Renormalization group approach to interacting polymerised manifolds

P. K. Mitter
Laboratoire de Physique mathématique*
Université Montpellier 2
Place E. Bataillon, Case 070
34095 Montpellier Cedex 05 France

B. Scoppola
Dipartimento di Matematica**
Universitá "La Sapienza" di Roma
Piazzale A. Moro 2
00185 Roma

* Laboratoire Associé au CNRS, UMR 5825
** Partially supported by CNR, G.N.F.M. and Short Term Mobility Program

ABSTRACT

We propose to study the infrared behaviour of polymerised (or tethered) random manifolds of dimension D interacting via an exclusion condition with a fixed impurity in d-dimensional Euclidean space in which the manifold is embedded. In this paper we take $D = 1$, but modify the underlying free Gaussian covariance (thereby changing the canonical scaling dimension of the Gaussian random field) so as to simulate a polymerised manifold with fractional dimension $D: \ 1 < D < 2$. We prove rigorously, via methods of Wilson's renormalization group, the convergence to a non Gaussian fixed point for $\varepsilon > 0$, sufficiently small. Here, $\varepsilon = 1 - \beta \frac{d}{2}$, where $-\beta/2$ is the canonical scaling dimension of the Gaussian embedding field. Although ε is small, our analysis is non-perturbative in ε. A similar model was studied earlier [CM] in the hierarchical approximation.

TABLE OF CONTENTS

"God is in the details": F. Dyson, Disturbing the Universe

§0: *Introduction*

Consider the Euclidean action

$$S(\phi) = \frac{1}{2}\int d^D x(\phi(x), -\Delta\phi(x)) + g\int d^D x\delta^{(d)}(\phi(x)) \tag{0.1}$$

where ϕ has values in $\mathbf{R}^d$, $(\ ,\ )$ is scalar product in $\mathbf{R}^d$, and the corresponding formal partition function:

$$Z = \int d\mu_C(\phi)e^{-g\int d^D x\delta^{(d)}(\phi(x))} \tag{0.2}$$

μ_C is a Gaussian measure with covariance $C = (-\Delta)^{-1}$.
ϕ can be considered as the embedding function of a D dimensional tethered or polymerised manifold (for tethered manifolds see [NPW]) in d-dimensional Euclidean space, and for $g > 0$, we have a repulsive interaction with a fixed impurity in the embedding space. It is easy to see that the coupling constant g has canonical (engineering) dimension

$$[g] = \varepsilon = D - (2-D)\frac{d}{2} \tag{0.3}$$

ϕ has canonical scaling dimension: $[\phi] = -(2-D)/2$ and the upper critical dimension of the embedding space is

$$d_c = \frac{2D}{2-D} \tag{0.4}$$

Such a model was considered in [DDG]. It was shown, for $1 < D < 2$ and $\varepsilon > 0$ sufficiently small, that there exists an ε-expansion in renormalization perturbation series, and that the infrared behaviour is governed by a non-Gaussian fixed point. The model with an ultraviolet cutoff was reconsidered in [CM] in the hierarchical approximation to Wilson's Renormalization group (henceforth called RG). It was shown, under the same conditions, that the iteration of the hierarchical RG transformations converge to a non-Gaussian fixed point independent of the ε-expansion.
In the present paper we consider a version of the above model and study the iterations of the exact RG with an ultraviolet cutoff in a finite but large volume eventually tending to infinity. The precise definition of the model and the RG iterations will be found in section

1.1. We shall simulate a fractional dimension D, $1 < D < 2$, by choosing $D = 1$ and modifying the covariance C to be

$$C = (-\Delta)^{(\delta-1)} F(-\Delta) \tag{0.5}$$

with $0 < \delta < 1/2$. Here F is an ultraviolet cutoff function which is positive, and in the momentum space $F(p^2)$ is of fast decrease.
The covariance C is so chosen (see section 1.1) that $C(x-y)$ is smooth and of compact support. In finite volume, the zero momentum mode $p = 0$ is automatically taken care of (for details see later). The delta function interaction is replaced (see [CM]) by its regularised version in finite volume:

$$V(\phi) = \int_\Lambda dx\ v(\phi(x)) \tag{0.6}$$

with

$$v(\phi(x)) = \left(\frac{\lambda}{2\pi}\right)^{\frac{d}{2}} e^{-\frac{\lambda}{2}|\phi(x)|^2} \tag{0.7}$$

where $|\cdot|$ is the norm in $\mathbf{R}^d$.
It is easy to see that the canonical scaling dimension of the field is

$$[\phi] = -\beta/2 \tag{0.8}$$

where

$$\beta = 1 - 2\delta > 0 \tag{0.9}$$

since $0 < \delta < 1/2$ by choice.
The coupling constant g has dimension

$$[g] = \varepsilon = 1 - \beta\frac{d}{2} \tag{0.10}$$

The upper critical dimension is

$$d_c = \frac{2}{\beta}$$

Note that the canonical scaling of the field is such that it simulates a Gaussian random field with covariance $(-\Delta)^{-1}$ in dimension $D = 1 + 2\delta$. We have $1 < D < 2$, and for δ close to $1/2$, D is close to 2. The idea of simulating fractional dimensions by changing the covariance is not new. It figures for example in [GK] and in the recent paper of [BDH3].

We shall hold $\beta > 0$ and very small and d very large so that $\varepsilon > 0$ is sufficiently small. Clearly in this configuration the critical embedding dimension d_c is very large and d close to d_c from below. Our main result is that *the exact RG iteractions converge to a non-Gaussian fixed point close to the unstable Gaussian fixed point*. The precise statement is to be found in Theorem 7.5 of section 7.

The formulation of RG iterations, as well as the method of analysis employed in this paper have been much influenced by the original paper of Brydges-Yau [BY], the Lausanne lectures of Brydges [B, Laus.], together with developments due to Brydges, Dimock and Hurd [BDH1,2,3]. For example, the functional integral is written in a polymer gas representation, the polymer algebra being that introduced in [BY]. However there are definite simplifications and differences. The simplifications stem from the use of "compact covariances", an idea suggested to us by David Brydges. This enables us to dispense with cluster or Mayer expansions. All polymer activities appearing in this paper are based on connected polymers. We can dispense with analyticity norms. As to the differences, they stem from the special form of the interaction. As a consequence we have that the growth of polymer activities is measured by a norm which employs a large fields regulator quite specific to this problem. Relevant terms are also extracted in a special way appropriate to this model. These matters are explained in detail in the subsequent sections.

§1: *The model, polymer gas representation and RG tranformations.*

1.1. The model

Let Λ_N be the closed interval $[-\frac{L^{N+1}}{2}, \frac{L^{N+1}}{2}]$ of length L^{N+1}. The model is described by the partition function

$$Z_0(\Lambda_N) = \int d\mu_{\mathbf{C}_N} z_0(\Lambda_N, \phi) \tag{1.1}$$

where

$$z_0(\Lambda_N, \phi) = e^{-V_0(\Lambda_N, \phi)} \tag{1.2}$$

$$V_0(\Lambda_N, \phi) = g_0 V_*(\Lambda_N, \phi) = g_0 \int_{\Lambda_N} dx \ v_*(\phi(x)) = g_0 \int_{\Lambda_N} dx \left(\frac{\lambda_*}{2\pi}\right)^{\frac{d}{2}} e^{-\frac{\lambda_*}{2}|\phi(x)|^2} \tag{1.3}$$

where each $\phi \in \mathbf{R}^d$ and $\lambda_* > 0$ will be fixed later (see below).
$\mu_{\mathbf{C}_N}$ is a gaussian measure with mean 0 and covariance $\mathbf{C}_N$. The components ϕ_j, $1 \le j \le d$, are independent gaussian random variables, and each component has covariance C_N. $d\mu_{\mathbf{C}_N}(\phi) = \otimes_{j=1}^d d\mu_{C_N}(\phi_j)$.

We now describe the covariance C_N.
Let $g(x)$ be a C^∞ function of compact support:

$$g(x) = 0 \quad \forall \ |x| \ge \frac{1}{2} \tag{1.4}$$

Choose for definiteness

$$g(x) = \begin{cases} e^{-\frac{1}{1-4x^2}} & \text{for } |x| \le 1/2 \\ 0 & \text{elsewhere} \end{cases} \tag{1.5}$$

Define

$$u(x) = (g * g)(x) \tag{1.6}$$

Then $u(x)$ is C^∞ and of compact support:

$$u(x) = 0 \quad \forall \ |x| \ge 1 \tag{1.7}$$

Define

$$\Gamma_L(x) = \int_1^L \frac{dl}{l} l^\beta u\left(\frac{x}{l}\right) \tag{1.8}$$

Note that $\Gamma_L(x)$ is C^∞ and of compact support:

$$\Gamma_L(x) = 0 \quad \forall\ |x| \geq L \tag{1.9}$$

Finally we define the covariance C_N by a truncated multiscale decomposition:

$$C_N(x) = \sum_{j=0}^N L^{j\beta} \Gamma_L(x/L^j) \tag{1.10}$$

with $\beta = 1 - 2\delta > 0$ as in the introduction.
From(1.10) C_N is C^∞ and of compact support:

$$C_N(x) = 0 \quad \forall |x| \geq L^{N+1} \tag{1.11}$$

From (1.8) we have

$$\Gamma_L(x) = \int \frac{dp}{2\pi} e^{ipx} \int_1^L dl\ l^\beta \hat{u}(lp) \tag{1.12}$$

$\hat{u}$ is of fast decrease, since u is C^∞ with compact support. Moreover, since $u = g * g$, $\hat{u} = |\hat{g}^2|$, so that $\hat{\Gamma}_L(p) \geq 0$. It follows that $\Gamma_L(x)$ defines a positive definite function:

$$\sum_{i,j=1}^n \Gamma_L(x_i - x_j) a_i \bar{a}_j \geq 0 \quad \forall n,\ x_1, \ldots, x_n \in \mathbf{R},\ a_1, \ldots, a_n \in \mathbf{C} \tag{1.13}$$

From the definition of C_N in (1.10), using (1.13) we have

$$\sum_{i,j=1}^n C_N(x_i - x_j) a_i \bar{a}_j \geq 0 \quad \forall n,\ x_1, \ldots, x_n \in \mathbf{R},\ a_1, \ldots, a_n \in \mathbf{C} \tag{1.14}$$

Thus $C_N(x)$ also defines a positive definite function. The positive definiteness of $C_N(x)$ together with its smoothness implies that there exists a gaussian measure of mean zero and covariance C_N which we call μ_{C_N}, realized on a Sobolev space $H_s(\Lambda_N)$, with $s > 1/2 + \sigma$ for any positive integer σ. The Sobolev embedding theorem implies that the sample fields ϕ are σ-times differentiable. For our purposes it is enough to fix $\sigma = 2$.

There exists another formula for C_N, derived from its definition, which is useful because it makes contact with the cutoff function F_N: from (1.8) it follows that

$$L^{j\beta}\Gamma_L(x/L^j) = \int_{L^j}^{L^{j+1}} \frac{dl}{l} l^\beta u\left(\frac{x}{l}\right) \tag{1.15}$$

Introducing this in (1.10) we get

$$C_N(x) = \int_1^{L^{N+1}} \frac{dl}{l} l^\beta u\left(\frac{x}{l}\right) \tag{1.16}$$

which we can rewrite as

$$C_N(x) = \int \frac{dp}{2\pi} e^{ipx} \frac{F_N(p^2)}{p^{2(1-\delta)}} \tag{1.17}$$

where the UV cutoff function in finite volume F_N has now the following form

$$F_N(p^2) = \int_{|p|}^{|p|L^{N+1}} dl\ l^\beta \hat{u}(l) = p^{2(1-\delta)} \int_1^{L^{N+1}} dl\ l^\beta \hat{u}(l|p|) \tag{1.18}$$

By our choice of g , $\hat{g}$ is an even function, and hence so is $\hat{u} = |\hat{g}|^2$. This justifies the replacement of p by p^2 in the above formula
$F_N(p^2)$ is of fast decrease because $\hat{u}$ is of fast decrease. We also see that in our finite volume covariance the zero mode at $p = 0$ is automatically regularized
To complete the definition of the model we specify the constant λ_* as

$$\lambda_* = \frac{\beta}{u(0)} \tag{1.19}$$

Note that

$$\gamma = \Gamma_L(0) = \frac{u(0)}{\beta}(L^\beta - 1) \tag{1.20}$$

we have

$$\lambda_* = \frac{L^\beta - 1}{\gamma} \tag{1.21}$$

Remark: if we had started with an arbitrary λ in the interaction (1.3), and we had performed a Renormalization Group transformation (defined below), then we would have obtained in the absence of quantum corrections:

$$\lambda' = \frac{L^\beta \lambda}{1 + \gamma\lambda}$$

Since $\beta > 0$, λ_* is an attractive fixed point. Choosing $\lambda = \lambda_*$ from the beginning is an updating of the Renormalization Group trajectory which simplifies the subsequent analysis.

From now on, a bound of the form $O(1)$ will mean a bound independent of L.
We have the following bound on the derivatives of $\Gamma_L(x)$:

LEMMA 1.1.1

For $0 \le \beta \le 1/4$ and all $k \ge 1$ we have

$$\sup_x |\partial^k \Gamma_L(x)| \le O(1) \tag{1.22}$$

$$\int_{\mathbf{R}} dx \ |\partial^k \Gamma_L(x)|^2 \le O(1) \tag{1.23}$$

Proof
The proof of (1.22) follows directly from (1.8) and from the fact that

$$\sup_x |\partial^k u(x)| \le O(1)$$

One has, taking k derivatives

$$\sup_x |\partial^k \Gamma_L(x)| \le O(1) \frac{1 - L^{\beta - k}}{k - \beta} \le O(1)$$

To prove (1.23) we have

$$\int_{\mathbf{R}} dx \ |\partial^k \Gamma_L(x)|^2 = 2 \int_1^L \frac{dl_1}{l_1} l_1^{\beta - k} \int_1^{l_1} \frac{dl_2}{l_2} l_2^{\beta - k} \int_{-l_2}^{l_2} dx \ (\partial^k u)(x/l_1)(\partial^k u)(x/l_2)$$

Using (1.22) the last integral can be bounded by $O(1) l_2$. Therefore we have

$$\int_{\mathbf{R}} dx \ |\partial^k \Gamma_L(x)|^2 \le O(1) \int_1^L \frac{dl_1}{l_1} l_1^{\beta - k} \int_1^{l_1} \frac{dl_2}{l_2} l_2^{\beta - k} l_2$$

and it is easy to see by direct computation that the integrals are bounded again by $O(1)$.

Q.E.D.

From now on we drop for simplicity the suffix L from Γ.
It is also natural to define the rescaled propagator

$$\mathcal{R}\Gamma(y) = L^{-\beta} \Gamma(yL) \tag{1.24}$$

We will also use sometimes the following notation

$$\bar{\Gamma}(y) = \lambda_* L^{-\beta} \Gamma(yL) = \lambda_* \mathcal{R}\Gamma(y) \tag{1.25}$$

Note that

$$\bar{\Gamma}(0) = 1 - L^{-\beta} \tag{1.26}$$

1.2. Renormalization Group transformation

It is easy to see from (1.10) that

$$C_N(x) = \Gamma(x) + L^\beta C_{N-1}(x/L) \tag{1.27}$$

Define the rescaled field $\mathcal{R}\phi(x)$

$$\mathcal{R}\phi(x) = L^{\beta/2}\phi(x/L) \tag{1.28}$$

From (1.27) and (1.28) we see that we can write

$$\int d\mu_{\mathbf{C}_N}(\phi) z_0(\Lambda_N, \phi) = \int d\mu_{\mathbf{C}_{N-1}}(\phi) z_1(\Lambda_{N-1}, \phi) \tag{1.29}$$

where

$$z_1(\Lambda_{N-1}, \phi) = \int d\mu_\Gamma(\zeta) z_0(\Lambda_N, \zeta + \mathcal{R}\phi) \tag{1.30}$$

This constitutes our Renormalization Group (RG) transformation, which can be iterated. After n steps ($0 \le n \le N$) we get

$$\int d\mu_{\mathbf{C}_N}(\phi) z_0(\Lambda_N, \phi) = \int d\mu_{\mathbf{C}_{N-n}}(\phi) z_n(\Lambda_{N-n}, \phi) \tag{1.31}$$

where

$$z_n(\Lambda_{N-n}, \phi) = \int d\mu_{\Gamma_L}(\zeta) z_{n-1}(\Lambda_{N-n+1}, \zeta + \mathcal{R}\phi) \tag{1.32}$$

After N steps we get $z_N(\Lambda_0, \phi)$ and measure $\mu_{\mathbf{C}_0}(\phi)$. Note that Λ_0 is the closed interval $[-L/2, L/2]$ and $C_0 = \Gamma$. Then we want to pass to the $N \to \infty$ limit. We would like to study the convergence of the iteration (1.32). This is awkward because the volume is changing with the iterations (see (1.31)). We can take N, n, $N-n$ very large. Then the

RG iterations can be viewed as those of a fixed map. The precise sense of the convergence of these iterations will be explained later.

1.3. The master formula for fluctuation integrals.

By a simple gaussian integration it is easy to see that, choosing suitably the constants β and d, the direction defined by the initial interaction (1.3) is relevant under the action of RG: with our definition of λ_* we have that

$$\int d\mu_\Gamma(\zeta) v_*(\zeta(x) + \mathcal{R}\phi(x)) = L^{-\alpha} v_*(L^{-\beta/2}\mathcal{R}\phi(x))$$

with $\alpha = \beta d/2$ chosen in such a way that $\alpha < 1$. Changing variables $x' = x/L$ we get

$$L^{-\alpha} \int dx\ v_*(L^{-\beta/2}\mathcal{R}\phi(x)) = L^{\varepsilon} \int dx\ v_*(\phi(x))$$

with $\varepsilon = 1 - \alpha > 0$.

We will see later that this direction is actually the only relevant one. In order to prove this it is often useful to define a modified fluctuation integration such that the result of this integration is already multiplied by a factor $L^{-\alpha} v_*(L^{-\beta/2}\mathcal{R}\phi(x))$. This is given by the following equality

LEMMA 1.3.1 (*Master formula*)

$$\int d\mu_\Gamma(\zeta) v_*(\zeta(x) + \mathcal{R}\phi(x)) F(\zeta + \mathcal{R}\phi) =$$

$$= L^{-\alpha} v_*(L^{-\beta/2}\mathcal{R}\phi(x)) \int d\mu_{\Sigma^x}(\zeta) F(\zeta + L^{-\beta} T^x \mathcal{R}\phi) \tag{1.33}$$

where

$$\Sigma^x(y - z) = \Gamma(y - z) - \lambda_* \Gamma(x - z)\Gamma(x - y) \tag{1.34}$$

$$T^x \mathcal{R}\phi(y) = L^{\beta}\mathcal{R}\phi(y) - \lambda_* \Gamma(y - x)\mathcal{R}\phi(x)$$

Note that by trivial algebraic manipulation, and by $L^\beta - 1 = \lambda_*\Gamma(0)$ we obtain

$$L^{-\beta} T^x \mathcal{R}\phi(y) = [\mathcal{R}\phi(y) - \mathcal{R}\phi(x)] + L^{-\beta}(1 + \lambda_*(\Gamma(0) - \Gamma(x - y)))\mathcal{R}\phi(x) \tag{1.35}$$

$$L^{-\beta} T^x \mathcal{R}\phi(x) = L^{-\beta}\mathcal{R}\phi(x) \tag{1.36}$$

The proof of Lemma 1.3.1 is obtained simply by gaussian integration.

1.4. Polymer expansion, [BY].

The polymerised version of the partition function (1.1) is obtained by expressing the volume Λ as a union of closed blocks Δ of size 1; since Λ is one-dimensional, the blocks are actually closed unit intervals.

We define first of all a *polymer* X as a union of blocks Δ. A *cell* may be the interior of a block, i.e. an open block, or a point of its boundary.

Then we define a commutative product, denoted $\circ$, on functions of sets containing polymers and cells, in the following way

$$(F_1 \circ F_2)(X) = \sum_{Y,Z : Y \overset{\circ}{\cup} Z = X} F_1(Y) F_2(Z) \tag{1.37}$$

where $X = Y \overset{\circ}{\cup} Z$ iff $X = Y \cup Z$ and $Y \cap Z = \emptyset$. The $\circ$ identity $\mathcal{I}$ is defined by

$$\mathcal{I} = \begin{cases} 1 & \text{if } X = \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{1.38}$$

The $\mathcal{E}$xponential is defined by

$$\mathcal{E}xp(K) = \mathcal{I} + K + K \circ K/2! + ... \tag{1.39}$$

This is the usual series for an exponential except that the product has been replaced by the $\circ$ product. The $\mathcal{E}$xponential with the $\circ$ product satisfies the usual properties of an exponential.

Moreover we define a *space filling* function $\square$ as

$$\square = \begin{cases} 1 & \text{if } X \text{ is a cell} \\ 0 & \text{otherwise} \end{cases} \tag{1.40}$$

Finally we denote, for X polymer or cell

$$V_0(X, \phi) = g_0 \int_X d^D x v_*(\phi(x)) \tag{1.41}$$

With these notations it is clear that, since

$$\mathcal{E}xp(\square)(X) = 1 \tag{1.42}$$

one has

$$Z_0(\Lambda_N) = \int d\mu_{\mathbf{C}_N} e^{-V_0(\phi, \Lambda_N)} \mathcal{E}xp(\square)(\Lambda_N) \tag{1.43}$$

In what follows it will be understood that in an expression of the form

$$e^{-V(\Lambda)}\mathcal{E}xp(\square + K)(\Lambda) \tag{1.44}$$

or

$$\mathcal{E}xp(\square + \hat{K})(\Lambda) \tag{1.45}$$

the functions $K, \hat{K}$, called *activities*, are supported only on polymers and vanish on cells. In particular an expression like (1.45) is often called *polymer gas*, because it can be written in the form

$$\mathcal{E}xp(\square + \hat{K})(\Lambda) = \sum_{N=0}^{\infty}(1/N!) \sum_{X_1,...,X_N} \hat{K}(X_1)...\hat{K}(X_N) \tag{1.46}$$

where $X_1, ..., X_N$ are all disjoint polymers in Λ. Since these are closed, they are separated by a distance of at least one.

We will consider often activities defined on connected polymers. For such activities the decomposition (1.46) is on connected subsets of X. Such activities will be called *connected* activities.

The polymer expansion given above is borrowed from [BY].

Let us conclude this subsection stating two useful lemmas about manipulations on $\mathcal{E}$xp. The (easy) proofs can be found in [BY] and [B,Laus.].

LEMMA 1.4.1

For any pair of polymer activities A, B

$$\mathcal{E}xp(\square + A)\mathcal{E}xp(\square + B) = \mathcal{E}xp(\square + A + B + A \vee B) \tag{1.47}$$

where the polymer activity $(A \vee B)$ is defined by

$$(A \vee B)(X) = \sum_{\{X_i\},\{Y_j\} \to X} \prod_i A(X_i) \prod_j B(Y_j) \tag{1.48}$$

with $\{X_i\}, \{Y_j\} \to X$ meaning that the sum is over the families of polymers $\{X_i\}, \{Y_j\}$ such that $(\cup_i X_i) \cup (\cup_j Y_j) = X$, the X's are disjoint, the Y's are disjoint, but the two families are overlap connected.

LEMMA 1.4.2

Let us define, for a polymer activity A, the quantity

$$A^+(X) = \sum_{N \geq 1} \frac{1}{N!} \sum_{X_1 ... X_N \to X} \prod_{j=1}^{N} A(X_j) \tag{1.49}$$

where $X_1 ... X_N \to X$ means that $X_1 ... X_N$ are distinct, overlap connected and such that $\cup_i X_i = X$.
Then we have

$$\prod_{X \subset Z} e^{A(X)} = \mathcal{E}xp(\square + (e^A - 1)^+)(Z) \tag{1.50}$$

1.5. RG strategy.

A single RG step has for us four parts. We describe them briefly here and we will fill in the details in later sections [3-5]. We begin with an expression of the form

$$e^{-V(\phi,\Lambda)} \mathcal{E}xp(\square + K)(\phi, \Lambda) \tag{1.51}$$

Here V is of the form (1.41), with g_0 replaced by g. Althought K is absent initially, it is necessarily generated in RG operations.
The structure of the activity K, together with bounds, will be exhibited in later sections. Suffice to say at this stage that K consists of an exact second order perturbation theory contribution plus a remainder.
Before we proceed further, let us rewrite (1.51) in the form

$$e^{-V(\phi,\Lambda)} \mathcal{E}xp(\square + K)(\phi, \Lambda) = \mathcal{E}xp(\square + \hat{K})(\phi, \Lambda) \tag{1.52}$$

$\hat{K}$ is a functional of K and V, given by a standard formula, given later (see section 3).

Step 1: Reblocking

We reblock (1.52) using the reblocking operator $\mathcal{B}$. The reblocking operator was introduced in [BY]. So far Λ has been paved with closed 1-blocks. Introduce a compatible paving of Λ on the next scale by closed L-blocks. Each closed L-block is a union of closed 1-blocks. For any 1-polymer X, let $\bar{X}$ be the smallest L-polymer containing X. If Z is a 1-polymer, LZ denotes a L-polymer. Then

$$(\mathcal{B}\hat{K})(LZ) = \sum_{N \geq 1} \frac{1}{N!} \sum_{\substack{X_1 ... X_N \text{ disjoint} \\ \{\bar{X}_j\} \text{ overlap connected} \\ \cup \bar{X}_j = LZ}} \prod_{j=1}^{N} \hat{K}(X_j) \tag{1.53}$$

We then have

$$\mathcal{E}xp(\square + \hat{K})(\Lambda, \phi) = \mathcal{E}xp_L(\square_L + \mathcal{B}\hat{K})(\Lambda, \phi) \tag{1.54}$$

where all the operations on the r. h. s. of (1.54) are on scale L.

We shall now let the RG act on (1.54). The RG action, as outlined before (see (1.30)), consists of a convolution with respect to the measure μ_{Γ_L} (called the fluctuation integration) followed by rescaling.

Step 2: Fluctuation integration

This is

$$(\mu_\Gamma * \mathcal{E}xp_L(\square_L + \mathcal{B}\hat{K}))(\Lambda, \mathcal{R}\phi)$$

with $\mathcal{R}\phi$ defined by (1.28)

The expansion of $\mathcal{E}xp_L(\square_L + \mathcal{B}\hat{K})$ gives a sum over products of L-polymer activities, where the L-polymers are closed and disjoint. They are thus separated from each other by a distance greater or equal to L. The fluctuation covariance Γ is of compact support by construction:

$$\Gamma(x-y) = 0 \quad \forall\ |x-y| \geq L$$

As a consequence

$$(\mu_\Gamma * \mathcal{E}xp_L(\square_L + \mathcal{B}\hat{K}))(\Lambda, \mathcal{R}\phi) = \mathcal{E}xp_L(\square_L + \mu_\Gamma * \mathcal{B}\hat{K})(\Lambda, \mathcal{R}\phi) \tag{1.55}$$

This leads to a considerable simplification in the RG analysis.

Our next step is

Step 3: Rescaling

We recall here (1.28) the definition of the rescaling operator $\mathcal{R}$ acting on the field ϕ:

$$\mathcal{R}\phi(x) = L^{\beta/2}\phi(x/L)$$

We have already defined also (see (1.24)) the rescaled fluctuation covariance

$$\mathcal{R}\Gamma(y) = L^{-\beta}\Gamma(yL)$$

For a polymer activity K we define

$$\mathcal{R}K(L^{-1}X, \phi) = K(X, \mathcal{R}\phi) \tag{1.56}$$

Note that

$$(\mu_\Gamma * K)(X, \mathcal{R}\phi) = (\mu_{\mathcal{R}\Gamma} * \mathcal{R}K)(L^{-1}X, \phi) \tag{1.57}$$

as it is easy to see.

Define

$$\mathcal{S} = \mathcal{R}\mathcal{B} \tag{1.58}$$

Then we have from (1.55)

$$\mathcal{E}xp_L(\square_L + \mu_\Gamma * \mathcal{B}\hat{K})(\Lambda, \mathcal{R}\phi) = \mathcal{E}xp(\square + (\mathcal{S}\hat{K})^\natural)(L^{-1}\Lambda, \phi) \tag{1.59}$$

Here $\natural$ denotes the convolution operation with respect to $\mu_{\mathcal{R}\Gamma}$.
On the r.h.s. of (1.59), $L^{-1}\Lambda$ stands for Λ shrunk by L^{-1} and paved by closed 1-blocks.
Our final step is the *extraction*:

Step 4: Extraction

This consists of picking up relevant parts F from $(\mathcal{S}\hat{K})^\natural$ and exponentiating them, in such a way that

$$\mathcal{E}xp(\square + (\mathcal{S}\hat{K})^\natural)(L^{-1}\Lambda, \phi) = e^{-V'(L^{-1}\Lambda,\phi)}\mathcal{E}xp(\square + K')(L^{-1}\Lambda, \phi) \tag{1.60}$$

We will find that

$$V'(F)(X) = g' \int_X dx\ v_*(\phi(x)) \tag{1.61}$$

with a new coupling constant g'.
K' is a functional of $\hat{K}$ and of the relevant part F:

$$K' = \mathcal{E}(\hat{K}, F) \tag{1.62}$$

An explicit formula for the extraction operator $\mathcal{E}$ is given in [B,Laus.] and we will put it to good use. The aim of the RG analysis is to control the discrete flows obtained by a large number of iterations $(V, K) \to (V', K')$.

§2. *Polymer activity norms and basic Lemmas.*

We want to define in this section the basic properties that we need on the activities K, and the appropriate norms to control them. In this section and throughout the paper we will use $D = 1$, but in most cases we will write D to remind the origin of the corresponding factors, either due to reblocking or rescaling.

2.1. Decay in X: the large set regulator Γ.

Let $K(X)$ be a connected polymer activity (with possible ϕ dependence suppressed). The decay of K in the "size" of X is controlled by a norm of the following type:

$$\|K\|_{\Gamma_n} = \sup_{\Delta} \sum_{\substack{X \supset \Delta \\ X \text{ connected}}} |K(X)|\Gamma_n(X) \tag{2.1}$$

Where the *large set regulators* are defined by

$$\Gamma_n(X) = 2^{n|X|}\Gamma(X) \tag{2.2}$$

$$\Gamma(X) = L^{(D+2)|X|} \tag{2.3}$$

and $|X|$ denotes the number of blocks in X. Because our fluctuation covariance is compactly supported it is sufficient to define the norm of the K only for connected X. This simplifies the definition of Γ with respect to [BY].
We define a *small set* as follows: a connected polymer X is a small set if $|X| \leq 2^D$.
Recall that the *L-closure* $\bar{X}$ of a polymer X is defined to be the smallest union of L-blocks containing X
The main result about Γ that we need in the next sections is the following statement

LEMMA 2.1.1

For each $p = 0, 1, 2, ...$ there is an O(1) constant c_p such that for L sufficiently large and for any polymer X

$$\Gamma_p(L^{-1}\bar{X}) \leq c_p\Gamma(X) \tag{2.4}$$

For any large set X a stronger bound is valid

$$\Gamma_p(L^{-1}\bar{X}) \leq c_p L^{-D-1}\Gamma(X) \tag{2.5}$$

Proof

For X small set one has (using $D=1$) $|X| \leq 2,\ |L^{-1}\bar{X}| \leq |X|$. This proves the (2.4) with $c_p = 2^{2p}$

For large sets we note that, for $L \geq 3$, $|X| \geq 3,\ |L^{-1}\bar{X}| \leq \frac{2}{3}|X|$. These relations imply:

$$\Gamma_p(L^{-1}\bar{X}) = 2^{p|L^{-1}\bar{X}|} L^{(D+2)|L^{-1}\bar{X}|} \leq 2^{p\frac{2}{3}|X|} L^{\frac{2}{3}(D+2)|X|} \leq 2^{p\frac{2}{3}|X|} L^{-\frac{1}{3}(D+2)|X|}\Gamma(X) \leq$$

$$\leq 2^{p\frac{2}{3}|X|} L^{-\frac{1}{3}|X|} L^{-\frac{1}{3}(D+1)|X|}\Gamma(X) \leq 2^{p\frac{2}{3}|X|} L^{-\frac{1}{3}|X|} L^{-D-1}\Gamma(X) \qquad Q.E.D.$$

2.2. Smoothness in the fields.

Functionals of ϕ are defined on the Banach space $\mathcal{C}^r(\Lambda)$ of r times continuously differentiable fields with the norm

$$\|f\|_{\mathcal{C}^r} = \sum_{l=0}^{r} \sup_x |\partial^l f(x)| \tag{2.6}$$

A derivative of a functional with respect to ϕ in the direction f is a linear functional $f \to DK(X,\phi;f)$ on this Banach space defined by

$$\frac{\partial}{\partial s} K(X, \phi + sf)\Big|_{s=0} = DK(X,\phi;f)$$

The size of a functional derivative is naturally measured by the norm

$$\|DK(X,\phi)\| = \sup[|DK(X,\phi;f)| : f \in \mathcal{C}^r(X), \|f\|_{\mathcal{C}^r(X)} \leq 1]$$

and $\|K(X,\phi)\| = |K(X,\phi)|$.
In the proof of the main theorem we will need to introduce the norm

$$\|K(X)\|_1 = \|K(X,\phi)\| + \|DK(X,\phi)\| \tag{2.7}$$

We have the obvious property:

LEMMA 2.2.1

For any polymers X_1, X_2 and for any activities K_1, K_2

$$\|K_1(X_1)K_2(X_2)\|_1 \leq \|K_1(X_1)\|_1 \|K_2(X_2)\|_1 \tag{2.8}$$

2.3. Growth in the fields: the large fields regulator G.

The growth of $K(X,\phi)$ as a function of ϕ and derivatives of ϕ is controlled by a *large fields regulator* $G(X,\phi)$.
The natural norm defined by G has the form

$$\|K(X)\|_G = \sup_{\phi\in\mathcal{C}^r} \|K(X,\phi)\| G^{-1}(X,\phi) \tag{2.9}$$

The functional $G(X,\phi)$ is chosen so as to satisfy the following inequality

$$G(X\cup Y,\phi) \geq G(X,\phi)G(Y,\phi) \quad \text{if } X\cap Y=\emptyset \tag{2.10}$$

The form of our interaction suggests the use of the following regulator

$$G_{\rho,k}(X,\phi) = \frac{1}{|X|}\int_X dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2} e^{\kappa\|\phi\|^2_{X,1,\sigma}} \tag{2.11}$$

with $0<\rho<1$, $\kappa>0$ and

$$\|\phi\|^2_{X,1,\sigma} = \sum_{1\leq\alpha\leq\sigma} \|\partial^\alpha\phi\|^2_X \tag{2.12}$$

where $\|\phi\|_X$ is the L^2 norm. We take σ large enough so that this norm can be used in Sobolev inequalities to control $\partial\phi$ pointwise.
Let us show that (2.10) is true for this choice.

LEMMA 2.3.1

$G_{\rho,k}$ satisfies (2.10).

Proof
It is enough to show

$$\frac{1}{|X\cup Y|}\int_{X\cup Y} dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2} \geq$$

$$\geq \frac{1}{|X|}\int_X dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2} \frac{1}{|Y|}\int_Y dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2}$$

Define:

$$a = \frac{1}{|X|}\int_X dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2}$$

$$b = \frac{1}{|Y|}\int_Y dy e^{-(\lambda_*/2)(1-\rho)|\phi(y)|^2}$$

$$p = \frac{|X\cup Y|}{|X|}\ ,\ q = \frac{|X\cup Y|}{|Y|}$$

Note that

$$0\leq a,b\leq 1$$

$$\frac{1}{p}+\frac{1}{q}=1$$

We have :

$$\frac{1}{|X\cup Y|}\int_{X\cup Y} dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2} = \frac{1}{p}a+\frac{1}{q}b=$$

$$=\frac{1}{p}(a^{\frac{1}{p}})^p+\frac{1}{q}(b^{\frac{1}{q}})^q$$

$$\geq a^{\frac{1}{p}}b^{\frac{1}{q}}\geq ab$$

where to go to the last line we have used $0\leq a,b\leq 1$.

Q.E.D.

For the norm (2.9) to be useful, we will need further properties for the regulator G. In particular to control the fluctuation step we will need that G is *stable* in the sense of the following lemma:

LEMMA 2.3.2 (*Stability of the large field regulator*)

Let $0<\rho<1/8\quad \rho=O(1)$ and $\kappa>0\quad \kappa=O(1)$ be both sufficiently small and independent of L. Let $\kappa/\rho<1$ and L be sufficiently large.
Then

$$(\mu_\Gamma * G_{\rho,\kappa})(X,\mathcal{R}\phi)\leq G^\sharp_{\rho,\kappa}(X,\mathcal{R}\phi) \tag{2.13}$$

with

$$G^\sharp_{\rho,\kappa}(X,\mathcal{R}\phi)=O(1)2^{|X|}\frac{L^{-\alpha}}{|X|}\int_X dx e^{-(\lambda_*/2)(1-\rho/L^{\beta/2})|L^{-\beta/2}\mathcal{R}\phi(x)|^2}e^{4\kappa\|\mathcal{R}\phi\|^2_{X,1,\sigma}} \tag{2.14}$$

The proof of this lemma, which is straightforward but rather long, is presented in Appendix A.
It is useful to note that from the scaling property of the field ϕ and the definition of β in (0.9) we have

$$\|\mathcal{R}\phi\|^2_{X,1,\sigma}\leq L^{\beta-1}\|\phi\|^2_{L^{-1}X,1,\sigma}$$

and for L sufficiently large

$$L^{\beta-1}4<1$$

since $\beta > 0$ but very small.
Note also that

$$L^{D-\alpha} = L^{\varepsilon} = O(1)$$

for ε sufficiently small (depending on L). Using these two facts it is easy to see that

$$G^{\sharp}_{\rho,\kappa}(X, \mathcal{R}\phi) \leq O(1) L^{-D} G(L^{-1}X, \phi)$$

which is the original form of the regulator up to the contractive factor L^{-D}.
Finally we remark that the stability of the large fields regulator can be stated analogously using the master formula: actually in section 5 the following form of the stability of the regulator will be used for $0 < \rho < 1/32$ and $\kappa/\rho < 1$ and L sufficiently large

$$\int d\mu_{\Sigma^{\bar{x}}}(\zeta) e^{(\lambda_*/2)4\rho|\zeta(\bar{x})+L^{-\beta}\mathcal{R}\phi(\bar{x})|^2} e^{4\kappa\|\zeta+L^{-\beta}T^{\bar{x}}\mathcal{R}\phi\|^2_{X,1,\sigma}} \leq$$

$$\leq e^{(\lambda_*/2)(4\rho/L^{\beta/2})L^{-\beta}|\mathcal{R}\phi(\bar{x})|^2} e^{8\kappa\|\mathcal{R}\phi\|^2_{L^{-1}X,1,\sigma}} \tag{2.15}$$

This can be derived from (2.13), (2.14) using the master formula in the following way

$$\int d\mu_{\Sigma^{\bar{x}}}(\zeta) e^{(\lambda_*/2)4\rho|\zeta(\bar{x})+L^{-\beta}\mathcal{R}\phi(\bar{x})|^2} e^{4\kappa\|\zeta+L^{-\beta}T^{\bar{x}}\mathcal{R}\phi\|^2_{X,1,\sigma}} = [L^{-\alpha} v_*(L^{-\beta/2}\mathcal{R}\phi(\bar{x}))]^{-1}\cdot$$

$$\cdot \int d\mu_{\Gamma}(\zeta) v_*(\zeta(\bar{x}) + \mathcal{R}\phi(\bar{x})) e^{(\lambda_*/2)4\rho|\zeta(\bar{x})+\mathcal{R}\phi(\bar{x})|^2} e^{4\kappa\|\zeta+\mathcal{R}\phi\|^2_{X,1,\sigma}} =$$

$$= [L^{-\alpha} v_*(L^{-\beta/2}\mathcal{R}\phi(\bar{x}))]^{-1} v_*(0) \int d\mu_{\Gamma}(\zeta) e^{-(\lambda_*/2)(1-4\rho)|\zeta(\bar{x})+\mathcal{R}\phi(\bar{x})|^2} e^{4\kappa\|\zeta+\mathcal{R}\phi\|^2_{X,1,\sigma}} \tag{2.16}$$

The right hand side of (2.16) is controlled using (2.14):

$$[L^{-\alpha} v_*(L^{-\beta/2}\mathcal{R}\phi(\bar{x}))]^{-1} v_*(0) \int d\mu_{\Gamma}(\zeta) e^{-(\lambda_*/2)(1-4\rho)|\zeta(\bar{x})+\mathcal{R}\phi(\bar{x})|^2} e^{4\kappa\|\zeta+\mathcal{R}\phi\|^2_{X,1,\sigma}} \leq$$

$$[L^{-\alpha} v_*(L^{-\beta/2}\mathcal{R}\phi(\bar{x}))]^{-1} v_*(0) L^{-\alpha} e^{-(\lambda_*/2)(1-4\rho/L^{\beta/2})|L^{-\beta/2}\mathcal{R}\phi(\bar{x})|^2} e^{8\kappa\|\mathcal{R}\phi\|^2_{X,1,\sigma}} \tag{2.17}$$

and (2.17) implies trivially (2.15).

One particular point which we have to take account of in this work is the fact that due to our expression of the large field regulator the usual relation (see e.g. BDH) $G(X, \phi) \geq 1$ is *not* true in our case. Therefore also the useful relation $G(X, \phi) \geq G(Y, \phi)$ if $X \supset Y$ is in general false. This implies that in many cases the reblocking step has to be evaluated in some detail. For the contributions due to small sets (see below, section 5) the following lemma is often used

LEMMA 2.3.3

$$\sum_{\substack{X \text{ s. s.} \\ \bar{X}=LZ}} G(L^{-1}X,\phi) \le O(1)L^D G(Z,\phi) \tag{2.18}$$

Proof

$$\sum_{\substack{X \text{ s. s.} \\ \bar{X}=LZ}} G(L^{-1}X,\phi) = \sum_{\substack{X \text{ s. s.} \\ \bar{X}=LZ}} \frac{1}{|L^{-1}X|}\int_{L^{-1}X} dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2} e^{\kappa\|\phi\|^2_{L^{-1}X,1,\sigma}} \tag{2.19}$$

First we observe that since X is a small set, so is Z. We have

$$\sum_{\substack{X \text{ s. s.} \\ \bar{X}=LZ}} G(L^{-1}X,\phi) \le O(1)L^D \Bigg[\sum_{\substack{\Delta \\ \bar{\Delta}=LZ}} \int_{L^{-1}\Delta} dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2}$$

$$+ \sum_{\substack{X:|X|=2, X \text{ conn.} \\ \bar{X}=LZ}} \int_{L^{-1}X} dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2}\Bigg] e^{\kappa\|\phi\|^2_{Z,1,\sigma}} \le$$

$$\le O(1)L^D \Bigg[\sum_{\substack{\Delta \\ \bar{\Delta}=LZ}} \int_{L^{-1}\Delta} dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2}$$

$$+ \sum_{\substack{\Delta_1,\Delta_2 \text{ conn.} \\ \overline{\Delta_1\cup\Delta_2}=LZ}} \left[\int_{L^{-1}\Delta_1} dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2} + \int_{L^{-1}\Delta_2} dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2}\right]\Bigg] e^{\kappa\|\phi\|^2_{Z,1,\sigma}} \le$$

$$\le O(1)L^D \Bigg[\sum_{\substack{\Delta \\ \Delta\subset LZ}} \int_{L^{-1}\Delta} dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2}$$

$$+ \sum_{\substack{\Delta_1 \\ \Delta_1\subset LZ}} \sum_{\substack{\Delta_2 \\ \Delta_1,\Delta_2 \text{ conn.}}} \int_{L^{-1}\Delta_1} dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2}\Bigg] e^{\kappa\|\phi\|^2_{Z,1,\sigma}} \le$$

$$\leq O(1)L^D\int_Z dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2}e^{\kappa\|\phi\|^2_{Z,1,\sigma}} \leq O(1)L^D G(Z,\phi) \tag{2.20}$$

In passing to the last line we have used

$$\sum_{\substack{\Delta_2 \\ \Delta_1,\Delta_2 \text{ conn.}}} 1 \leq 2$$

and the fact that $|Z| \leq 2$.
Q.E.D

2.4. Norms.

Now we have all the ingredients to construct norms on K. We define the norms

$$\|K(X)\|_{G,1} = \|K(X)\|_G + \|DK(X)\|_G \tag{2.21}$$

$$\|K\|_{G,1,\Gamma} = \|\|K(X)\|_{G,1}\|_\Gamma \tag{2.22}$$

However sometimes it will be useful to define L^∞ norms on certain activities in the following way

$$\|K(X)\|_{\infty,1} = \|K(X)\|_\infty + \|DK(X)\|_\infty \tag{2.23}$$

where

$$\|K(X)\|_\infty = \sup_{\phi\in\mathcal{C}^r} |K(X,\phi)| \tag{2.24}$$

2.5. Basic estimates on generic integrated activities.

In this subsection we state some bounds valid for integrated activities with initial norm small enough. These bounds will be used often in the next sections.

LEMMA 2.5.1

Let K be an activity such that $\|K\|_{G,1,\Gamma} = O(\varepsilon^q)$, with $q \geq 1/10$, and let us define $\mathcal{S}_{\geq k}K$ by restricting the sum on N in (1.53) to $N \geq k$. Then we have

$$\mathcal{S}_{\geq k}K(Z,\phi) = \sum_{\substack{X \\ \bar{X}=LZ}} (\mathcal{R}\bar{K}_k)(L^{-1}X,\phi) \tag{2.25}$$

with $\bar{K}_k$ defined by

$$\bar{K}_k(X,\phi) = \sum_{N\geq k} \frac{1}{N!} \sum_{\substack{X_1,\dots,X_N \text{ disj.}\\ \cup_j X_j = X \\ \{\bar{X}_j\} \text{ ov conn}}} \prod_{j=1}^{N} K(X_j) \tag{2.26}$$

Then one has for any integer $p \geq 0$ and $\varepsilon \geq 0$ sufficiently small

$$\| (\mathcal{S}_{\geq k} K)^{\natural} \|_{G,1,\Gamma_p} \leq O(1)^k L^{(\beta/2+kD)} \|K\|^k_{G,1,\Gamma} \tag{2.27}$$

with $O(1)$ depending on p.

Proof

The action of the fluctuation operator is controlled by the stability of the large fields regulator:

$$|(\mu_\Gamma * \bar{K})(X, \mathcal{R}\phi)| \leq L^{-D} O(1) 2^{|X|} G(L^{-1}X, \phi) \|\bar{K}(X)\|_G \tag{2.28}$$

then

$$| (\mathcal{S}_{\geq k} K)^{\natural} (Z,\phi)| \leq \sum_{\substack{X \\ \bar{X}=LZ}} L^{-D} O(1) 2^{|X|} G(L^{-1}X,\phi) \|\bar{K}_k(X)\|_G \leq$$

$$\leq L^{-D} O(1) e^{\kappa \|\phi\|^2_{Z,1,\sigma}} |Z| \frac{1}{|Z|} \int_Z dx e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2} \sum_{\substack{X \\ \bar{X}=LZ}} \frac{2^{|X|}}{|L^{-1}X|} \chi_{L^{-1}X}(x) \|\bar{K}_k(X)\|_G \tag{2.29}$$

where $\chi_{L^{-1}X}(x)$ is the characteristic function of the set $L^{-1}X$. Using the trivial bounds

$$\frac{1}{|L^{-1}X|} \chi_{L^{-1}X}(x) \leq L^D, \qquad |Z| \leq 2^{|Z|}$$

we have

$$\| (\mathcal{S}_{\geq k} K)^{\natural} (Z)\|_G \leq O(1) 2^{|Z|} \sum_{\substack{X \\ \bar{X}=LZ}} 2^{|X|} \|\bar{K}_k(X)\|_G \tag{2.30}$$

Performing the same bound on the functional derivative we have

$$\| (\mathcal{S}_{\geq k} K)^{\natural} (Z)\|_{G,1} \leq O(1) 2^{|Z|} L^{\beta/2} \sum_{\substack{X \\ \bar{X}=LZ}} 2^{|X|} \|\bar{K}_k(X)\|_{G,1} \tag{2.31}$$

and therefore

$$\| (\mathcal{S}_{\geq k} K)^{\natural} (Z)\|_{G,1} \Gamma_p(Z) \leq O(1) L^{\beta/2} \sum_{\substack{X \\ \bar{X}=LZ}} \|\bar{K}_k(X)\|_{G,1} \Gamma_{p+2}(L^{-1}\bar{X}) \tag{2.32}$$

Defining (see also [BY])

$$\|\bar{K}_k\|^{(1)}_{G,1,\Gamma_{p+2}} = \sup_{\Delta} \sum_{\substack{X \\ L^{-1}\bar{X}\cap\Delta\neq\emptyset}} \|\bar{K}_k(X)\|_{G,1}\Gamma_{p+2}(L^{-1}\bar{X}) \tag{2.33}$$

we easily obtain from (2.32)

$$\| (\mathcal{S}_{\geq k}K)^{\natural} \|_{G,1,\Gamma_p} \leq O(1)L^{\beta/2}\|\bar{K}_k\|^{(1)}_{G,1,\Gamma_{p+2}} \tag{2.34}$$

Now we use the BY argument (Lemma 7.1 [BY]) and we obtain

$$\|\bar{K}_k\|^{(1)}_{G,1,\Gamma_{p+2}} \leq \sum_{N\geq k} O(1)^N(\|K\|^{(1)}_{G,1,\Gamma_{p+3}})^N \tag{2.35}$$

From (2.4) of Lemma 2.1.1

$$\Gamma_{p+3}(L^{-1}\bar{X}) \leq O(1)\Gamma(X) \tag{2.36}$$

and using again BY we obtain finally

$$\|K\|^{(1)}_{G,1,\Gamma_{p+3}} \leq O(1)L^D\|K\|_{G,1,\Gamma} \tag{2.37}$$

hence

$$\| (\mathcal{S}_{\geq k}K)^{\natural} \|_{G,1,\Gamma_p} \leq O(1)L^{\beta/2}\sum_{N\geq k} O(1)^N(L^D)^N(\|K\|_{G,1,\Gamma})^N \tag{2.38}$$

Using the bound $\|K(X)\|_{G,1,\Gamma} \leq O(1)\varepsilon^q$ to control the sum over N we have the lemma

Q.E.D.

Lemma 2.5.1 obviously implies the following

COROLLARY 2.5.2

For the linearized scaling operator $\mathcal{S}_1$ the following bound holds

$$\| (\mathcal{S}_1K)^{\natural} \|_{G,1,\Gamma_p} \leq O(1)L^{(\beta/2+D)}\|K\|_{G,1,\Gamma} \tag{2.39}$$

for any integer $p\geq 1$, $O(1)$ depends on p.

We now define the linearized scaling operator restricted to contributions from large sets by

$$\mathcal{S}_1^{(l.s.)}K(Z,\phi) = \sum_{\substack{X \text{ conn. large set} \\ \bar{X}=LZ}} (\mathcal{R}K)(L^{-1}X,\phi) \tag{2.40}$$

We have the following result.

LEMMA 2.5.3

$$\| \left(\mathcal{S}_1^{(l.s.)} K \right)^{\natural} \|_{G,1,\Gamma_p} \leq O(1) L^{-(1-\beta/2)} \|K\|_{G,1,\Gamma} \tag{2.41}$$

for any integer $p \geq 0$, $O(1)$ depends on p.

Proof
Repeat the proof of lemma 2.5.1 using

$$\Gamma_{p+3}(L^{-1}\bar{X}) \leq O(1) L^{-D-1} \Gamma(X) \quad \text{for } X \text{ large set} \tag{2.42}$$

which comes from (2.5) of Lemma 2.1.1 instead of (2.36).

Q.E.D.

Let now F be a polymer activity supported on small sets, such that for every integer $p \geq 0$ and for $q \geq 1/10$

$$\|F\|_{G,1,\Gamma_p} \leq O(\varepsilon^q) \tag{2.43}$$

and

$$\|F\|_{\infty,1,\Gamma_p} \leq O(\varepsilon^q) \tag{2.44}$$

with $O(1)$ depending on p.
We then have for $\varepsilon \geq 0$ sufficiently small

LEMMA 2.5.4

$$\|e^{-F} - 1\|_{G,1,\Gamma_p} \leq O(1) \|F\|_{G,1,\Gamma_p} \tag{2.45}$$

$$\|e^{-F} - 1 - F\|_{G,1,\Gamma_p} \leq O(1) \|F\|_{G,1,\Gamma_p} \|F\|_{\infty,1,\Gamma_p} \tag{2.46}$$

Remark:
Lemma 2.5.4 remains true if the G-norm is replaced by the L^∞ norm, by the same proof.

Proof

$$\|e^{-F(X)} - \sum_{l=0}^{k} \frac{1}{l!}(-F(X))^l\|_1 \leq \sum_{N \geq k+1} \frac{1}{N!} \|F(X)\|_1^N \tag{2.47}$$

whence:

$$\|e^{-F(X)} - \sum_{l=0}^{k} \frac{1}{l!}(-F(X))^l\|_{G,1} \leq \sum_{N\geq k+1} \frac{1}{N!}\|F(X)\|_{G,1}\|F(X)\|_{\infty,1}^{N-1} \leq$$

$$\leq \sum_{N\geq k+1} \frac{1}{N!}\|F(X)\|_{G,1}(\|F(X)\|_{\infty,1}\Gamma(X))^{N-1} \tag{2.48}$$

Let $\Delta \subset X$ be any block in X. Then

$$\|F(X)\|_{\infty,1}\Gamma(X) \leq \sum_{Y\cap\Delta\neq\emptyset} \|F(Y)\|_{\infty,1}\Gamma(Y) \leq \sup_{\Delta'} \sum_{Y\cap\Delta'\neq\emptyset} \|F(Y)\|_{\infty,1}\Gamma(Y) = \|F\|_{\infty,1,\Gamma} \tag{2.49}$$

From (2.48) and (2.49) we get:

$$\|e^{-F(X)} - \sum_{l=0}^{k} \frac{1}{l!}(-F(X))^l\|_{G,1} \leq \sum_{N\geq k+1} \frac{1}{N!}\|F(X)\|_{G,1}\|F\|_{\infty,1,\Gamma}^{N-1} \tag{2.50}$$

or

$$\|e^{-F} - \sum_{l=0}^{k} \frac{1}{l!}(-F)^l\|_{G,1,\Gamma} \leq$$

$$\leq \|F\|_{G,1,\Gamma}\|F\|_{\infty,1,\Gamma}^{k} \sum_{N\geq k+1} \frac{1}{N!}\|F\|_{\infty,1,\Gamma}^{N-1-k} \tag{2.51}$$

Because of the smallness of $\|F\|_{\infty,1,\Gamma}$ the series is bounded by $O(1)$. Hence

$$\|e^{-F} - \sum_{l=0}^{k} \frac{1}{l!}(-F)^l\|_{G,1\Gamma} \leq O(1)\|F\|_{G,1_G}\|F\|_{\infty,1,\Gamma}^{k} \tag{2.52}$$

Setting $k = 0, 1$ in (2.52) we prove the lemma.

Q.E.D.

LEMMA 2.5.5

For any integers $k \geq 1$ and $p \geq 0$, and with $O(1)$ dependent on p

$$\|(e^{-F} - 1)^{+}_{\geq k}\|_{G,1,\Gamma_p} \leq O(1)^k \|F\|_{G,1,\Gamma_{p+3}}\|F\|_{\infty,1,\Gamma_{p+3}}^{k-1} \tag{2.53}$$

$$\|(e^{-F} - 1)^{+}_{\geq k}\|_{\infty,1,\Gamma_p} \leq O(1)^k \|F\|_{\infty,1,\Gamma_{p+1}}^{k} \tag{2.54}$$

Proof

$$(e^{-F}-1)^{+}_{\geq k}(X)=\sum_{N\geq k}\frac{1}{N!}\sum_{\substack{X_1,..,X_N:\cup_j X_j=X\\ \{X_j\}\text{ overlap conn.}}}\prod_{j=1}^{N}(e^{-F}-1)(X_j) \tag{2.55}$$

then

$$\|(e^{-F}-1)^{+}_{\geq k}(X)\|_1\leq\sum_{N\geq k}\frac{1}{N!}\sum_{\substack{X_1,..,X_N:\cup_j X_j=X\\ \{X_j\}\text{ overlap conn.}}}\prod_{j=1}^{N}\|(e^{-F}-1)(X_j)\|_1\leq$$

$$\sum_{N\geq k}\frac{1}{N!}\sum_{\substack{X_1,..,X_N:\cup_j X_j=X\\ \{X_j\}\text{ overlap conn.}}}G(X_1)\|(e^{-F}-1)(X_1)\|_{G,1}\prod_{j=2}^{N}\|(e^{-F}-1)(X_j)\|_{\infty,1} \tag{2.56}$$

We estimate for $X_1\subset X$

$$G(X_1)\leq\sum_{Y\subset X}G(Y) \tag{2.57}$$

and then

$$G(Y)=\frac{1}{|Y|}\int_Y dxe^{-\frac{\lambda_*}{2}(1-\rho)|\phi(x)|^2}e^{\kappa\|\phi\|^2_{Y,1,\sigma}}\leq\int_Y dxe^{-\frac{\lambda_*}{2}(1-\rho)|\phi(x)|^2}e^{\kappa\|\phi\|^2_{X,1,\sigma}}\leq$$

$$\leq|X|\frac{1}{|X|}\int_X dxe^{-\frac{\lambda_*}{2}(1-\rho)|\phi(x)|^2}e^{\kappa\|\phi\|^2_{X,1,\sigma}} \tag{2.58}$$

so that

$$G(Y)\leq 2^{|X|}G(X) \tag{2.59}$$

also

$$\sum_{Y\subset X}1\leq 2^{|X|} \tag{2.60}$$

Hence from (2.57)-(2.60) we get

$$G(X_1)\leq 2^{2|X|}G(X)\leq G(X)\prod_{j=1}^{N}2^{2|X_j|} \tag{2.61}$$

Putting (2.61) in (2.56) we get

$$\|(e^{-F}-1)^{+}_{\geq k}(X)\|_{G,1} \leq$$

$$\leq \sum_{N\geq k}\frac{1}{N!}\sum_{\substack{X_1,..,X_N:\cup_j X_j=X\\ \{X_j\}\text{ overlap conn.}}} 2^{2|X_1|}\|(e^{-F}-1)(X_1)\|_{G,1}\prod_{j=2}^{N}2^{2|X_j|}\|(e^{-F}-1)(X_j)\|_{\infty,1} \quad (2.62)$$

Since the $\{X_j\}$ are overlap connected

$$\Gamma_p(X) \leq \prod_{j=1}^{N}\Gamma_p(X_j)$$

and hence

$$\|(e^{-F}-1)^{+}_{\geq k}\|_{G,1,\Gamma_p} \leq \sum_{N\geq k}\frac{1}{N!}\sup_{\Delta}\sum_{X\cap\Delta\neq\emptyset}\sum_{\substack{X_1,..,X_N:\cup_j X_j=X\\ \{X_j\}\text{ overlap conn.}}}$$

$$\|(e^{-F}-1)(X_1)\|_{G,1}\Gamma_{p+2}(X_1)\prod_{j=2}^{N}\|(e^{-F}-1)(X_j)\|_{\infty,1}\Gamma_{p+2}(X_j) \quad (2.63)$$

We now estimate the r.h.s. of (2.63) by the spanning tree argument in the proof of Lemma 5.1 of [BY]. We then get

$$\|(e^{-F}-1)^{+}_{\geq k}\|_{G,1,\Gamma_p} \leq \sum_{N\geq k}O(1)^N\|(e^{-F}-1)\|_{G,1,\Gamma_{p+3}}\|(e^{-F}-1)\|^{N-1}_{\infty,1,\Gamma_{p+3}} =$$

$$= O(1)^k\|(e^{-F}-1)\|_{G,1,\Gamma_{p+3}}\|(e^{-F}-1)\|^{k-1}_{\infty,1,\Gamma_{p+3}}\sum_{N\geq k}O(1)^{N-k}\|(e^{-F}-1)\|^{N-k}_{\infty,1,\Gamma_{p+3}} \quad (2.64)$$

Now use lemma 2.5.4 and the remark following it

$$\|(e^{-F}-1)^{+}_{\geq k}\|_{G,1,\Gamma_p} \leq O(1)^k\|F\|_{G,1,\Gamma_{p+3}}\|F\|^{k-1}_{\infty,1,\Gamma_{p+3}}\sum_{N\geq k}O(1)^{N-k}\|F\|^{N-k}_{\infty,1,\Gamma_{p+3}} \quad (2.65)$$

By assumption (2.44) the series converges to $O(1)$. Hence:

$$\|(e^{-F}-1)^{+}_{\geq k}\|_{G,1,\Gamma_p} \leq O(1)^k\|F\|_{G,1,\Gamma_{p+3}}\|F\|^{k-1}_{\infty,1,\Gamma_{p+3}} \quad (2.66)$$

This proves (2.53). The proof of (2.54) is the same, except that we do not need the estimate (2.61), so that on the r.h.s. of (2.54) we have the norm with Γ_{p+1} instead of Γ_{p+3}.

Q.E.D.

2.6. Lemmas on increments

Let $K,\ K'$ be two polymer activities satisfing the hypothesis of lemma 2.5.1, namely

$$\|K\|_{G,1,\Gamma} \leq O(\varepsilon^q) \qquad \|K'\|_{G,1,\Gamma} \leq O(\varepsilon^q) \tag{2.67}$$

for some $q \geq \frac{1}{10}$ and $\varepsilon \geq 0$ sufficiently small.
Define the increments

$$\Delta K = K - K' \qquad \Delta(\mathcal{S}_{\geq k} K) = \mathcal{S}_{\geq k} K' - \mathcal{S}_{\geq k} K \tag{2.68}$$

Then we have

LEMMA 2.6.1

For any integer $p \geq 0$

$$\|\Delta(\mathcal{S}_{\geq k} K)^{\natural}\|_{G,1,\Gamma_p} \leq O(1)^k L^{(\beta/2+kD)} \varepsilon^{(k-1)q} \|\Delta K\|_{G,1,\Gamma} \tag{2.69}$$

with $O(1)$ depending on p.

Let now F, F' be two polymer activities supported on small sets, such that for every integer $p \geq 0$

$$\|F\|_{G,1,\Gamma_p} \leq O(\varepsilon^q) \qquad \|F'\|_{G,1,\Gamma_p} \leq O(\varepsilon^q) \tag{2.70}$$

and

$$\|F\|_{\infty,1,\Gamma_p} \leq O(\varepsilon^q) \qquad \|F'\|_{\infty,1,\Gamma_p} \leq O(\varepsilon^q) \tag{2.71}$$

for some $q \geq \frac{1}{10}$, and $\varepsilon \geq 0$ sufficiently small. Define increments as before
We then have

LEMMA 2.6.2

$$\|\Delta(e^{-F} - 1)\|_{G,1,\Gamma_p} \leq O(1) \|\Delta F\|_{G,1,\Gamma_p} \tag{2.72}$$

$$\|\Delta(e^{-F} - 1 - F)\|_{G,1,\Gamma_p} \leq O(1) \varepsilon^q \|\Delta F\|_{G,1,\Gamma_p} \tag{2.73}$$

Remark:

Lemma 2.6.2 remains true if the G-norm is replaced by the L^∞ norm, by the same proof.

LEMMA 2.6.3

Given two activities satisfying (2.70), (2.71), for any integers $k \geq 1$ and $p \geq 0$, and with $O(1)$ dependent on p

$$\|\Delta(e^{-F}-1)^+_{\geq k}\|_{G,1,\Gamma_p} \leq O(1)^k \varepsilon^{(k-1)q} \|\Delta F\|_{G,1,\Gamma_{p+3}} \tag{2.74}$$

$$\|\Delta(e^{-F}-1)^+_{\geq k}\|_{\infty,1,\Gamma_p} \leq O(1)^k \varepsilon^{(k-1)q} \|\Delta F\|_{\infty,1,\Gamma_{p+1}} \tag{2.75}$$

We shall only prove lemma 2.6.1, the proofs of lemmas 2.6.2, 2.6.3 being similar

Proof of lemma 2.6.1

$$\Delta(\mathcal{S}_{\geq k}K)^\natural = (\mathcal{S}_{\geq k}(K+\Delta K))^\natural - (\mathcal{S}_{\geq k}K)^\natural =$$

$$= \int_0^1 dt \frac{\partial}{\partial t}(\mathcal{S}_{\geq k}(K+t\Delta K))^\natural = \int_0^1 dt \left(\frac{\partial}{\partial t}\mathcal{S}_{\geq k}(K+t\Delta K)\right)^\natural \tag{2.76}$$

From the proof of lemma 2.5.1 we have that

$$K \longrightarrow \mathcal{S}_{\geq k}(K)$$

is an analytic map between the Banach spaces with norms $\|\cdot\|_{G,1,\Gamma}$ and $\|\cdot\|_{G,1,\Gamma_p}$ respectively.
Define

$$K(t) = K + t\Delta K \tag{2.77}$$

Then $\mathcal{S}_{\geq k}(K(t))$ is analytic in t. By the Cauchy integral formula

$$\left[\frac{\partial}{\partial t}\mathcal{S}_{\geq k}(K(t))\right]^\natural = \frac{1}{2\pi i}\oint_\gamma dz \frac{[\mathcal{S}_{\geq k}(K(z))]^\natural}{(z-t)^2} \tag{2.78}$$

where we choose the closed contour γ in $\mathbf{C}$ as follows

$$\gamma:\ z-t = Re^{i\theta}, \quad 0 \leq \theta \leq 2\pi, \quad R = \frac{\varepsilon^q}{\|\Delta K\|_{G,1,\Gamma}} \tag{2.79}$$

With this choice of γ, and $0 \leq t \leq 1$, for $z \in \gamma$

$$K(z) = K + \left(t + \frac{\varepsilon^q}{\|\Delta K\|_{G,1,\Gamma}} e^{i\theta}\right)\Delta K \tag{2.80}$$

Clearly $K(z)$ satisfies the hypothesis of lemma 2.5.1 under the hypothesis (2.68)

$$\|K(z)\|_{G,1,\Gamma} \leq O(\varepsilon^q) \tag{2.81}$$

Hence from (2.78) we have the Cauchy estimate

$$\left\| \left[\frac{\partial}{\partial t} \mathcal{S}_{\geq k}(K(t)) \right]^{\natural} \right\|_{G,1,\Gamma_p} \leq \|\Delta K\|_{G,1,\Gamma} \varepsilon^{-q} \sup_{z \in \gamma} \left\| [\mathcal{S}_{\geq k}(K(z))]^{\natural} \right\|_{G,1,\Gamma_p} \leq$$

$$\leq \|\Delta K\|_{G,1,\Gamma} \varepsilon^{-q} \sup_{z \in \gamma} \|K(z)\|^k_{G,1,\Gamma} O(1)^k L^{\beta/2+kD} \leq$$

$$\leq O(1)^k L^{\beta/2+kD} \varepsilon^{(k-1)q} \|\Delta K\|_{G,1,\Gamma} \tag{2.82}$$

where in the last two lines we used lemma 2.5.1 and (2.81) respectively.
We use the estimate (2.82) in (2.76) to finish the proof.

Q.E.D.

§3. *Estimates for a generic RG step. Remainder estimates.*

In this section we present the detailed structure of the initial form of the partition functional and the form of the activities produced by a RG step.

3.1. Some manipulation on the starting partition functional

The starting partition functional $z_0(\Lambda,\phi)$ has the initial expression

$$z_0(\Lambda,\phi) = e^{-V_0}\mathcal{E}xp(\square) \tag{3.1}$$

where the initial activity V_0 is

$$V_0(X) = g_0 V_*(X) = g_0 \int_X dx \; v_*(\phi(x)) = g_0 \int_X dx \left(\frac{\lambda_*}{2\pi}\right)^{\frac{d}{2}} e^{-\frac{\lambda_*}{2}|\phi(x)|^2} \tag{3.2}$$

with $\lambda_* = \frac{\beta}{2u(0)}$.
However it is convenient to write (3.1) in a form suitable for iteration as

$$e^{-V_0}\mathcal{E}xp(\square + K_0) \tag{3.3}$$

where initially $K_0 \equiv 0$.
We want to show that the structure of (3.3) is reproduced also after the generic step of RG. We start therefore from an expression of the form

$$e^{-V(\phi)}\mathcal{E}xp(\square + K) \tag{3.4}$$

where in (3.4) the activity V is $V = gV_*$, with $g = O(\varepsilon)$, and

$$0 < \varepsilon < L^{-10(D+2)} \tag{3.5}$$

with L sufficiently large. The bounds

$$\|V(\Delta)\|_{G,1} = O(\varepsilon) \tag{3.6}$$

$$\|V(\Delta)\|_{\infty,1} = O(\varepsilon) \tag{3.7}$$

obviously hold. We assume for the connected activity K the following structure

$$K = \mathcal{I} + r \tag{3.8}$$

where $\mathcal{I}$ is an activity exactly computed in second order perturbation theory, supported only on small sets and satisfying the following bounds

$$\|\mathcal{I}\|_{G,1,\Gamma_6} \le \varepsilon^{7/4} \tag{3.9}$$

$$\|(\mathcal{S}_1\mathcal{I})^{\natural}\|_{G,1,\Gamma} \le L^{-\beta/4}\varepsilon^{7/4} \tag{3.10}$$

The structure of $\mathcal{I}$ and the above bounds will be given in section 4 devoted to second order perturbation theory.
r satisfies the following inductive bound

$$\|r\|_{G,1,\Gamma_6} \le \varepsilon^{5/2+\eta}, \quad 0 < \eta \le 1/20 \tag{3.11}$$

As outlined above (see (1.52)) it is convenient first of all to rewrite the partition functional

$$e^{-gV_*}\mathcal{E}xp(\square + K) = \mathcal{E}xp(\square + \hat{K}) \tag{3.12}$$

A suitable explicit expression of the connected activity $\hat{K}$ is provided by the following lemma.

LEMMA 3.1.1

$$\hat{K} = P^+ + K + P^+ \vee K \tag{3.13}$$

where

$$P(X) = \begin{cases} e^{-gV_*(\Delta)} - 1 & \text{for } X = \Delta \\ 0 & \text{otherwise} \end{cases} \tag{3.14}$$

and, following (1.49)

$$P^+(X) = \sum_{N\ge 1} \frac{1}{N!} \sum_{\substack{\Delta_1\ldots\Delta_N:\ conn. \\ \cup_j\Delta_j = X}} \prod_{j=1}^{N} P(\Delta_j) \tag{3.15}$$

Proof

$$e^{-gV_*} = \prod_{\Delta\in X}((e^{-gV_*(\Delta)} - 1) + 1) = \mathcal{E}xp(\square + P^+)(X) \tag{3.16}$$

$$e^{-gV_*}\mathcal{E}xp(\square + K) = \mathcal{E}xp(\square + P^+)\mathcal{E}xp(\square + K) \tag{3.17}$$

and the proof follows from lemma 1.4.1

Q.E.D.

3.2. Extraction of the second order activities

It is convenient now to separate out the second order contributions from $\hat{K}$. We write

$$\hat{K} = \hat{Q} + \hat{r} \tag{3.18}$$

with

$$\hat{Q}(X) = \begin{cases} -gV_*(\Delta) + \frac{g^2}{2}(V_*(\Delta))^2 + \mathcal{I}(\Delta) & \text{for } |X| = 1,\ X \equiv \Delta \\ \frac{g^2}{2} \sum\limits_{\substack{\Delta_1, \Delta_2 \\ \Delta_1 \cup \Delta_2 = X}} V_*(\Delta_1) V_*(\Delta_2) + \mathcal{I}(X) & \text{for } |X| = 2,\ X \text{ connected} \\ 0 & \text{otherwise} \end{cases} \tag{3.19}$$

and

$$\hat{r} = \Pi + r + P^+ \vee K \tag{3.20}$$

where Π is given by

$$\Pi(X) = \begin{cases} \frac{-g^3}{2}(V_*(\Delta))^3 \int_0^1 ds (1-s)^2 \exp(-sgV_*(\Delta)) & \text{for } |X| = 1,\ X \equiv \Delta \\ \frac{1}{2} \sum\limits_{\substack{\Delta_1, \Delta_2 \\ \Delta_1 \cup \Delta_2 = X}} \sigma(\Delta_1, \Delta_2) & \text{for } |X| = 2,\ X \text{ connected} \\ P^+_{\geq 3}(X) & \text{otherwise} \end{cases} \tag{3.21}$$

with

$$\sigma(\Delta_1, \Delta_2) = -g^3 V_*(\Delta_1)(V_*(\Delta_2))^2 \int_0^1 ds (1-s) \exp(-sgV_*(\Delta_2)) + (\Delta_1 \rightleftharpoons \Delta_2) +$$

$$+ g^4 \prod_{j=1}^{2} (V_*(\Delta_j))^2 \int_0^1 ds (1-s) \exp(sgV_*(\Delta_j)) \tag{3.22}$$

$$P^+_{\geq 3}(X) = \sum_{N \geq 3} \frac{1}{N!} \sum_{\substack{\Delta_1 \ldots \Delta_N:\ conn. \\ \cup_j \Delta_j = X}} \prod_{j=1}^{N} P(\Delta_j) \tag{3.23}$$

3.3. Bound on the remainder $\hat{r}$

We prove now the following result.

LEMMA 3.3.1

$$\|\hat{r}\|_{G,1,\Gamma} \leq O(1)\varepsilon^{5/2+\eta} \tag{3.24}$$

Proof

In order to prove lemma 3.3.1 let us list some preliminary trivial results: recall first that for any polymers X_1, X_2 and for any activity J_1, J_2

$$\|J_1(X_1)J_2(X_2)\|_1 \le \|J_1(X_1)\|_1 \|J_2(X_2)\|_1 \tag{3.25}$$

(see section 2)
This implies easily:

$$\|P(\Delta)\|_1 \le O(1)\|V(\Delta)\|_1 \tag{3.26}$$

(trivial, from (3.25) and expansion of exponential)

$$\|P(\Delta)\|_{\infty,1} \le O(1)\|V(\Delta)\|_{\infty,1} \le O(1)\varepsilon \tag{3.27}$$

(from (3.26) and (3.7))

$$\|P(\Delta)\|_{G,1} \le O(1)\|V(\Delta)\|_{G,1} \le O(1)\varepsilon \tag{3.28}$$

(from (3.26) and (3.6))

$$\|\Pi(X)\|_{G,1} \le O(1)\varepsilon^3 \quad \text{for } |X| \le 2 \tag{3.29}$$

(from (3.25), (3.7) and (3.6))
To give a bound on P^+ and on $P^+_{\ge 3}$ we use lemmas 2.5.4 and 2.5.5 and we obtain

$$\|P^+\|_{\infty,1,\Gamma} \le O(1)\varepsilon^{9/10} \tag{3.30}$$

and

$$\|P^+_{\ge 3}\|_{G,1,\Gamma} \le O(1)\varepsilon^{27/10} \tag{3.31}$$

Therefore we have, using (3.31) and (3.29) and again the smallness of ε (3.5)

$$\|\Pi\|_{G,1,\Gamma} \le O(1)\varepsilon^{27/10} \le \frac{1}{8}L^{-1}\varepsilon^{5/2+\eta} \tag{3.32}$$

Finally we consider $P^+ \vee K$. By definition of $\vee$ and using the [BY] spanning tree argument we obtain

$$\|P^+ \vee K\|_{G,1,\Gamma} \le \sum_{N,M\ge 1} (O(1))^{N+M} \|P^+\|^N_{\infty,1,\Gamma_3} \|K\|^M_{G,1,\Gamma_3} \tag{3.33}$$

Note that from (3.8), (3.9) and (3.11) it follows that

$$\|K\|_{G,1,\Gamma_3} \le \varepsilon^{7/4} \tag{3.34}$$

Then from (3.30) and (3.34), and summing the series, we obtain

$$\|P^+ \vee K\|_{G,1,\Gamma} \leq O(1)\varepsilon^{9/10+7/4} \leq \varepsilon^{5/2+\eta} \tag{3.35}$$

Putting toghether (3.11), (3.35), and (3.32) we obtain (3.24)

Q.E.D.

Finally it is useful to note

LEMMA 3.3.2

$$\|\hat{K}\|_{G,1,\Gamma} \leq O(1)\varepsilon^{9/10} \tag{3.36}$$

$$\|\hat{K} + gV_*\|_{G,1,\Gamma} \leq O(1)\varepsilon^{7/4} \tag{3.37}$$

The proof is simply obtained by the definition of $\hat{K}$ (3.18), by the estimate (3.6) with the smallness of ε, by (3.9) and by the lemma 3.3.1 above.

3.4. The action of RG

Reblocking-rescaling

The connected activity $\hat{K}$ is given by (3.18), (3.19), (3.20). It is convenient to separate out the second order term $\mathcal{I}$ from $\hat{Q}$. So we define $\hat{Q}^{(0)}(X)$, supported on connected sets $|X| \leq 2$ by:

$$\hat{Q}^{(0)}(X) = \begin{cases} -gV_*(\Delta) + \frac{g^2}{2}(V_*(\Delta))^2 & \text{for } |X| = 1,\ X \equiv \Delta \\ \frac{g^2}{2} \sum\limits_{\substack{\Delta_1,\Delta_2 \\ \Delta_1 \cup \Delta_2 = X}} V_*(\Delta_1)V_*(\Delta_2) & \text{for } |X| = 2,\ X \text{ connected} \\ 0 & \text{otherwise} \end{cases} \tag{3.38}$$

Now it is easy to see that we can express the reblocked-rescaled activity $\mathcal{S}\hat{K}$ as:

$$\mathcal{S}\hat{K} = \mathcal{S}_1\hat{Q}^{(0)} + g^2\mathcal{S}_2(V_*) + \mathcal{S}_1\mathcal{I} + \mathcal{S}_1\hat{r} + \tilde{r} \tag{3.39}$$

where

$$\tilde{r} = \mathcal{S}_{\geq 3}\hat{K} + \mathcal{S}_2(\hat{K} + gV_*) + \mathcal{S}_2(gV_*, \hat{K} + gV_*) \tag{3.40}$$

Remarks

1) In (3.39), (3.40) V_* is supported on single blocks.

2) $\mathcal{S}_1$ is the linearized reblocking-rescaling, $\mathcal{S}_2$ is the quadratic part of $\mathcal{S}$ and $\mathcal{S}_{\geq 3}$ stands for $\mathcal{S} - \mathcal{S}_1 - \mathcal{S}_2$. In the last term of (3.40), the quadratic reblocking sum has for each term one factor gV_* and the other $\hat{K} + gV_*$.

3) Note that $\tilde{r}$ is formally of $O(\varepsilon^3)$. We shall estimate it after performing the fluctuation integration.

Fluctuation integration
We have

$$\mu_{\mathcal{R}\Gamma} * \mathcal{E}xp(\square + \mathcal{S}\hat{K}) = \mathcal{E}xp(\square + (\mathcal{S}\hat{K})^{\natural}) \tag{3.41}$$

Then

$$(\mathcal{S}\hat{K})^{\natural} = -gL^{\varepsilon}V_* + \frac{g^2}{2}(\mathcal{R}\tilde{Q})^{\natural} + (\mathcal{S}_1\mathcal{I})^{\natural} + (\mathcal{S}_1\hat{r})^{\natural} + \tilde{r}^{\natural} \tag{3.42}$$

where V_* above is again supported on single blocks and the connected activity $\tilde{Q}$ is supported on L polymers LZ such that $|Z| \le 2$ and has the following expression

$$\tilde{Q}(L\Delta) = (V_*(L\Delta))^2$$

$$\tilde{Q}(L\Delta \cup L\Delta') = V_*(L\Delta)V_*(L\Delta') + (\Delta \rightleftharpoons \Delta') \tag{3.43}$$

The first three terms of (3.42) are contributions up to second order in perturbation theory. They will be treated in more detail in the next section.

Preliminary extraction
It is convenient to extract the first order term and to rewrite our partition functional in the following way

$$\mathcal{E}xp(\square + (\mathcal{S}\hat{K})^{\natural})(L^{-1}\Lambda) = e^{-L^{\varepsilon}gV_*(L^{-1}\Lambda)}\mathcal{E}xp(\square + \tilde{K})(L^{-1}\Lambda) \tag{3.44}$$

where the activity $\tilde{K}$ is described by the following lemma

LEMMA 3.4.1

$$\tilde{K} = (\mathcal{S}\hat{K})^{\natural} + \tilde{P}^{+} + \tilde{P}^{+} \vee (\mathcal{S}\hat{K})^{\natural} \tag{3.45}$$

where

$$\tilde{P} = \begin{cases} e^{-L^{\varepsilon}gV_*(\Delta)} - 1 & \text{for } X = \Delta \\ 0 & \text{otherwise} \end{cases} \tag{3.46}$$

Proof
By (3.44)

$$\mathcal{E}xp(\square + \tilde{K}) = e^{L^{\varepsilon}gV_*}\mathcal{E}xp(\square + (\mathcal{S}\hat{K})^{\natural}) \tag{3.47}$$

Then (3.45) follows as in lemma 3.1.1

Q.E.D.

We now isolate out the terms proportional to g and g^2 in (3.45). Introduce the notation $\tilde{P}^+_{(\leq 2)}$ (to be distinguished from $\tilde{P}^+_{\leq 2}$) to represent the sum of contributions proportional to g and g^2 in $\tilde{P}^+$. Clearly

$$\tilde{P}^+_{(\leq 2)}(X) = \begin{cases} L^\varepsilon g V_*(\Delta) + \frac{1}{2}L^{2\varepsilon}g^2(V_*(\Delta))^2 & \text{for } |X|=1,\ X\equiv\Delta \\ \frac{1}{2}L^{2\varepsilon}g^2 \sum\limits_{\substack{\Delta_1,\Delta_2 \\ \Delta_1\cup\Delta_2=X}} V_*(\Delta_1)V_*(\Delta_2) & \text{for } |X|=2,\ X \text{ connected} \\ 0 & \text{otherwise} \end{cases} \tag{3.48}$$

Note also

$$(\tilde{P}^+ \vee (\mathcal{S}\hat{K})^\natural)_{(\leq 2)}(X) = \begin{cases} -L^{2\varepsilon}g^2(V_*(\Delta))^2 & \text{for } |X|=1,\ X\equiv\Delta \\ -L^{2\varepsilon}g^2 \sum\limits_{\substack{\Delta_1,\Delta_2 \\ \Delta_1\cup\Delta_2=X}} V_*(\Delta_1)V_*(\Delta_2) & \text{for } |X|=2,\ X \text{ connected} \\ 0 & \text{otherwise} \end{cases} \tag{3.49}$$

Adding (3.48) and (3.49) we get

$$(\tilde{P}^+ + \tilde{P}^+ \vee (\mathcal{S}\hat{K})^\natural)_{(\leq 2)} = L^\varepsilon g V_* - \frac{1}{2}L^{2\varepsilon}g^2\tilde{Q} \tag{3.50}$$

with V_* supported on single blocks and $\tilde{Q}$ defined in (3.43). Hence returning to (3.45) we obtain

$$\tilde{K} = \frac{g^2}{2}\int_0^1 ds\frac{\partial}{\partial s}(\mathcal{R}\tilde{Q})^{s\natural} + (\mathcal{S}_1\mathcal{I})^\natural + (\mathcal{S}_1\hat{r})^\natural + \tilde{r}^\natural + \bar{r} \tag{3.51}$$

where the new remainder $\bar{r}$ is

$$\bar{r} = (\tilde{P}^+ - \tilde{P}^+_{(\leq 2)}) + (\tilde{P}^+ \vee (\mathcal{S}\hat{K})^\natural - (\tilde{P}^+ \vee (\mathcal{S}\hat{K})^\natural)_{\leq 2} \tag{3.52}$$

Remarks

1) $\bar{r}$ is formally $O(\varepsilon^3)$ and toghether with $\tilde{r}^\natural$ needs no further extraction. Their norms will be estimated in the following subsection 3.5.

2) $\tilde{K}$ needs further extractions, namely from first and third term in (3.51). The first extracted term, denoted $F_{\tilde{Q}}$, is the perturbative relevant part of the first term of (3.51). It will be computed explicitely in the following section 4, and its irrelevant part will be $\mathcal{I}$. Note that in section 4 it will be also proved that $(\mathcal{S}_1\mathcal{I})^\natural$ needs no extraction, since its norm, by exact computations, goes down by a contracting factor. The second extracted term, denoted $F_{\hat{r}}$, is the relevant part of $(\mathcal{S}_1\hat{r})^\natural$. Althought $\hat{r}$ is $O(\varepsilon^{5/2+\eta})$, an extraction

has to be performed because the linear reblocking for small sets produces a factor L^D, and therefore a contractive factor has to be obtained. This extraction, and the control of the obtained remainder, will be the subject of section 5.

3.5. Bounds on irrelevant remainders

We prove now the following results.

LEMMA 3.5.1

$$\|\bar r\|_{G,1,\Gamma_p} \le L^{-\beta/2}\varepsilon^{5/2+\eta} \tag{3.53}$$

Proof
The first addend of $\bar r$, namely the term $(\tilde P^+ - \tilde P^+_{(\le 2)})$, has the same form as Π in (3.21) with V_* substituted by $-L^\varepsilon V_*$. Therefore, since $L^\varepsilon = O(1)$, one can obtain the bound

$$(\tilde P^+ - \tilde P^+_{(\le 2)}) \le L^{-1}\varepsilon^{5/2+\eta} \tag{3.54}$$

along the same lines as for Π. To control the term $(\tilde P^+ \vee (\mathcal S\hat K)^\natural - (\tilde P^+ \vee (\mathcal S\hat K)^\natural)_{\le 2}$ it is enough to estimate $\tilde P^+_{\ge 2} \vee (\mathcal S\hat K)^\natural$ and $\tilde P \vee (\mathcal S(\hat K + gV_*))^\natural$. We have

$$\|\tilde P^+_{\ge 2} \vee (\mathcal S\hat K)^\natural\|_{G,1,\Gamma_p} \le \sum_{N,M\ge 1} (O(1))^{N+M} \|\tilde P^+_{\ge 2}\|^N_{\infty,G,1,\Gamma_{p+3}} \|(\mathcal S\hat K)^\natural\|^M_{G,1,\Gamma_{p+3}} \tag{3.55}$$

By lemma 2.5.1 and using the fact that from the condition of the smallness of ε (3.5) and $D=1$ it turns out $L \le \varepsilon^{-1/30}$ we have

$$\|(\mathcal S\hat K)^\natural\|^M_{G,1,\Gamma_{p+3}} \le O(1) L^{\beta/2} L^D \|\hat K\|^M_{G,1,\Gamma} \le L^2\varepsilon^{9/10} \le \varepsilon^{9/10-2/30} \le \varepsilon^{8/10} \tag{3.56}$$

It is easy to see, as in the proof of (3.31), that

$$\|\tilde P^+_{\ge 2}\|^N_{G,1,\Gamma_{p+3}} \le \varepsilon^{18/10} \tag{3.57}$$

This gives

$$\|\tilde P^+_{\ge 2} \vee (\mathcal S\hat K)^\natural\|_{G,1,\Gamma_p} \le \varepsilon^{26/10} \le L^{-1}\varepsilon^{5/2+\eta} \tag{3.58}$$

We obtain in the same way

$$\|\tilde P \vee (\mathcal S(\hat K + gV_*))^\natural\|_{G,1,\Gamma_p} \le O(1)\varepsilon^{9/10}\varepsilon^{7/4} L^{D+\beta/2} \le L^2\varepsilon^{53/20} \le L^{-1}\varepsilon^{5/2+\eta} \tag{3.59}$$

From (3.54), (3.58) and (3.59) we obtain the lemma.

Q.E.D.

LEMMA 3.5.2

$$\|\tilde{r}^\natural\|_{G,1,\Gamma_p} \leq L^{-\beta/2}\varepsilon^{5/2+\eta} \tag{3.60}$$

Proof

$\tilde{r}$ is given in (3.40). There are three terms. By lemma 2.5.1 the contribution of the first is bounded by

$$\|(\mathcal{S}_{\geq 3}\hat{K})^\natural\|_{G,1,\Gamma_p} \leq O(1)L^{\beta/2+3D}\|\hat{K}\|^3_{G,1,\Gamma} \leq$$
$$\leq \frac{L^{4-\beta/2}}{3}\varepsilon^{27/10} \leq \frac{1}{3}\varepsilon^{27/10-4/30}L^{-\beta/2} \leq \frac{1}{3}L^{-\beta/2}\varepsilon^{5/2+\eta} \tag{3.61}$$

From the second, by lemma 3.3.2, we get

$$\|(\mathcal{S}_2(\hat{K}+gV_*))^\natural\|_{G,1,\Gamma_p} \leq O(1)L^{\beta/2+2D}\|\hat{K}+gV_*\|^2_{G,1,\Gamma} \leq$$
$$\leq \frac{L^{3-\beta/2}}{3}e^{14/4} \leq \frac{1}{3}L^{-\beta/2}\varepsilon^{14/4-1/10} \leq \frac{1}{3}L^{-\beta/2}\varepsilon^{5/2+\eta} \tag{3.62}$$

whilst from the third we obtain

$$\|(\mathcal{S}_2(gV_*,\hat{K}+gV_*))^\natural\|_{G,1,\Gamma_p} \leq O(1)L^{\beta/2+2D}\|\hat{K}+gV_*\|_{G,1,\Gamma}\|\hat{K}\|_{G,1,\Gamma} \leq$$
$$\leq \frac{L^{3-\beta/2}}{3}e^{7/4}e^{9/10} \leq \frac{1}{3}L^{-\beta/2}\varepsilon^{53/20-1/10} \leq \frac{1}{3}L^{-\beta/2}\varepsilon^{5/2+\eta} \tag{3.63}$$

summing (3.61), (3.62) and (3.63) we obtain the proof.

Q.E.D.

§4. *RG step to second order. Relevant and irrelevant terms. Estimates.*

4.1. The starting second order activity

In this section we consider the contribution to the partition functional up to the second order

$$\mathcal{E}xp(\square + \hat{K}_{\leq 2}) \equiv \mathcal{E}xp(\square + \hat{Q}) \tag{4.1}$$

As we showed in (3.19) the second order activity $\hat{Q}$ is given by

$$\hat{Q}(X) = \begin{cases} -gV_*(\Delta) + \frac{g^2}{2}(V_*(\Delta))^2 + \mathcal{I}(\Delta) & \text{for } |X| = 1,\ X \equiv \Delta \\ \frac{g^2}{2} \sum\limits_{\substack{\Delta_1, \Delta_2 \\ \Delta_1 \cup \Delta_2 = X}} V_*(\Delta_1)V_*(\Delta_2) + \mathcal{I}(X) & \text{for } |X| = 2,\ X \text{ connected} \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

It is convenient in the following computations to use the obvious representation

$$V_*(\Delta) = \left(\frac{\lambda_*}{2\pi}\right)^{d/2} \int_\Delta d^D x \int \frac{d^d k}{(2\pi\lambda_*)^{d/2}} e^{-\frac{|k|^2}{2\lambda_*}} e^{ik\cdot\phi(x)} \tag{4.3}$$

The irrelevant second order activity $\mathcal{I}(X) = \mathcal{I}_k(X)$ depends actually on the number k of iterations of RG so far performed. We assume here inductively that $\mathcal{I}_0(X) = 0$ and for $k \geq 1$

$$\mathcal{I}_k(X) = \sum_{l=1}^{k} g_{k-l}^2 \bar{\mathcal{I}}_l(X) \tag{4.4}$$

In the above $g = g_k$ and we assume inductively that $g_j = O(\varepsilon)$ for all $0 \leq j \leq k$.
We will see in a moment that the action of RG will give us an irrelevant second order activity of the form $\mathcal{I}_{k+1}(X)$. The activities $\bar{\mathcal{I}}_l(X)$ are supported on polymer X such that $|X| \leq 2$, and are defined by the following expression:

$$\bar{\mathcal{I}}_l(\Delta) = L^{2l\varepsilon} \frac{1}{2} \left(\frac{\lambda_*}{2\pi}\right)^d \int_0^1 ds \frac{\partial}{\partial s} \int_\Delta d^D x_1 \int \frac{d^d k_1}{(2\pi\lambda_*)^{d/2}} \int_\Delta d^D x_2 \int \frac{d^d k_2}{(2\pi\lambda_*)^{d/2}}$$

$$\int_0^1 dt \frac{\partial}{\partial t} \exp\left[-\frac{1}{2\lambda_*}(k, I_l(s,t)k) + i(k_1 \cdot \phi(x_1) + tk_2 \cdot (\phi(x_2) - \phi(x_1)))\right] \tag{4.5}$$

$$\bar{\mathcal{I}}_l(\Delta\cup\Delta') = L^{2l\varepsilon}\frac{1}{2}\left(\frac{\lambda_*}{2\pi}\right)^d \int_0^1 ds\frac{\partial}{\partial s}\int_\Delta d^Dx_1\int\frac{d^dk_1}{(2\pi\lambda_*)^{d/2}}\int_{\Delta'} d^Dx_2\int\frac{d^dk_2}{(2\pi\lambda_*)^{d/2}}$$

$$\int_0^1 dt\frac{\partial}{\partial t}\exp\left[-\frac{1}{2\lambda_*}(k, I_l(s,t)k) + i(k_1\cdot\phi(x_1) + tk_2\cdot(\phi(x_2)-\phi(x_1)))\right] + (\Delta\rightleftharpoons\Delta') \quad (4.6)$$

where $k\equiv(k_1,k_2)$ and

$$I_l(s,t) = \begin{pmatrix} 1 & -tC_l(s) \\ -tC_l(s) & 2\left[C_l(s) + (1-t^2)D_1\Gamma(x_2-x_1)\right]\end{pmatrix} \tag{4.7}$$

with

$$C_l(s) = \left[\frac{(1-s\bar{\Gamma}(L^{(l-1)}(x_2-x_1)))}{L^{\beta(l-1)}} + \sum_{p=0}^{l-2}\frac{\bar{\Gamma}(0)-\bar{\Gamma}(L^p(x_2-x_1))}{L^{p\beta}}\right]$$

where the sums over p are void if $l=1$, and with

$$D_1\bar{\Gamma}(x_2-x_1) = \sum_{j=1}^{\infty} L^{j\beta}\left[\bar{\Gamma}(0) - \bar{\Gamma}((x_2-x_1)/L^j)\right] \tag{4.8}$$

Note that $D_1\bar{\Gamma}(x)\geq 0$ and that the series converges by virtue of the estimate

$$0 \leq L^{j\beta}(\bar{\Gamma}(0)-\bar{\Gamma}(x/L^j)) \leq O(1)L^{-(j-1)(2-\beta)}\left|x\right|^2$$

We will compute explicitly in the rest of this section the evolution of these terms under RG transformation.

4.2. Reblocking

First we consider the reblocking of $\hat{Q}$ up to the second order: it is easy to see that

$$(\mathcal{B}\hat{Q})_{(\leq 2)}(LZ) = \sum_{\substack{X\\ \bar{X}=LZ}}\hat{Q}(X) + \frac{1}{2}\sum_{\substack{X_1,X_2\ \mathrm{disj}\\ \bar{X}_1,\bar{X}_2\mathrm{overlap\ conn.}\\ \bar{X}_1\cup\bar{X}_2=LZ}}\hat{Q}(X_1)\hat{Q}(X_2) =$$

$$= \sum_{\substack{X\\ \bar{X}=LZ}}\hat{Q}(X) + \frac{1}{2}\sum_{\substack{\Delta_1,\Delta_2\ \mathrm{disj}\\ \bar{\Delta}_1,\bar{\Delta}_2\mathrm{overlap\ conn.}\\ \bar{\Delta}_1\cup\bar{\Delta}_2=LZ}} V(\Delta_1)V(\Delta_2) \tag{4.9}$$

This gives, in the case $|Z|=1$, i.e. $Z\equiv\Delta$,

$$(\mathcal{B}\hat{Q})_{(\leq 2)}(L\Delta) = -V(L\Delta) + \frac{V(L\Delta)^2}{2} + \sum_{\substack{X \\ \bar{X}=L\Delta}} \mathcal{I}_k(X) \tag{4.10}$$

while, for $|Z| = 2$, i.e. $Z \equiv (\Delta \cup \Delta')$,

$$(\mathcal{B}\hat{Q})_{(\leq 2)}(L(\Delta \cup \Delta')) = +\frac{V(L\Delta)V(L\Delta')}{2} + (\Delta \rightleftharpoons \Delta') + \mathcal{I}_k(\Delta_1 \cup \Delta_2) \tag{4.11}$$

where in (4.11), since $D = 1$, $\Delta_1 \subset L\Delta$ and $\Delta_2 \subset L\Delta'$ are uniquely defined by the fact that they have to be overlap connected, and therefore the relation $\Delta_1 \cap \Delta_2 = L\Delta \cap L\Delta'$ has to be fullfilled. No symmetrization is necessary in Δ_1, Δ_2 because $\mathcal{I}_k(\Delta_1 \cup \Delta_2)$ is already symmetrized. Note that, exploiting the compactness of the propagator $\bar{\Gamma}$, the reblocking for the irrelevant activities $\mathcal{I}_k$ can be written in the following form

$$\sum_{\substack{X \\ \bar{X}=L\Delta}} \mathcal{I}_k(X) = \mathcal{B}\mathcal{I}_k(L\Delta) = \sum_{l=1}^{k} g_{k-l}^2 \mathcal{B}\bar{\mathcal{I}}_l(L\Delta) \tag{4.12}$$

$$\mathcal{I}_k(\Delta_1 \cup \Delta_2) = \mathcal{B}\mathcal{I}_k(L(\Delta \cup \Delta')) = \sum_{l=1}^{k} g_{k-l}^2 \mathcal{B}\bar{\mathcal{I}}_l(L(\Delta \cup \Delta')) \tag{4.13}$$

with

$$\mathcal{B}\bar{\mathcal{I}}_l(L\Delta) = L^{2l\varepsilon}\frac{1}{2}\left(\frac{\lambda_*}{2\pi}\right)^d \int_0^1 ds \frac{\partial}{\partial s}\int_{L\Delta} d^D x_1 \int \frac{d^d k_1}{(2\pi\lambda_*)^{d/2}} \int_{L\Delta} d^D x_2 \int \frac{d^d k_2}{(2\pi\lambda_*)^{d/2}}$$
$$\int_0^1 dt \frac{\partial}{\partial t} \exp\left[-\frac{1}{2\lambda_*}(k, I_l(s,t)k) + i(k_1 \cdot \phi(x_1) + t k_2 \cdot (\phi(x_2) - \phi(x_1)))\right] \tag{4.14}$$

$$\mathcal{B}\bar{\mathcal{I}}_l(L(\Delta \cup \Delta'))) = L^{2l\varepsilon}\frac{1}{2}\left(\frac{\lambda_*}{2\pi}\right)^d \int_0^1 ds \frac{\partial}{\partial s}\int_{L\Delta} d^D x_1 \int \frac{d^d k_1}{(2\pi\lambda_*)^{d/2}} \int_{L\Delta'} d^D x_2 \int \frac{d^d k_2}{(2\pi\lambda_*)^{d/2}}$$
$$\int_0^1 dt \frac{\partial}{\partial t} \exp\left[-\frac{1}{2\lambda_*}(k, I_l(s,t)k) + i(k_1 \cdot \phi(x_1) + t k_2 \cdot (\phi(x_2) - \phi(x_1)))\right] + (\Delta \rightleftharpoons \Delta') \tag{4.15}$$

4.3. Rescaling, integration and preliminary extraction

Now we rescale and we integrate the fluctuating field and we obtain immediately (3.42) up to the second order in g:

$$(\mathcal{S}\hat{Q})^{\natural}_{(\leq 2)}(\Delta) = -gL^{\varepsilon}V_* + \frac{g^2}{2}(\mathcal{R}\tilde{Q})^{\natural} + (\mathcal{S}_1\mathcal{I}_k)^{\natural} \tag{4.16}$$

where the V_* is supported only on single blocks and $\tilde{Q}$ is defined by

$$\tilde{Q}(L\Delta) = (V_*(L\Delta))^2$$
$$\tilde{Q}(L\Delta \cup L\Delta') = V_*(L\Delta)V_*(L\Delta') + (\Delta \rightleftharpoons \Delta') \tag{4.17}$$

Following again section 3 we obtain for the contribution up to the second order $\tilde{K}_{(2)}$ of the activity $\tilde{K}$ defined in (3.51) the following expression

$$\tilde{K}_{(2)} = \frac{g^2}{2}\int_0^1 ds \frac{\partial}{\partial s}(\mathcal{R}\tilde{Q})^{s\natural} + (\mathcal{S}_1\mathcal{I})^{\natural} \equiv \tilde{K}_{\tilde{Q}} + (\mathcal{S}_1\mathcal{I}_k)^{\natural} \tag{4.18}$$

where, using (1.57) and the representation (4.3) we have

$$\tilde{K}_{\tilde{Q}}(\Delta) = L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{2\pi}\right)^d \int_0^1 ds \frac{\partial}{\partial s}\int_{\Delta} d^D x_1 \int \frac{d^d k_1}{(2\pi\lambda_*)^{d/2}} \int_{\Delta} d^D x_2 \int \frac{d^d k_2}{(2\pi\lambda_*)^{d/2}}$$

$$\int d\mu_{s\mathcal{R}\Gamma}(\zeta) \exp\left[-\frac{|k_1|^2 + |k_2|^2}{2\lambda_*} + i\left(k_1\cdot(\zeta(x_1)+\phi(x_1)) + k_2\cdot(\zeta(x_2)+\phi(x_2))\right)\right] \tag{4.19}$$

$$\tilde{K}_{\tilde{Q}}(\Delta\cup\Delta') = L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{2\pi}\right)^d \int_0^1 ds \frac{\partial}{\partial s}\int_{\Delta} d^D x_1 \int \frac{d^d k_1}{(2\pi\lambda_*)^{d/2}} \int_{\Delta'} d^D x_2 \int \frac{d^d k_2}{(2\pi\lambda_*)^{d/2}}$$

$$\int d\mu_{s\mathcal{R}\Gamma}(\zeta) \exp\left[-\frac{|k_1|^2 + |k_2|^2}{2\lambda_*} + i\left(k_1\cdot(\zeta(x_1)+\phi(x_1)) + k_2\cdot(\zeta(x_2)+\phi(x_2))\right)\right] +$$

$$+(\Delta \rightleftharpoons \Delta') \tag{4.20}$$

(4.19), (4.20) are obtained writing $\tilde{Q}$ in terms of the representation (4.3) and performing the change of variables $k \to L^{-\beta/2}k$, $x \to Lx$.
The gaussian integral with respect to the measure $\mu_{s\mathcal{R}\Gamma}$ appearing in (4.19), (4.20) is easily done:

$$\tilde{K}_{\tilde{Q}}(\Delta) = L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{2\pi}\right)^d \int_0^1 ds \frac{\partial}{\partial s}\int_{\Delta} d^D x_1 \int \frac{d^d k_1}{(2\pi\lambda_*)^{d/2}} \int_{\Delta} d^D x_2 \int \frac{d^d k_2}{(2\pi\lambda_*)^{d/2}}$$

$$\exp\left[-\frac{1}{2\lambda_*}(k,\sigma_1 k)+i(k_1\cdot\phi(x_1)+k_2\cdot\phi(x_2))\right] \tag{4.21}$$

$$\tilde K_{\tilde Q}(\Delta\cup\Delta')=L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{2\pi}\right)^d\int_0^1 ds\frac{\partial}{\partial s}\int_\Delta d^Dx_1\int\frac{d^dk_1}{(2\pi\lambda_*)^{d/2}}\int_{\Delta'}d^Dx_2\int\frac{d^dk_2}{(2\pi\lambda_*)^{d/2}}$$

$$\exp\left[-\frac{1}{2\lambda_*}(k,\sigma_1 k)+i(k_1\cdot\phi(x_1)+k_2\cdot\phi(x_2))\right]+(\Delta\rightleftharpoons\Delta') \tag{4.22}$$

where the matrix σ_1 is given by

$$\sigma_1=\begin{pmatrix}1 & s\bar\Gamma(x_2-x_1)\\ s\bar\Gamma(x_2-x_1) & 1\end{pmatrix} \tag{4.23}$$

In order to extract the relevant part from (4.21), (4.22) it is useful to perform the following change of variables $k_1\to k_1-k_2$, $k_2\to k_2$ obtaining

$$\tilde K_{\tilde Q}(\Delta)=L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{2\pi}\right)^d\int_0^1 ds\frac{\partial}{\partial s}\int_\Delta d^Dx_1\int\frac{d^dk_1}{(2\pi\lambda_*)^{d/2}}\int_{\Delta}d^Dx_2\int\frac{d^dk_2}{(2\pi\lambda_*)^{d/2}}$$

$$\exp\left[-\frac{1}{2\lambda_*}(k,T\sigma_1 k)+i(k_1\cdot\phi(x_1)+k_2\cdot(\phi(x_2)-\phi(x_1)))\right] \tag{4.24}$$

$$\tilde K_{\tilde Q}(\Delta\cup\Delta')=L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{2\pi}\right)^d\int_0^1 ds\frac{\partial}{\partial s}\int_\Delta d^Dx_1\int\frac{d^dk_1}{(2\pi\lambda_*)^{d/2}}\int_{\Delta'}d^Dx_2\int\frac{d^dk_2}{(2\pi\lambda_*)^{d/2}}$$

$$\exp\left[-\frac{1}{2\lambda_*}(k,T\sigma_1 k)+i(k_1\cdot\phi(x_1)+k_2\cdot(\phi(x_2)-\phi(x_1)))\right]+(\Delta\rightleftharpoons\Delta') \tag{4.25}$$

where the matrix $T\sigma_1$ is given by

$$T\sigma_1=\begin{pmatrix}1 & -(1-s\bar\Gamma(x_2-x_1))\\ -(1-s\bar\Gamma(x_2-x_1)) & 2(1-s\bar\Gamma(x_2-x_1))\end{pmatrix} \tag{4.26}$$

In order to give the explicit expression of $(\mathcal{S}_1\mathcal{I}_k)^\natural$ and to prove the iterative form of $\mathcal{I}_k$ (4.4) it is useful the following lemma

LEMMA 4.3.1

$$(\mathcal{S}_1\bar{\mathcal{I}}_l)^\natural=\bar{\mathcal{I}}_{l+1} \tag{4.27}$$

Proof
Let us write the proof for the single block contribution only. The proof for the couple of adjacent blocks is identical. Integrating and rescaling (4.14) we obtain

$$(\mathcal{S}_1\bar{\mathcal{I}}_l)^\natural(\Delta,\phi) = ((\mathcal{B}_1\bar{\mathcal{I}}_l)^\sharp)(L\Delta,\mathcal{R}\phi) =$$
$$= L^{2l\varepsilon}\frac{1}{2}(\frac{\lambda_*}{2\pi})^d\int_0^1 ds\frac{\partial}{\partial s}\int_{L\Delta} d^Dx_1\int\frac{d^dk_1}{(2\pi\lambda_*)^{d/2}}\int_{L\Delta} d^Dx_2\int\frac{d^dk_2}{(2\pi\lambda_*)^{d/2}}\cdot$$
$$\cdot\int_0^1 dt\frac{\partial}{\partial t}e^{-\frac{1}{2\lambda_*}(k,I_l(s,t)k)}e^{i(k_1\cdot L^{\beta/2}\phi(x_1/L)+tk_2\cdot L^{\beta/2}(\phi(x_2/L)-\phi(x_1/L)))}e^{-\frac{1}{2}(k,\Gamma(t)k)}$$

with the matrix $\Gamma(t)$ given by

$$\Gamma(t)=\begin{pmatrix}\Gamma(0) & -t(\Gamma(0)-\Gamma(x_2-x_1))\\ -t(\Gamma(0)-\Gamma(x_2-x_1)) & 2t^2(\Gamma(0)-\Gamma(x_2-x_1))\end{pmatrix}$$

Adding ${\lambda_*}^{-1}I_l(s,t)$ and $\Gamma(t)$, performing the change of variables $k\to L^{-\beta/2}k$, $x\to Lx$ and some elementary manipulations, we obtain the proof.

Q.E.D.

4.4. Extraction

Now we define the relevant part $F_{\tilde{Q}}(Z,\phi)=\mathcal{L}\tilde{K}_{\tilde{Q}}(Z,\phi)$ in the following way

$$\mathcal{L}\tilde{K}_{\tilde{Q}}(\Delta)=L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{2\pi}\right)^d\int_\Delta d^Dx_1\int\frac{d^dk_1}{(2\pi\lambda_*)^{d/2}}\int_\Delta d^Dx_2\int\frac{d^dk_2}{(2\pi\lambda_*)^{d/2}}$$
$$\int_0^1 ds\frac{\partial}{\partial s}\exp\left[-\frac{1}{2\lambda_*}(k,R_1k)+i(k_1\cdot\phi(x_1))\right] \tag{4.28}$$

$$\mathcal{L}\tilde{K}_{\tilde{Q}}(\Delta\cup\Delta',\phi)=L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{2\pi}\right)^d\int_\Delta d^Dx_1\int\frac{d^dk_1}{(2\pi\lambda_*)^{d/2}}\int_{\Delta'} d^Dx_2\int\frac{d^dk_2}{(2\pi\lambda_*)^{d/2}}$$
$$\int_0^1 ds\frac{\partial}{\partial s}\exp\left[-\frac{1}{2\lambda_*}(k,R_1k)+i(k_1\cdot\phi(x_1))\right]+(\Delta\rightleftharpoons\Delta') \tag{4.29}$$

where the matrix R_1 is given by

$$R_1=\begin{pmatrix}1 & 0\\ 0 & 2[1-s\bar{\Gamma}(x_2-x_1)+D_1\bar{\Gamma}(x_2-x_1)]\end{pmatrix} \tag{4.30}$$

It is convenient to perform the gaussian integral in the k's variables obtaining for (4.28) and (4.29) the following expression

$$F_{\tilde{Q}}(\Delta,\phi) = L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{2\pi}\right)^d \int_\Delta d^D x_1 \int_\Delta d^D x_2 \int_0^1 ds \frac{\partial}{\partial s}$$

$$[2(1 - s\bar{\Gamma}(x_2 - x_1) + D_1\bar{\Gamma}(x_2 - x_1))]^{-d/2} e^{-\frac{\lambda_*}{2}|\phi(x_1)|^2} \tag{4.31}$$

$$F_{\tilde{Q}}(\Delta \cup \Delta',\phi) = L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{2\pi}\right)^d \int_\Delta d^D x_1 \int_{\Delta'} d^D x_2 \int_0^1 ds \frac{\partial}{\partial s}$$

$$[2(1 - s\bar{\Gamma}(x_2 - x_1) + D_1\bar{\Gamma}(x_2 - x_1))]^{-d/2} e^{-\frac{\lambda_*}{2}|\phi(x_1)|^2} + (\Delta \rightleftharpoons \Delta') \tag{4.32}$$

Following [BDH2] , we want to write $F_{\tilde{Q}}(Z,\phi)$ in terms of the sets where the dependence from the field ϕ is localized. In other words, we want to write the decomposition

$$F_{\tilde{Q}}(Z,\phi) = \sum_{\Delta \subset Z} F_{\tilde{Q}}(Z,\Delta,\phi) \tag{4.33}$$

where in $F_{\tilde{Q}}(Z,\Delta,\phi)$ appear only fields defined in Δ. The explicit expression for the relevant contribution $F_{\tilde{Q}}(Z,\Delta,\phi)$ is therefore

$$F_{\tilde{Q}}(Z,\Delta,\phi) = \int_\Delta dx_1 v_*(\phi(x_1)) f_{\tilde{Q}}(Z,\Delta) \tag{4.34}$$

where

$$f_{\tilde{Q}}(\Delta,\Delta) = L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{4\pi}\right)^{d/2} \int_\Delta dx_2 \int_0^1 ds \frac{\partial}{\partial s}[1 - s\bar{\Gamma}(x_2 - x_1) + D_1\Gamma(x_2 - x_1)]^{-d/2} \tag{4.35}$$

$$f_{\tilde{Q}}(\Delta \cup \Delta',\Delta) = L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{4\pi}\right)^{d/2} \int_{\Delta'} dx_2 \int_0^1 ds \frac{\partial}{\partial s}[1 - s\bar{\Gamma}(x_2 - x_1) + D_1\Gamma(x_2 - x_1)]^{-d/2} \tag{4.36}$$

By definition

$$V'_{F_{\tilde{Q}}}(\Delta,\phi) = - \sum_{\substack{Z \supset \Delta \\ |Z| \le 2,\ \mathrm{conn}}} F_{\tilde{Q}}(Z,\Delta,\phi) = -\int_\Delta dx_1 v_*(\phi(x_1)) \sum_{\substack{Z \supset \Delta \\ |Z| \le 2}} f_{\tilde{Q}}(Z,\Delta) =$$

$$= -\int_\Delta dx_1 v_*(\phi(x_1)) f_{\tilde{Q}}(\Lambda,\Delta) \tag{4.37}$$

where

$$f_{\tilde{Q}}(\Lambda,\Delta) = L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{4\pi}\right)^{d/2}\int_\Lambda dx_2 \int_0^1 ds\frac{\partial}{\partial s}[1-s\bar\Gamma(x_2-x_1)+D_1\bar\Gamma(x_2-x_1)]^{-d/2} \quad (4.38)$$

and we have used the fact that $\bar\Gamma(y)$ vanishes for $|y|\geq 1$. Now we can use the translation invariance for $f_{\tilde{Q}}(\Lambda,\Delta,\phi)$ and we obtain

$$V'_{F_{\tilde{Q}}}(\Delta,\phi) = -V_*(\Delta,\phi)L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{4\pi}\right)^{d/2}\int dy\int_0^1 ds\frac{\partial}{\partial s}[1-s\bar\Gamma(y)+D_1\Gamma(y)]^{-d/2} \quad (4.39)$$

and therefore

$$V'(\Delta,\phi) = L^\varepsilon g V_*(\Delta,\phi)+V'_{F_{\tilde{Q}}}(\Delta,\phi) = (L^\varepsilon g - b_1 g^2)V_*(\Delta,\phi) \quad (4.40)$$

with

$$b_1 = L^{2\varepsilon}\frac{1}{2}\left(\frac{\lambda_*}{4\pi}\right)^{d/2}\int dy\int_0^1 ds\frac{\partial}{\partial s}[1-s\bar\Gamma(y)+D_1\bar\Gamma(y)]^{-d/2} \quad (4.41)$$

In order to give a good estimate of b_1 we have to study the behaviour for short distances of the covariance $\bar\Gamma$.

4.5. Asymptotic behaviour of the propagator $\bar\Gamma$

The asymptotic behaviour for small $|y|$ of $\bar\Gamma(y)$ is described by the following lemma

LEMMA 4.5.1

For $1/L\leq y\leq 1/2$ we have

$$O(1)y^\beta \leq |\bar\Gamma(0)-\bar\Gamma(y)| \leq O(1)y^\beta \quad (4.42)$$

Proof

$$\bar\Gamma(0)-\bar\Gamma(y) = -L^{-\beta}\lambda_*\int_1^L\frac{dl}{l}\,l^\beta\int_0^1 dt\frac{yL}{l}u'\left(\frac{tyL}{l}\right) = -y^\beta\lambda_*\int_{1/Ly}^{1/y}\frac{dl}{l}l^{(\beta-1)}\int_0^1 dtu'\left(\frac{t}{l}\right)$$

we have therefore to show that in the region $1/L\leq y\leq 1/2$

$$\left|\int_{1/Ly}^{1/y}\frac{dl}{l}l^{(\beta-1)}\int_0^1 dtu'\left(\frac{t}{l}\right)\right| = O(1)$$

We observe first that

$$\int_{1/Ly}^{1/y} \frac{dl}{l} l^{(\beta-1)} \int_0^1 dtu'\left(\frac{t}{l}\right) =$$

$$= \int_{1/Ly}^{1} \frac{dl}{l} l^{(\beta-1)} \int_0^1 dtu'\left(\frac{t}{l}\right) + \int_1^{1/y} \frac{dl}{l} l^{(\beta-1)} \int_0^1 dtu'\left(\frac{t}{l}\right)$$

To obtain the upper bound we bound

$$\left|\int_0^1 dtu'\left(\frac{t}{l}\right)\right| \leq O(1)l \quad \text{for } l \leq 1$$

$$\left|\int_0^1 dtu'\left(\frac{t}{l}\right)\right| \leq O(1) \quad \text{for } l \geq 1$$

therefore

$$\left|\int_{1/Ly}^{1/y} \frac{dl}{l} l^{(\beta-1)} \int_0^1 dtu'\left(\frac{t}{l}\right)\right| \leq O(1)\left[\int_{1/Ly}^{1} \frac{dl}{l} l^{\beta} + \int_1^{1/y} \frac{dl}{l} l^{(\beta-1)}\right] \leq O(1)$$

The lower bound is obtained simply observing that since $u'(x) < 0$ for all $x > 0$ the integrand has always the same sign, and therefore

$$\left|\int_{1/Ly}^{1/y} \frac{dl}{l} l^{(\beta-1)} \int_0^1 dtu'\left(\frac{t}{l}\right)\right| \geq \int_1^2 \frac{dl}{l} l^{(\beta-1)} \left|\int_{1/2}^1 dtu'\left(\frac{t}{l}\right)\right| = O(1) \qquad Q.E.D.$$

It will be useful in the following to define the quantity

$$\tilde{a}(L,\varepsilon) = \int_0^1 dy(1 - \bar{\Gamma}(y) + D_1\bar{\Gamma}(y))^{-d/2} \tag{4.43}$$

For such quantity we have

LEMMA 4.5.2

$$\tilde{a}(L,\varepsilon) = O(\ln L) \tag{4.44}$$

Proof
First observe that from (4.8) and the remark following it we have

$$0 \leq D_1\bar{\Gamma}(y) \leq O(1)\left|y\right|^2$$

Write

$$\tilde{a}(L,\varepsilon) = \int_0^{1/L} dy(1-\bar{\Gamma}(y)+D_1\bar{\Gamma}(y))^{-d/2} + \int_{1/L}^1 dy(1-\bar{\Gamma}(y)+D_1\bar{\Gamma}(y))^{-d/2} \equiv$$

$$\equiv \tilde{a}_<(L,\varepsilon) + \tilde{a}_>(L,\varepsilon)$$

Now we can estimate $\tilde{a}_<(L,\varepsilon)$ simply observing that $1-\bar{\Gamma}(y) \geq L^{-\beta}$ and $D_1\bar{\Gamma}(y) \geq 0$

$$\tilde{a}_<(L,\varepsilon) \leq L^{\beta d/2}(1/L) = L^{-\varepsilon} \leq 1$$

$\tilde{a}_>(L,\varepsilon)$ is estimated using lemma 4.5.1, the dominant contribution coming from the region $1/L \leq y \leq 1/2$.
Write

$$\tilde{a}_>(L,\varepsilon) = \int_{1/L}^1 dy(L^{-\beta}+\bar{\Gamma}(0)-\bar{\Gamma}(y)+D_1\bar{\Gamma}(y))^{-d/2}$$

Then we have

$$\tilde{a}_>(L,\varepsilon) \leq \int_{1/L}^1 dy(L^{-\beta}+\bar{\Gamma}(0)-\bar{\Gamma}(y))^{-d/2} \leq$$

$$\leq \int_{1/L}^1 dy(L^{-\beta}+O(1)y^{\beta})^{-d/2}$$

where in the last step we have used Lemma 4.5.1
On the other hand,

$$\tilde{a}_>(L,\varepsilon) \geq \int_{1/L}^1 dy(L^{-\beta}+\bar{\Gamma}(0)-\bar{\Gamma}(y)+O(1)\left|y\right|^2)^{-d/2} \geq$$

$$\geq \int_{1/L}^1 dy(L^{-\beta}+O(1)y^{\beta}+O(1)\left|y\right|^2))^{-d/2} \geq$$

$$\geq \int_{1/L}^1 dy(L^{-\beta}+O(1)y^{\beta})^{-d/2}$$

where in the last step we have used $\left|y\right|^2 \leq y^{\beta}$ in the region of integration since β is positive but very small.
Hence for large L,

$$\tilde{a}_>(L,\varepsilon) = O(1)\int_{1/L}^1 \frac{dy}{y^{\beta d/2}} = O(1)\frac{1}{\varepsilon}(1-L^{-\varepsilon}) = O(1)\frac{L^{\varepsilon}-1}{\varepsilon}L^{-\varepsilon} = O(\ln L) \qquad Q.E.D.$$

4.6. Second order coefficient of the relevant part

b_1 is controlled by the following lemma.

LEMMA 4.6.1

$$b_1 = O(\ln L),\ \ b_1 > 0$$

Proof
That $b_1 > 0$ follows from its definition. Note that the function whose s-derivative is to be taken is is independent of s for $y \geq 1$ since in this region $\bar{\Gamma}(y)$ vanishes. This together with the evenness of $\bar{\Gamma}(y)$ implies for ε sufficiently small

$$b_1 = O(1)\int_0^1 dy \int_0^1 ds \frac{\partial}{\partial s}(1 - s\bar{\Gamma}(y) + D_1\bar{\Gamma}(y))^{-d/2} =$$

$$= O(1)\int_0^1 dy(1 - \bar{\Gamma}(y) + D_1\bar{\Gamma}(y))^{-d/2} - O(1)\int_0^1 dy(1 + D_1\bar{\Gamma}(y))^{-d/2}$$

Since $0 \leq D_1\bar{\Gamma}(y) \leq O(1)y$, the second integral is bounded by O(1). The first integral is estimated by Lemma 4.5.2 and we are done.

Q.E.D.

From lemma 4.6.1 and the definition (4.31), (4.32) it is immediate to show that

LEMMA 4.6.2

$$\|F_{\tilde{Q}}\|_{G,1,\Gamma_p} \leq O(1)\varepsilon^{7/4} \qquad \|F_{\tilde{Q}}\|_{\infty,1,\Gamma_p} \leq O(1)\varepsilon^{7/4}$$

for any integer $p \geq 1$, with $O(1)$ depending on p.

4.7. Second order irrelevant part

From the explicit expression of $\tilde{K}_{\tilde{Q}}$ given in (4.24) and (4.25) and of $F_{\tilde{Q}}$ given in (4.31) and (4.32) we obtain

$$(\tilde{K}_{\tilde{Q}} - F_{\tilde{Q}})(\Delta) = L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{2\pi}\right)^d \int_0^1 ds \frac{\partial}{\partial s}\int_\Delta d^D x_1 \int \frac{d^d k_1}{(2\pi\lambda_*)^{d/2}} \int_\Delta d^D x_2 \int \frac{d^d k_2}{(2\pi\lambda_*)^{d/2}}$$

$$\int_0^1 dt \frac{\partial}{\partial t} \exp\left[-\frac{1}{2\lambda_*}(k, I_1(s,t)k) + i(k_1 \cdot \phi(x_1) + tk_2 \cdot (\phi(x_2) - \phi(x_1)))\right] \tag{4.45}$$

$$(\tilde{K}_{\tilde{Q}}-F_{\tilde{Q}})(\Delta\cup\Delta') = L^{2\varepsilon}\frac{g^2}{2}\left(\frac{\lambda_*}{2\pi}\right)^d\int_0^1 ds\frac{\partial}{\partial s}\int_\Delta d^Dx_1\int\frac{d^dk_1}{(2\pi\lambda_*)^{d/2}}\int_{\Delta'}d^Dx_2\int\frac{d^dk_2}{(2\pi\lambda_*)^{d/2}}$$

$$\int_0^1 dt\frac{\partial}{\partial t}\exp\left[-\frac{1}{2\lambda_*}(k,I_1(s,t)k)+i(k_1\cdot\phi(x_1)+tk_2\cdot(\phi(x_2)-\phi(x_1)))\right]+(\Delta\rightleftharpoons\Delta') \tag{4.46}$$

Therefore $(\tilde{K}_{\tilde{Q}}-F_{\tilde{Q}}) = g^2\bar{\mathcal{I}}_1$, and by lemma 4.3.1 the induction on (4.4) is proved. The bounds (3.9) and (3.10) are now easy consequences of the following lemma 4.7.1, lemma 4.3.1 and the smallness of the coupling constants (stated after (4.4)).

LEMMA 4.7.1

For any positive integer $p\geq 1$ and with $O(1)$ p-dependent

$$\|\bar{\mathcal{I}}_l\|_{G,1,\Gamma_p}\leq O(1)L^{3\beta-l\beta/2}L^{2(D+2)} \tag{4.47}$$

Proof

We perform first of all in (4.5), (4.6) the gaussian integral in the k variables and we obtain

$$\bar{\mathcal{I}}_l(\Delta) = L^{2l\varepsilon}\frac{1}{2}\left(\frac{\lambda_*}{2\pi}\right)^d\int_0^1 ds\frac{\partial}{\partial s}\int_0^1 dt\frac{\partial}{\partial t}$$

$$\int_\Delta d^Dx_1\int_\Delta d^Dx_2\exp\left[-\frac{1}{2}\lambda_*(\phi,I_l(s,t)^{-1}\phi)\right](\det I_l(s,t))^{-d/2} \tag{4.48}$$

$$\bar{\mathcal{I}}_l(\Delta\cup\Delta') = L^{2l\varepsilon}\frac{1}{2}\left(\frac{\lambda_*}{2\pi}\right)^d\int_0^1 ds\frac{\partial}{\partial s}\int_0^1 dt\frac{\partial}{\partial t}$$

$$\int_\Delta d^Dx_1\int_{\Delta'} d^Dx_2\exp\left[-\frac{1}{2}\lambda_*(\phi,I_l(s,t)^{-1}\phi)\right](\det I_l(s,t))^{-d/2}+(\Delta\rightleftharpoons\Delta') \tag{4.49}$$

where $\phi = (\phi(x_1),\phi(x_2)-\phi(x_1))$.
Then we observe that the derivative with respect to s produces a factor $\bar{\Gamma}(L^{(l-1)}(x_2-x_1))$, and due to the compact support of the covariance this implies that

$$\left|\bar{\mathcal{I}}_l(X)\right|\leq L^{2l\varepsilon}\frac{1}{2}\left(\frac{\lambda_*}{2\pi}\right)^d\int_\Delta d^Dx_1\int_{\delta_l}d^Dx_2 2\sup_{0\leq s\leq 1}$$

$$\left|\int_0^1 dt\frac{\partial}{\partial t}\left[\exp\left[-\frac{1}{2}\lambda_*(\phi,I_l(s,t)^{-1}\phi)\right](\det I_l(s,t))^{-d/2}\right]\right| \tag{4.50}$$

where $\delta_l\equiv\{x_2:|x_2-x_1|\leq L^{-(l-1)}\}$

Then we consider the derivative

$$\frac{\partial}{\partial t} \exp\left[-\frac{1}{2}\lambda_*(\phi, I_l(s,t)^{-1}\phi)\right] (\det I_l(s,t))^{-d/2} \tag{4.51}$$

We will denote $I_t = \frac{\partial}{\partial t} \det I_l(s,t)$ and we compute explicitely the inverse matrix $I_l(s,t)^{-1}$:

$$I_l(s,t)^{-1} = \frac{1}{\det I_l(s,t)} J_l(s,t) \tag{4.52}$$

with

$$J_l(s,t) = \begin{pmatrix} 2\left[C_l(s) + (1-t^2)D_1\bar{\Gamma}(x_2 - x_1)\right] & tC_l(s) \\ tC_l(s) & 1 \end{pmatrix} \tag{4.53}$$

obtaining for (4.51)

$$\frac{\partial}{\partial t} \exp\left[-\frac{\lambda_*}{2}(\phi, I_l(s,t)^{-1}\phi)\right] (\det I_l(s,t))^{-d/2} =$$

$$= \frac{e^{-\frac{\lambda_*}{2}(\phi, I_l(s,t)^{-1}\phi)}}{(\det I_l(s,t))^{d/2+1}} \left[I_t \left(\frac{\lambda_*}{2} \frac{(\phi, J_l(s,t)\phi)}{\det I_l(s,t)} - (d/2) \right) - \left(\frac{\lambda_*}{2} \frac{\partial}{\partial t}(\phi, J_l(s,t)\phi) \right) \right] \tag{4.54}$$

In order to bound (4.54) we compute first of all explicitely

$$\det I_l(s,t) = 2\left[C_l(s) + (1-t^2)D_1\bar{\Gamma}(x_2 - x_1)\right] - t^2(C_l(s))^2 \tag{4.55}$$

and we list some preliminary useful bounds:

$$\left| \exp\left[-\frac{1}{2}\lambda_*(\phi, I_l(s,t)^{-1}\phi)\right] \right| \le e^{-\frac{1}{2}\lambda_*|\phi(x_1)|^2} \tag{4.56}$$

Proof of (4.56)
From (4.52), (4.53) it is easy to see that

$$I_l(s,t)^{-1} = E_1 + \tilde{I}_l(s,t)^{-1}$$

where

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \tilde{I}_l(s,t)^{-1} = \frac{1}{\det I_l(s,t)} \begin{pmatrix} t^2 C_l(s)^2 & tC_l(s) \\ tC_l(s) & 1 \end{pmatrix}$$

and the proof follows since $\tilde{I}_l(s,t)^{-1}$ is positive definite. In fact the displayed matrix is manifestly positive definite and from (4.59) below $\det I_l(s,t) \ge 0$.

$$C_l(s) \geq \begin{cases} O(1)L^{-\beta l} & \text{for } |x_2 - x_1| < L^{-l} \\ O(1)|x_2 - x_1|^{\beta} & \text{for } L^{-l} \leq |x_2 - x_1| \leq L^{-l+1} \end{cases} \tag{4.57}$$

Proof of (4.57)
The first summand in the definition of $C_l(s)$ after (4.7) can be bounded by

$$\frac{(1 - s\bar{\Gamma}(L^{(l-1)}(x_2 - x_1)))}{L^{\beta(l-1)}} \geq \begin{cases} O(1)L^{-\beta l} & \text{for } |x_2 - x_1| < L^{-l} \\ O(1)|x_2 - x_1|^{\beta} & \text{for } L^{-l} \leq |x_2 - x_1| \leq L^{-l+1} \end{cases}$$

repeating the proof of lemma 4.5.1. The second summand is positive. Thus (4.57) is proved.

$$0 \leq C_l(s) \leq O(1)L^{-\beta(l-1)} \quad \text{for } |x_2 - x_1| \leq L^{-l+1} \tag{4.58}$$

and $O(1) = 1$ for $l = 1$
Proof of (4.58)
We first observe that

$$0 \leq \frac{(1 - s\bar{\Gamma}(L^{(l-1)}(x_2 - x_1)))}{L^{\beta(l-1)}} \leq L^{-\beta(l-1)}$$

which is trivial
Then we observe, by a simple Taylor expansion

$$0 \leq \bar{\Gamma}(0) - \bar{\Gamma}(x) \leq O(1)|x|^2 L^{(2-\beta)}$$

obtaining

$$0 \leq \frac{\bar{\Gamma}(0) - \bar{\Gamma}(L^p(x_2 - x_1))}{L^{p\beta}} \leq O(1)|x_2 - x_1|^2 L^{(2-\beta)(p+1)}$$

and for $|x_2 - x_1| \leq L^{-l+1}$ we obtain

$$0 \leq \frac{\bar{\Gamma}(0) - \bar{\Gamma}(L^p(x_2 - x_1))}{L^{p\beta}} \leq L^{-\beta(l-1)} L^{-(2-\beta)(l-p-2)}$$

This proves the convergence of the second sum in $C_l(s)$, and the fact that the sum is dominated by the therm with $p = l - 2$, giving the proof of (4.58).

$$\det I_l(s,t) \geq \begin{cases} O(1)L^{-\beta l} & \text{for } |x_2 - x_1| < L^{-l} \\ O(1)|x_2 - x_1|^{\beta} & \text{for } L^{-l} \leq |x_2 - x_1| \leq L^{-l+1} \end{cases} \tag{4.59}$$

Proof of (4.59)
We note that

$$D_1\bar{\Gamma}(x_2 - x_1) \geq 0$$

so that

$$\det I_l(s,t) \geq C_l(s)(2 - C_l(s))$$

Now the proof follows from (4.57) and (4.58).

$$|I_t| \leq O(1)L^{-2\beta(l-1)} \quad \text{for } |x_2 - x_1| \leq L^{-l+1} \tag{4.60}$$

Proof of (4.60)
Differentiate (4.55), then use (4.58) and

$$0 \leq D_1\Gamma(x_2 - x_1) \leq O(1)|x_2 - x_1|^2$$

which follows from (4.57) and the remark following it to obtain the proof.

$$|(\phi, J_l(s,t)\phi)| \leq O(1)L^{-\beta(l-1)} e^{(\kappa/2)\|\phi\|^2_{1,\sigma,X}} e^{\frac{1}{2}\lambda_*(\rho/2)|\phi(x_1)|^2} \quad \text{for } |x_2 - x_1| \leq L^{-l+1} \tag{4.61}$$

Proof of (4.61)
From the explicit expression of $J_l(s,t)$ (4.53) it is easy to find the following bound

$$|(\phi, J_l(s,t)\phi)| \leq O(1)|\phi(x_1)|^2(C_l(s) + D_1\bar{\Gamma}) +$$
$$+O(1)\left[|\phi(x_1)||\phi(x_2) - \phi(x_1)|C_l(s) + |\phi(x_2) - \phi(x_1)|^2\right]$$

Then from the mean value theorem and Sobolev embedding we bound

$$|\phi(x_2) - \phi(x_1)| \leq |x_2 - x_1|\|\phi\|_{1,\sigma,X}$$

Finally, since k and ρ are small but $O(1)$, we have

$$\|\phi\|_{1,\sigma,X} \leq O(1)e^{(\kappa/4)\|\phi\|^2_{1,\sigma,X}} \qquad |\phi(x_1)| \leq O(1)e^{\frac{1}{2}\lambda_*(\rho/4)|\phi(x_1)|^2}$$

and we obtain

$$|(\phi, J_l(s,t)\phi)| \leq O(1)((C_l(s) + D_1\bar{\Gamma}) + |x_2 - x_1|)e^{(\kappa/2)\|\phi\|^2_{1,\sigma,X}} e^{\frac{1}{2}\lambda_*(\rho/2)|\phi(x_1)|^2}$$

that gives the proof of (4.61) by (4.58) and the estimate on $D_1\bar{\Gamma}$ given in the proof of (4.60) .

$$\left|\frac{\partial}{\partial t}(\phi, J_l(s,t)\phi)\right| \le$$

$$\le O(1)L^{-(\beta+1)(l-1)}e^{(\kappa/2)\|\phi\|^2_{1,\sigma,X}}e^{\frac{\lambda_*}{2}(\rho/2)|\phi(x_1)|^2} \text{ for } 0\le|x_2-x_1|\le L^{-l+1} \tag{4.62}$$

Proof of (4.62)
We start from the bound

$$\left|\frac{\partial}{\partial t}(\phi, J_l(s,t)\phi)\right| \le O(1)(C_l(s)|\phi(x_1)||\phi(x_2)-\phi(x_1)| + D_1\bar{\Gamma}(x_2-x_1)|\phi(x_1)|^2)$$

then we proceed as in (4.61) and we obtain the proof.
Using the bounds (4.56)-(4.62) it is now an easy task to bound the explicit expression (4.54). We have, defining $g_{\rho,\kappa}(X,\phi)$ by

$$G_{\rho,k}(X,\phi) = \frac{1}{|X|}\int_X dx_1 g_{\rho,k}(x_1,X,\phi)$$

the following bound

$$\left|\frac{\partial}{\partial t}\exp\left[-\frac{1}{2}\lambda_*(\phi, I_l(s,t)^{-1}\phi)\right](\det I_l(s,t))^{-d/2}\right| \le$$

$$\le \begin{cases} O(1)L^{\beta l(d/2+2)}L^{-3\beta(l-1)}g_{\rho,\kappa}(x_1,X,\phi) & \text{for } |x_2-x_1|<L^{-l} \\ O(1)|x_2-x_1|^{-\beta(d/2+2)}L^{-3\beta(l-1)}g_{\rho,\kappa}(x_1,X,\phi) & \text{for } L^{-l}\le|x_2-x_1|\le L^{-l+1} \end{cases} \tag{4.63}$$

Now we come back to (4.50) and obtain

$$|\bar{\mathcal{I}}_l(X)| \le O(1)\left[\int_0^{L^{-l}} dy L^{\beta l(d/2+2)}L^{-3\beta(l-1)}G_{\rho,k}(X,\phi) + \right.$$

$$\left.\int_{L^{-l}}^{L^{-l+1}} dy|y|^{-\beta(d/2+2)}L^{-3\beta(l-1)}G_{\rho,k}(X,\phi)\right] \le$$

$$\le O(1)\left[L^{l(\beta d/2-1)}L^{-\beta(l-3)} + \int_{L^{-l}}^{L^{-l+1}} dy|y|^{-\beta(d/2)}L^{-\beta(l-3)}\right]G_{\rho,k}(X,\phi) \tag{4.64}$$

Observe that $\beta(d/2)=1-\varepsilon$. We obtain for the integral in (4.64), for ε sufficiently small,

$$\int_{L^{-l}}^{L^{-l+1}} dy|y|^{-\beta(d/2)} = \frac{1}{\varepsilon}L^{-l\varepsilon}(L^\varepsilon - 1) = O(\ln L) \tag{4.65}$$

Hence we have

$$\|\bar{\mathcal{I}}_l(X)\|_G \leq O(1) L^{3\beta - l\beta/2} \tag{4.66}$$

Evaluating the functional derivative $D\bar{\mathcal{I}}_l(X)$ we obtain the same bound, so that we have

$$\|\bar{\mathcal{I}}_l(X)\|_{G,1} \leq O(1) L^{3\beta - l\beta/2} \tag{4.67}$$

and we obtain finally the lemma by the smallness of the set X.

Q.E.D.

§5. *RG action on the remainder. Relevant and irrelevant terms. Estimates.*

This section is devoted to the activity $(\mathcal{S}_1\hat{r})^\natural$, which we encountered earlier in (3.51) of section 3. We need to extract relevant terms from the contributions from small sets. We define the relevant and irrelevant terms and give suitable bounds. We also control the remainder contribution to the flow of the effective coupling constant. The contributions from large sets of course need no subtractions since we easily obtain a contractive bound for them.

5.1. Linear reblocking, small set contributions

Let X be a small set. We can write

$$\hat{r}(X,\phi) = \frac{1}{|X|}\int_X dx_1 \hat{r}(X,\phi) \tag{5.1}$$

and define

$$\hat{r}_*(X,\phi(\bar{x}),\phi) = e^{(\lambda_*/2)|\phi(\bar{x})|^2}\hat{r}(X,\phi) \tag{5.2}$$

where $\bar{x}$ is the midpoint of the polymer X. We have therefore

$$\hat{r}(X,\phi) = \frac{1}{|X|}\int_X dx_1 e^{-(\lambda_*/2)|\phi(\bar{x})|^2}\hat{r}_*(X,\phi(\bar{x}),\phi) \tag{5.3}$$

Now we consider the contribution to the linear reblocking of the activity $\hat{r}$ restricted to small sets, denoted by $\hat{r}^{s.s}$. We have

$$\mathcal{B}_1\hat{r}^{(s.s.)}(LZ,\phi) = \sum_{\substack{X \text{ small set} \\ \bar{X}=LZ}} \frac{1}{|X|}\int_X dx_1 e^{-(\lambda_*/2)|\phi(\bar{x})|^2}\hat{r}_*(X,\phi(\bar{x}),\phi) \tag{5.4}$$

and after rescaling and convolution integration, using the master formula in Lemma 1.3.1,

$$\left(\mu_{\bar{\Gamma}} * (\mathcal{S}_1\hat{r}^{(s.s.)})\right)(Z,\phi) = \sum_{\substack{X \text{ small set} \\ \bar{X}=LZ}} L^{-\alpha}\frac{1}{|X|}\int_X dx_1 e^{-(\lambda_*/2)L^{-\beta}|\mathcal{R}\phi(\bar{x})|^2}$$

$$\left(\mu_{\Sigma^{\bar{x}}} * \hat{r}_*\right)\left(X, L^{-\beta}\mathcal{R}\phi(\bar{x}), L^{-\beta}T^{\bar{x}}\mathcal{R}\phi\right) \tag{5.5}$$

we will write (5.5) in compact notation

$$\left(\mathcal{S}_1\hat{r}^{(s.s.)}\right)^\natural(Z,\phi) = \sum_{\substack{X \text{ s. s.} \\ \bar{X}=LZ}} (\mathcal{R}\hat{r})^\natural(L^{-1}X,\phi) =$$

$$= \sum_{\substack{X \text{ s. s.} \\ \bar{X}=LZ}} \frac{L^{-\alpha}}{|X|} \int_X dx_1 e^{-(\lambda_*/2)L^{-\beta}|\mathcal{R}\phi(\bar{x})|^2} \hat{r}_*^{\#\Sigma_{\bar{x}}}(X, L^{-\beta}\mathcal{R}\phi(\bar{x}), L^{-\beta}T^{\bar{x}}\mathcal{R}\phi) \tag{5.6}$$

Then we define the relevant part in the following way

$$\mathcal{L}\left(\mathcal{S}^{(1)}\hat{r}^{(s.s.)}\right)^{\natural}(Z,\phi) = \sum_{\substack{X \text{ s. s.} \\ \bar{X}=LZ}} \mathcal{L}(\mathcal{R}\hat{r})^{\natural}(L^{-1}X,\phi) =$$

$$= \sum_{\substack{X \text{ s. s.} \\ \bar{X}=LZ}} \frac{L^{-\alpha}}{|X|} \int_X dx_1 e^{-(\lambda_*/2)L^{-\beta}|\mathcal{R}\phi(x_1)|^2} \hat{r}_*^{\#\Sigma_{\bar{x}}}(X,0,0) \tag{5.7}$$

so that the irrelevant term is

$$(1-\mathcal{L})\left(\mathcal{S}^{(1)}\hat{r}^{(s.s.)}\right)^{\natural}(Z,\phi) = \sum_{\substack{X \text{ s. s.} \\ \bar{X}=LZ}} (1-\mathcal{L})(\mathcal{R}\hat{r})^{\natural}(L^{-1}X,\phi) \tag{5.8}$$

with

$$(1-\mathcal{L})(\mathcal{R}\hat{r})^{\natural}(L^{-1}X,\phi) =$$

$$= \frac{L^{-\alpha}}{|X|} \int_X dx_1 e^{-(\lambda_*/2)L^{-\beta}|\mathcal{R}\phi(x_1)|^2} \Big[-\hat{r}_*^{\#\Sigma_{\bar{x}}}(X,0,0)+$$

$$+e^{-(\lambda_*/2)L^{-\beta}\left(|\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\right)} \hat{r}_*^{\#\Sigma_{\bar{x}}}(X, L^{-\beta}\mathcal{R}\phi(\bar{x}), L^{-\beta}T^{\bar{x}}\mathcal{R}\phi)\Big] =$$

$$= \frac{L^{-\alpha}}{|X|} \int_X dx_1 e^{-(\lambda_*/2)L^{-\beta}|\mathcal{R}\phi(x_1)|^2} \int_0^1 dt \frac{\partial}{\partial t}$$

$$\left[e^{-t^2(\lambda_*/2)L^{-\beta}\left(|\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\right)} \hat{r}_*^{\#\Sigma_{\bar{x}}}(X, tL^{-\beta}\mathcal{R}\phi(\bar{x}), tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)\right] \tag{5.9}$$

Then we perform the derivative with respect to t obtaining

$$(1-\mathcal{L})(\mathcal{R}\hat{r})^{\natural}(L^{-1}X,\phi) = \frac{L^{-\alpha}}{|X|} \int_X dx_1 e^{-\frac{\lambda_*}{2}L^{-\beta}|\mathcal{R}\phi(x_1)|^2} \int_0^1 dt$$

$$\left[\frac{\partial}{\partial t}\left(e^{-\frac{\lambda_*}{2}t^2L^{-\beta}\left(|\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\right)}\right) \hat{r}_*^{\#\Sigma_{\bar{x}}}(X, tL^{-\beta}\mathcal{R}\phi(\bar{x}), tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)+\right.$$

$$\left.+\left(e^{-\frac{\lambda_*}{2}t^2L^{-\beta}\left(|\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\right)}\right) \frac{\partial}{\partial t}\hat{r}_*^{\#\Sigma_{\bar{x}}}(X, tL^{-\beta}\mathcal{R}\phi(\bar{x}), tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)\right] =$$

$$= \frac{L^{-\alpha}}{|X|}\int_X dx_1 e^{-(\lambda_*/2)L^{-\beta}|\mathcal{R}\phi(x_1)|^2}\int_0^1 dt$$

$$\left[\frac{\partial}{\partial t}\left(e^{-\frac{\lambda_*}{2}t^2L^{-\beta}\left(|\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\right)}\right)\hat{r}_*^{\#\Sigma_{\bar{x}}}(X,tL^{-\beta}\mathcal{R}\phi(\bar{x}),tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)+\right.$$

$$+\left(e^{-\frac{\lambda_*}{2}t^2L^{-\beta}\left(|\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\right)}\right)\left[(D\hat{r}_*^{\#\Sigma_{\bar{x}}})(X,tL^{-\beta}\mathcal{R}\phi(\bar{x}),tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi;(L^{-\beta}\mathcal{R}\phi(\bar{x}),0))+\right.$$

$$\left.\left.+(D\hat{r}_*^{\#\Sigma_{\bar{x}}})(X,tL^{-\beta}\mathcal{R}\phi(\bar{x}),tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi;(0,L^{-\beta}T^{\bar{x}}\mathcal{R}\phi))\right]\right] \tag{5.10}$$

where the derivatives with respect to the field $\phi(\bar{x})$ are ordinary partial derivatives, and we will denote them hereafter with ∂, while variational derivatives with respect to the field ϕ have norms computed in $C^1(X)$ topology. We denote with $D_{(2)}K$ the first variational derivative in the direction of ϕ in the activity $K(X,\phi(\bar{x}),\phi)$. We can write the following bound for the activity $(1-\mathcal{L})(\mathcal{R}\hat{r})^\natural(L^{-1}X,\phi)$:

$$\left|(1-\mathcal{L})(\mathcal{R}\hat{r})^\natural(L^{-1}X,\phi)\right| \le \frac{L^{-\alpha}}{|X|}\int_X dx_1 e^{-(\lambda_*/2)L^{-\beta}|\mathcal{R}\phi(x_1)|^2}$$

$$\int_0^1 dt\left[L^{-1/2}e^{(\lambda_*/2)L^{1/2-\beta}\left||\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\right|}\left|(\hat{r}_*^{\#\Sigma_{\bar{x}}})(X,tL^{-\beta}\mathcal{R}\phi(\bar{x}),tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)\right|+\right.$$

$$+\left(e^{-\frac{\lambda_*}{2}t^2L^{-\beta}\left(|\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\right)}\right)\left[|L^{-\beta}\mathcal{R}\phi(\bar{x})|\left|(\partial\hat{r}_{*R}^{\#\Sigma_{\bar{x}}})(X,tL^{-\beta}\mathcal{R}\phi(\bar{x}),tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)\right|+\right.$$

$$\left.\left.+\left\|L^{-\beta}T^{\bar{x}}\mathcal{R}\phi\right\|_{C^1(X)}\left\|(D_{(2)}\hat{r}_*^{\#\Sigma_{\bar{x}}})(X,tL^{-\beta}\mathcal{R}\phi(\bar{x}),tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)\right\|\right]\right] \tag{5.11}$$

vwhere in the second line we used the trivial inequality

$$|xe^x| \le \frac{1}{\alpha}\, e^{\alpha|x|} \quad \forall\alpha > 2$$

We can give a suitable estimate of (5.11) by means of the following lemmas

LEMMA 5.1.1

$$\left||\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\right| \le O(1)L^{-(1-\beta)}\left(\|\phi\|^2_{L^{-1}X,1,\sigma}+L^{-\beta}|\mathcal{R}\phi(x_1)|^2\right)$$

Proof

$$\left||\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\right| \le ||\mathcal{R}\phi(\bar{x})|-|\mathcal{R}\phi(x_1)||\,||\mathcal{R}\phi(\bar{x})|+|\mathcal{R}\phi(x_1)||$$

Using the mean value theorem, a Sobolev inequality and the assumption that X is a small set

$$\big||\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\big|\le\Big(|\bar{x}-x_1|L^{-(1-\beta/2)}\sup_{x\in L^{-1}X}\nabla\phi(x)\Big)\big(2|\mathcal{R}\phi(x_1)|+2\|\phi\|_{L^{-1}X,1,\sigma}\big)\le$$

$$\le|X|L^{-(1-\beta)}\|\phi\|_{L^{-1}X,1,\sigma}\left(2L^{-\beta/2}|\mathcal{R}\phi(x_1)|+2\|\phi\|_{L^{-1}X,1,\sigma}\right)$$

and the lemma easily follows.

LEMMA 5.1.2

$$\left\|L^{-\beta}T^{\bar{x}}\mathcal{R}\phi\right\|_{C^1(X)}\le$$

$$\le O(1)L^{-\beta/2}\frac{1}{\sqrt{1-t^2}}\left(\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{\rho}}\right)e^{(\lambda_*/2)(\rho/4)(1-t^2)L^{-\beta}|\mathcal{R}\phi(x_1)|^2}e^{(\kappa/4)(1-t^2)\|\phi\|^2_{L^{-1}X,1,\sigma}} \tag{5.12}$$

Proof
Recall that

$$T^{\bar{x}}\mathcal{R}\phi(x)=L^{\beta}\mathcal{R}\phi(x)-\lambda_*\Gamma(x-\bar{x})\mathcal{R}\phi(\bar{x})$$

By trivial algebraic manipulation, and the property $L^\beta-1=\lambda_*\Gamma(0)$ we obtain

$$L^{-\beta}T^{\bar{x}}\mathcal{R}\phi(x)=[\mathcal{R}\phi(x)-\mathcal{R}\phi(\bar{x})]+L^{-\beta}(1+\lambda_*(\Gamma(0)-\Gamma(x-\bar{x})))\mathcal{R}\phi(\bar{x})$$

then we observe that because of Lemma 5.1.1

$$|\Gamma(0)-\Gamma(x-\bar{x})|\le|x-\bar{x}|\sup_x|\nabla\Gamma(x)|\le O(1)|X|$$

$$|\mathcal{R}\phi(x)-\mathcal{R}\phi(\bar{x})|\le L^{-(1-\beta/2)}|x-\bar{x}|\sup_x|\nabla\phi(x/L)|\le|X|L^{-(1-\beta/2)}\sup_{x\in L^{-1}X}|\nabla\phi(x)|$$

and we have therefore

$$\sup_{x\in X}|L^{-\beta}T^{\bar{x}}\mathcal{R}\phi(x)|\le|X|[L^{-(1-\beta/2)}\sup_{x\in L^{-1}X}|\nabla\phi(x)|+O(1)L^{-\beta}|\mathcal{R}\phi(\bar{x})|]$$

Analogously, for the first derivative

$$L^{-\beta}\nabla_x T^{\bar{x}}\mathcal{R}\phi(x) = L^{-(1-\beta/2)}\nabla\phi(x/L) - \lambda_*\nabla\Gamma(x-\bar{x})\mathcal{R}\phi(\bar{x})$$

and therefore

$$\sup_{x\in X}|L^{-\beta}\nabla_x T^{\bar{x}}\mathcal{R}\phi(x)| \le L^{-(1-\beta/2)}\sup_{x\in L^{-1}X}|\nabla\phi(x)| + O(1)L^{-\beta}|\mathcal{R}\phi(\bar{x})|$$

Combining the two relations above we obtain

$$\left\|L^{-\beta}T^{\bar{x}}\mathcal{R}\phi\right\|_{C^1(X)} \le |X|[L^{-(1-\beta/2)}\sup_{x\in L^{-1}X}|\nabla\phi(x)| + O(1)L^{-\beta}|\mathcal{R}\phi(\bar{x})|]$$

Moreover,

$$|\mathcal{R}\phi(\bar{x})| \le |\mathcal{R}\phi(x_1)| + |X|[L^{-(1-\beta/2)}\sup_{x\in L^{-1}X}|\nabla\phi(x)$$

Now we exploit the estimates

$$\sup_{x\in L^{-1}X}|\nabla\phi(x)| \le \frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{k}}e^{(\kappa/4)(1-t^2)\|\phi\|^2_{L^{-1}X,1,\sigma}}$$

and

$$L^{-\beta/2}|\mathcal{R}\phi(x_1)| \le O(1)\frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{\rho}}e^{(\lambda_*/2)(\rho/4)(1-t^2)L^{-\beta}|\mathcal{R}\phi(x_1)|^2}$$

and X is a small set
The lemma has been proved.

Q.E.D

LEMMA 5.1.3

$$|L^{-\beta}\mathcal{R}\phi(\bar{x})| \le O(1)\frac{1}{\sqrt{\rho}}\frac{1}{\sqrt{k}}\frac{L^{-\beta/2}}{\sqrt{1-t^2}}e^{(\lambda_*/2)(\rho/4)(1-t^2)L^{-\beta}|\mathcal{R}\phi(x_1)|^2}e^{(\kappa/4)(1-t^2)\|\phi\|^2_{L^{-1}X,1,\sigma}} \tag{5.13}$$

The proof follows the above lines.
ρ, k are chosen sufficiently small but are of O(1) in L.
Then we obtain from (5.11) using the above lemmas,

$$\left|(1-\mathcal{L})(\mathcal{R}\hat{r})^\natural(L^{-1}X,\phi)\right| \le O(1)L^{-\beta/2}\frac{L^{-\alpha}}{|X|}\int_X dx_1 e^{-\frac{\lambda_*}{2}L^{-\beta}(1-\frac{\rho}{4})|\mathcal{R}\phi(x_1)|^2}e^{\frac{k}{4}\|\phi\|^2_{L^{-1}X,1,\sigma}}$$

$$\int_0^1 dt \frac{1}{\sqrt{1-t^2}} e^{(\lambda_*/2)L^{-\beta}(1-t^2)(\rho/4)|\mathcal{R}\phi(x_1)|^2} e^{(\kappa/4)(1-t^2)\|\phi\|^2_{L^{-1}X,1,\sigma}}$$

$$\Big[\big|(\hat{r}_*^{\#\Sigma_{\bar{x}}})(X, tL^{-\beta}\mathcal{R}\phi(\bar{x}), tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)\big| +$$

$$+\Big|(\partial\hat{r}_{*R}^{\#\Sigma_{\bar{x}}})(X, tL^{-\beta}\mathcal{R}\phi(\bar{x}), tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)\Big| +$$

$$+\big\|(D_{(2)}\hat{r}_*^{\#\Sigma_{\bar{x}}})(X, tL^{-\beta}\mathcal{R}\phi(\bar{x}), tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)\big\|\Big] \tag{5.14}$$

In order to bound the activities in (5.14) in terms of the norms introduced in section 2 we introduce the following intermediate regulator

$$G_{*\rho}(X, \phi(\bar{x}), \phi) = e^{(\lambda_*/2)(1+\rho)|\phi(\bar{x})|^2} G(X, \phi) \tag{5.15}$$

where $G(X, \phi)$ is defined in (2.11). For $G_{*\rho}(X, \phi(\bar{x}), \phi)$ the following lemma holds

LEMMA 5.1.4

Let β be small enough but $O(1)$ independent of L. Let X be a small set. Then

$$G_{*\rho}(X, \phi(\bar{x}), \phi) \leq e^{(\lambda_*/2)4\rho|\phi(\bar{x})|^2} e^{2\kappa\|\phi\|^2_{X,1,\sigma}} \tag{5.16}$$

Proof

We plug in the definition of $G(X, \phi)$ and observe that

$$e^{-(\lambda_*/2)(1-\rho)|\phi(x)|^2} e^{(\lambda_*/2)(1+\rho)|\phi(\bar{x})|^2} = e^{-(\lambda_*/2)(|\phi(x)|^2-|\phi(\bar{x})|^2)} e^{(\lambda_*/2)\rho(|\phi(x)|^2+|\phi(\bar{x})|^2)} \leq$$

$$\leq e^{(\lambda_*/2)|\phi(x)-\phi(\bar{x})||\phi(x)+\phi(\bar{x})|} e^{(\lambda_*/2)\rho(|\phi(x)|^2+|\phi(\bar{x})|^2)}$$

Recall that x and $\bar{x}$ belong to the small set X. Then by using the Sobolev inequality we have

$$|\phi(x) - \phi(\bar{x})| \leq 2\|\phi\|_{X,1,\sigma}$$

$$|\phi(x) + \phi(\bar{x})| \leq 2|\phi(\bar{x})| + 2\|\phi\|_{X,1,\sigma}$$

$$|\phi(x)|^2 \leq 2|\phi(\bar{x})|^2 + 4\|\phi\|^2_{X,1,\sigma}$$

Then, using elementary inequalities we get

$$G_{*\rho}(X,\phi(\bar{x}),\phi) \le e^{(\lambda_*/2)4\rho|\phi(\bar{x})|^2} e^{(\kappa+\lambda_*(1/\rho+2+2\rho))\|\phi\|^2_{X,1,\sigma}}$$

and lemma 5.1.4 follows choosing β small enough to give $\lambda_* \le k/(1/\rho+2+2\rho)$

Q.E.D.

We will also need :

LEMMA 5.1.5

$$\|\hat{r}_*(X)\|_{G_{*\rho}} \le \|\hat{r}(X)\|_G \tag{5.17}$$

$$\|(D_{(2)}\hat{r}_*)(X)\|_{G_{*\rho}} \le \|(D\hat{r})(X)\|_G \tag{5.18}$$

$$\|(\partial\hat{r}_*)(X)\|_{G_{*\rho}} \le \frac{O(1)}{\sqrt{\rho}}\|\hat{r}(X)\|_G \tag{5.19}$$

Proof

It follows immediately from the definition of $G_{*\rho}$ and of $\hat{r}_*$

Q.E.D.

We observe now that the derivatives and the fluctuation integral commute, and denoting

$$J = \begin{cases} \hat{r}_* \\ \partial\hat{r}_* \\ D_{(2)}\hat{r}_* \end{cases}$$

we have

$$|(J^{\#\Sigma^{\bar{x}}})(X, tL^{-\beta}\mathcal{R}\phi(\bar{x}), tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)| \le$$

$$\le \int d\mu_{\Sigma^{\bar{x}}}(\zeta) G_{*\rho}(X, \zeta(\bar{x}) + tL^{-\beta}\mathcal{R}\phi(\bar{x}), \zeta + tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)\|J(X)\|_{G_{*\rho}} \le$$

$$\le \int d\mu_{\Sigma^{\bar{x}}}(\zeta) e^{(\lambda_*/2)4\rho|\zeta(\bar{x})+tL^{-\beta}\mathcal{R}\phi(\bar{x})|^2} e^{4\kappa\|\zeta+tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi\|^2_{X,1,\sigma}}\|J(X)\|_{G_{*\rho}} \tag{5.20}$$

In passing to the last line we have used Lemma 5.1.4
Now using the stability of the large fields regulator in the form (2.15) we have

$$|(J^{\#\Sigma^{\bar{x}}})(X, tL^{-\beta}\mathcal{R}\phi(\bar{x}), tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)| \leq$$

$$\leq e^{(\lambda_*/2)(\rho/8)t^2L^{-\beta}|\mathcal{R}\phi(\bar{x})|^2} e^{(\kappa/8)t^2\|\phi\|^2_{L^{-1}X,1,\sigma}} \|J(X)\|_{G_{*\rho}} \tag{5.21}$$

Using again lemma 5.1.1 we obtain

$$|(J^{\#\Sigma^{\bar{x}}})(X, tL^{-\beta}\mathcal{R}\phi(\bar{x}), tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)| \leq$$

$$\leq e^{(\lambda_*/2)(\rho/4)t^2L^{-\beta}|\mathcal{R}\phi(x_1)|^2} e^{(\kappa/4)t^2\|\phi\|^2_{L^{-1}X,1,\sigma}} \|J(X)\|_{G_{*\rho}} \tag{5.22}$$

Finally returning to (5.14) and using (5.22) together with the fact that $\int_0^1 dt(1-t^2)^{-1/2} = O(1)$ we obtain

$$\left|(1-\mathcal{L})(\mathcal{R}\hat{r})^\natural(L^{-1}X, \phi)\right| \leq$$

$$\leq O(1)L^{-\beta/2}\frac{L^{-\alpha}}{|X|}\int_X dx_1 e^{-(\lambda_*/2)(1-\rho/2)L^{-\beta}|\mathcal{R}\phi(x_1)|^2} e^{(\kappa/2)\|\phi\|^2_{L^{-1}X,1,\sigma}}$$

$$\left[\|\hat{r}_*(X)\|_{G_{*\rho}} + \|(\partial\hat{r}_*)(X)\|_{G_{*\rho}} + \|(D_{(2)}\hat{r}_*)(X)\|_{G_{*\rho}}\right] \tag{5.23}$$

and then from (5.17)-(5.19), performing the rescaling, using $L^{-\alpha} = O(1)L^{-D}$ for ε small enough, and Lemma 5.1.5 we get

$$\left|(1-\mathcal{L})(\mathcal{R}\hat{r})^\natural(L^{-1}X, \phi)\right| \leq O(1)L^{-\beta/2}L^{-D}G(L^{-1}X, \phi)\|\hat{r}(X)\|_{G,1} \tag{5.24}$$

Now we want to obtain an analogous estimate for the derivative $(D(1-\mathcal{L})(\mathcal{R}\hat{r})^\natural)(L^{-1}X, \phi)$ By definition we have

$$(D(1-\mathcal{L})(\mathcal{R}\hat{r})^\natural)(L^{-1}X, \phi; f) = \left.\frac{d}{ds}\right|_{s=0} (1-\mathcal{L})(\mathcal{R}\hat{r})^\natural(L^{-1}X, \phi + sf) =$$

$$= \frac{L^{-\alpha}}{|X|}\int_X dx_1 e^{-(\lambda_*/2)L^{-\beta}|\mathcal{R}\phi(x_1)|^2}\left\{-\frac{\lambda_*}{2}f(x_1/L)(L^{-\beta/2}\mathcal{R}\phi)(x_1)\right.$$

$$\int_0^1 dt\frac{\partial}{\partial t}\left[\left(e^{-\frac{\lambda_*}{2}t^2L^{-\beta}\left(|\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\right)}\right)\hat{r}_*^{\#\Sigma_{\bar{x}}}(X, tL^{-\beta}\mathcal{R}\phi(\bar{x}), tL^{-\beta}T^{\bar{x}}\mathcal{R}\phi)\right] +$$

$$+\lambda_*(f(\frac{\bar{x}}{L})\phi(\frac{\bar{x}}{L}) - f(\frac{x_1}{L})\phi(\frac{x_1}{L}))e^{-\frac{\lambda_*}{2}L^{-\beta}\left(|\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\right)}\hat{r}_*^{\#\Sigma_{\bar{x}}}(X, L^{-\beta}\mathcal{R}\phi(\bar{x}), L^{-\beta}T^{\bar{x}}\mathcal{R}\phi) +$$

$$+e^{-\frac{\lambda_*}{2}L^{-\beta}\left(|\mathcal{R}\phi(\bar{x})|^2-|\mathcal{R}\phi(x_1)|^2\right)}\left[(D_{(2)}\hat{r}_*^{\#\Sigma})(X,L^{-\beta}\mathcal{R}\phi(\bar{x}),L^{-\beta}T^{\bar{x}}\mathcal{R}\phi;(0,L^{-\beta}T^{\bar{x}}\mathcal{R}\phi))+\right.$$

$$\left.+L^{-\beta}(\mathcal{R}f(\bar{x}))(\partial\hat{r}_*^{\#\Sigma^{\bar{x}}})(X,L^{-\beta}\mathcal{R}\phi(\bar{x}),L^{-\beta}T^{\bar{x}}\mathcal{R}\phi;(L^{-\beta}\mathcal{R}\phi(\bar{x}),0))\right]\Bigg\} \tag{5.25}$$

We note that

$$|f(\frac{\bar{x}}{L})\phi(\frac{\bar{x}}{L})-f(\frac{x_1}{L})\phi(\frac{x_1}{L})|\leq|(f(\frac{\bar{x}}{L})-f(\frac{x_1}{L}))\phi(\frac{x_1}{L})|+|f(\frac{\bar{x}}{L})(\phi(\frac{\bar{x}}{L})-\phi(\frac{x_1}{L}))|\leq$$

$$\leq\frac{O(1)}{L}\|f\|_{C^1(L^{-1}X)}(|\phi(\frac{x_1}{L})|+\|\phi\|_{L^{-1}X,1,\sigma}) \tag{5.26}$$

and

$$\|L^{-\beta}T^{x_1}\mathcal{R}f\|_{C^1(X)}\leq O(1)L^{-\beta/2}\|f\|_{C^1(L^{-1}X)}$$

which is proved as (5.12).
Now we can proceed to the estimate of (5.25) along the same lines as the proof of (5.24). Recalling the definition of the norm of functional derivatives given in section 2, we obtain

$$\left\|(D(1-\mathcal{L})(\mathcal{R}\hat{r})^{\natural})(L^{-1}X,\phi)\right\|\leq O(1)L^{-\beta/2}L^{-D}G(L^{-1}X,\phi)\left[\|\hat{r}(X)\|_G+\|(D\hat{r})(X)\|_G\right] \tag{5.27}$$

The results (5.24) and (5.27) can be written in terms of the norms introduced in section 2 in the following way

$$\left\|(1-\mathcal{L})(\mathcal{R}\hat{r})^{\natural}(L^{-1}X,\phi)\right\|_1\leq O(1)L^{-\beta/2}L^{-D}G(L^{-1}X,\phi)\|\hat{r}(X)\|_{G,1} \tag{5.28}$$

Now we go back to (5.8). Using (5.28) we obtain

$$\|(1-\mathcal{L})\left(\mathcal{S}_1\hat{r}^{(s.s.)}\right)^{\natural}(Z,\phi)\|_1\leq O(1)L^{-\beta/2}L^{-D}\sum_{\substack{X\ \text{s. s.}\\ \bar{X}=LZ}}G(L^{-1}X,\phi)\|\hat{r}(X)\|_{G,1} \tag{5.29}$$

This implies, exploiting Lemma 2.1.1

$$\|(1-\mathcal{L})\left(\mathcal{S}_1\hat{r}^{(s.s.)}\right)^{\natural}(Z,\phi)\|_1\Gamma_p(Z)\leq$$

$$\leq O(1)L^{-\beta/2}L^{-D}\sum_{\substack{X\ \text{s. s.}\\ \bar{X}=LZ}}G(L^{-1}X,\phi)\|\hat{r}(X)\|_{G,1}\Gamma(X)\leq$$

$$\leq O(1)L^{-\beta/2}L^{-D}\|\hat r\|_{G,1,\Gamma}\sum_{\substack{X \text{ s. s.}\\ \bar X=LZ}} G(L^{-1}X,\phi) \tag{5.30}$$

Now we use lemma 2.3.3 to get

$$\|(1-\mathcal{L})\left(\mathcal{S}_1\hat r^{(s.s.)}\right)^\natural(Z,\phi)\|_1\Gamma_p(Z)\leq O(1)L^{-\beta/2}\|\hat r\|_{G,1,\Gamma}G(Z,\phi) \tag{5.31}$$

Observing now that

$$\sum_{\substack{Z \text{ s. s.}\\ Z\supset\Delta}} 1=O(1)$$

we have proved the following proposition

PROPOSITION 5.1.6

The contribution $(1-\mathcal{L})\left(\mathcal{S}_1\hat r^{(s.s.)}\right)^\natural$ due to small set activities and linear reblocking satisfies the following bound for any $p\geq 0$,

$$\|(1-\mathcal{L})\left(\mathcal{S}_1\hat r^{(s.s.)}\right)^\natural\|_{G,1,\Gamma_p}\leq O(1)L^{-\beta/2}\|\hat r\|_{G,1,\Gamma} \tag{5.32}$$

5.2. Linear reblocking, large set contributions

Now we consider the contribution $\left(\mathcal{S}_1\hat r^{(l.s.)}\right)^\natural$ due to large sets in linear reblocking. After reblocking and rescaling we obtain for such terms

$$\mathcal{S}_1\hat r^{(l.s.)}(Z,\phi)=\sum_{\substack{X \text{ large set}\\ \bar X=LZ}}(\mathcal{R}K)(L^{-1}X,\phi) \tag{5.33}$$

For such contributions we have the following lemma :

LEMMA 5.2.1

$$\|\left(\mathcal{S}_1\hat r^{(l.s.)}\right)^\natural\|_{G,1,\Gamma}\leq O(1)L^{-(1-\beta/2)}\|\hat r\|_{G,1,\Gamma} \tag{5.34}$$

Proof
This is nothing but Lemma 2.5.3 of section 2.

5.3. Bound for the relevant part.

We prove some preliminary lemmas on the bounds of the relevant part. The relevant terms $F_{\hat r}(Z,\phi)$ are defined by (see (5.7))

$$F_{\hat{r}}(Z,\phi) = \sum_{\substack{X \text{ s. s.} \\ \bar{X}=LZ}} \mathcal{L}(\mathcal{R}\hat{r})^{\natural}(L^{-1}X,\phi) \tag{5.35}$$

Since $|\bar{X}| \le |X|$, Z is a small set.
Then we have

LEMMA 5.3.1 For any integer $p \ge 0$

$$\|F_{\hat{r}}\|_{G,1,\Gamma_p} \le O(1)\|\hat{r}\|_{G,1,\Gamma} \tag{5.36}$$

where O(1) depends on p.

Proof
X is a small set. From (5.7) we have

$$\mathcal{L}(\mathcal{R}\hat{r})^{\natural}(L^{-1}X,\phi) = \frac{L^{-\alpha}}{|L^{-1}X|}\int_{L^{-1}X} dx_1 e^{-(\lambda_*/2)|\phi(x_1)|^2}\hat{r}_*^{\#\Sigma^{\bar{x}}}(X,0,0) \tag{5.37}$$

and it is easy to see, using (5.21) as well as (from Lemma 5.1.5)

$$\|(\hat{r}_*)(X)\|_{G_{*\rho}} \le O(1)\|\hat{r}(X)\|_G \tag{5.38}$$

that the following inequality holds

$$\|\mathcal{L}(\mathcal{R}\hat{r})^{\natural}(L^{-1}X)\|_G \le O(1)L^{-D}\|\hat{r}(X)\|_G \tag{5.39}$$

Analogously, for the functional derivative

$$\|(D\mathcal{L}(\mathcal{R}\hat{r})^{\natural})(L^{-1}X)\|_G \le O(1)L^{-D}\|\hat{r}(X)\|_G \tag{5.40}$$

and we have therefore

$$\|\mathcal{L}(\mathcal{R}\hat{r})^{\natural}\|_{G,1,\Gamma_p} \le O(1)L^{-D}\|\hat{r}\|_{G,1,\Gamma} \tag{5.41}$$

Applying lemma 2.3.3 finishes the proof

Q.E.D

LEMMA 5.3.2

$$\|F_{\hat{r}}\|_{\infty,1,\Gamma_p} \le O(1)\|\hat{r}\|_{G,1,\Gamma} \tag{5.42}$$

Proof

From (5.37) and (5.38) we have

$$|\mathcal{L}(\mathcal{R}\hat{r})^{\natural}(L^{-1}X,\phi)| \leq O(1)L^{-D}\|\hat{r}(X)\|_G \tag{5.43}$$

and similarly, for the functional derivative

$$\|(D\mathcal{L}(\mathcal{R}\hat{r})^{\natural})(L^{-1}X,\phi)\| \leq O(1)L^{-D}\|\hat{r}(X)\|_G \tag{5.44}$$

Applying again lemma 2.3.3 we have the proof.

Q.E.D

We want to write $F_{\hat{r}}(Z,\phi)$ in terms of the sets where the dependence from the field ϕ is localized. In other words, we want to write the decomposition

$$F_{\hat{r}}(Z,\phi) = \sum_{\Delta\subset Z} F_{\hat{r}}(Z,\Delta,\phi) \tag{5.45}$$

where in $F_{\hat{r}}(Z,\Delta,\phi)$ appear only fields defined in Δ.

$F_{\hat{r}}(Z,\Delta,\phi)$ is given by

$$F_{\hat{r}}(Z,\Delta,\phi) = \sum_{\Delta_1:\bar{\Delta}_1=L\Delta} \sum_{\substack{X:\ s.s.\\ \bar{X}=LZ\\ X\supset\Delta_1}} L^{-\alpha}\int_{\Delta_1} dx_1 e^{-\frac{\lambda_*}{2}L^{-\beta}|\mathcal{R}\phi(x_1)|^2} f_{\hat{r}}(X) \tag{5.46}$$

where

$$f_{\hat{r}}(X) = \frac{1}{|X|}\hat{r}_*^{\#\Sigma_{\bar{x}}}(X,0,0) \tag{5.47}$$

From the above expressions it is easy to check that (5.45) is satisfied.
Now, following [BDH 1], we define the contribution to the local effective potential :

$$V'_{F_{\hat{r}}}(\Delta,\phi) = -\sum_{\substack{Z\supset\Delta\\ |Z|\leq 2,\ \mathrm{conn}}} F_{\hat{r}}(Z,\Delta,\phi) \tag{5.48}$$

and from (5.46)

$$\sum_{Z\supset\Delta} F_{\hat{r}}(Z,\Delta,\phi) = \sum_{\Delta_1:\bar{\Delta}_1=L\Delta} \sum_{Z\supset\Delta} \sum_{\substack{X:\ s.s.\\ \bar{X}=LZ\\ X\supset\Delta_1}} L^{-\alpha}\int_{\Delta_1} dx_1 e^{-\frac{\lambda_*}{2}L^{-\beta}|\mathcal{R}\phi(x_1)|^2} f_{\hat{r}}(X) \tag{5.49}$$

It is immediate to see that this can be rewritten as

$$\sum_{Z\supset\Delta} F_{\hat{r}}(Z,\Delta,\phi) = \sum_{\Delta_1:\bar{\Delta}_1=L\Delta} L^{-\alpha}\int_{\Delta_1} dx_1 e^{-\frac{\lambda_*}{2}L^{-\beta}|\mathcal{R}\phi(x_1)|^2} \sum_{\substack{X:\ s.s.\\ X\supset\Delta_1}} f_{\hat{r}}(X) \tag{5.50}$$

Now we want to prove the following lemma

LEMMA 5.3.3

$$V'_{F_{\hat r}}(\Delta,\phi)=\xi_{\hat r}V_*(\Delta,\phi) \tag{5.51}$$

with

$$|\xi_{\hat r}|\le O(1)\|\hat r\|_{G,1,\Gamma}\le O(1)\varepsilon^{5/2+\eta} \tag{5.52}$$

Proof
By translation invariance

$$\sum_{\substack{X:\ s.s.\\ X\supset\Delta_1}} f_{\hat r}(X)=\sum_{\substack{X:\ s.s.\\ X\supset\Delta_1}}\frac{1}{|X|}\hat r_*^{\#\Sigma_{\bar x}}(X,0,0) \tag{5.53}$$

with $\bar x$ midpoint of X is independent of Δ_1, i.e. Δ_1 can be taken any unit block and the sum does not change. Therefore we define

$$\sum_{\substack{X:\ s.s.\\ X\supset\Delta_1}}\frac{1}{|X|}\hat r_*^{\#\Sigma_{\bar x}}(X,0,0)=\xi_{\hat r}L^{-\varepsilon}$$

and (5.50) can be rewritten

$$\sum_{Z\supset\Delta}F_{\hat r}(Z,\Delta,\phi)=L^{-\alpha}\int_{L\Delta}dx_1e^{-\frac{\lambda_*}{2}L^{-\beta}|\mathcal{R}\phi(x_1)|^2}\xi_{\hat r}L^{-\varepsilon} \tag{5.54}$$

Performing the rescaling in (5.54) we obtain (5.51).
To prove the bound (5.52) we observe

$$|\xi_{\hat r}|\le O(1)\sum_{\substack{X:\ s.s.\\ X\supset\Delta_1}}\frac{1}{|X|}|\hat r_*^{\#\Sigma_{\bar x}}(X,0,0)|\le O(1)\sum_{\substack{X:\ s.s.\\ X\supset\Delta_1}}O(1)\|\hat r(X)\|_G \tag{5.55}$$

where in the last step we used (5.22) followed by Lemma 5.1.5, (5.17).
From (5.55) we have

$$|\xi_{\hat r}|\le O(1)(\sum_{\substack{Z\ \text{s. s.}\\ Z\supset\Delta_1}}1)\|\hat r\|_{G,1,\Gamma}\le$$

$$\le O(1)\|\hat r\|_{G,1,\Gamma}$$

and by lemma 3.3.1 we obtain the proof.

Q.E.D.

§6 *Extraction estimates.*

From lemma 3.1.1, then reblocking-rescaling followed by fluctuation integration, and then the preliminary extraction provided by lemma 3.4.1, we obtained a single step of RG in the form

$$e^{-gV_*(\Lambda)}\mathcal{E}xp(\square + K)(\Lambda) \longrightarrow e^{-gL^{\varepsilon}V_*(L^{-1}\Lambda)}\mathcal{E}xp(\square + \tilde{K})(L^{-1}\Lambda) \tag{6.1}$$

with $\tilde{K}$ given by (3.45). In sections 4-5 respectively we obtained the relevant parts $F_{\tilde{Q}}$ (see (4.28)) and $F_{\hat{r}}$ (see (5.35)). They are supported on small sets and have the following local representations:

$$F_{\tilde{Q}}(Z) = \sum_{\Delta\subset Z} F_{\tilde{Q}}(Z,\Delta) \tag{6.2}$$

$$F_{\hat{r}}(Z) = \sum_{\Delta\subset Z} F_{\hat{r}}(Z,\Delta) \tag{6.3}$$

The corresponding local potentials are

$$V'_{\tilde{Q}}(L^{-1}\Lambda) = \sum_{\Delta\subset L^{-1}\Lambda} V'_{\tilde{Q}}(\Delta) \tag{6.4}$$

$$V'_{\tilde{Q}}(\Delta) = \sum_{\substack{Z\supset\Delta \\ Z\subset L^{-1}\Lambda}} F_{\tilde{Q}}(Z,\Delta) = -L^{\varepsilon}b_1 g^2 V_*(\Delta) \tag{6.5}$$

with $b_1 = O(\ln L) > 0$ (see lemma 4.6.1)

$$V'_{\hat{r}}(L^{-1}\Lambda) = \sum_{\Delta\subset L^{-1}\Lambda} V'_{\hat{r}}(\Delta) \tag{6.6}$$

$$V'_{\hat{r}}(\Delta) = \sum_{\substack{Z\supset\Delta \\ Z\subset L^{-1}\Lambda}} F_{\hat{r}}(Z,\Delta) = \xi_{\hat{r}} V_*(\Delta) \tag{6.7}$$

with $|\xi_{\hat{r}}| \le O(1)\varepsilon^{5/2+\eta}$ (see lemma 5.3.3). These lemmas are valid under the assumptions (3.5) and (3.11) on g and on the remainder r.
Define the total contribution to the local effective potential, due to $F_{\tilde{Q}}$ and $F_{\hat{r}}$ as

$$V'_F = V'_{\tilde{Q}} + V'_{\hat{r}} \tag{6.8}$$

where the total relevant part is

$$F = F_{\tilde{Q}} + F_{\hat{r}} \tag{6.9}$$

Then the local effective potential $gL^{\varepsilon}V_*(L^{-1}\Lambda)$ can be replaced by

$$V' = gL^{\varepsilon}V_* + V'_F \tag{6.10}$$

provided we replace the activity $\tilde{K}$ with a new activity $\mathcal{E}(\tilde{K})$. This procedure known as *extraction* (adopting the terminology of BDH) is given by the following proposition:

PROPOSITION 6.1 (Extraction)

$$e^{-gL^{\varepsilon}V_*(L^{-1}\Lambda)}\mathcal{E}xp(\square + \tilde{K})(L^{-1}\Lambda) = e^{-V'(L^{-1}\Lambda)}\mathcal{E}xp(\square + \mathcal{E}(\tilde{K}))(L^{-1}\Lambda) \tag{6.11}$$

where

$$\mathcal{E}(\tilde{K}) = (\tilde{K} - F) + (e^{-F} - 1 - F) + (e^{-F} - 1)^{+}_{\geq 2} + (e^{-F} - 1)^{+} \vee \tilde{K} \tag{6.12}$$

Before we prove this proposition let us observe:

$$V' = g'V_* \tag{6.13}$$

where

$$g' = L^{\varepsilon}g(1 - b_1 g) + \xi_{\hat{r}} \tag{6.14}$$

as follows from (6.8) and (6.10).

Proof of proposition 6.1
$\mathcal{E}(\tilde{K})$ is determined by

$$\mathcal{E}xp(\square + \mathcal{E}(\tilde{K}))(L^{-1}\Lambda) = e^{V'_F(L^{-1}\Lambda)}\mathcal{E}xp(\square + \tilde{K})(L^{-1}\Lambda) \tag{6.15}$$

From (6.2), (6.3) and (6.9) we have

$$F(Z) = \sum_{\Delta\subset Z} F(Z,\Delta) \tag{6.16}$$

Observe that

$$\sum_{Z\subset L^{-1}\Lambda} F(Z) = \sum_{Z\subset L^{-1}\Lambda}\sum_{\Delta\subset Z} F(Z,\Delta) = \sum_{\Delta\subset L^{-1}\Lambda}\sum_{\substack{Z\supset\Delta \\ Z\subset L^{-1}\Lambda}} F(Z,\Delta) =$$

$$= \sum_{\Delta\subset L^{-1}\Lambda} -V'_F(\Delta) = -V'_F(L^{-1}\Lambda) \tag{6.17}$$

Hence

$$e^{V'_F(L^{-1}\Lambda)} = \prod_{Z\subset L^{-1}\Lambda} e^{-F(Z)} = \prod_{Z\subset L^{-1}\Lambda} ((e^{-F(Z)}-1)+1) = \mathcal{E}xp(\square + (e^{-F}-1)^+)(L^{-1}\Lambda) \tag{6.18}$$

Using this, from (6.15)

$$\mathcal{E}xp(\square + \mathcal{E}(\tilde{K})) = \mathcal{E}xp(\square + (e^{-F}-1)^+)\mathcal{E}xp(\square + \tilde{K}) \tag{6.19}$$

and from lemma 1.4.1

$$\begin{aligned}\mathcal{E}(\tilde{K}) = \tilde{K} + (e^{-F}-1)^+ + (e^{-F}-1)^+ \vee \tilde{K} = \\ = (\tilde{K}-F) + (e^{-F}-1-F) + (e^{-F}-1)^+_{\geq 2} + (e^{-F}-1)^+ \vee \tilde{K}\end{aligned} \tag{6.20}$$

Q.E.D.

Note that from the definition of $\tilde{K}$ given in (3.51) and from (5.35)

$$\mathcal{E}(\tilde{K}) = \mathcal{I}_{k+1} + r' \tag{6.21}$$

where

$$\begin{aligned}r' = (1-\mathcal{L})\left(\mathcal{S}_1\hat{r}^{(s.s.)}\right)^\natural + \left(\mathcal{S}_1\hat{r}^{(l.s.)}\right)^\natural + \tilde{r}^\natural + \bar{r} + \\ +(e^{-F}-1-F) + (e^{-F}-1)^+_{\geq 2} + (e^{-F}-1)^+ \vee \tilde{K}\end{aligned} \tag{6.22}$$

We wish to estimate r'. This is provided by the following proposition.

PROPOSITION 6.2

$$\|r'\|_{G,1,\Gamma_6} \leq \frac{1}{L^{\beta/4}}\varepsilon^{5/2+\eta} \tag{6.23}$$

Proof

We recall that we have already proved

$$\|(1-\mathcal{L})\left(\mathcal{S}^{(1)}\hat{r}^{(s.s.)}\right)^\natural\|_{G,1,\Gamma_6} \leq O(1)L^{-\beta/2}\|\hat{r}\|_{G,1,\Gamma} \tag{6.24}$$

(see proposition 5.1.6)

$$\|\left(\mathcal{S}_1\hat{r}^{(l.s.)}\right)^\natural\|_{G,1,\Gamma_6} \leq O(1)L^{-(1-\beta/2)}\|\hat{r}\|_{G,1,\Gamma} \leq O(1)L^{-\beta/2}\|\hat{r}\|_{G,1,\Gamma} \tag{6.25}$$

(see lemma 5.2.1)

$$\|\tilde{r}^{\natural}\|_{G,1,\Gamma_6} \leq L^{-\beta/2}\varepsilon^{5/2+\eta} \tag{6.26}$$

(see lemma 3.5.2)

$$\|\bar{r}\|_{G,1,\Gamma_6} \leq L^{-\beta/2}\varepsilon^{5/2+\eta} \tag{6.27}$$

(see lemma 3.5.1)
We prove the following lemmas before returning to the proof of proposition 6.2

LEMMA 6.3

$$\|e^{-F} - 1 - F\|_{G,1,\Gamma_6} \leq L^{-\beta/2}\varepsilon^{5/2+\eta} \tag{6.28}$$

Proof
From lemmas 4.6.2, 5.3.1, 5.3.2 we have

$$\|F\|_{G,1,\Gamma_p} \leq O(1)\varepsilon^{7/4} \qquad \|F\|_{\infty,1,\Gamma_p} \leq O(1)\varepsilon^{7/4} \tag{6.29}$$

for any integer $p \geq 0$, with $O(1)$ depending on p.
From lemma 2.5.4

$$\|e^{-F} - 1 - F\|_{G,1,\Gamma_6} \leq O(1)\|F\|_{G,1,\Gamma_6}\|F\|_{\infty,1,\Gamma_6} \tag{6.30}$$

putting in (6.30) the bounds (6.29) we have

$$\|e^{-F} - 1 - F\|_{G,1,\Gamma_6} \leq O(1)\varepsilon^{14/4} \tag{6.31}$$

and the lemma follows from the smallness of ε.

Q.E.D.

LEMMA 6.4

$$\|(e^{-F} - 1)^{+}_{\geq 2}\|_{G,1,\Gamma_6} \leq L^{-\beta/2}\varepsilon^{5/2+\eta} \tag{6.32}$$

Proof
From lemma 2.5.5, following the same lines of lemma 6.3

Q.E.D.

LEMMA 6.5

$$\|(e^{-F}-1)^+\vee\tilde{K}\|_{G,1,\Gamma_6}\le L^{-\beta/2}\varepsilon^{5/2+\eta} \tag{6.33}$$

Proof

$$\|(e^{-F}-1)^+\vee\tilde{K}\|_{G,1,\Gamma_6}\le\sum_{N,M\ge 1}O(1)^{N+M}\|(e^{-F}-1)^+\|^N_{\infty,1,\Gamma_9}\|\tilde{K}\|^M_{G,1,\Gamma_9} \tag{6.34}$$

From the expression of $\tilde{K}$ (3.51), corollary 2.5.2, bound (3.9) proved by means of lemma 4.7.1, and lemmas 3.5.1, 3.5.2 we have

$$\|\tilde{K}\|_{G,1,\Gamma_9}\le O(1)\varepsilon^{7/4} \tag{6.35}$$

From lemma 2.5.5, and (6.29)

$$\|(e^{-F}-1)^+\|_{\infty,1,\Gamma_9}\le O(1)\varepsilon^{7/4} \tag{6.36}$$

Then

$$\|(e^{-F}-1)^+\vee\tilde{K}\|_{G,1,\Gamma_6}\le O(1)\varepsilon^{14/4} \tag{6.37}$$

and the lemma follows from the smallness of ε.

Q.E.D.

Now we come back to the proof of proposition 6.2.
From (6.24) - (6.27) and lemmas 6.3 - 6.5 we have

$$\|r'\|_{G,1,\Gamma_6}\le O(1)\frac{1}{L^{\beta/2}}\varepsilon^{5/2+\eta} \tag{6.38}$$

and proposition 6.2 follows for L large enough.

Q.E.D.

From (6.14) the evolved coupling constant g' at the end of one RG step is

$$g'=L^{\varepsilon}g(1-b_1g)+\xi_{\hat{r}}$$

In the absence of the remainder contribution $\xi_{\hat{r}}$, the approximate flow has the fixed point $\bar{g}$

$$\bar g = L^\varepsilon \bar g (1 - b_1 \bar g) \tag{6.39}$$

whence

$$\bar g = \frac{L^\varepsilon - 1}{L^\varepsilon b_1} \tag{6.40}$$

From lemma 4.6.1 we have

$$0 < b_1 = O(\ln L) \tag{6.41}$$

From the smallness of ε (3.5) we have

$$\bar g = O(\varepsilon) \tag{6.42}$$

Assume

$$|g - \bar g| \le \varepsilon^{3/2} \tag{6.43}$$

Then, since $\bar g = O(\varepsilon)$, we have $g = O(\varepsilon)$ and the hypotesis at the beginning of section 3 is satisfied. We have, under the assumption (6.43) above

PROPOSITION 6.6

The evolved coupling constant g' satisfies

$$|g' - \bar g| \le \varepsilon^{3/2} \tag{6.44}$$

so that the closed ball centered at $\bar g$ of radius $\varepsilon^{3/2}$ is stable under RG iteration.
Also,

$$|g' - g| \le \varepsilon^{1/2} \varepsilon^{3/2} \tag{6.45}$$

Proof
From (6.14) and an elementary calculation using property (6.39) we have

$$g' - \bar g = (g' - \bar g)(1 - L^\varepsilon b_1 g) + \xi_{\hat r}$$

g is $O(\varepsilon)$, b_1 is $O(\ln L)$, hence for ε sufficiently small

$$0 < 1 - L^\varepsilon b_1 g \le 1 - O(\varepsilon)$$

By assumption (6.43) above and by lemma 5.3.3

$$|\xi_{\hat r}| \le \varepsilon^{5/2+\eta}$$

we have

$$|g' - \bar{g}| \leq \varepsilon^{3/2}(1 - O(\varepsilon) + O(\varepsilon^{1+\eta})) \leq \varepsilon^{3/2}$$

for ε sufficiently small.
The next part of the proposition follows by writing

$$g' - g = (g - \bar{g})(-L^{\varepsilon} b_1 g) + \xi_{\hat{r}}$$

and estimating as before.

Q.E.D.

§7 *Convergence to a non Gaussian fixed point*

The partition functional at the n-th step of the RG can be parametrized by

$$(g_n, \mathcal{I}_n, r_n)$$

in volume L^{N+1-n}. A further RG transformation is a map

$$(g_n, \mathcal{I}_n, r_n) \longrightarrow (g_{n+1}, \mathcal{I}_{n+1}, r_{n+1}) \tag{7.1}$$

where the volume changes to $L^{N+1-(n+1)}$. In order to discuss the convergence of the sequence of RG transformations we shall consider N, n very large with $N \gg n$, so that we are effectively in "infinite" volume. With this hypothesis, the sequence of transformations (7.1) are iterations of a fixed mapping.
Let us now recall some results of the preceeding sections.
By hypothesis $\mathcal{I}_0 = r_0 = 0$ and $|g_0 - \bar{g}| \leq \varepsilon^{3/2}$. By the structure of irrelevant terms in second order perturbation theory given in section 4, and by results established there, we can write for any n

$$\mathcal{I}_n = \sum_{l=1}^{n} g_{n-l}^2 \bar{\mathcal{I}}_l \tag{7.2}$$

$$\bar{\mathcal{I}}_l = (\mathcal{S}_1 \bar{\mathcal{I}}_{l-1})^\natural \tag{7.3}$$

$$\|\bar{\mathcal{I}}_l\|_{G,1,\Gamma_p} \leq O(1) L^{3\beta - l\beta/2} L^{2(D+2)} \tag{7.4}$$

for all integers $p \geq 1$, with $O(1)$ depending on p.
Note that for $n \geq 1$:

$$\|\mathcal{I}_n - \mathcal{I}_{n-1}\|_{G,1,\Gamma_p} \leq L^{-n\beta/4} \varepsilon^{7/4} \tag{7.5}$$

where we have used the smallness of ε.
By proposition 6.2, 6.6, the closed ball:

$$B = \left\{ g, r \,\middle|\, |g - \bar{g}| \leq \varepsilon^{3/2}, \|r\|_{G,1,\Gamma_6} \leq \varepsilon^{5/2+\eta} \right\} \tag{7.6}$$

is stable under RG iteractions. Moreover from (7.2), (7.4), and using $g_n = O(\varepsilon)$ for all n, we get easily

$$\|\mathcal{I}_n\|_{G,1,\Gamma_p} \le O(\varepsilon^2) L^{2(D+2)} L^{3\beta} \sum_{l=1}^{n} L^{-l\beta/4} \le \varepsilon^{7/4} \tag{7.7}$$

We thus have

PROPOSITION 7.1

For any n

$$|g_n - \bar{g}| \le \varepsilon^{3/2} \tag{7.8}$$

$$\|r_n\|_{G,1,\Gamma_6} \le \varepsilon^{5/2+\eta} \tag{7.9}$$

$$\|\mathcal{I}_n\|_{G,1,\Gamma_6} \le \varepsilon^{7/4} \tag{7.10}$$

Define, for any sequence $\{a_n\}$, the increments $\Delta a_n = a_{n+1} - a_n$, and make the following *inductive hypothesis*:

For all $j = 1, 2, ..., n$,

$$|\Delta g_{j-1}| \le k_*^j \varepsilon^{3/2} \tag{7.11}$$

$$\|\Delta r_{j-1}\|_{G,1,\Gamma_6} \le k_*^j \varepsilon^{5/2+\eta} \tag{7.12}$$

where

$$k_* = 1 - \varepsilon \ln L + 2\varepsilon^{1+\eta/2} \tag{7.13}$$

Clearly $0 < k_* < 1$.
Note that the inductive hypothesis is true for $j = 1$. In fact, for $j = 1$ (7.11) follows from proposition 6.6, and (7.12) follows from proposition 6.2 if we use $L^{-\beta/4} \le k_*$, for sufficiently large L. Remember $r_0 = 0$, $\eta > 0$ and sufficiently small, say $\eta = 1/20$ as in section 3. Our task will be to prove that (7.11) and (7.12) are true for $j = n + 1$. To this end we first note some preliminary estimates. These are increment version of lemma 3.3.1, (6.35), and lemmas 3.5.1, 3.5.2. They are summarized in the following lemma:

LEMMA 7.2

For any integer $p \ge 0$ and for $O(1)$ depending on p

$$\|\Delta \hat{r}_{n-1}\|_{G,1,\Gamma} \le O(1) k_*^n \varepsilon^{5/2+\eta} \tag{7.14}$$

$$\|\Delta \tilde{K}_{n-1}\|_{G,1,\Gamma_p} \le O(1) k_*^n \varepsilon^{7/4} \tag{7.15}$$

$$\|\Delta \bar{r}_{n-1}\|_{G,1,\Gamma_p} \le O(1) \frac{k_*^n}{L^{\beta/2}} \varepsilon^{5/2+\eta} \tag{7.16}$$

$$\|(\Delta\tilde{r}_{n-1})^{\natural}\|_{G,1,\Gamma_p} \le O(1)\frac{k_*^n}{L^{\beta/2}}\varepsilon^{5/2+\eta} \tag{7.17}$$

The quantities $\hat{r}$, $\tilde{K}$, $\bar{r}$, $\tilde{r}$ are those introduced in section 3.
We omit the proof of lemma 7.2. The proof is straightforward. We have to use proposition 7.1 and the inductive hypothesis. We use lemmas 2.6.1, 2.6.2, 2.6.3. For (7.15) we use also (7.5) and $L^{-\beta/4} \le k_*$. Then follow the lines of the proofs of lemma 3.3.1, (6.35), and lemmas 3.5.1, 3.5.2.
Now turn to the extracted activities. By definition (see (6.22))

$$r_{n+1} = (1-\mathcal{L})\left(\mathcal{S}_1\hat{r}_n^{(s.s.)}\right)^{\natural} + \left(\mathcal{S}_1\hat{r}_n^{(l.s.)}\right)^{\natural} + \tilde{r}_n^{\natural} + \bar{r}_n +$$

$$+(e^{-F_n}-1-F_n) + (e^{-F_n}-1)^{+}_{\ge 2} + (e^{-F_n}-1)^{+} \vee \tilde{K}_n \tag{7.18}$$

Hence

$$\Delta r_n = (1-\mathcal{L})\left(\mathcal{S}_1\Delta\hat{r}_{n-1}^{(s.s.)}\right)^{\natural} + \left(\mathcal{S}_1\Delta\hat{r}_{n-1}^{(l.s.)}\right)^{\natural} + (\Delta\tilde{r}_{n-1})^{\natural} + \Delta\bar{r}_{n-1} +$$

$$+\Delta(e^{-F_{n-1}}-1-F_{n-1}) + \Delta(e^{-F_{n-1}}-1)^{+}_{\ge 2} +$$

$$+(\Delta(e^{-F_{n-1}}-1)^{+}) \vee \tilde{K}_n + (e^{-F_{n-1}}-1)^{+}) \vee \Delta\tilde{K}_{n-1} \tag{7.19}$$

PROPOSITION 7.3

$$\|\Delta r_n\|_{G,1,\Gamma_6} \le O(1)k_*^{n+1}\varepsilon^{5/2+\eta} \tag{7.20}$$

Proof
We estimate each term on the r. h. s. of (7.19).
By proposition 5.1.6 and linearity

$$\|(1-\mathcal{L})\left(\mathcal{S}_1\Delta\hat{r}_{n-1}^{(s.s.)}\right)^{\natural}\|_{G,1,\Gamma_6} \le \frac{O(1)}{L^{\beta/2}}\|\Delta\hat{r}_{n-1}\|_{G,1,\Gamma} \le \frac{O(1)}{L^{\beta/2}}k_*^n\varepsilon^{5/2+\eta} \tag{7.21}$$

where in the last step we have used lemma 7.2.

$$\|\left(\mathcal{S}_1\Delta\hat{r}_{n-1}^{(l.s.)}\right)^{\natural}\|_{G,1,\Gamma_6} \le \frac{O(1)}{L^{\beta/2}}\|\Delta\hat{r}_{n-1}\|_{G,1,\Gamma} \le \frac{O(1)}{L^{\beta/2}}k_*^n\varepsilon^{5/2+\eta} \tag{7.22}$$

by lemmas 2.5.3 and 7.2.

$$\|(\Delta\tilde{r}_{n-1})^{\natural}\|_{G,1,\Gamma_6} \le O(1)\frac{k_*^n}{L^{\beta/2}}\varepsilon^{5/2+\eta} \tag{7.23}$$

by lemma 7.2.

$$\|\Delta(e^{-F_{n-1}}-1-F_{n-1})\|_{G,1,\Gamma_6} \le O(1)\frac{k_*^n}{L^{\beta/2}}\varepsilon^{5/2+\eta} \tag{7.24}$$

Proof of (7.24)

$$F_n = F_{\tilde{Q}_n} + F_{\hat{r}_n}$$

Consider first $F_{\tilde{Q}_n}$ with $g = g_n = O(\varepsilon)$ by proposition 7.1. ¿From lemma 4.6.2, for $j = n-1, n$ and any integer $p \ge 0$

$$\|F_{\tilde{Q}_j}\|_{G,1,\Gamma_p} \le O(1)\varepsilon^{7/4} \qquad \|F_{\tilde{Q}_j}\|_{\infty,1,\Gamma_p} \le O(1)\varepsilon^{7/4}$$

Then, by lemmas 5.3.1, 5.3.2, for $j = n-1, n$ and any integer $p \ge 0$

$$\|F_{\hat{r}_j}\|_{G,1,\Gamma_p} \le O(1)\|\hat{r}_j\|_{G,1,\Gamma} \le O(1)\varepsilon^{5/2+\eta} \qquad \|F_{\hat{r}_j}\|_{\infty,1,\Gamma_p} \le O(1)\|\hat{r}_j\|_{G,1,\Gamma} \le O(1)\varepsilon^{5/2+\eta}$$

where in the last step we used lemma 3.3.1
Hence

$$\|F_j\|_{G,1,\Gamma_p} \le O(1)\varepsilon^{7/4} \qquad \|F_j\|_{\infty,1,\Gamma_p} \le O(1)\varepsilon^{7/4} \tag{7.25}$$

Hence, by lemma 2.6.2

$$\|\Delta(e^{-F_{n-1}}-1-F_{n-1})\|_{G,1,\Gamma_6} \le O(1)\varepsilon^{7/4}\|\Delta F_{n-1}\|_{G,1,\Gamma_6} \tag{7.26}$$

By (4.31), (4.32)

$$\|\Delta F_{\tilde{Q}_{n-1}}\|_{G,1,\Gamma_p} \le O(\ln L)L^{2(D+2)}|g_n^2 - g_{n-1}^2| \le O(1)k_*^n\varepsilon^{7/4} \tag{7.27}$$

where we used in the last step the inductive hypothesis. By lemma 5.3.2, linearity, and lemma 7.2

$$\|\Delta F_{\hat{r}_{n-1}}\|_{G,1,\Gamma_p} \le O(1)\|\Delta\hat{r}_{n-1}\|_{G,1,\Gamma} \le O(1)k_*^n\varepsilon^{5/2+\eta} \tag{7.28}$$

From (7.27), (7.28) we obtain

$$\|\Delta F_{n-1}\|_{G,1,\Gamma_6} \le O(1)k_*^n\varepsilon^{7/4} \tag{7.29}$$

Putting (7.29) in (7.26) and using the smallness of ε we obtain the proof of (7.24)

We have now the estimate

$$\|\Delta(e^{-F_{n-1}}-1)^+_{\ge 2}\|_{G,1,\Gamma_6} \le \frac{O(1)}{L^{\beta/2}}k_*^n\varepsilon^{5/2+\eta} \tag{7.30}$$

This follows from (7.25), (7.29) and lemma 2.6.3

Next we have the estimate:

$$\|(\Delta(e^{-F_{n-1}}-1)^+)\vee\tilde{K}_n\|_{G,1,\Gamma_6}\le O(1)\|(\Delta(e^{-F_{n-1}}-1)^+)\|_{\infty,1,\Gamma_9}\|\tilde{K}_n\|_{G,1,\Gamma_9}\le$$

$$\le O(1)\|\Delta F_{n-1}\|_{\infty,1,\Gamma_9}\|\tilde{K}_n\|_{G,1,\Gamma_9}\le O(1)k_*^n\varepsilon^{14/4}\le\frac{O(1)}{L^{\beta/2}}k_*^n\varepsilon^{5/2+\eta}\tag{7.31}$$

where we have used lemma 2.6.3 with $k=1$, and then (7.29), (6.35).

Finally, we have:

$$\|(e^{-F_{n-1}}-1)^+\vee\Delta\tilde{K}_{n-1}\|_{G,1,\Gamma_6}\le O(1)\|(e^{-F_{n-1}}-1)^+\|_{\infty,1,\Gamma_9}\|\Delta\tilde{K}_{n-1}\|_{G,1,\Gamma_9}\le$$

$$\le O(1)k_*^n\varepsilon^{14/4}\le\frac{O(1)}{L^{\beta/2}}k_*^n\varepsilon^{5/2+\eta}\tag{7.32}$$

where we have used (6.36) and lemma 7.2
Adding up the estimates given in (7.21)-(7.24) and (7.30)-(7.32) we obtain

$$\|\Delta r_n\|_{G,1,\Gamma_6}\le\frac{O(1)}{L^{\beta/2}}k_*^n\varepsilon^{5/2+\eta}\tag{7.33}$$

and from this we obtain the proof for L large enough.

Q.E.D.

PROPOSITION 7.4

$$|\Delta g_n|\le k_*^{n+1}\varepsilon^{5/2+\eta}\tag{7.34}$$

Proof

$$g_{n+1}=L^\varepsilon g_n(1-b_1g_n)+\xi_{\hat{r}_n}$$

whence

$$\Delta g_n=\Delta g_{n-1}(L^\varepsilon-L^\varepsilon b_1(g_n+g_{n-1}))+\Delta\xi_{\hat{r}_{n-1}}\tag{7.35}$$

Using the definition of the approximate fixed point

$$\bar{g}=\frac{L^\varepsilon-1}{L^\varepsilon b_1}$$

We have

$$L^\varepsilon - L^\varepsilon b_1(g_n + g_{n-1}) = 2 - L^\varepsilon - L^\varepsilon b_1(g_n - \bar g) - L^\varepsilon b_1(g_{n-1} - \bar g)$$

Using the fact that $L^\varepsilon - L^\varepsilon b_1(g_n + g_{n-1}) > 0$ for ε sufficiently small we have

$$0 \le L^\varepsilon - L^\varepsilon b_1(g_n + g_{n-1}) \le 1 - \varepsilon \ln L + L^\varepsilon b_1|g_n - \bar g| + L^\varepsilon b_1|g_{n-1} - \bar g| \le 1 - \varepsilon \ln L + 2L^\varepsilon b_1 \varepsilon^{3/2} \tag{7.36}$$

We have also by the inductive hypothesis

$$|\Delta g_{n-1}| \le k_*^n \varepsilon^{3/2} \tag{7.37}$$

$$|\Delta \xi_{\hat r_{n-1}}| \le O(1)\|\Delta \hat r_{n-1}\|_{G,1,\Gamma} \le O(1) k_*^n \varepsilon^{5/2+\eta} \tag{7.38}$$

where we also used linearity and lemma 5.3.3.
Putting (7.36), (7.37) and (7.38) in (7.35) we get

$$|\Delta g_n| \le \varepsilon^{3/2}(1 - \varepsilon \ln L + 2L^\varepsilon b_1 \varepsilon^{3/2} + O(1)\varepsilon^{1+\eta}) k_*^n \le$$

$$\le k_*^{n+1} \varepsilon^{3/2}$$

which gives the proof of (7.34).

Q.E.D.

Propositions 7.3 and 7.4 imply that the inductive hypothesis (7.11), (7.12) is actually true for all $j \ge 1$. This, together with (7.5) and proposition 7.1 implies:

THEOREM 7.5

Let $\varepsilon > 0$ and sufficiently small. Then the sequence $(g_n, \mathcal{I}_n, r_n)$ converges, as $n \longrightarrow \infty$ to the fixed point $(g_\infty, \mathcal{I}_\infty, r_\infty)$. Moreover

$$|g_\infty - \bar g| \le \varepsilon^{3/2}$$
$$\|r_\infty\|_{G,1,\Gamma_6} \le \varepsilon^{5/2+\eta}$$
$$\|\mathcal{I}_\infty\|_{G,1,\Gamma_6} \le \varepsilon^{7/4}$$

Since $\bar g = O(\varepsilon)$, we have $g_\infty \ne 0$ so that the fixed point is non Gaussian.

Acknowledgements
We are grateful to David Brydges for many interesting conversations and suggestions during the course of this work. We thank Marzio Cassandro for having participated at an early stage of this project and for his continued interest. P.K.M thanks Gerard Menessier for his mathematical vigilance and for having cooperated on an earlier version of the proof of lemma 2.3.2. He thanks the INFN, Sezione di Roma, for having supported his visits to Rome during the course of this work.

Appendix A

In this appendix we will prove lemma 2.3.2 on the stability of the large field regulator.

Proof

Using the master formula given in Lemma 1.3.1, we have

$$(\mu_\Gamma * G_{\rho,\kappa})(X,\phi) = \frac{L^{-\alpha}}{|X|}\int_X dx_1 e^{-(\lambda_*/2)L^{-\beta}|\phi(x_1)|^2}$$

$$\int d\mu_{\Sigma^{x_1}}(\zeta) e^{(\rho\lambda_*/2)|\zeta(x_1)+L^{-\beta}\phi(x_1)|^2} e^{\kappa\|\zeta+L^{-\beta}T^{x_1}\phi\|^2_{X,1,\sigma}} \le$$

$$\le \frac{L^{-\alpha}}{|X|}\int_X dx_1 e^{-(\lambda_*/2)L^{-\beta}|\phi(x_1)|^2}$$

$$\left(\int d\mu_{\Sigma^{x_1}}(\zeta) e^{\rho\lambda_*|\zeta(x_1)+L^{-\beta}\phi(x_1)|^2}\right)^{1/2}\left(\int d\mu_{\Sigma^{x_1}}(\zeta) e^{2\kappa\|\zeta+L^{-\beta}T^{x_1}\phi\|^2_{X,1,\sigma}}\right)^{1/2} \qquad (A1.1)$$

Observing that

$$\sigma = \Sigma^{x_1}(x_1,x_1) = \gamma - \lambda_* L^{-\beta}\gamma^2$$

so that

$$\lambda_*\sigma = 1 - \frac{1}{L^\beta}$$

we easily obtain the bound

$$\int d\mu_{\Sigma^{x_1}}(\zeta) e^{\rho\lambda_*|\zeta(x_1)+L^{-\beta}\phi(x_1)|^2} \le$$

$$\le e^{2\rho\lambda_* L^{-2\beta}|\phi(x_1)|^2}\int d\mu_{\Sigma^{x_1}}(\zeta) e^{2\rho\lambda_*|\zeta(x_1)|^2} \le$$

$$\le (1-4\rho)^{-d/2} e^{2\rho\lambda_* L^{-2\beta}|\phi(x_1)|^2} \qquad (A1.2)$$

hence for $0<\rho<\frac{1}{8}$ we get from (A1.1)

$$(\mu_\Gamma * G_{\rho,\kappa})(X,\phi) \le 2^{d/4}\frac{L^{-\alpha}}{|X|}\int_X dx_1 e^{-(\lambda_*/2)\left(1-\frac{2\rho}{L^\beta}\right)L^{-\beta}|\phi(x_1)|^2}.$$

$$\cdot \left(\int d\mu_{\Sigma^{x_1}}(\zeta) e^{2\kappa \|\zeta + L^{-\beta} T^{x_1} \phi\|^2_{X,1,\sigma}} \right)^{1/2} \tag{A1.3}$$

We shall now estimate the last integral in (A1.3). First we observe:

$$\zeta + L^{-\beta} T^{x_1} \phi = \zeta + \phi - \lambda_* L^{-\beta} \Gamma(\cdot, x_1) \phi(x_1)$$

Then we have

$$\int d\mu_{\Sigma^{x_1}}(\zeta) e^{2\kappa \|\zeta + L^{-\beta} T^{x_1}\phi\|^2_{X,1,\sigma}} \leq e^{4\kappa \lambda_*^2 L^{-2\beta} \|\Gamma(\cdot,x_1)\|^2_{X,1,\sigma} |\phi(x_1)|^2} \int d\mu_{\Sigma^{x_1}}(\zeta) e^{4\kappa \|\zeta + \phi\|^2_{X,1,\sigma}} \tag{A1.4}$$

We now make the following claim.

Claim A.1

For $\kappa > 0$ sufficiently small, independent of L, and any $x_1 \in X$

$$\int d\mu_{\Sigma^{x_1}}(\zeta) e^{4\kappa \|\zeta + \phi\|^2_{X,1,\sigma}} \leq 2^{|X|} e^{8\kappa \| \ \phi\|^2_{X,1,\sigma}} \tag{A1.5}$$

Observe also that, from lemma 1.1.1,

$$\|\Gamma(\cdot, x_1)\|^2_{X,1,\sigma} \leq \sum_{1 \leq \alpha \leq \sigma} \int_X dx |\partial^\alpha \Gamma(x - x_1)|^2 \leq \sum_{1 \leq \alpha \leq \sigma} \int_{\mathbf{R}} dx |\partial^\alpha \Gamma(x - x_1)|^2 \leq O(1) \tag{A1.6}$$

Using (A1.5) and (A1.6) we get from (A1.4)

$$\int d\mu_{\Sigma^{x_1}}(\zeta) e^{2\kappa \|\zeta + L^{-\beta} T^{x_1}\phi\|^2_{X,1,\sigma}} \leq 2^{|X|} e^{O(1)\kappa \lambda_*^2 L^{-2\beta} |\phi(x_1)|^2} e^{8\kappa \|\phi\|^2_{X,1,\sigma}} \tag{A1.7}$$

and using (A1.7) we get from (A1.3)

$$(\mu_\Gamma * G_{\rho,\kappa})(X, \phi) \leq O(1) \frac{L^{-\alpha}}{|X|} \int_X dx_1 e^{-(\lambda_*/2)\left(1 - \frac{2\rho}{L^\beta}(1 + O(1)\frac{\kappa}{\rho}\lambda_*)\right) L^{-\beta} |\phi(x_1)|^2} 2^{|X|} e^{4\kappa \|\phi\|^2_{X,1,\sigma}} \tag{A1.8}$$

We have chosen $\kappa > 0$, $O(1)$ in L, sufficiently small and $0 < \rho < 1/8$. We choose $\rho \geq \kappa$. Then we get (2.14) for L sufficiently large. So lemma 2.3.2 will have been proved provided we prove the claim A.1.

Proof of Claim A.1

The proof is along the lines of that of Lemma 3 in [BDH2], the only difference being that we have the "covariance" Σ^{x_1} instead of the covariance Γ.

Recall

$$\Sigma^{x_1}(x,y) = \Gamma(x-y) - \lambda_* L^{-\beta}\Gamma(x-x_1)\Gamma(y-x_1) \tag{A1.9}$$

Define for $t \in [0,1]$

$$\mathcal{G}_t(X,\phi) = e^{U_t(X,\phi)}$$

where

$$U_t(X,\phi) = t\ln 2|X| + 4\kappa(1+t)\|\phi\|^2_{X,1,\sigma} \tag{A1.10}$$

We have to prove

$$\int_0^1 ds \frac{\partial}{\partial s}\left(\mu_{(1-s)\Sigma^{x_1}} * \mathcal{G}_s\right)(X,\phi) \geq 0$$

It is sufficient to prove that the integrand is non-negative.
Thus we have to prove, for $s \in [0,1]$

$$\frac{\partial}{\partial s}U_s - \frac{1}{2}\Delta_{\Sigma^{x_1}}U_s - \frac{1}{2}\Sigma^{x_1}\left(\frac{\partial U_s}{\partial\phi}, \frac{\partial U_s}{\partial\phi}\right) \geq 0 \tag{A1.11}$$

Here

$$\Delta_{\Sigma^{x_1}}U_s = \int_X dx \int_X dy \Sigma^{x_1}(x,y)\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)}U_s =$$

$$= \Delta_\Gamma U_s - \lambda_* L^{-\beta}\left(\int_X dx \Gamma(x-x_1)\frac{\delta}{\delta\phi(x)}\right)^2 U_s \tag{A1.12}$$

and

$$\Sigma^{x_1}\left(\frac{\partial U_s}{\partial\phi}, \frac{\partial U_s}{\partial\phi}\right) = \int_X dx \int_X dy \Sigma^{x_1}(x,y)\frac{\delta U_s}{\delta\phi(x)}\frac{\delta U_s}{\delta\phi(y)} =$$

$$= \Gamma\left(\frac{\partial U_s}{\partial\phi}, \frac{\partial U_s}{\partial\phi}\right) - \lambda_* L^{-\beta}\left(\int_X dx \Gamma(x-x_1)\frac{\delta U_s}{\delta\phi(x)}\right)^2 \tag{A1.13}$$

$$\frac{\partial U_s}{\partial s} = \ln 2|X| + 4\kappa\|\phi\|^2_{X,1,\sigma} \tag{A1.14}$$

We have

$$\left|\frac{1}{2}\Delta_\Gamma U_s\right| \leq O(1)\kappa|X| \tag{A1.15}$$

where $O(1)$ is independent of L. The latter follows from the fact that the Sobolev norm starts with one derivative and lemma 1.1.1.

$$\left(\int_X dx\Gamma(x-x_1)\frac{\delta}{\delta\phi(x)}\right)^2 U_s \le 8\kappa(1+s)\|\Gamma(\cdot,x_1)\|^2_{X,1,\sigma} \le O(1)\kappa \tag{A1.16}$$

with $O(1)$ independent of L, by (A1.6). Using (A1.15) and (A1.16), we get from (A1.12)

$$\left|\frac{1}{2}\Delta_{\Sigma^{x_1}} U_s\right| \le O(1)\kappa|X| \tag{A1.17}$$

It is easy to see that

$$\left|\Gamma\left(\frac{\partial U_s}{\partial\phi},\frac{\partial U_s}{\partial\phi}\right)\right| \le 32\kappa^2 \sum_{1\le\alpha_1,\alpha_2\le\sigma}\int_X dy|\partial^{\alpha_1}\phi(y)|\ |((\partial^{\alpha_1+\alpha_2}\Gamma)*\partial^{\alpha_2}\phi)(y)| \le$$

$$\le 32\kappa^2 \sum_{1\le\alpha_1,\alpha_2\le\sigma} \|\partial^{\alpha_1}\phi\|_{L^2(X)}\|(\partial^{\alpha_1+\alpha_2}\Gamma)*\partial^{\alpha_2}\phi\|_{L^2(X)} \le$$

$$\le 32\kappa^2\left(\sup_{2\le j\le 2\sigma}\|\partial^j\Gamma\|_{L^1(\mathbf{R})}\right)\left(\sum_{1\le\alpha\le\sigma}\|\partial^\alpha\phi\|_{L^2(X)}\right)^2 \le$$

$$\le O(1)\kappa^2\left(\sup_{2\le j\le 2\sigma}\|\partial^j\Gamma\|_{L^1(\mathbf{R})}\right)\|\phi\|^2_{X,1,\sigma} \tag{A1.18}$$

where to pass to the next from the last line we have used Young's convolution inequality. Now, using the compact support of the kernel function u

$$\sup_{2\le j\le 2\sigma}\|\partial^j\Gamma\|_{L^1(\mathbf{R})} \le \sup_{2\le j\le 2\sigma}\int_1^L \frac{dl}{l} l^{\beta-j}\int_{-l}^{l} dx\ |(\partial^j u)(x/l)| \le$$

$$\le 2\sup_{2\le j\le 2\sigma}\int_1^L \frac{dl}{l} l^{\beta-j+1}\|\partial^j u\|_\infty \le O(1) \tag{A1.19}$$

where $O(1)$ is independent on L.
From (A1.18) and (A1.19) we get

$$\left|\Gamma\left(\frac{\partial U_s}{\partial\phi},\frac{\partial U_s}{\partial\phi}\right)\right| \le O(1)\kappa^2\|\phi\|^2_{X,1,\sigma} \tag{A1.20}$$

Next we have

$$\left(\int_X dx\Gamma(x-x_1)\frac{\delta U_s}{\delta\phi(x)}\right)^2 \le 32\kappa^2\left(\sum_{1\le\alpha\le\sigma}\int_X dx\partial^\alpha\Gamma(x-x_1)\partial^\alpha\phi(x)\right)^2 \le$$

$$\leq 32\kappa^2 \left(\sum_{1\leq\alpha\leq\sigma} \left(\int_X dx (\partial^\alpha \Gamma(x-x_1))^2 \right)^{1/2} \left(\int_X dx |\partial^\alpha \phi(x)|^2 \right)^{1/2} \right)^2 \leq$$

$$\leq O(1)\kappa^2 \left(\sup_{1\leq\alpha\leq\sigma} \int_{\mathbf{R}} (\partial^\alpha \Gamma(x))^2 \right) \|\phi\|^2_{X,1,\sigma} \leq$$

$$\leq O(1)\kappa^2 \|\phi\|^2_{X,1,\sigma} \tag{A1.21}$$

where $O(1)$ is independent on L and we have used lemma 1.1.1. Using (A1.20) and (A1.21) we get from (A1.13)

$$\left| \Sigma^{x_1} \left(\frac{\partial U_s}{\partial \phi}, \frac{\partial U_s}{\partial \phi} \right) \right| \leq O(1)\kappa^2 \|\phi\|^2_{X,1,\sigma} \tag{A1.22}$$

Putting together the estimates (A1.14), (A1.17) and (A1.22) we get

$$\frac{\partial}{\partial s} U_s - \frac{1}{2}\Delta_{\Sigma^{x_1}} U_s - \frac{1}{2}\Sigma^{x_1} \left(\frac{\partial U_s}{\partial \phi}, \frac{\partial U_s}{\partial \phi} \right) \geq$$

$$\geq (\ln 2 - O(1)\kappa)|X| + (4\kappa - O(1)\kappa^2)\|\phi\|^2_{X,1,\sigma} \geq 0 \tag{A1.23}$$

for $\kappa > 0$ small enough and independent of L, since the $O(1)$ are independent of L. This completes the proof of the claim, and hence of lemma 2.3.2.

Q.E.D.

References

[B] D. Brydges, Functional integrals and their applications, Technical report, Ecole Polytechnique Federale de Lausanne (1992).

[BDH1] D. Brydges, J. Dimock and T.R. Hurd, Comm. Math. Phys., **172**, 143-186 (1995).

[BDH2] D. Brydges, J. Dimock and T.R. Hurd, Estimates on Renormalization Group Transformation, Preprint (1996)

[BDH3] D. Brydges, J. Dimock and T.R. Hurd, Comm. Math. Phys., **198**, 111-156 (1998)

[BY] D. Brydges and H.T. Yau, Comm. Math. Phys., **129**, 351-392 (1990).

[CM] M. Cassandro and P.K. Mitter, Nucl. Phys., **B422**, 634-674 (1994).

[DDG] F. David, B. Duplantier and E. Guitter, Phys. Rev. Lett. **70**, 2205 (1993).

[DH1] J. Dimock and T.R. Hurd, J. Stat. Phys, **66**, 1277-1318 (1992).

[DH2] J. Dimock and T.R. Hurd, Comm. Math. Phys., **156**, 547-580 (1993).

[GK] K. Gawedski and A. Kupiainen, Nucl. Phys., **B262**, 33 (1985).

[KW] J.Kogut and K.G. Wilson, Phys. Rep., **12**, 75 (1974).

[NPW] D.R. Nelson, T. Piron and S. Weinberg, eds., Statistical mechanics of membranes and surfaces (World Scientific, Singapore, 1989)