [0, 2\pi a_n^{-1}).$$

The vector t is called the *quasimomentum*. The operator $H(t)$, $t \in K$, acts in $L_2(Q)$. It is described by the formula (1) and the quasiperiodic conditions:

$$u(x_1, ..., x_{j-1}, a_j, x_{j+1}...x_n) = \exp(it_ja_j)u(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n), \tag{5}$$

$$j = 1, 2, ..., n.$$

The derivatives with respect to x_j must also satisfy the analogous conditions. The operator $H(t)$, $t \in K$, has a discrete semibounded below spectrum $\Lambda(t)$:

$$\Lambda(t) = \cup_{n=1}^{\infty} \lambda_n(t), \ \lambda_n(t) \to_{n\to\infty} \infty.$$

The spectrum Λ of the operator H is the union of the spectra of the operators $H(t)$: $\Lambda = \cup_{t\in K}\Lambda(t) = \cup_{n\in N, t\in K}\lambda_n(t)$. The functions $\lambda_n(t)$ are continuous, so Λ has a band structure:

$$\Lambda = \cup_{n=1}^{\infty}[q_n, Q_n], \ q_n = \min_{t\in K} \lambda_n(t), \ Q_n = \max_{t\in K} \lambda_n(t).$$

The eigenfunctions of $H(t)$ and H are simply related. Extending all the eigenfunctions of the operators $H(t)$ quasiperiodically (see (5)) to R^n, we obtain the complete system of eigenfunctions of H. In the case of $V = 0$, the eigenvalues and eigenfunctions of corresponding operators $H_0(t)$, $t \in K$, are naturally indexed by points of Z^n: $\lambda_m^0(t) = p_m^{2l}(t)$, $\psi_m^0(t, x) = \exp i(\vec{p}_m(t), x)$, $m \in Z^n$, here and below $\vec{p}_m(t) = \vec{p}_m(0) + t$, $\vec{p}_m(0)$ being given by (3), and $p_m^{2l}(t) = |\vec{p}_m(t)|^{2l}$.

Let $N_V(\lambda, t)$ be the counting function of eigenvalues, i.e., $N_V(\lambda, t)$ is the number of eigenvalues $\lambda_n(t)$ of the operator $H(t)$, which are not greater than λ:

$$N_V(\lambda, t) = \#\{n : \lambda_n(t) \leq \lambda\} = \sum_{n\in N} \chi(\lambda - \lambda_n(t)), \tag{6}$$

$\chi(\cdot)$ being the Heaviside function. In the case of $V = 0$, $N_0(\lambda, t)$ is the number of lattice points $\vec{p}_m(t)$ in the sphere $\{x : |x| \leq \lambda^{\frac{1}{2l}}\}$. Further we will use the notation $\lambda = k^{2l}$. The asymptotic of the number of the lattice points in the sphere $\{x : |x| \leq k\}$ when $k \to \infty$ depends significantly on t and properties of the lattice. Thus, even $N_0(k^{2l}, t)$ depends on k and t in a nontrivial way, while perturbation V makes this dependence even more sophisticated (see f.e. [B, C, Ch, DT, Sk1, Sk2, Y]). The proof of the estimate $N_V(k^2, t) = \omega_n k^n + O_t(k^{n-1})$, for the case $l = 1$ one can find in [H4], in the latter formula $\omega_n = w_n |K|^{-1}$, w_n, $|K|$ being the volumes of the unit sphere and of the cell K, correspondingly. The proof of the better estimate $N_V(k^2, t) = \omega_n k^n + O_t(k^{n-1-\eta})$ is given in [V].

The density of states $D_V(k^{2l})$ is the integral of $N_V(k^{2l}, t)$ over all t:

$$D_V(k^{2l}) = \int_K N_V(k^{2l}, t) dt. \tag{7}$$

The physical sense of the density of states in the case of the Schrödinger operator (n=3, l=1) is the following: $D_V(\lambda)$ is the limit number of the possible energy levels of a particle in a body in the interval $(-\infty, \lambda)$ divided by the volume of the body as the volume expands to infinity (see f.e. [Z]). M.A. Shubin [Sh1, Sh2, RoShSo] gave the mathematical justification of the passage to the limit for the case of almost periodic (in particular, periodic) potentials. The perturbation methods applied by M.A. Shubin provide the asymptotic estimate for the density of states as $k \to \infty$: [1]

$$D_V(k^{2l}, t) = D_0(k^{2l}, t) + O(k^{n-2l}), \tag{8}$$

where, as it is well known, $D_0(k^{2l}) = \omega_n k^n$. B. Helffer, A. Mohamed [HM] improved this estimate for the case of periodic C^∞-potentials and $l = 1$:

$$D_V(k^2, t) = \omega_n k^n + O(1) + O(k^{n-3+\varepsilon}), \quad \varepsilon > 0. \tag{9}$$

They used advanced methods of microlocal analysis.

This paper has two goals. The first goal is to present a short proof of stronger estimates for $D_V(k^{2l}, t)$ by using perturbation methods. We consider the general case $n \geq 2$, $l \geq 1$, and V satisfying (4). The following estimate is proved in Section 2:

$$D_V(k^{2l}, t) = \omega_n k^n + O(k^{-\xi} \ln k), \quad \xi = 4l - n - 1. \tag{10}$$

[1] Results for other nonperiodic cases can be found f.e. in [G, Ki, Sa].

Clearly (10) is stronger than (8): $\xi > 2l - n$. Comparing (9) and (10) for $l = 1$, we see that (10) is stronger than (9), the biggest difference is in the case $n = 2$: $O(k^{-1} \ln k)$ instead of $O(1)$.

The second goal is to obtain the second term of the asymptotic expansion for the special case $n = 3$, $l = 1$, V satisfying (4) with $\nu = 900$ (Section 3). We prove that

$$D_V(k^2, t) = \omega_3 k^3 + d_V^0 + O(k^{-\zeta}), \quad \zeta < 1/130, \tag{11}$$

where d_V^0 is a constant which can be expressed as the sum of the integrals of the densities of states of some one-dimensional Schrödinger operators (Section 3.1). The number $1/130$ in the estimate (11) arises for technical reasons and could be improved by somewhat longer considerations.

2 The general case $n \geq 2$, $l \geq 1$.

Theorem 1 *Let $n \geq 2$, $l \geq 1$ and V satisfy the condition (4). Then, the density of states of the operator H satisfies the following asymptotic as $k \to \infty$:*

$$D_V(k^{2l}) = \omega_n k^n + O(k^{-\xi} \ln k), \quad \xi = 4l - n - 1. \tag{12}$$

Let us describe the main steps in the proof of Theorem 1. First, we represent operator $H(t)$ in the form

$$H(t) = H_0(t) + (I-P)V(I-P) + PV(I-P) + (I-P)VP + PVP, \tag{13}$$

where $H_0(t)$ corresponds to $V = 0$, $P = P(t)$ is a finite-dimensional diagonal projection. We prove that

$$N_{(I-P)V(I-P)}(k^{2l}, t) = N_0(k^{2l}, t), \tag{14}$$

i.e., the operators $H_0(t)$ and $\tilde{H}(t) = H_0(t) + (I-P)V(I-P)$ have the same number of eigenvalues which are not greater than k^{2l}. The equality (14) holds due to the special choice of the diagonal projection P. Indeed, $\tilde{H}(t)$ is a direct sum of PH_0 and $(I-P)H_0(t) + (I-P)V(I-P)$. Thus, $(I-P)V(I-P)$, in fact, perturbs only $(I-P)H_0$. We construct P in such a way that $(I-P)H_0$ does not contain eigenvalues $p_m^{2l}(t)$ of $H_0(t)$, which are "in danger" to intersect the point $\lambda = k^{2l}$ under the perturbation $(I-P)V(I-P)$. The simplest way would be to chose P: $P_{mm} = 1$, when $|p_m^{2l}(t) - k^{2l}| \leq \|V\|$, however it would result, eventually, only in the estimate (8). So, we will choose P in a somewhat more subtle way. Since $(I-P)V(I-P)$ perturbs only $(I-P)H_0$, the relation (14) holds.

The second step: considering that $PV(I-P) + (I-P)VP + PVP$ is a finite-dimensional operator, we get

$$|N_V(k^{2l}, t) - N_{(I-P)V(I-P)}(k^{2l}, t)| \leq 2\#(t),$$

where $\#(t)$ is the dimension of $P(t)$. Using (14), we obtain

$$|N_V(k^{2l}, t) - N_0(k^{2l}, t)| \leq 2\#(t). \tag{15}$$

In the third part of the proof we use counting arguments to show that

$$\int_K \#(t)dt < ck^{-\xi} \ln k, \quad c = c(\|V\|_*, \nu, a_{\max}, n, l), \tag{16}$$

here and below $a_{\max} = \max_i a_i$, $a_{\min} = \min_i a_i$. Integrating both parts of (15), we obtain (12).

Let us introduce some notations:

$$\sigma_{mq}(z,t) = \sqrt{|p_m^{2l}(t) - z||p_{m+q}^{2l}(t) - z|}, \quad m, q \in Z^n,$$
$$\sigma_m(z,t) = \min_{q \in Z^n \setminus \{0\}} |q|^\nu \sigma_{mq}(z,t), \tag{17}$$

ν being a parameter in the condition (4). We define the diagonal projection P as follows:

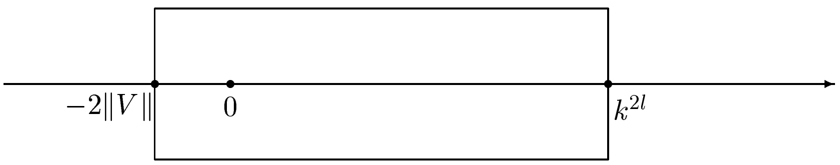
$$P_{mm} = \begin{cases} 1, & \text{if } \sigma_m(k^{2l}, t) < 4\|V\|_*, \\ 0, & \text{otherwise;} \end{cases} \tag{18}$$

$\|V\|_*$ being given by (4).

Lemma 1 *If P is defined by (18), then*

$$N_{(I-P)V(I-P)}(k^{2l}, t) = N_0(k^{2l}, t). \tag{19}$$

Proof. We consider the following contour on the complex plane:



The contour C_0

It is clear that $N_{(I-P)V(I-P)}(k^{2l},t)$ is given by the integral:

$$N_{(I-P)V(I-P)}(k^{2l},t) = \mathrm{Tr} \oint_{C_0} (\tilde{H}(t)-z)^{-1} dt,$$

if there is no eigenvalue at k^{2l}. Let us formally expand the resolvent in the perturbation series with respect to H_0:

$$(\tilde{H}(t)-z)^{-1} = (H_0(t)-z)^{-1} + \sum_{r=1}^{\infty} (H_0(t)-z)^{-1/2} \tilde{A}^r (H_0(t)-z)^{-1/2}, \quad (20)$$

$$\tilde{A} = (I-P)(H_0(t)-z)^{-1/2} V (H_0(t)-z)^{-1/2} (I-P).$$

To prove the convergence of the series, it is enough to check that

$$\|\tilde{A}(z)\| < 1/2, \ \ z \in C_0. \quad (21)$$

This estimate is obvious on the left side of the contour and on the horizontal parts. Let us consider the right side. It is easy to see that

$$\|\tilde{A}\| \le \max_{m:P_{mm}=0} \sum_{q\in Z^n\setminus\{0\}} \frac{|v_q|}{\sigma_{mq}}.$$

Using the definition (18) of P, we get

$$\|\tilde{A}\| \le \sum_{q\in Z^n\setminus\{0\}} \frac{|v_q||q|^{\nu}}{4\|V\|_*} < 1/4.$$

Thus, we obtain (21). Note that

$$\|(I-P)(H_0-z)^{-1/2}\| = \left(\min_{m\in Z^n,\ m:P_{mm}=0} |p_m^{2l}(t)-z| \right)^{-1/2}. \quad (22)$$

If $\sigma_m > 4\|V\|_*$, then, obviously, $|p_m^{2l}(t)-k^{2l}| > 0$. Thus, $(I-P)(H_0-z)^{-1/2}$ is bounded. Estimate (21) provides convergence of the series for the resolvent on the contour C_0. Taking into account that $(H_0-z)^{-1}$ is a Sturm-Liouville operator, we obtain that the series converges in the trace class. Thus, $N_{\alpha(I-P)V(I-P)}(k^{2l},t)$ is a continuous function of αV, $0 \le \alpha \le 1$. Considering that $N_{(I-P)V(I-P)}(k^{2l},t)$ and $N_0(k^2,t)$ are integers, we get (19). □

Note that the difference between $H(t)$ and $\tilde{H}(t)$ is a finite dimensional operator: $\tilde{H}(t) - H(t) = (I-P)VP + PV(I-P) + PVP$. The dimension of this operator is not greater than $2\#(t)$, where $\#(t)$ is the dimension of P. According to the general perturbation theory (see f.e. [Kato]):

$$|N_V(k^{2l},t) - N_{(I-P)V(I-P)}(k^{2l},t)| \leq 2\#(t). \tag{23}$$

Lemma 2 *The function $\#(t)$ satisfies the following estimate:*

$$\int_K \#(t)dt < ck^{-\xi}\ln k, \quad c = c(\|V\|_*, \nu, a_{\max}, l, n). \tag{24}$$

<u>Proof.</u> Let $\#(\Omega,t)$ be the number of points $\vec{p}_m(t)$ in a measurable set $\Omega \subset R^n$. It is easy to see that

$$\int_K \#(\Omega,t)dt = V(\Omega),$$

where V is the volume. Let us define Ω in R^n, as follows:

$$\Omega = \cup_{q\in Z^n\setminus\{0\}}\Omega_q, \quad \Omega_q = \left\{x : ||x|^{2l} - k^{2l}||||x+\vec{p}_q(0)|^{2l} - k^{2l}| < F(q)\right\}, \tag{25}$$

$F(q) = \left(4|q|^{-\nu}\|V\|_*\right)^2$. By the definition of P, $P_{mm} = 1$ if and only if $\vec{p}_m(t)$ is in Ω. Hence, $\#(t) = \#(\Omega,t)$. Thus, it suffices to prove the estimate:

$$V(\Omega) < ck^{-\xi}\ln k. \tag{26}$$

Let us represent Ω_q in the form $\Omega_q = \Omega_q^{>} \cup \Omega_q^{<}$, where

$$\Omega_q^{>} = \{x \in \Omega_q : ||x+\vec{p}_q(0)|^{2l} - k^{2l}| \geq ||x|^{2l} - k^{2l}|\}, \tag{27}$$

and, correspondingly,

$$\Omega_q^{<} = \{x \in \Omega_q : ||x+\vec{p}_q(0)|^{2l} - k^{2l}| \leq ||x|^{2l} - k^{2l}|\}.$$

It is easy to see that under the parallel shift $x \to x + \vec{p}_q(0)$ the set $\Omega_q^{>}$ is transformed into $\Omega_{-q}^{<}$. Thus, $V(\Omega_q^{<}) = V(\Omega_{-q}^{>})$. Considering (25), we see that

$$V(\Omega) \leq 2\sum_{q\in Z^n\setminus\{0\}} V(\Omega_q^{>}).$$

To prove (26), it suffices to obtain the estimate

$$V(\Omega_q^{>}) < c|q|^{-n-\delta}k^{-\xi}\ln k, \quad \delta > 0. \tag{28}$$

for all $q \neq 0$. Let us prove (28). Using (25) and (27), it is easy to see that $\Omega_q^>$ lies in the spherical shell $\hat{S} = \{x : ||x|^{2l} - k^{2l}|^2 \leq (4|q|^{-\nu}\|V\|_*)^2\} = \{x : ||x| - k| \leq 4|q|^{-\nu}\|V\|_* k^{-2l+1}\}$. Let us split $\hat{S}$ into the spherical shells:

$$\hat{S} \subset \cup_{m=0}^M S_m(k), \quad M = \left[\ln\left(12|q|^{\nu+1}\|V\|_* k^{2l-1}\right)\right],$$

where

$$S_0 = \{x : ||x| - k| < |q|^{-2\nu-1}k^{-n+1-\xi}\}, \tag{29}$$

$$S_m = \{x : |q|^{-2\nu-1}k^{-n+1-\xi}e^{(m-1)} \leq ||x| - k| < |q|^{-2\nu-1}k^{-n+1-\xi}e^m\},$$

$m = 1, ..., M$. Then, $V(\Omega_q^>) \leq \sum_{m=0}^M V(\Omega_q^> \cap S_m)$. Clearly,

$$V(\Omega_q^> \cap S_0) \leq V(S_0) < c|q|^{-2\nu-1}k^{-\xi} < c|q|^{-n-\delta}k^{-\xi},\ \delta = 2\nu + 1 - n > 0. \tag{30}$$

Let us consider the intersection of a spherical shell S_m, $m > 0$, with the plane layer $T_{q,m} = \left\{x : ||x + \vec{p}_q(0)|^2 - |x|^2| < |q|\left(26\|V\|_*\right)^2 ke^{-m}\right\}$. We prove that

$$\Omega_q^> \cap S_m \subset \Omega_q^> \cap S_m \cap T_{q,m}. \tag{31}$$

Suppose it is not so. Then there is $x \in \Omega_q^> \cap S_m$, such that $x \notin T_{q,m}$. From the definitions of S_m and $T_{q,m}$, we see that

$$||x|^2 - k^2| \geq |q|^{-2\nu-1}k^{-n+2-\xi}e^{m-1}, \tag{32}$$

$$||x|^2 - k^2| < 2|q|^{-2\nu-1}k^{-n+2-\xi}e^m, \tag{33}$$

$$||x + \vec{p}_q(0)|^2 - |x|^2| \geq |q|\left(26\|V\|_*\right)^2 ke^{-m}. \tag{34}$$

Using (33), (34) and the inequality $m \leq M$, it is easy to check that

$$||x + p_q(0)|^2 - k^2| \geq |q|\left(13\|V\|_*\right)^2 ke^{-m}. \tag{35}$$

Multiplying (32) and (35), we get $||x|^2 - k^2|||x + p_q(0)|^2 - k^2| > \left(6|q|^{-\nu}\|V\|_* k^{-2l+2}\right)^2$. This contradicts to the fact that $x \in \Omega_q$ (see (25)). Thus, (31) holds. Therefore,

$$V(\Omega_q^> \cap S_m) \leq V(S_m \cap T_{q,m}). \tag{36}$$

From the definitions of $\hat{S}$ and $T_{q,m}$ it follows that $S_m \cap T_{q,m} = \emptyset$ if $|q| > c_1 k$, $c_1 = 50(1 + \|V\|_*)$, hence it is enough to consider $|q| \leq c_1 k$. Let us check that

$$V(S_m \cap T_{q,m}) < c|q|^{-n-\delta}k^{-\xi},\ \delta > 0. \tag{37}$$

It is not difficult to estimate the volume of $S_m \cap T_{q,m}$, as the volume of the intersection of a spherical shell with a plane layer. [2] If $n \geq 3$, then

$$V(S_m \cap T_{q,m}) < ca_m b_{mq} k^{n-2}, \tag{38}$$

a_m and b_{mq} being the widths of S_m and $T_{q,m}$, correspondingly, $a_m = |q|^{-2\nu-1} k^{-n+1-\xi} e^m$, $b_{mq} < 8\pi a_{\max} \left(26\|V\|_*\right)^2 k e^{-m}$. Thus, $V(S_m \cap T_{q,m})) < c|q|^{-2\nu-1} k^{-\xi}$. We see that (37) holds for $\delta = 2\nu + 1 - n$, $n \geq 3$. If $n = 2$ and $0 < |q| < k/2$, we can use the estimate (38). However, the estimate (38) is not valid when $|q| \approx k$. For $2k > |q| \geq k/2$ we use the estimate: $V(S_m \cap T_{q,m}) < ca_n \sqrt{b_{mq} k} \leq ck^{-\xi} |q|^{-2\nu-1} e^{m/2}$. Taking into account that $m \leq M$, $M \approx (2l + \nu) \ln k$, we get

$$V(S_m \cap T_{q,m}) \leq ck^{-\xi - 3\nu/2 - 1 + l} \leq ck^{-\xi - 2 - \delta}, \quad \delta = 3\nu/2 - l - 1. \tag{39}$$

Using (38) for $|q| \leq k/2$ and (39) for $|q| > k/2$, we get (37) in the case $n = 2$. Considering (30), (36), (37) and summarizing the estimates (37) over m, we get $V(\Omega_q^>) \leq V(S_0) + \sum_{m=1}^{M} V(S_m \cap T_{q,m}) \leq (M+1)|q|^{-n-\delta} k^{-\xi} \leq c|q|^{-n-\delta} k^{-\xi} \ln k$. Thus, we have obtained (28). $\square$

The proof of Theorem 1. By definition, $D_V(k^{2l}) = \int_K N_V(k^{2l}, t)dt$. Using Lemmas 1 and 2, and estimate (23), we obtain

$$|D_V(k^{2l}) - D_0(k^{2l})| \leq 2 \int_K \#(t)dt \leq ck^{-\xi} \ln k. \ \square$$

3 The Schrödinger Operator in R^3.

3.1 The main results.

First, we consider the case of a trigonometric polynomial:

$$V(x) = \sum_{m \in Z^3,\ 0 < |m| < R_0} v_m \exp i(\vec{p}_m(0), x), \quad R_0 < \infty. \tag{40}$$

Before formulating the main theorems let us represent $V(x)$ as a sum of potentials V_q, each changing only in one direction. To do that, we start with

[2] f.e. Lemma 2.3 in [K] gives the estimates for the measures of the intersections of the sphere $|x| = k$ with the spherical layers $T_{q,m}$, depending on q and the width of $T_{q,m}$. Integrating the estimates over the radius of the sphere we get the estimates for $V(S_m \cap T_{q,m})$.

the definition of a set $\Gamma(R_0)$: let us consider $m : m \in Z^3, 0 < |m| < R_0$. In this set some of the m are scalar multipliers of others. Let us keep from every family of the scalar multipliers only the minimal representative, i.e., the representative having the minimal length. We denote by $\Gamma(R_0)$ the union of these minimal representatives. In other words, each $m : m \in Z^3, 0 < |m| < R_0$ can be uniquely represented in the form $m = rq$, where $r \in Z \setminus \{0\}$, $q \in \Gamma(R_0)$. It is easy to see that

$$V = \sum_{q \in \Gamma(R_0)} V_q, \tag{41}$$

where V_q depends only on $(x, \vec{p}_q(0))$,

$$V_q = \sum_{r \in Z \ ,0<|rq|<R_0} v_{rq} \exp ir(\vec{p}_q(0), x).$$

We consider the operators $H_q = H_0 + V_q$. Let $\delta D_{V_q}(k^2) = D_{V_q}(k^2) - D_0(k^2)$, and

$$d_V(k^2) = \sum_{q \in \Gamma(R_0)} \delta D_{V_q}(k^2). \tag{42}$$

Theorem 2 *If V is a trigonometric polynomial (40), then the density of states of operator H satisfies the following asymptotic formula as $k \to \infty$:*

$$D_V(k^2) = \omega_3 k^3 + d_V(k^2) + O(k^{-\zeta}), \tag{43}$$

where $\zeta > 1/105$, and $d_V(k^2)$, given by (42), satisfies the estimate:

$$|d_V(k^2)| < c\|V\|_*(1 + \|V\|_*), \quad c = c(a_{\max}, a_{\min}). \tag{44}$$

Estimate (44) is uniform in R_0 when $\|V\|_*$ stays finite, while $O(k^{-\zeta})$ is not uniform (for details, see the proof of the theorem in Section 3.3). The asymptotic formula uniform in R_0 we will be given in Theorem 4.

We are going to show that d_V is close to a constant d_V^0 and to obtain the formula for d_V^0 . First, we consider $H_q = -\Delta + V_q$. Let us direct x_1 along $\vec{p}_q(0)$ (the density of states D_{V_q} is invariant with respect to this operation [Sh1, Sh2, RoShSo]). Obviously, by separation of variables, the spectral study of H_q can be reduced to that of the one-dimensional periodic Schrödinger operator :

$$H_q^{(1)} = -\frac{d^2}{dx_1^2} + V_q^{(1)}(x_1), \tag{45}$$

$$V_q^{(1)}(x_1)=\sum_{r\in Z,0<|rq|<R_0} v_{rq}\exp\left(irp_q(0)x_1\right),\quad p_q(0)=|\vec{p}_q(0)|.$$

Let $\rho_{V_q}(k^2)$ be the integral density of states of $H_q^{(1)}$, and $\delta\rho_{V_q}=\rho_{V_q}-\rho_0$. It is easy to show (Section 3.3) that

$$\delta D_{V_q}=\pi\int_{-\infty}^{k^2}\delta\rho_{V_q}(y)dy, \tag{46}$$

$\delta\rho_{V_q}(y)$ satisfying the estimate:

$$|\delta\rho_{V_q}(y)|<c_0(V_q)y^{-3/2},$$

for $\sqrt{y}>10\|V_q\|p_q^{-1}(0)$, here and below

$$c_0(V_q)=c\|V_q\|_*(a_{\min}^{-2}+\|V_q\|_*).\quad c\neq c(y,V_q,a_1,a_2,a_3), \tag{47}$$

Therefore,

$$\left|\delta D_{V_q}-\pi\int_{-\infty}^{\infty}\delta\rho_{V_q}(y)dy\right|<c_0(V_q)k^{-1},\quad k>10\|V_q\|p_q^{-1}(0). \tag{48}$$

Using the definition (42) of d_V, we will get

$$\left|d_V^0-d_V\right|<c_0(V)k^{-1}, \tag{49}$$

where

$$d_V^0=\sum_{q\in\Gamma(R_0)}\int_{-\infty}^{\infty}\delta\rho_{V_q}(y)dy. \tag{50}$$

Note that d_V^0 is a constant with respect to k^2. We will show that

$$|d_V^0|<c\|V\|_*(1+\|V\|_*),\quad c=c(a_{\max},a_{\min}). \tag{51}$$

The next theorem follows from Theorem 2 and relations (49), (51).

Theorem 3 *The density of states of operator H satisfies the following asymptotic formula as $k\to\infty$:*

$$D_V(k^2)=\omega_3k^3+d_V^0+O(k^{-\zeta}), \tag{52}$$

where d_V^0 is a constant given by (50), d_V^0 satisfies estimate (51) and $\zeta>1/105$.

If V satisfies (4) with $\nu = 900$, then we can pass to the limit in the definitions of $d_V(k^2)$ and d_V^0 as $R_0 \to \infty$: $d_V(k^2) = \sum_{q \in \Gamma(\infty)} \delta D_{V_q}(k^2)$ and $d_V^0 = \sum_{q \in \Gamma(\infty)} \int_{-\infty}^{\infty} \delta \rho_{V_q}(y) dy$, where $\Gamma(\infty) = \lim_{R_0 \to \infty} \Gamma(R_0) = \cup_{R_0=1}^{\infty} \Gamma(R_0)$.

Theorem 4 *If V satisfies (4) with $\nu = 900$, then relations (43), (44) and (51), (52) hold, where $|O(k^{-\zeta})| < c_1 k^{-\zeta}$, $\zeta > 1/130$, $c_1 = c_1(\|V\|_*, a_{\max}, a_{\min})$.*

Theorems 2 – 4 are in correspondence with the result of G.Eskin, J.Ralston, E.Trubowitz [ERT1, ERT2, ERT3] that isospectral potentials must have the same directional components V_q.

We will start our considerations with the case of a trigonometric polynomial. We want to be able to extend proofs to the case of a smooth potential. For this reason everywhere below the only estimate for R_0 which we use is $R_0 < k^{\delta/8}$, $\delta \leq 1/105$.

3.2 The spectra of the operators H_q.

The purpose of this section is to show that eigenvalues $\lambda_m^q(t)$ of $H_q(t)$ can be represented in the form: $\lambda_m^q(t) = p_m^{2l}(t) + \Delta\Lambda_q(t)_{mm}$, where the shift $\Delta\Lambda_q(t)_{mm}$ of $p_m^{2l}(t)$ under the perturbation V_q admits the estimate (64).

Let us consider the matrix of $H_q(t)$ in l_2^3, l_2^3 being the space of square-summable sequences with indices in Z^3. We associate with each i in Z^3 the diagonal projection P_i^q in l_2^3:

$$(P_i^q)_{mm} = \begin{cases} 1, & \text{if } \vec{p}_i(0) - \vec{p}_m(0) = l\vec{p}_q(0),\ l \in Z, \\ 0, & \text{otherwise.} \end{cases} \tag{53}$$

Obviously, $P_i^q = P_j^q$ if $\vec{p}_i(0) - \vec{p}_j(0) = l\vec{p}_q(0)$, $l \in Z$. It is clear that there exists a minimal subset J_q^0 of Z^3 such that

$$\sum_{i \in J_q^0} P_i^q = I. \tag{54}$$

Note that $(V_q)_{rn} = 0$ if $\vec{p}_r(0) - \vec{p}_n(0) \neq l\vec{p}_q(0)$, $l \in Z$, because V_q depends only on $(x, \vec{p}_q(0))$. This at once gives us that $P_i^q V_q P_i^q = P_i^q V_q = V_q P_i^q$, and, therefore,

$$P_i^q H_q(t) P_i^q = P_i^q H_q(t) = H_q(t) P_i^q. \tag{55}$$

Considering relations (54) and (55) we get $H_q(t) = \sum_{i \in J_q^0} P_i^q H_q(t) P_i^q$.

We establish an isometric isomorphism between $P_i^q l_2^3$ and l_2^1, l_2^1 being the space of square-summable sequences with indices in Z. Denote by δ_m, $m \in Z^3$, the element of l_2^3 given by the formula $\{\delta_m\}_n = \delta_{mn}$, and by δ_l^1, $l \in Z$, the analogous element of l_2^1 given by the formula $\{\delta_l^1\}_r = \delta_{lr}$, δ_{mn}, δ_{lr} being the Kronecker symbols. To construct an isometric isomorphism we represent $\vec{p}_i(t)$, $i \in J_q^0$, in the form of a linear combination of $\vec{p}_q(0)$ and a vector in the orthogonal complement of $\vec{p}_q(0)$: $\vec{p}_i(t) = \tau_i(2\pi)^{-1}\vec{p}_q(0) + \vec{d_i^q}$, $\tau_i \in R$, $\tau_i = \tau_i(t,q)$, $(\vec{d_i^q}(t), \vec{p}_q(0)) = 0$. Thus, if $\delta_m \in P_i^q l_2^3$, then $\vec{p}_m(t)$ is uniquely representable in the form

$$\vec{p}_m(t) = \vec{d_i^q}(t) + (\tau_i + 2\pi l)(2\pi)^{-1}\vec{p}_q(0), \quad l \in Z, \tag{56}$$

where it can be assumed without loss of generality that $0 \le \tau_i < 2\pi$. From this,

$$(P_i^q)_{mm} = \begin{cases} 1, & \text{if } \vec{p}_m(t) = \vec{d_i^q}(t) + (\tau_i + 2\pi l)(2\pi)^{-1}\vec{p}_q(0), \quad l \in Z, \\ 0, & \text{otherwise.} \end{cases} \tag{57}$$

An isomorphism between $P_i^q l_2^3$ and l_2^1 is now established in the natural way with the help of the formula (56): $\delta_m \leftrightarrow \delta_l^1$, $\delta_m \in P_i^q l_2^3$, $\delta_l^1 \in l_2^1$, and it follows from (56) that

$$l = [(\vec{p}_m(t), \vec{p}_q(0))p_q(0)^{-2}], \tag{58}$$

$$\tau_i = 2\pi(\vec{p}_m(t), \vec{p}_q(0))p_q(0)^{-2} - 2\pi l. \tag{59}$$

It is easy to verify that the operator $P_i^q H_q P_i^q$ is equivalent to the operator $H_1(\tau_i) + |(\vec{d_i^q}(t)|^2 I$; here the operator $H_1(\tau_i)$ is given by the matrix

$$H_1(\tau_i)_{lp} = (\tau_i + 2\pi l)^2 a_q^{-2}\delta_{lp} + v_{(l-p)q}, \quad l, p \in Z, \quad a_q = (2\pi)p_q(0)^{-1}. \tag{60}$$

This is the matrix representation of a periodic Schrödinger operator (45) on the real axis. Each operator $H_1(\tau)$, $\tau \in [0, 2\pi)$, has a discrete spectrum. It is well known that its eigenvalues $\lambda_l(\tau)$ can be enumerated by integers in such a way that

$$|\lambda_l(\tau) - (2\pi l + \tau)^2 a_q^{-2}| < c_0(V_q)(l p_q(0))^{-2}, \tag{61}$$

when $l p_q(0) > 10\|V_q\| p_q^{-1}(0)$, $c_0(V_q)$ being given by (47). The functions $\lambda_l(\tau)$ are piecewise continuous. Let $\Delta\lambda_l^q(\tau_i)$ be the shift of the eigenvalue under the perturbation: $\Delta\lambda_l^q(\tau_i) = \lambda_l^q(\tau_i) - (\tau_i + 2\pi l)^2 a_q^{-2}$. The spectrum of the operator $P_i^q H_q P_i^q$ can be represented in the form:

$$\{\lambda_l^q(\tau_i) + |(\vec{d}_i^q(t)|^2\}_{l\in Z} =$$
$$= \{\Delta\lambda_l^q(\tau_i) + (\tau_i + 2\pi l)^2(2\pi)^{-2}p_q(0)^2 + |(\vec{d}_i^q(t)|^2\}_{l\in Z} =$$
$$= \{\Delta\lambda_{l(m)}^q(\tau_i) + p_m^2(t)\}_{m:\delta_m\in P_i^q l_2^3} =$$
$$= \{\Delta\lambda_{l(m)}^q(\tau_m) + p_m^2(t)\}_{m:\delta_m\in P_i^q l_2^3}; \tag{62}$$

here τ_m can be computed from the formula

$$\tau_m = 2\pi(\vec{p}_m(t),\vec{p}_q(0))p_q(0)^{-2} - 2\pi[(\vec{p}_m(t),\vec{p}_q(0))p_q(0)^{-2}], \tag{63}$$

and, as it is easily seen, τ_m coincides with τ_i for all $m : \delta_m \in P_i^q l_2^3$, and $l(m)$ is given by the formula (58). The spectrum $\Lambda_q(t)$ of the operator H_q is the union of the spectra of the operators $P_i^q H_q(t) P_i^q$:

$$\Lambda_q(t) = \cup_{i\in J_q^0}\{\Delta\lambda_{l(m)}^q(\tau_m) + p_m^2(t)\}_{m:\delta_m\in P_i^q l_2^3} = \{\Delta\lambda_{l(m)}^q(\tau_m) + p_m^2(t)\}_{m\in Z^3}.$$

Introducing the diagonal matrix $\Delta\Lambda_q(t)$ of shifts of the eigenvalues, $\Delta\Lambda_q(t)_{mm} = \Delta\lambda_{l(m)}^q(\tau_m)$, we get that the diagonal matrix $\Lambda_q(t)$ of eigenvalues of the operator $H_q(t)$ is $H_0(t) + \Delta\Lambda_q(t)$. Using (58) and (61), we obtain:

$$|\Delta\Lambda_q(t)_{mm}| < c_0(V_q)|(\vec{p}_m(t),\vec{p}_q(0))|^{-2}, \tag{64}$$

when $|(\vec{p}_m(t),\vec{p}_q(0))| > 10\|V_q\|$, where $c_0(V_q)$ is given by (47).

3.3 The densities of states of the operators H_q.

Lemma 3 *The functions $\delta D_{V_q}(k^2)$ admit representation (46), estimates (48) and*

$$|\delta D_{V_q}(k^2)| < c\|V_q\|_*(1 + \|V_q\|_*), \quad c = c(a_{\max}, a_{\min}). \tag{65}$$

Proof. Let us calculate $D_{V_q}(k^2)$ in terms of the density of states of the corresponding one-dimensional Schrödinger operator (45). By rotating the coordinates, we direct x_1 along $\vec{p}_q(0)$. It does not change the density of states, [Sh1, Sh2, RoShSo]. By separation of variables, we get $D_{V_q}(k^2) =$

$$= \sum_{n_2,n_3\in Z, n_1\in N}\int_K \chi\left(k^2 - (t_2 + 2\pi n_2)^2 - (t_3 + 2\pi n_3)^2 - \lambda_{n_1}(t_1)\right)dt_1$$

$$= 2\pi \sum_{n_1 \in N} \int_0^\infty \int_0^{2\pi} \chi(k^2 - \rho^2 - \lambda_{n_1}(t_1))\rho d\rho dt_1,$$

where $\lambda_{n_1}(t_1)$ are Bloch eigenvalues of the operator (45). Changing the variable $y = k^2 - \rho^2$, we get (46). The regular perturbation theory arguments give:

$$N_0^{(1)}(y - \|V_q\|) \le N_{V_q}^{(1)}(y) \le N_0^{(1)}(y + \|V_q\|),$$

where $N_{V_q}^{(1)}$ is the counting function (6) of the operator (45), $\|V_q\|$ is the norm of V_q in the class of bounded operators, $\|V_q\| < \|V_q\|_*$. Considering (61) it is easy to get more precise estimate for large y:

$$N_0^{(1)}(y - c_0(V_q)y^{-1}) \le N_{V_q}^{(1)}(y) \le N_0^{(1)}(y + c_0(V_q)y^{-1}),$$

when $\sqrt{y} > 10\|V_q\|p_q^{-1}(0)$. Taking into account that $N_0(y) = \sqrt{y}$ and integrating the last estimates, we get $\left|\delta\rho_{V_q}(y)\right| < \|V_q\|(y + \|V_q\|)^{-1/2}$ and $\left|\delta\rho_{V_q}(y)\right| < c_0(V_q)y^{-3/2}$. [3] Using the second estimate, we get (48). Using (46) and combining both estimates, we get (65). □

Lemma 4 *If V is a trigonometric polynomial with $R_0 < k^\delta$, $0 < \delta < 1/105$, then the function $d_V(k^2)$, defined by (42), satisfies the estimates (44) and*

$$|d_V - d_V^0| < c_0(V)k^{-1}, \quad c \ne c(V). \tag{66}$$

d_V^0 being the constant given by (50).

Corollary. Estimate (51) holds.

Proof. Summarizing (65) and (48) over all q and taking into account that

$$\sum_q \|V_q\|_*^j \le \|V\|_*^j, \quad j = 1, 2, \tag{67}$$

we obtain (44) and (66). □

Let us introduce the notations:

$$\delta\chi_q(\vec{p}_m(t), k) = \chi\left(k^2 - p_m^2(t) - \Delta\Lambda_q(t)_{mm}\right) - \chi\left(k^2 - p_m^2(t)\right), \tag{68}$$

[3] The complete asymptotic expansion of ρ_V for $V \in C^\infty$ is given in [ScSh].

$$P(V_q,k^2,M)=\int_K \sum_{m\in Z^3:|(\vec p_m(0),\vec p_q(0))|<M,\ |p_m^2(t)-k^2|\leq \|V_q\|} \delta\chi_q(\vec p_m(t),k)dt. \quad (69)$$

It turns out that $P(V_q,k^2,M)$ is a good approximation for $\delta D_{V_q}(k^2)$ as $M\to\infty$:

Lemma 5 *If $M>8\pi a_{\min}^{-1}p_q(0)+10\|V_q\|$, then the following estimate holds:*

$$\left|\delta D_{V_q}(k^2)-P(V_q,k^2,M)\right|<c_0(V_q)p_q^{-1}(0)M^{-1}. \quad (70)$$

Proof. By definition, $\delta D_{V_q}(k^2)=\int_K \delta N_{V_q}(k^2,t)dt$, where $\delta N_{V_q}(k^2,t)=\sum_{m\in Z^3}\delta\chi_q(\vec p_m(t),k)$. Let us break the area of summation and, correspondingly, the sum into the two: $\delta N_{V_q}(k^2,t)=\Sigma_1+\Sigma_2$,

$$\Sigma_1(M,k,t)=\sum_{m\in Z^3:|(\vec p_m(0),\vec p_q(0))|<M}\delta\chi_q(\vec p_m(t),k),$$

$$\Sigma_2(M,k,t)=\sum_{m\in Z^3:|(\vec p_m(0),\vec p_q(0))|\geq M}\delta\chi_q(\vec p_m(t),k).$$

Thus,

$$\delta D_{V_q}(k^2)=\int_K \Sigma_1 dt+\int_K \Sigma_2 dt. \quad (71)$$

We consider Σ_1. Note that $|\Delta\Lambda_q(t)_{mm}|\leq\|V_q\|$ and, therefore, taking into account (68), we see that

$$\delta\chi(\vec p_m(t),k)=0,\ \text{when } |k^2-p_m^2(t)|>\|V_q\|. \quad (72)$$

Using notation (69), we easily obtain:

$$\int_K \Sigma_1(M,k,t)dt=P(V_q,k^2,M). \quad (73)$$

Let us consider Σ_2. Taking into account (64), we get

$$\delta\chi(\vec p_m(t),k)=0,\ \text{when } |k^2-p_m^2(t)|>c_0(V_q)|(\vec p_m(t),\vec p_q(0))|^{-2}, \quad (74)$$

and, therefore,

$$\Sigma_2=\sum_{m\in Z^3:|(\vec p_m(0),\vec p_q(0))|\geq M,\ |k^2-p_m^2(t)|\leq c_0(V_q)|(\vec p_m(t),\vec p_q(0))|^{-2}}\delta\chi_q(\vec p_m(t),k).$$

Considering that $|\delta\chi_q| \leq 1$ and $|(t, \vec{p}_q(0))| \leq M/2$, we obtain: $|\Sigma_2(M,k,t)| \leq \#(\Omega_{1q}, t)$, where $\Omega_{1q} \subset R^3$,

$$\Omega_{1q}(M) = \{x : ||x|^2 - k^2| \leq c_0(V_q)|(x, \vec{p}_q(0))|^{-2},\ |(x, \vec{p}_q(0))| \geq M/2\}.$$

It is easy to show that $V(\Omega_{1q}(M)) < c_0(V_q)p_q^{-1}(0)M^{-1}$. Thus,

$$\int_K |\Sigma_2(M,k,t)|\, dt \leq \int_K \#(\Omega_{1q}, t)dt = V(\Omega_{1q}(M)) < c_0(V_q)p_q^{-1}(0)M^{-1}. \tag{75}$$

Using (71), (73) and (75), we obtain (70). □

3.4 Auxiliary Operator $\hat{H}(t)$.

Our considerations are based on the expansion of the resolvent in a perturbation series. In [K] it was proved that in the case of the Schrödinger operator in R^3 the perturbation series converges not with respect to the free operator $H_0(t)$, but with respect to some auxiliary operator $\hat{H}(t)$. Let us describe this operator for a trigonometric polynomial (40) with $R_0 < k^{\delta/8}$, $0 < \delta < 1/105$.

We consider the following sets in Z^3:

$$\begin{gathered}
\Pi(k^{1/5}) = \cup_{q \in \Gamma(R_0)} \Pi_q(k^{1/5}), \\
\Pi_q(k^{1/5}) = \{m :\mid (\vec{p}_m(0), \vec{p}_q(0)) \mid < k^{1/5}\}, \\
T(k) = \cup_{q,q' \in \Gamma(R_0), q \neq q'} T_{qq'}, \\
T_{qq'} = \Pi_q(k^{1/5}) \cap \Pi_{q'}(k^{3/5}) = \\
= \{m : |(\vec{p}_m(0), \vec{p}_q(0))| < k^{1/5}, |(\vec{p}_m(0), \vec{p}_{q'}(0))| < k^{3/5}\}.
\end{gathered} \tag{76}$$

We define the diagonal projection P_q and the auxiliary operator $\hat{H}$ as follows:

$$(P_q)_{mm} = \begin{cases} 1, & \text{if } m \in \Pi_q(k^{1/5}) \setminus T(k), \\ 0, & \text{otherwise;} \end{cases} \tag{77}$$

$$\hat{H}(t) = H_0(t) + \sum_{q \in \Gamma(R_0)} P_q V_q P_q. \tag{78}$$

It is clear that $\hat{H}(t)$ has a block structure. Each block $H_0(t)P_q + P_q V_q P_q$ is the "piece" of $H_q(t)$, corresponding to $m \in \Pi_q(k^{1/5}) \setminus T(k)$. From the definition of $T(k)$ it follows that the blocks do not intersect. Because each block is big enough, it is easy to show, that its eigenvalues $\hat{\lambda}_m(t)$ are close to corresponding ones of H_q, namely, (see f.e. [K]):

$$\hat{\lambda}_m(t) = p_m^2(t) + \Delta\Lambda_q(t)_{mm} + O(c_0(V_q)k^{-2/5}), \tag{79}$$

$$\text{if } m \in \Pi_q(k^{1/5} - M_q) \setminus T(k),$$

$$\hat{\lambda}_m(t) = p_m^2(t) + \Delta\Lambda_q(t)_{mm} + O(c_0(V_q)k^{-1/5}) \tag{80}$$

$$\text{if } m \in \Pi_q(k^{1/5}) \setminus \left(\Pi_q(k^{1/5} - M_q) \cup T(k)\right),$$

$M_q \approx 10R_0p_q(0)$. Thus, the eigenvalues $\hat{\lambda}_m(t)$ of $\hat{H}(t)$ can be enumerated by indexes $m \in Z^3$, $\hat{\lambda}_m(t)$ satisfies (79), (80) if $m \in \Pi_q(k^{1/5}) \setminus T(k)$ for some q, and $\hat{\lambda}_m(t) = p_m^2(t)$ otherwise.

Let $\hat{D}_V(k^2)$ be the density of states for the operator $\hat{H}(t)$.

Lemma 6 *When $k^{1/5} > 8\pi a_{\min}^{-1}p_q(0) + 10\|V_q\|$, the following relations hold:*

$$\hat{D}_V(k^2) = D_0(k^2) + d_V(k^2) + f(k), \tag{81}$$

$$|f(k)| < cR_0^3\|V\|_*(1 + \|V\|_*)k^{-1/5}, \quad c = c(a_{\max}, a_{\min}).$$

Proof. By definition, $\hat{D}_V(k^2) = \int_K \hat{N}_V(k^2,t)dt$, where $\hat{N}_V(k^2,t) = \sum_{m\in Z^3}\chi(k^2 - \hat{\lambda}_m(t))$. Considering (68) it is easy to see that $\hat{N}_V(k^2,t) = N_0(k^2,t) + \sigma_1 + \sigma_2 + \sigma_3$, where

$$\sigma_1 = \sum_{q\in\Gamma(R_0)}\sigma_{1q}, \quad \sigma_{1q} = \sum_{m\in\Pi_q(k^{1/5})\setminus T(k)} \delta\chi_q(\vec{p}_m(t), k),$$

$$\sigma_2 = \sum_{q\in\Gamma(R_0)}\sum_{m\in\Pi_q(k^{1/5}-M_q)\setminus T(k)} \delta\chi_q'(\vec{p}_m(t), k),$$

$$\sigma_3 = \sum_{q\in\Gamma(R_0)}\sum_{m\in\Pi_q(k^{1/5})\setminus\left(\Pi_q(k^{1/5}-M_q)\cup T(k)\right)} \delta\chi_q'(\vec{p}_m(t), k),$$

$$\chi_q'(\vec{p}_m(t), k) = \chi\left(k^2 - \hat{\lambda}_m(t)\right) - \chi\left(k^2 - p_m^2(t) - \Delta\Lambda_q(t)_{mm}\right).$$

We represent $\sigma_{1q}(t)$ in the form $\sigma_{1q} = \sigma_{1q}' - \sigma_{1q}''$, where

$$\sigma_{1q}' = \sum_{m\in\Pi_q(k^{1/5})}\delta\chi_q(\vec{p}_m(t), k), \quad \sigma_{1q}'' = \sum_{m\in\Pi_q(k^{1/5})\cap T(k)}\delta\chi_q(\vec{p}_m(t), k).$$

Considering (69) and (72), we see that $\int_K \sigma'_{1q}(t)dt = P(V_q, k^2, k^{1/5})$. To estimate σ''_{1q}, let us note that $\Pi_q(k^{1/5}) \cap T(k) \subset \cup_{q' \in \Gamma(R_0), q' \neq q} T_{qq'}$, $T_{qq'}$ being given by (76). Again, taking into account (72) and the inequalities $|\delta \chi_q| \leq 1$, $|(t, \vec{p}_q(0))| < k^{1/5}$, we get $|\sigma''_{1q}| \leq \#(\Omega_{3q}, t)$, where $\Omega_{3q} = \cup_{q' \in \Gamma(R_0), q' \neq q} \Omega_{qq'}$, $\Omega_{qq'} = \{x : |k^2 - |x|^2| \leq \|V_q\|, \ |(x, \vec{p}_q(0))| \leq 2k^{1/5}, \ |(x, \vec{p}_{q'}(0))| \leq 2k^{3/5}\}$. Thus, $\left|\int_K \sigma''_{1q}(t)dt\right| \leq V(\Omega_{3q}) < cR_0^3 \|V_q\| p_q(0)^{-1} k^{-1/5}$, and, hence,

$$\left|\int_K \sigma_{1q}(t)dt - P(V_q, k^2, k^{1/5})\right| < cR_0^3 \|V_q\| k^{-1/5}.$$

Summarizing this estimate over q and using (70), (42) and (67), we obtain

$$\left|\int_K \sigma_1 dt - d_V(k^2)\right| < cR_0^3 \|V\|_* (1 + \|V\|_*) k^{-1/5}, c = c(a_{\max}, a_{\min}). \qquad (82)$$

Considering estimate (79) and the definition of χ'_q, we get

$$|\sigma_2| \leq \#(\Omega_4, t), \quad \Omega_4 = \cup_{q \in \Gamma(R_0)} \Omega_{4q},$$

$$\Omega_{4q}(k^2, c_0(V_q)k^{-2/5}, 2k^{1/5}) =$$
$$\{x \in R^3 : |k^2 - |x|^2 - \Delta_q(x)| < c_0(V_q)k^{-2/5}, |(x, \vec{p}_q(0))| < 2k^{1/5}\},$$

the function $\Delta_q(x)$ being defined by two relations: $x = \vec{p}_m(t)$ (a unique representation) and $\Delta_q(x) = \Delta\Lambda_q(t)_{mm}$. In the next lemma we will prove that $V(\Omega_{4q}) < c_0(V_q) p_q(0)^{-1} k^{-1/5}$. Therefore, $V(\Omega_4) \leq c_0(V) R_0^3 k^{-1/5}$. This gives us $|\int_K \sigma_2(t)dt| \leq \int_K \#(\Omega_4, t) = V(\Omega_4) \leq c_0(V) R_0^3 k^{-1/5}$. Similarly, considering (80), we prove that $|\int_K \sigma_3 dt| < c_0(V) R_0^3 k^{-1/5}$. Using the formulae for the integrals of σ_i, $i = 1, 2, 3$, we get (81). □

Lemma 7 *If $L_1, L_2 = o(k)$, then the volume of the set $\Omega_{4q}(k^2, L_1, L_2)$:*

$$\Omega_{4q}(k^2, L_1, L_2) = \{x \in R^3 : |k^2 - |x|^2 - \Delta_q(x)| < L_1, |(x, \vec{p}_q(0))| \leq L_2\} \quad (83)$$

satisfies the estimate

$$V(\Omega_{4q}) < cp_q(0)^{-1} L_1 L_2, \quad c \neq c(k, L_1, L_2). \qquad (84)$$

<u>Proof.</u> Let us introduce a cylindrical coordinates (z, ρ, ϑ), z being directed along $\vec{p}_q(0)$. Using the fact that $\Delta\Lambda_q(x)$ depends only on z, $\Delta\Lambda_q(x) \equiv \varphi(z)$, we get the following representation for Ω_{4q}: $\Omega_{4q} = \{(z, \rho, \vartheta) : |\rho^2 - \rho_0^2| < cL_1, |z| \leq L_2 p_q(0)^{-1}\}$, where $\rho_0^2 = k^2 - z^2 - \varphi(z)$. From this we easily obtain (84). □

3.5 Proof of the Main Result.

First, we prove that there is a rich subset $B_1(k)$ of K, such that $N_V(k^2,t) = \hat{N}_V(k^2,t)$ for all t in $B_1(k)$ and k big enough (Lemma 9). The set $K \setminus B_1(k)$ has an asymptotically zero measure as $k \to \infty$, and, if $t \in K \setminus B_1(k)$, then $|N_V(k^2,t) - \hat{N}_V(k^2,t)| < \#(\Omega_0,t)$, where the step-function $\#(\Omega_0,t)$ satisfies the estimate $\int_{K\setminus B_1} \#(\Omega_0,t)dt < cR_0^2 k^{-\delta}$, $\delta < 1/105$. (Lemma 10). Combining Lemmas 6, 9 and 10, we get Theorem 2. Theorem 3 follows from Theorem 2 and Lemma 4. Passing to the limit as $R_0 \to \infty$, we obtain Theorem 4. Lemma 8 describes the properties of the set $B_1(k)$.

Let us consider the set $\Omega_0 = \Omega_1 \cup \Omega' \cup M \cup T_1$, where Ω_1 is the spherical shell $\Omega_1 = \{||x|^2 - k^2| \le k^{-1-\delta}\}$, Ω' is defined by (25) with $F_q = k^{-4\delta}$, $M = \cup_{q\in\Gamma(R_0)}\Omega_{4q}(k^2, k^{-1/5-\delta}, k^{1/5})$, $T_1 = \cup_{q,q'\in\Gamma(R_0)}\tilde{\Omega}_{qq'}$, $\tilde{\Omega}_{qq'} = \{x : ||x|^2 - k^2| < k^{1/5-10\delta}, |(x,\vec{p}_q(0))| < k^{1/5}, |(x,\vec{p}_{q'}(0))| < k^{3/5}\}$, Ω_{4q} being given by formula (83). Obviously, $V(\Omega_1) < ck^{-\delta}$. Considering like in Lemma 2, we get $V(\Omega') < R_0^3 k^{-4\delta}\ln k$. Using Lemma 7, we get $V(M) < R_0^2 k^{-\delta}$. It is easy to show (see f.e. [K]) that $V(\Omega_{qq'}) < cR_0 p_q(0)^{-1} p_{q'}(0)^{-1} k^{-10\delta}$, and, therefore, $V(T_1) \le V(T_1) < cR_0^5 k^{-10\delta}$. Adding above estimates and taking into account that $R_0 < k^{\delta/8}$, we get $V(\Omega_0) < cR_0^2 k^{-\delta}$ and, therefore,

$$\int_K \#(\Omega_0,t) < cR_0^2 k^{-\delta}, \quad c = c(a_{\max}, a_{\min}). \tag{85}$$

This estimate means that $\#(\Omega_0,t)$ differs from zero only on the set with a volume of order $R_0^2 k^{-\delta}$.

Let $B_1(k,\delta,V) \subset K$ is the set of t such that $\#(\Omega_0,t) = 0$.

Lemma 8 *If $t \in B_1(k,\delta,V)$, then*

$$|p_m^2(t) - k^2| > k^{-1-\delta} \text{ for all } m \in Z^n, \tag{86}$$

$$\min_{q\in Z^3 0<|q|<R_0} |p_m^2(t) - k^2||p_{m+q}^2(t) - k^2| > k^{-4\delta}, \text{ for all } m \in Z^n, \tag{87}$$

$$|k^2 - p_m^2(t) - \Delta\Lambda_q(t)_{mm}| > k^{-1/5-\delta} \text{ for all } m \in \Pi_q(k^{1/5}) \text{and } q \in \Gamma(R_0) \tag{88}$$

$$|k^2 - p_m^2(t)| > k^{1/5-10\delta} \text{ for all } m \in T(k). \tag{89}$$

Proof. Let us prove (86). Suppose it does not hold. This means that $\vec{p}_m(t) \in \Omega_1 \subset \Omega_0$, therefore $\#(\Omega_0,t) \ge 1$, i.e., $t \notin B_1$, what contradicts to the hypothesis of the lemma. Conditions (87) – (89) can be proved in the analogous way. □

Lemma 9 *If* $t \in B_1(k,\delta,V)$*, then for sufficiently large* k*,* $k^{\delta/8} > R_0 + \max_m |v_m| + c(a_{\max}, a_{\min})$,

$$N_V(k^2,t) = \hat{N}_V(k^2,t). \tag{90}$$

Proof. To construct the convergent series for the resolvent we take the unperturbed operator to be not $H_0(t)$ but $\hat{H}(t)$ defined by formulae (77) and (78). It is convenient to reduce $\hat{H}(t)$ to diagonal form, $\hat{H}(t) \to U\hat{H}(t)U^* \equiv H_0 + \Delta\hat{\Lambda}(t)$, $U = U(t)$, and then to consider the operator $H(t)$ in this representation: $UH(t)U^* = H_0 + \Delta\hat{\Lambda}(t) + B$, where $B = UWU^*$, $W = V - \sum_{q\in\Gamma(R_0)} P_q V_q P_q$. We expand the resolvent of $UH(t)U^*$ in the formal series:

$$U(H(t)-z)^{-1}U^* = (H_0 + \Delta\hat{\Lambda}(t) - z)^{-1} + \tag{91}$$

$$\sum_{r=1}^{\infty}(H_0 + +\Delta\hat{\Lambda}(t)-z)^{-1/2}A_1^r(H_0 + +\Delta\hat{\Lambda}(t)-z)^{-1/2},$$

$$A_1(z,t) = (H_0(t) + \Delta\hat{\Lambda}(t) - z)^{-1/2}B(t)(H_0(t) + \Delta\hat{\Lambda}(t) - z)^{-1/2}.$$

Lemma 4.14 in [K] proves that under conditions (86) – (89) and $R_0 < k^{\delta/8}$, $\max_m |v_m| < k^{\delta/8}$, $k^{\delta/8} > c(a_{\max}, a_{\min})$, the following estimates hold:

$$\|A_1(k^2,t)\| < k^{4\delta}, \quad \|A_1^3(k^2,t)\| < k^{-1/5+21\delta}. \tag{92}$$

The estimates (92) provides the convergence of the series (91) for $z \in C_0$. The series converges in the trace class since $(H_0 + \Delta\hat{\Lambda}(t) - z)^{-1}$ is a Sturm-Liouville operator. This means that the number of eigenvalues inside the contour is a continuous function of αB, $0 \le \alpha \le 1$, i.e., the relation (90) holds. □

Lemma 10 *If* $t \in K \setminus B_1$*, then for sufficiently large* k*,* $k^{\delta/8} > R_0 + \max_m |v_m| + c(a_{\max}, a_{\min})$*, the following estimate holds:*

$$|N_V(k^{2l},t) - \hat{N}_V(k^{2l},t)| \le 2\#(\Omega_0,t). \tag{93}$$

Proof. Let $t \in K \setminus B_1$. Then there is a positive number of different indexes m such that $\vec{p}_m(t) \in \Omega_0$. Let us introduce the diagonal projection P' with elements equal to 1 for such m:

$$P'_{mm} = \begin{cases} 1, & \text{if } \vec{p}_m(t) \in \Omega_0; \\ 0, & \text{otherwise.} \end{cases} \tag{94}$$

Let us consider the operator $H'(t) = H_0(t) + \Delta\hat{\Lambda}(t) + (I - P')B(I - P')$. This means that in the matrix of the operator $UHU^* = H_0(t) + \Delta\hat{\Lambda}(t) + B$ we replace all columns and rows, corresponding to $m : \vec{p}_m(t) \in \Omega_0$, by zeros with the exception of the diagonal elements, i.e., we delete from B all columns and rows, which can break convergence of the perturbation series for the resolvent. Let $N_V'(K^2, t)$ be the number of eigenvalues for $H'(t)$. Considering as in Lemma 4.14 [K], we show that the perturbation series converges and

$$\hat{N}_V(k^2, t) = N_V'(k^2, t). \tag{95}$$

Note that the difference between $UH(t)U^*$ and $H'(t)$ is a finite dimensional operator: $UH(t)U^* - H'(t) = (I - P')BP' + P'B(I - P') + P'BP'$. The dimension of this operator is not greater than $2\#(\Omega_0, t)$. According to the general perturbation theory (see f.e. [Kato]): $|N_V(k^2, t) - N_V'(k^2, t)| \leq 2\#(\Omega_0, t)$. Combining the last estimate with (95), we get (93). $\square$

The proof of Theorem 2. Clearly, $D_V(k^2) = \int_{B_1(k)} N_V(k^2, t)dt + \int_{K \setminus B_1} N_V(k^2, t)dt$. Using Lemmas 9 and 10, and estimate (85), we obtain $|D_V(k^2) - \hat{D}_V(k^2)| \leq 2\int_{K\setminus B_1} \#(\Omega_0, t)dt = 2V(\Omega_0) \leq cR_0^2 k^{-\delta}$. Considering (81), we get

$$\left| D_V(k^2) - D_0(k^2) + d_V(k^2) \right| < cR_0^2 k^{-\delta} + cR_0^3 \|V\|_* (1 + \|V\|_*) k^{-1/5}, \tag{96}$$

$c = c(a_{\max}, a_{\min})$. Note that the only restriction on δ is $1/5 - 21\delta > 0$ (see (92)), therefore (43) holds. The estimate (44) is proved in Lemma 4. $\square$

The proof of Theorem 3. It follows immediately from Theorem 2 and estimate (66). $\square$

The proof of Theorem 4. Suppose V satisfies (4) with $\nu = 900$. Let V_0 be a trigonometric polynomial with $R_0 = k^{\delta/8}$, $\delta \leq 1/105$, which is the truncation of the Fourier series of V. Theorems 2 and 3 hold for V_0. Clearly, $\|V - V_0\| < \|V\|_* R_0^{-\nu}$. Taking into account (65), it is not difficult to show that $|d_V - d_{V_0}| < cR_0^{-\nu}\|V\|_*(1 + \|V\|_*)$. Using this estimate and (44) for d_{V_0}, we get (44) for d_V.

Obviously, $N_{V_0}(k^2 - \|V - V_0\|, t) \leq N_V(k^2, t) \leq N_{V_0}(k^2 + \|V - V_0\|, t)$. Integrating the last inequalities and taking into account the above estimate for $\|V - V_0\|$, we get $|D_V - D_{V_0}| < 12\pi^2 \|V\|_* R_0^{-\nu} k$. Using (96) for V_0 we get $|D_V - D_0 - d_{V_0}| < 12\pi^2\|V\|_* R_0^{-\nu} k + cR_0^2 k^{-\delta} + cR_0^3 \|V\|_*(1 + \|V\|_*)k^{-1/5}$. From this estimate and above estimate for $|d_V - d_{V_0}|$, we obtain: $|D_V - D_0 - d_V| < c_1 k^{-3\delta/4}$, $\delta < 1/105$, hence $|D_V - D_0 - d_V| < c_1 k^{-\zeta}$, $\zeta > 1/130$, $c_1 = c_1(\|V\|_*, a_{\max}, a_{\min})$. Similarly, we get $|D_V - D_0 - d_V^0| < c_1 k^{-\zeta}$. $\square$

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