COMPLEX RESONANCES IN ACOUSTIC WAVEGUIDES

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Abstract. We consider a two-dimensional infinitely long acoustic waveguide formed by two parallel lines containing an arbitrarily shaped obstacle. The existence of trapped modes that are the eigenfunctions of the Laplace operator in the corresponding domain subject to Neumann boundary conditions was proved by Evans, Levitin & Vassiliev (1994) for obstacles symmetric about the centreline of the waveguide. In our paper we deal with the situation when the obstacle is shifted with respect to the centreline and study the resulting complex resonances. We are particularly interested in those resonances which are perturbations of (real) eigenvalues. We study how an eigenvalue becomes a complex resonance moving from the real axis into the upper half–plane as the obstacle is shifted from its original position. The shift of the eigenvalue along the imaginary axis is predicted theoretically and the result is compared with numerical computations. The total number of resonances lying inside a sequence of expanding circles is also calculated numerically.

1. Introduction

The existence of trapped modes in an infinitely long acoustic waveguide containing an obstacle of a fairly general shape symmetric about the centreline of the waveguide was proved by Evans, Levitin & Vassiliev (1994). The trapped modes under consideration are the eigenfunctions of the Laplace operator in an unbounded domain (which is the strip without the obstacle) subject to Neumann boundary conditions. The corresponding eigenvalues are embedded in the continuous spectrum. The vibration modes studied in [1] are antisymmetric with respect to the centreline and describe acoustic waves decaying at infinity. Further rigorous mathematical results concerning the existence or non–existence of trapped modes were obtained by Davies & Parnovski (1998).

There is also a substantial bibliography devoted to the numerical study of trapped modes. For example, Evans & Linton (1991) provided

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a technique for computing trapped modes in an acoustic waveguide containing a symmetrically placed rectangular obstacle and presented numerical evidence for the existence of such modes. Callan, Linton & Evans (1991) considered a waveguide with a disc–shaped obstacle and followed the method due to Ursell to establish the existence of trapped modes. The construction of the corresponding solutions in [3] was based on its expansion into a series of Bessel functions. Finally, Maniar & Newman (1997) proposed a technique for finding eigenvalues and corresponding vibration modes for an acoustic waveguide containing a long array of discs symmetric about the centreline. In the above mentioned papers the special shape of the obstacle has been taken into account and essentially used by the authors when proving theoretical results and elaborating numerical procedures.

We study complex resonances occurring in a waveguide which contains a smooth obstacle. The problem is posed in Section 2 where a mathematically rigorous statement is given. Further on we concentrate on resonances which originate from the embedded eigenvalues. According to [1], such eigenvalues exist for obstacles symmetric about the centreline of a waveguide, and the aim of our paper is to track the movement of these eigenvalues into the complex plane as the obstacle is shifted from the centreline by a small distance δ. Section 3 contains the main result of the paper, which is an asymptotic ($\delta \to 0$) formula for the imaginary part of the complex resonance. This formula is proved in Section 4. Numerical results illustrating our asymptotic formula are presented in Section 5. Section 6 is devoted to the numerical computation of the total number of complex resonances inside a sequence of expanding circles.

2. Statement of the problem

Consider an infinite two-dimensional strip $(-\infty,\infty)\times(-1,1)\subset\mathbb{R}^2$, and let $\mathcal{O}$ be a compact subset of this strip with smooth boundary γ. We assume that the acoustic medium occupies the domain $G := (-\infty,\infty)\times(-1,1)\setminus\mathcal{O}$ (see Figure 1). The boundary Γ of the domain G consists of three parts: $\Gamma=\Gamma_1\cup\Gamma_2\cup\gamma$, where $\Gamma_1 := \{y=-1\}$, $\Gamma_2 := \{y=1\}$.

We consider the following boundary value problem in G:

$$\Delta u+\omega^2 u=0\,, \tag{2.1}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial G} = 0\,. \tag{2.2}$$

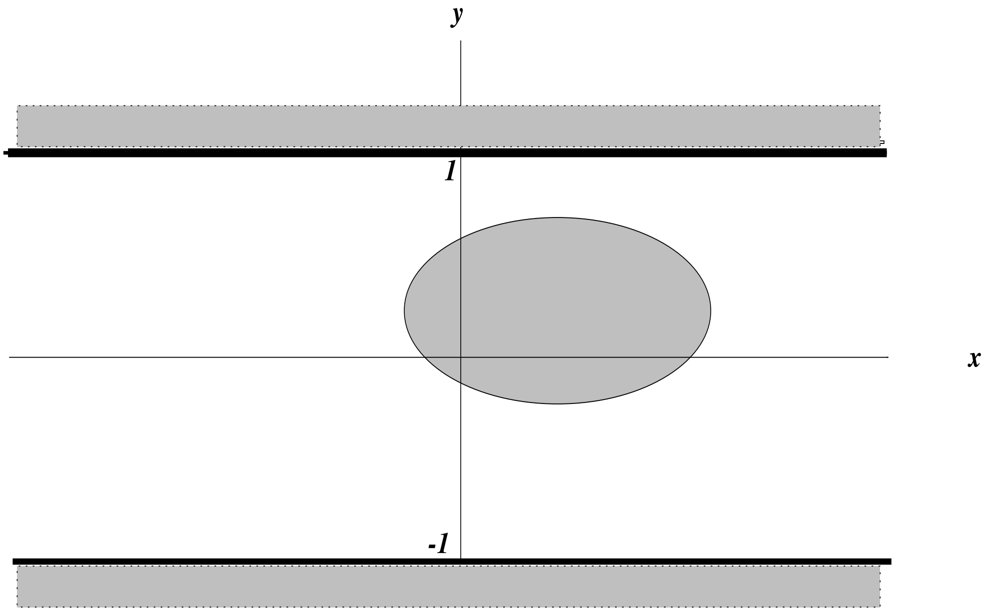

FIGURE 1. Waveguide with an obstacle

Here $u(x, y)$ is the acoustic potential (potential of displacements), ω is the spectral parameter (vibration frequency), and $\frac{\partial}{\partial n}$ denotes the outer (with respect to G) normal derivative.

Let $R(\omega) = (-\Delta - \omega^2)^{-1} : L^2(G) \to L^2(G)$ be the resolvent of the Neumann Laplacian. Initially, $R(\omega)$ is defined for $\mathrm{Im}\omega < 0$ (on the so called *physical sheet*) and forms a holomorphic family of operators. This family cannot be meromorphically continued onto the whole $\mathbb{C}$. It turns out, however, that a meromorphic extension is possible if we view our resolvent as an operator $C_0^\infty(G) \to \mathcal{D}'(G)$. Actually, we get a meromorphic continuation onto a complicated Riemann surface Λ, see, *e.g.,* [6]. The surface Λ has infinitely many sheets, namely, each of the points $\pm\pi(k+1)/2$ $(k \in \mathbb{N})$ is a second order branch point. The poles of $R(\omega)$ on Λ are called *resonances* of the problem (2.1), (2.2). Of course, eigenvalues are among resonances; also, one can show that resonances are continuous (even real analytic) with respect to perturbations of the operator. At present little is known about the possible location, number, or even existence of resonances in this setting.

The definition of resonances as poles of the meromorphic continuation of the resolvent, although being quite straightforward, has a serious disadvantage. Namely, it is quite difficult to construct the resolvent of our problem. Therefore, we shall give another definition of resonances, as the poles of the *scattering matrix*. This definition, though more complicated, is easier to deal with.

Let us first define the scattering matrix. For $\mathrm{Im}\omega < 0$ set $\xi_k = -\sqrt{(\pi k/2)^2 - \omega^2}$, $k = 0, 1, \dots$, so that $\mathrm{Re}\xi_k < 0$. Let $\eta_0(y) \equiv \frac{\sqrt{2}}{2}$, $\eta_k(y) = \cos\left(\frac{\pi k(y+1)}{2}\right)$, $k = 1, 2, \dots$. Let us denote by $\sharp$ and $\flat$ indices taking the values $+$ or $-$. Put

$$e_j^\sharp(x) = \begin{cases} \exp(-\xi_j|x|), & \sharp x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tilde{e}_j^\sharp(x) = \begin{cases} \exp(\xi_j|x|), & \sharp x > 0, \\ 0, & \text{otherwise}. \end{cases}$$

Let us choose a $d > 0$ such that $\mathcal{O} \subset \{|x| < d\}$, and consider the set $u_j^\sharp(x, y; \omega)$ ($\sharp = \pm$, $j = 0, 1, \dots$) of solutions to (2.1), (2.2) which, for $|x| > d$, have the following form:

$$u_j^\sharp(x, y; \omega) = e_j^\sharp(x)\eta_j(y) + \sum_{\flat=\pm}\sum_{k=0}^{\infty} T_{jk}^{\sharp\flat}(\omega)\tilde{e}_k^\flat(x)\eta_k(y). \tag{2.3}$$

Here the $T_{jk}^{\sharp\flat}(\omega)$ are some (complex–valued) functions of ω. Formulae (2.1)–(2.3) define the $u_j^\sharp$, and, therefore, the $T_{jk}^{\sharp\flat}(\omega)$, uniquely (this follows from the properties of the resolvent).

Let μ be a real number from the interval $(\pi(m-1)/2, \pi m/2)$, $m \in \mathbb{N}$. Then the limits

$$u_j^\sharp(x, y; \mu) := \lim_{\omega \to \mu,\, \mathrm{Im}\omega < 0} u_j^\sharp(x, y; \omega)$$

exist. Moreover, for $j \le m$ the functions $u_j^\sharp(x, y; \mu)$ are bounded and are called *generalised eigenfunctions of the continuous spectrum.* They are called generalised because they do not lie in $L^2(G)$. However, one can easily extend the definition of the L^2–inner product to pairs

$$(\text{generalised eigenfunction}, \text{proper eigenfunction})\,.$$

It turns out that any generalised eigenfunction is orthogonal to any proper eigenfunction.

The infinite matrix

$$\mathbf{T}(\omega) := \left\{T_{jk}^{\sharp\flat}(\omega)\right\}_{\sharp,\flat=\pm;\, j,k=0,1,\dots}$$

is called the (extended) *scattering matrix* of problem (2.1), (2.2); see [6] and [7] for more details. It can be shown that $\mathbf{T}(\omega)$ admits a meromorphic continuation onto Λ. Moreover, it turns out that any resonance $\omega \in \Lambda$ is either

(i) an eigenvalue (and, therefore, $\omega \in \mathbb{R}$),

or

(ii) a pole of the scattering matrix (and $\omega \in \Lambda \setminus \mathbb{R}$).

So far we do not seem to have simplified the problem at all: finding the poles of an infinite matrix is not an easy problem. However, if we restrict our attention to a fixed sheet of Λ, then we will have to follow only finitely many entries of $\mathbf{T}(\omega)$. In particular, if we are interested in resonances lying on the first *non–physical sheet* (which can be identified with the upper half–plane $\mathrm{Im}\omega > 0$ accessed from the lower half–plane through the interval $(-\pi/2, \pi/2)$), then they coincide with the poles of the 2×2 matrix $T(\omega) := \{T_{00}^{\sharp\flat}(\omega)\}_{\sharp,\flat=\pm}$ or even with the poles of its determinant $t(\omega) := T_{00}^{++}(\omega)T_{00}^{--}(\omega) - T_{00}^{+-}(\omega)T_{00}^{-+}(\omega)$. The functional equation for the scattering matrix [6] implies that a point $\omega = \overline{\omega_*}$, $\mathrm{Im}\overline{\omega_*} > 0$, is a pole of $t(\omega)$ iff $t(\omega_*) = 0$. Therefore, instead of looking for poles of $t(\omega)$ in the upper half–plane (the lower half–plane $\mathrm{Im}\omega < 0$ is pole–free), we can look for zeros of $t(\omega)$ in the lower half–plane.

In our numerical computations we restrict ourselves to the consideration of obstacles symmetric about the vertical axis (though the main result covers the general case). Then all solutions to (2.1) are linear combinations of those satisfying $u(x, y) = \pm u(-x, y)$. Therefore, it suffices to consider the right half of G for which $x > 0$, and set either the Neumann or the Dirichlet boundary condition at $x = 0$, respectively. The problem of finding resonances also splits into two problems on $G^+ := \{(x, y) \in G,\ x > 0\}$ with the corresponding conditions on the new part of the boundary. Indeed, let $u_0^D(x, y; \omega)$ and $u_0^N(x, y; \omega)$ be solutions to (2.1), (2.2) considered in G^+ with additional boundary conditions

$$u_0^D(0, y; \omega) = \frac{\partial}{\partial x} u_0^N(0, y; \omega) = 0, \tag{2.4}$$

which for $x > d$ have the following form:

$$u_0^D(x, y; \omega) = \exp(-\xi_0 x)\eta_0(y) + \sum_{k=0}^{\infty} T_k^D(\omega) \exp(\xi_k x)\eta_k(y), \tag{2.5}$$

and, similarly,

$$u_0^N(x, y; \omega) = \exp(-\xi_0 x)\eta_0(y) + \sum_{k=0}^{\infty} T_k^N(\omega) \exp(\xi_k x)\eta_k(y). \tag{2.6}$$

Here $T_k^D(\omega)$ and $T_k^N(\omega)$ are complex–valued functions. Conditions (2.1), (2.2), (2.4)–(2.6) determine the functions u_0^D and u_0^N uniquely. It is easily seen that the symmetry of the problem implies that for $x > 0$ we have:

$$u_0^+ = \frac{u_0^N + u_0^D}{2}, \qquad u_0^- = \frac{u_0^N - u_0^D}{2}.$$

For the coefficients of the scattering matrix this means that

$$T_{00}^{++} = T_{00}^{--} = \frac{T_0^N + T_0^D}{2},$$

and

$$T_{00}^{+-} = T_{00}^{-+} = \frac{T_0^N - T_0^D}{2}.$$

Thus, the determinant t factorises, $t(\omega) = T_0^N(\omega)T_0^D(\omega)$, and, consequently, the set of zeros of t is the union of the zeros of T_0^N and T_0^D. In what follows we consider both problems (2.1), (2.2), (2.4)–(2.6) simultaneously and use the notation S for either T_0^N or T_0^D.

Naturally, instead of the infinite domain G^+ we work in a truncated domain $G^+(a) := \{(x,y) \in G^+,\ x < a\}$ with appropriate boundary conditions on the right side. To obtain an explicit boundary condition equivalent to (2.5) or (2.6) at some sufficiently large $x = a$, let us first consider equation (2.1) in $G^+(a)$. We set boundary conditions (2.2) and one of (2.4) at the corresponding parts of $\partial G^+(a)$ and represent the solution to this problem for $x \geq a$ as

$$u(x,y;\omega) = \exp(i\omega x) + S(\omega)\exp(-i\omega x) + \sqrt{2}\sum_{k=1}^{\infty} T_k(\omega)\eta_k(y)\exp(\xi_k x) \tag{2.7}$$

(recall that here $\operatorname{Im}\omega < 0$). The function $S(\omega)$ will be called the *scattering coefficient*.

As has been discussed above, $S(\omega_*) = 0$ iff $\overline{\omega_*}$ is a complex resonance of the problem. In this case $u(x,y;\omega_*)$ will be called a *resonance eigenfunction*. Let us emphasise that a resonance eigenfunction always grows exponentially at infinity. The coefficient $S(\omega)$ is uniquely defined by relation (2.7). For certain obstacles we will evaluate numerically the function S and find its zeros in the lower complex half–plane.

Remark. Sometimes not u itself, but its complex conjugate is called the resonance eigenfunction. This is motivated by the fact that $\overline{u}$ is a solution of our problem at the resonance $\overline{\omega_*}$ subject to a properly understood *radiation condition*.

Let us now replace (2.7) by an equivalent boundary condition at $x = a$. Denote the value of $u(x,y;\omega)$ and its normal derivative at the vertical interval $x = a$ by

$$u_a(y) = u\Big|_{x=a}, \qquad v_a(y) = \frac{\partial u}{\partial x}\Big|_{x=a},$$

respectively (of course, these functions depend also on ω). Given the expansion (2.7), its coefficients can be obtained by means of differentiating (2.7) with respect to x and integrating the result against $\eta_k(y)$ over $[-1, 1]$:

$$T_k = \frac{1}{\sqrt{2}\,\xi_k \exp(\xi_k a)} \int_{-1}^{1} v_a(y)\eta_k(y)dy, \qquad k = 1, 2, \dots;$$

similarly, the scattering coefficient is given by

$$S = \left(\exp(i\omega a) - \frac{1}{2i\omega}\int_{-1}^{1} v_a(y)dy\right)\exp(i\omega a). \tag{2.8}$$

Here we use the fact that the system of functions $\eta_k(y)$, $y \in [-1, 1]$, $k = 0, 1, \dots$, is orthonormal. Thus, the function $u_a(y)$ is expressed via $v_a(y)$ as follows:

$$u_a(y) = \sum_{k=1}^{\infty} \frac{\eta_k(y)}{\xi_k}\int_{-1}^{1} v_a(t)\eta_k(t)dt + \frac{i}{2\omega}\int_{-1}^{1} v_a(t)dt + 2\exp(i\omega a). \tag{2.9}$$

Note that the series in (2.9) converges. Indeed, integration by parts

$$\int_{-1}^{1} v_a(t)\eta_k(t)dt = -\frac{2}{\pi k}\int_{-1}^{1} v_a'(t)\sin\left(\frac{\pi k(t+1)}{2}\right)dt$$

immediately implies uniform convergence. Moreover, repeated integration by parts shows that the terms of the series vanish faster than any given inverse power of k.

We have two problems in $G^+(a)$:

$$\Delta u + \omega^2 u = 0, \qquad (x, y) \in G^+(a),$$

$$\frac{\partial u}{\partial n} = 0, \qquad (x, y) \in (\gamma \cup \Gamma_1 \cup \Gamma_2) \cap G^+(a),$$

$$u \;=\; \mathcal{F}_{a,\omega}\left[\frac{\partial u}{\partial n}\right] + 2e^{i\omega a}, \qquad x = a,$$

$$u\Big|_{x=0} \;=\; 0, \qquad (\mathbf{D}) \qquad \text{or} \qquad \frac{\partial u}{\partial n}\Big|_{x=0} \;=\; 0; \qquad (\mathbf{N})$$

the explicit formula for the operator $\mathcal{F} = \mathcal{F}_{a,\omega}$ is given by (2.9). We shall call these problems (**D**) and (**N**), respectively. Each of these problems has a unique solution provided $\mathrm{Im}\omega \le 0$ (we work in the closure of the physical sheet) and ω is neither an eigenvalue nor a branch point.

3. Main result

Consider an obstacle with smooth boundary γ_0 of length L given by

$$x = X(s), \quad y = Y(s), \qquad 0 \le s \le L,$$

where X, Y are smooth L-periodic functions. Here the parameter s is the variable arc length, so that $X'^2(s) + Y'^2(s) \equiv 1$. We assume that the normal vector $(-Y', X')$ is outer with respect to the domain G (in other words, it points inside the obstacle). By $K(s) := X'Y'' - Y'X''$ we denote the curvature of γ_0 (of course, for a convex obstacle $K(s) \ge 0$).

We shall denote the domain G (see Section 1) by G_0, and call the self–adjoint eigenvalue problem (2.1), (2.2) in G_0 the *unperturbed problem.* We assume that the unperturbed problem has an eigenvalue $\omega_0 \in (0, \pi/2)$ corresponding to an eigenfunction $u_0(x, y)$. A sufficient condition for the existence of such an eigenvalue is, for example, γ_0 being symmetric about $y = 0$ with $X'(s) \le 0$ for $0 \le s \le L/2$ (see [1]). Other conditions (sufficient as well as necessary) can be found in [2]. We will assume that the eigenvalue ω_0 is simple, and that $\iint_{G_0} |u_0|^2 dx dy = 1$.

Let us now shift the obstacle by distance δ in the vertical direction. We denote the domain G in this case by G_δ and study the resonance problem in this domain; we shall call this problem the *perturbed problem.* We denote the boundaries of the unperturbed and perturbed obstacles by γ_0 and γ_δ respectively. Since resonances are continuous (even real–analytic) under perturbations, for small δ there is a unique resonance ω_δ, $\mathrm{Im}\omega_\delta \ge 0$, of the perturbed problem located in a small neighbourhood of ω_0 and such that $\omega_\delta \to \omega_0$ as $\delta \to 0$. Our aim is to find an asymptotic formula for $\mathrm{Im}\omega_\delta$ as $\delta \to 0$.

Let $u_0^+(x, y; \omega_0)$ and $u_0^-(x, y; \omega_0)$ be the two generalised eigenfunctions of the continuous spectrum of the unperturbed problem. According to Section 2 they are defined as solutions of

$$\Delta u + \omega_0^2 u = 0\,,$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial G_0} = 0$$

singled out by the conditions

$$u_0^+ = \exp(i\omega_0 x) + T_{00}^{++} \exp(-i\omega_0 x) + O(\exp(-\varepsilon\,|x|)) \quad \text{as} \quad x \to +\infty,$$

$$u_0^+ = T_{00}^{+-} \exp(i\omega_0 x) + O(\exp(-\varepsilon\,|x|)) \quad \text{as} \quad x \to -\infty,$$

or

$$u_0^- = \exp(-i\omega_0 x) + T_{00}^{--} \exp(i\omega_0 x) + O(\exp(-\varepsilon\,|x|)) \quad \text{as} \quad x \to -\infty,$$

$$u_0^- = T_{00}^{-+} \exp(-i\omega_0 x) + O(\exp(-\varepsilon\,|x|)) \quad \text{as} \quad x \to +\infty$$

(here ε is some positive constant) and the orthogonality relations

$$\iint_{G_0} u_0^{\pm}(x,y;\omega_0)\overline{u_0(x,y)}\,dx\,dy \;=\; 0\,.$$

The meaning of the conditions at infinity formulated above becomes clear if we introduce time dependence according to

$$v_0^{\pm}(t,x,y;\omega_0) := \exp(i\omega_0 t)u_0^{\pm}(x,y;\omega_0).$$

Then v_0^+ has the following interpretation. We have an incoming wave $\exp(i\omega_0(t+x))$ travelling from $x=+\infty$ which is being scattered by the obstacle. As a result, part of the wave, $T_{00}^{+-}\exp(i\omega_0(t+x))$, goes through towards $x=-\infty$, and another part, $T_{00}^{++}\exp(i\omega_0(t-x))$, is reflected back towards $x=+\infty$. Of course, $|T_{00}^{+-}|^2+|T_{00}^{++}|^2=1$ (conservation of energy). Similarly, v_0^- has the following interpretation. We have an incoming wave $\exp(i\omega_0(t-x))$ travelling from $x=-\infty$ which is being scattered by the obstacle. Part of the wave, $T_{00}^{-+}\exp(i\omega_0(t-x))$, goes through towards $x=+\infty$, and another part, $T_{00}^{--}\exp(i\omega_0(t+x))$, is reflected back towards $x=-\infty$. Of course, $|T_{00}^{-+}|^2+|T_{00}^{--}|^2=1$.

Given our eigenfunction $u_0(x,y)$ let us define the following function on γ_0:

$$F_{u_0}(s) := X'\left(\frac{\partial^2 u_0}{\partial s^2}+\omega_0^2 u_0\right) - KY'\frac{\partial u_0}{\partial s},$$

where $u_0=u_0(s)=u_0(X(s),Y(s))$. Put

$$\alpha_{\pm} := \frac{1}{4\omega_0}\int_0^L F_{u_0}(s)\,\overline{u_0^{\pm}(s)}\,ds\,, \tag{3.1}$$

where $u_0^{\pm}(s)=u_0^{\pm}(X(s),Y(s))$.

The main result of our paper is the following

Theorem 3.1.

$$\mathrm{Im}\omega_\delta = (|\alpha_+|^2+|\alpha_-|^2)\delta^2+O(\delta^3) \quad \text{as} \quad \delta\to 0. \tag{3.2}$$

Remark. Suppose that the obstacle is symmetric about $y=0$. Then symmetry arguments immediately imply that ω_δ is an even function of δ. Therefore, in the expansion of ω_δ only even powers will be present, and one can change $O(\delta^3)$ in (3.2) to $O(\delta^4)$.

4. Proof of the main result

Since ω_δ is analytic as a function of δ, we may express it as

$$\omega_\delta \sim \sum_{k=0}^{\infty}\Omega_k\delta^k. \tag{4.1}$$

with some constants Ω_k (of course, $\Omega_0=\omega_0$).

As we have mentioned above, the complex conjugate to the resonance $\overline{\omega}_\delta$ satisfies the equation $t(\overline{\omega_\delta}) = 0$, where $t(\,\cdot\,)$ is the 2×2 scattering determinant of the perturbed problem. Therefore, the 2×2-matrix $T(\overline{\omega}_\delta)$ has a nontrivial kernel. Let us denote by

$$\begin{pmatrix} c_\delta^+ \\ c_\delta^- \end{pmatrix}$$

the family of nontrivial vectors from the null spaces of $T(\overline{\omega}_\delta)$. Of course, the choice of c_δ^+, c_δ^- is not unique, therefore we have to normalise this vector. The fact that it belongs to the null space of $T(\overline{\omega}_\delta)$ is equivalent to saying that

$$w_\delta(x,y) := c_\delta^+ u_0^+(x,y;\overline{\omega_\delta}) + c_\delta^- u_0^-(x,y;\overline{\omega_\delta})$$

enjoys the following behaviour at infinity:

$$w_\delta(x,y) = \begin{cases} c_\delta^+ \exp(i\overline{\omega_\delta}x) + O(\exp(-\varepsilon\,|x|)) & \text{as} \quad x\to+\infty, \\ c_\delta^- \exp(-i\overline{\omega_\delta}x) + O(\exp(-\varepsilon\,|x|)) & \text{as} \quad x\to-\infty \end{cases}$$

with some $\varepsilon > 0$.

We choose the vector

$$\begin{pmatrix} c_\delta^+ \\ c_\delta^- \end{pmatrix}$$

in such a way that c_δ^+ and c_δ^- are analytic functions of δ, and $w_\delta(x,y) \to u_0(x,y)$ as $\delta\to 0$ uniformly over any compact. This normalisation is still not unique: one can multiply c_δ^+ and c_δ^- by any function $f(\delta)$ such that $f(0)=1$. However, this non-uniqueness will not be of any concern to us. After such normalisation we have:

$$c_\delta^\pm = \sum_{k=1}^{\infty} C_k^\pm \delta^k.$$

with some constants $C_k^\pm$ (note that $\lim_{\delta\to 0} c_\delta^\pm = 0$).

Let us introduce more notation: put

$$\theta_\delta(x) := \begin{cases} c_\delta^+ \exp(i\overline{\omega_\delta}x), & x\ge 0, \\ c_\delta^- \exp(-i\overline{\omega_\delta}x), & x<0, \end{cases}$$

$$h_\delta(x,y) := w_\delta(x,y) - \theta_\delta(x),$$

$$\zeta(x) := \begin{cases} C_1^+ \exp(i\omega_0 x), & x\ge 0, \\ C_1^- \exp(-i\omega_0 x), & x<0. \end{cases}$$

The function h_δ is defined over $(x,y)\in G_\delta$. However, we can smoothly continue it so that it will be defined over $\tilde{G} := \cup_{0\le\delta\le\delta_0} G_\delta$. In fact, this continuation can be carried out in such a way that the following asymptotic expansion is uniform over $\tilde{G}$:

$$h_\delta(x,y) \sim \sum_{k=0}^{\infty} H_k(x,y)\delta^k.$$

Note that each H_k vanishes exponentially as $x \to \infty$, and this decay is uniform in δ; moreover, we obviously have $H_0 = u_0$

Let us fix an arbitrary sufficiently large number a, and put $G_\delta(a) := \{(x,y) \in G_\delta,\ |x| < a\}$. From the definition of $w_\delta(x,y)$ it follows that it satisfies the Helmholtz equation in G_δ with $\omega = \overline{\omega_\delta}$, subject to Neumann boundary conditions. Therefore, when we apply Green's formula to $w_\delta(x,y)$ and $\overline{w_\delta(x,y)}$, we get

$$(\overline{\omega_\delta^2} - \omega_\delta^2)\iint_{G_\delta(a)} |w_\delta(x,y)|^2 dx\,dy \;=\; \iint_{G_\delta(a)} (\overline{w_\delta}(\Delta w_\delta) - (\Delta\overline{w_\delta})w_\delta)dx\,dy$$
$$= \int_{\partial G_\delta(a)} \left(\overline{w_\delta}\frac{\partial w_\delta}{\partial n} - w_\delta\frac{\partial\overline{w_\delta}}{\partial n}\right) d\sigma = \int_{|x|=a} \left(\overline{w_\delta}\frac{\partial w_\delta}{\partial n} - w_\delta\frac{\partial\overline{w_\delta}}{\partial n}\right) d\sigma$$

($d\sigma$ is the element of length of the boundary). Equivalently,

$$\int_{-1}^{1} \left(\overline{w_\delta}\frac{\partial w_\delta}{\partial x} - w_\delta\frac{\partial\overline{w_\delta}}{\partial x}\right)\Bigg|_{x=a} dy - \int_{-1}^{1} \left(\overline{w_\delta}\frac{\partial w_\delta}{\partial x} - w_\delta\frac{\partial\overline{w_\delta}}{\partial x}\right)\Bigg|_{x=-a} dy$$
$$+ (\overline{\omega_\delta^2} - \omega_\delta^2)\iint_{G_\delta(a)} |w_\delta(x,y)|^2 dx\,dy \;=\; 0. \tag{4.2}$$

Let us expand the LHS of (4.2) into powers of δ. It is easy to see that the coefficient at δ^0 is automatically zero. The coefficient at δ^1 is

$$-4i\Omega_0 \mathrm{Im}\Omega_1 \iint_{G_0(a)} |u_0(x,y)|^2 dx\,dy \;+ O(e^{-\varepsilon a}) = 0.$$

Going to the limit as $a \to +\infty$ we conclude that $\mathrm{Im}\Omega_1 = 0$.

Let us write down the coefficient at δ^2 in the LHS of (4.2):

$$4i\Omega_0(|C_1^+|^2 + |C_1^-|^2) - 4i\Omega_0\mathrm{Im}\Omega_2 \iint_{G_0(a)} |u_0(x,y)|^2 dx\,dy + O(e^{-\varepsilon a}) \;=\; 0$$

(here we used the fact that we already know that $\mathrm{Im}\Omega_1 = 0$). Going to the limit as $a \to +\infty$ we conclude that

$$\mathrm{Im}\Omega_2 = |C_1^+|^2 + |C_1^-|^2.$$

To prove our theorem, it remains to show that $|C_1^+|^2 + |C_1^-|^2 = |\alpha_+|^2 + |\alpha_-|^2$. Let us denote $u_1(x,y) = \zeta(x) + H_1(x,y)$ (so that $w_\delta = u_0 + \delta u_1 + O(\delta^2)$). Then u_1 satisfies the Helmholtz equation in G_0 with $\omega = \omega_0$ subject to the homogeneous Neumann boundary condition on the upper and lower sides. Keeping in mind that w_δ satisfies the homogeneous Neumann boundary condition on γ_δ, it is an easy exercise in differential geometry to show that

$$\left.\frac{\partial u_1}{\partial n}\right|_{\gamma_0} = F_{u_0}. \tag{4.3}$$

We now apply Green's formula to the functions u_1 and $\overline{u_0^+}$ and get

$$0 = (\overline{\omega_0^2} - \omega_0^2)\iint_{G_0(a)} \overline{u_0^+} u_1 dx\, dy \;=\; \iint_{G_0(a)} \left(\overline{u_0^+}\big(\Delta u_1\big) - \left(\Delta \overline{u_0^+}\right) u_1\right) dx\, dy$$
$$= \int_{\partial G_0(a)} \left(\overline{u_0^+}\frac{\partial u_1}{\partial n} - u_1 \frac{\partial \overline{u_0^+}}{\partial n}\right) d\sigma \;=\; \int_{\gamma_0} \overline{u_0^+} F_{u_0} d\sigma + 4iC_1^+\omega_0 + O(e^{-\varepsilon a}),$$

which implies

$$C_1^+ = -\frac{1}{4i\omega_0}\int_{\gamma_0} \overline{u_0^+} F_{u_0} d\sigma \;=\; i\alpha_+\,. \tag{4.4}$$

Analogously,

$$C_1^- = -\frac{1}{4i\omega_0}\int_{\gamma_0} \overline{u_0^-} F_{u_0} d\sigma \;=\; i\alpha_-\,. \tag{4.5}$$

This completes the proof of Theorem 3.1.

5. Numerical results

5.1. **Finite difference approximation.** We approximate the boundary value problem (2.1), (2.2), (2.9) by finite differences using an approach similar to that in [8]. It is based on the integration method and proves to be equivalent to the variational principle when applied to self-adjoint boundary value problems. In implementing this method we deal with integral relations instead of the differential equation (2.1). The approximation of the boundary condition (2.9) requires the introduction of additional variables compared to problems with simple (Dirichlet or Neumann, say) boundary conditions. Looking for complex resonances, we have to solve numerically a non-self-adjoint boundary value problem and hence, to face certain difficulties typical for such problems. Nevertheless, the chosen numerical method allowed us to obtain sufficiently accurate and reliable results.

In the rest of the paper we study the case when the obstacle is a disc, $\mathcal{O} := \{(x,y) :\ x^2 + (y-\delta)^2 \le R^2\}$, $|\delta| + R < 1$, which is not necessarily symmetric about the centreline of the waveguide, and present the results of computations for this problem. Taking advantage of the symmetry with respect to x we consider only the right-hand half of the waveguide (as proposed in Section 2). We are mostly interested in the resonance generated by the lowest eigenvalue. The corresponding generalised eigenfunction is symmetric about the vertical axis, and therefore is a solution to problem (**N**) introduced in Section 2. Later on, in Section 6, we deal with the numerical solution of both (**N**) and (**D**).

When studying complex resonances for different radii R and different shifts δ we are primarily interested in comparing our numerical results with the eigenvalues corresponding to the centred obstacle. The first eigenvalue for a centred disc can be found in [3], [5].

To derive a finite difference approximation for the original boundary value problem (2.1), (2.2), (2.9) we use a rectangular mesh in $G^+(a)$, as suggested in [8]. The mesh formed by straight lines parallel to the coordinate axes need not to be uniform in any direction. The curve $\gamma \cap G^+(a)$ (that is, a half-circle) is approximated by a polygonal boundary such that the (boundary) mesh points are equidistant. Such uniform selection of points at this part of the boundary generates a (non-uniform) mesh in the part of $G^+(a)$ influenced by the obstacle. Namely, in the subdomains of $G^+(a)$: $\{(x,y) : x < R\}$ and $\{(x,y) : |y-\delta| < R,\ x > R\}$ the mesh is non-uniform. In the remaining rectangular parts we use a uniform mesh formed by equidistant lines $x = x_i$, $y = y_j$. Denote the number of horizontal mesh lines by n, the number of vertical lines by m, and the total number of mesh points by L. Each mesh point (x_i, y_j) is associated with a certain closed polygonal mesh region r_{ij}. Equation (2.1) is integrated over each r_{ij}, which due to Green's formula results in L relations

$$\int_{\partial r_{ij}} u_x dy - u_y dx \, + \, \omega^2 \int_{r_{ij}}\int u dx dy = 0. \tag{5.1}$$

The line integrals in (5.1) are approximated by means of central differences (see [8] for details). Thus, we obtain L finite difference equations containing L unknown variables $u_{ij} := u(x_i, y_j)$. We have Neumann boundary conditions everywhere on $\partial G^+(a)$ except for the interval $x = a$. Obviously, putting the line integrals along the appropriate parts of $\partial G^+(a)$ to be zero we take into account the boundary conditions set on these parts as well as the differential equation itself. (Clearly, problem (**D**) is handled in quite a similar way.) Since we impose the specific non-uniform condition (2.9) involving the values of the normal derivative of u at $x = a$, we have to introduce n additional variables $v_j = v_a(y_j)$, $j = 1, \dots, n$. If we again approximate the integrals in (2.9) using central differences and replace the series by a finite sum, the discrete boundary condition takes the form

$$u_{mj} = \sum_{k=1}^{k_0} \frac{\eta_k(y_j)}{\xi_k} \sum_{l=1}^{n} \tau_l \eta_k(y_l) v_l + \frac{i}{2\omega} \sum_{l=1}^{n} \tau_l v_l + 2\exp(i\omega a), \quad j = 1, \dots, n,$$

where $\tau_1 = (y_2 - y_1)/2$, $\tau_s = (y_{s+1} - y_{s-1})/2$, $2 \le s \le n-1$, $\tau_n = (y_n - y_{n-1})/2$. Note that, since the first sum in the above relation does not converge as $k_0 \to \infty$, the convergence of a finite difference solution to that of the original boundary value problem is understood as the existence of the repeated limit $\lim\limits_{k_0\to\infty}\lim\limits_{L\to\infty} u_{ij} = u(x,y)$, whereas the double limit $\lim\limits_{k_0,L\to\infty} u_{ij} = u(x,y)$ does not necessarily exist. Taking this into account, we choose $k_0 \ll L$ (an optimal k_0 providing convergent results has been determined experimentally). Changing the order of summation in the above formulae and multiplying both parts of the j-th equation by τ_j, we finally obtain n relations approximating boundary condition (2.9):

$$\begin{gathered}\sum_{l=1}^{n} \phi_{jl} v_l - \tau_j u_{mj} = \gamma_j,\\ \phi_{jl} = \frac{i\tau_j\tau_l}{2\omega} + \sum_{k=1}^{k_0} \frac{\eta_k(y_j)\eta_k(y_l)\tau_j\tau_l}{\xi_k},\\ \gamma_j = -2\tau_j \exp(i\omega a), \qquad j = 1, \dots, n.\end{gathered} \tag{5.2}$$

Since we use the so-called five-point formula when approximating the differential equation along with the boundary conditions and replace the derivatives in integral relation (2.9) by central differences, it can be concluded that the described approximation is of second order. As is easily seen, this approach is equivalent to the Ritz method when applied to a self-adjoint problem. Basic theorems on the convergence rate for our problem are still applicable. The only delicate question mentioned above concerns approximating the series in (2.9) by a finite sum with a proper number of terms.

As mentioned before, in order to use expansion (2.7) it formally suffices to take the value of a (the length of the truncated domain $G^+(a)$) greater than the obstacle size R. In practice, the interval $x = a$ should not be too close to the obstacle. The parameter a was varied during computations and, thus, chosen experimentally. To be more precise, we increased the value of a gradually until the results for two successive values became close enough (see subsection 5.5 for the actual values of a for different discs).

5.2. **Solving the linear algebraic system.** Our approach to the finite difference approximation of (2.1) along with the corresponding boundary conditions leads to a discrete problem that can be written as a system of $L + n$ linear algebraic equations:

$$\mathbf{Au} = \mathbf{f}. \tag{5.3}$$

Here $\mathbf{u} = \|u_{11}, u_{21}, \dots, u_{mn}, v_1, \dots, v_n\|^T$, that is, we enumerate the mesh points from left to right, top to bottom and add the n variables v_j. The matrix of the system (5.3) can be represented as $\mathbf{A} = \left\| \begin{matrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{matrix} \right\|$, where $\mathbf{A}_{11} \in \mathbb{C}^{L\times L}$, $\mathbf{A}_{12} = \mathbf{A}_{21}^T \in \mathbb{C}^{L\times n}$, $\mathbf{A}_{22} \in \mathbb{C}^{n\times n}$. The explicit form of $\mathbf{A}_{11}$ can be found in [8]; the block $\mathbf{A}_{12}$ contains only n nonzero coefficients $-\tau_j$, $j = 1, \dots, n$; the right lower block is defined by (5.2): $\mathbf{A}_{22} = \|\phi_{jl}\|$; the right-hand side $\mathbf{f} = \|0, \dots, 0, \gamma_1, \dots, \gamma_n\|^T$.

Note that the matrix $\mathbf{A}$ is symmetric though complex. It is still possible to reduce (5.3) to a system with a Hermitian matrix which is more convenient to process. Namely, if we represent $\mathbf{A} = \mathbf{A}_1 + i\mathbf{A}_2$, $\mathbf{u} = \mathbf{u}_1 + i\mathbf{u}_2$, $\mathbf{f} = \mathbf{f}_1 + i\mathbf{f}_2$, then, as is easily seen, (5.3) is equivalent to

$$\left\| \begin{matrix} -\mathbf{A}_2 & \mathbf{A}_1 \\ \mathbf{A}_1 & \mathbf{A}_2 \end{matrix} \right\| \left\| \begin{matrix} \mathbf{u}_2 \\ \mathbf{u}_1 \end{matrix} \right\| = \left\| \begin{matrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{matrix} \right\|. \tag{5.4}$$

The matrix of (5.4) which we denote by $\mathbf{B}$ is real and symmetric, and, therefore, special numerical methods can be applied to solve this system. In practice it is (5.4) that we solve numerically. The matrix $\mathbf{B}$, though Hermitian, is not positive definite, and hence relaxation methods do not, in general, converge. (When testing the procedure, several relaxation methods were examined and proved to be inapplicable to our particular system (5.4).) In order to solve the system we first use the Jacobi rotation method which allows us to diagonalise $\mathbf{B}$ and find the corresponding orthonormal basis. After we have found the eigenvalues and eigenvectors of $\mathbf{B}$, (5.4) is solved in the usual way. It should be remarked that the Jacobi rotation method is known to be just several times slower than matrix inversion methods, and it can work even better in certain cases than, say, the Gauss method. This is exactly what happens in our case. Though $\mathbf{B}$ turns out to be ill-conditioned, our solution is computed up to a required accuracy, which is easily checked by substitution. The Jacobi rotation method proves to be a reliable way of solving (5.4).

5.3. **Finding complex resonances.** After the solution of the approximate problem (5.3) has been found, we calculate the resonances. To be more precise, we compute the zeros of the scattering coefficient $S(\omega)$ in the lower half-plane, keeping in mind that $S(\omega) = 0$ iff $\overline{\omega}$ is a resonance. The function $S(\omega)$ is given explicitly in terms of the solution u according to formula (2.8). We substitute the corresponding components $v_1, \dots, v_n$ of the mesh solution $\mathbf{u}$ into (2.8) using a standard second order approximate formula for the integral. We have already mentioned that the function $S(\omega)$ is analytic in the lower half-plane and, therefore, its zeros can be found via the argument principle. The number N of zeros of $S(\omega)$ (counting their multiplicities) lying inside a bounded domain $\Omega \subset \mathbb{C}$ with smooth boundary is given by the formula

$$N = \frac{1}{2\pi} \oint_{\partial\Omega} d\mathrm{Arg} S(\omega)\,, \tag{5.5}$$

provided that the function S is analytic in Ω, continuous up to $\partial\Omega$, and nonzero on $\partial\Omega$. Applications of the argument principle were considered in [9] where an appropriate computation scheme was proposed. Following that procedure, we choose a closed contour in the lower half-plane and a number of sufficiently close points $\omega_1, \omega_2, \dots, \omega_l = \omega_1$ on it. For these values of the spectral parameter the system (5.4) is successively solved as we move along the contour in the positive (anti-clockwise) direction. At each step as soon as the solution to (5.4) is calculated, the value of the scattering coefficient $S(\omega_k)$ is found as described above. The contour integral in (5.5) is computed by summing the $\Delta \mathrm{Arg} S(\omega_k) \Delta\omega_k$, $k = 1, \dots, l-1$.

Note that this procedure can be generalised. Namely, not only can we compute the number of zeros s_j of an analytic function in a particular domain but also the sum of these zeros, the sum of their squares, cubes, etc., by means of calculating the corresponding contour integrals:

$$\sum_{j=1}^{N} s_j^p = \frac{1}{2\pi i} \oint_{\partial\Omega} \omega^p d\mathrm{Ln} S(\omega)\,, \qquad p = 0, 1, \dots .$$

This would allow us to locate all the zeros in the chosen domain by solving the resulting nonlinear system. In practice we restrict ourselves to the two basic computational tasks: to calculate those particular resonances which can be regarded as small perturbations of the eigenvalues and to find the distribution of resonances without locating them. For the former problem the initial approximations are known, that is, the eigenvalues computed for a range of discs given in [3], [5]. Therefore, being interested in the resonances originating from these eigenvalues, we start from the known results for $\delta = 0$ and choose sufficiently small contours containing only the required zeros of the scattering coefficient. Thus, the numerical technique given in [9] provides reliable results which are presented below.

5.4. **Calculating corrections to eigenvalues.** Our numerical calculations are primarily aimed at verifying the main theoretical result, that is, Theorem 3.1. To this end we need to calculate the coefficients $\alpha_\pm$ appearing in the right-hand side of (3.2). It is convenient for us to evaluate these coefficients via the function u_1 which is the first order correction to the trapped mode u_0 (see Section 4 for details), rather than use the original definition (3.1). Recall that u_1 is a solution of the partial differential equation

$$\Delta u_1 + \omega_0^2 u_1 = 0, \qquad (x, y) \in G_0, \tag{5.6}$$

subject to homogeneous Neumann conditions on the sides of the waveguide

$$\frac{\partial u_1}{\partial n} = 0, \qquad (x, y) \in \Gamma_1, \Gamma_2, \tag{5.7}$$

the non-homogeneous Neumann condition (4.3) on the boundary of the unperturbed obstacle, and the radiation conditions

$$\frac{\partial u_1}{\partial x} \mp i\omega_0 u_1 \to 0 \quad \text{as} \quad x \to \pm\infty. \tag{5.8}$$

Then, according to (4.4) and (4.5),

$$u_1 \sim i\alpha_\pm \exp(\pm i\omega_0 x) \quad \text{as} \quad x \to \pm\infty. \tag{5.9}$$

The solution u_1 of the problem (5.6), (5.7), (4.3), (5.8) is non-unique as one can always add the eigenfunction u_0 multiplied by an arbitrary constant. This non-uniqueness has no effect on the asymptotic behaviour of u_1 as $x \to \pm\infty$ and, consequently, no effect on the constants $\alpha_\pm$. However, for numerical calculations it is better to deal with a problem which has a unique solution. We achieve uniqueness by imposing the orthogonality condition

$$\iint_{G_0} u_1(x, y)\overline{u_0(x, y)}\, dx\, dy \;=\; 0\,.$$

As our unperturbed ($\delta = 0$) obstacle is symmetric about the centreline of the waveguide and the eigenfunction u_0 is odd in y, the above orthogonality condition is equivalent to the condition that u_1 is even in y:

$$u_1(x, -y) = u_1(x, y). \tag{5.10}$$

The unperturbed obstacle is symmetric about the vertical axis and the eigenfunction u_0 with which we are working is even in x, therefore the (unique) solution u_1 of the problem (5.6), (5.7), (4.3), (5.8), (5.10) turns out to be even in x:

$$u_1(-x, y) = u_1(x, y). \tag{5.11}$$

The latter implies $\alpha_+ = \alpha_-$.

Before solving the problem (5.6), (5.7), (4.3), (5.8), (5.10) we have to obtain the corresponding eigenfunction involved into (4.3). Given the first eigenvalue ω_0 of the unperturbed problem (see Section 3) calculated by Callan *et al.* [3], we use the following approach to compute u_0.

The eigenfunction we are looking for is the solution of (2.1), (2.2) decaying at infinity. To get a more convenient boundary condition at infinity we represent $u_0(x,y)$ as

$$u_0(x,y)=\sum_{k=1}^{\infty} b_k\eta_k(y)\exp(\xi_k x), \qquad x\ge a, \tag{5.12}$$

where a is an arbitrary number greater than the radius of the disc R. Formula (5.12) is obtained from (2.7) by omitting the two oscillating terms (recall that we are looking at the case when $\omega=\omega_0$ is real). Of course, all the terms in (5.12) decay exponentially at infinity. By analogy with (2.9), we write an integral boundary condition at $x=a$ having the form $u_0(a,y)=\mathcal{F}_0[\frac{\partial u_0}{\partial x}]$. The finite difference approach does not differ from that designed for the problem (2.1), (2.2), (2.9). Naturally, instead of the non-homogeneous system (5.3) we get a homogeneous system

$$\mathbf{A}_0\mathbf{u}_0=0$$

where the matrix $\mathbf{A}_0$ is real and symmetric. This matrix $\mathbf{A}_0$ is easily obtained from the matrix $\mathbf{A}$ of (5.3); actually, $\mathbf{A}_0=\operatorname{Re}\mathbf{A}|_{\omega=\omega_0}$. We know in advance that the matrix $\mathbf{A}_0$ is degenerate and that the eigenvector corresponding to the zero eigenvalue of $\mathbf{A}_0$ is the set of values of u_0 at the mesh points up to the chosen accuracy. For $|x|>a$ we extend the eigenfunction u_0 recovering the coefficients b_k of the expansion (5.12) from the values of $u_0(a,y)$, and, finally, calculate $\|u_0\|$. (Recall that in Theorem 3.1 it is assumed that $\|u_0\|=1$.) Note that the same technique is applicable if we are looking for the eigenvalues of (2.1), (2.2), i.e., such values of ω that $\det\mathbf{A}(\omega)=0$.

Now that we have computed the eigenfunction we proceed to solving the boundary value problem (5.6), (5.7), (4.3), (5.8), (5.10). First, we pose a boundary condition at $x=a$ replacing the condition (5.8) at infinity. The proper boundary condition at $x=a$ is the following:

$$\begin{aligned}u_1(a,y)=-\sum_{k=1}^{\infty}\frac{\cos(\pi k(y+1))}{\sqrt{(\pi k)^2-\omega_0^2}}\int_{-1}^{1}\frac{\partial u_1}{\partial x}(a,t)\cos(\pi k(t+1))dt\\ -\frac{i}{2\omega_0}\int_{-1}^{1}\frac{\partial u_1}{\partial x}(a,t)dt.\end{aligned} \tag{5.13}$$

Condition (5.13) is, of course, similar to (2.9). The problem (5.6), (5.7), (4.3), (5.10), (5.13) in the truncated domain is solved numerically in the same way as described in subsections 5.1, 5.2. Namely, we use a similar mesh to approximate the problem and handle the boundary condition (5.13) in the same way. Note that the boundary value problem under consideration differs from (2.1), (2.2), (2.9) only by a particular form of boundary conditions at γ_0 and $x=a$. Thus, the basic numerical procedure described in subsections 5.1, 5.2 (with slight modifications) applies to the calculation of generalised eigenfunctions, eigenfunctions and their first order (with respect to δ) correction terms. This can be regarded as an advantage of the numerical method designed for the problem as a whole.

After the correction u_1 has been found we calculate the coefficients $\alpha_\pm$. Using for this purpose the asymptotic expansion (5.9) is not convenient, so we calculate these coefficients using the equivalent formula

$$\alpha_\pm = \mp\frac{\exp(\mp i\omega_0 a)}{2\omega_0}\int_{-1}^{1}\frac{\partial u_1}{\partial x}(a,y)\,dy\,.$$

Having found the coefficients $\alpha_\pm$, we proceed to the numerical verification of Theorem 3.1.

5.5. **Results of calculations.** In this subsection we present results of calculations obtained by means of techniques described in subsections 5.1–5.4. Recall that our numerical calculations concern the case of a disc of radius R shifted by distance δ from the centreline of the waveguide.

Tables 1 and 2 give the value of the resonance ω_δ for different R and δ. The eigenvalue ω_0 for different R can be found in [5, Table 1]; it corresponds to an eigenfunction symmetric about $x = 0$. The resonance eigenfunction is also symmetric about $x = 0$ (cf. (5.11)), that is, it is a solution of the problem (**N**), see end of Section 2.

Tables 1 and 2 give also the values of the parameter a used in calculations. Recall that a determines the size of the truncated domain $G^+(a)$. Choosing a particular value of a is, to a large extent, a matter of convenience. Theoretically, the resulting resonances ω_δ do not depend on a, and numerically the dependence turns out to be very insignificant.

In the series (5.2) we took account of $k_0 = 3$ terms. The solution of the problem (**N**) was calculated up to a global accuracy 10^{-4} provided by the finite difference approximation for the actual mesh size. The procedure for finding the zeros of the scattering coefficient $S(\omega)$ was more precise.

Table 3 gives the value of $(|\alpha_+|^2 + |\alpha_-|^2)\delta^2$ for different R and δ.

Comparing the results of Table 3 with those of Tables 1, 2 we can judge how well formula (3.2) really works. Examination of these tables shows that our numerical results are in good agreement with Theorem 3.1. In order to make this agreement more transparent we give Figure 2 which plots the dependence of $\mathrm{Im}\omega_\delta$ on δ for $R = 0.5$.

6. Estimates for the number of resonances

In this section we present results of computations of the total number of resonances within a disc $\{|\omega| < \rho\}$ in the complex ω–plane. We base our calculations on the use of formula (5.5) with

$$\Omega = \{|\omega| < \rho\} \cap \{\mathrm{Im}\omega < 0\}$$

(recall that the scattering coefficient $S(\omega)$ has no zeros in the upper half-plane). As the scattering coefficient possesses the property $S(\omega) = \bar{S}(-\bar{\omega})$, it suffices to calculate the integral (5.5) along the part of the contour $\partial\Omega$ which lies, say, in the right half-plane. Thus, we take the arc of a circle

$$\{|\omega| = \rho\} \cap \{\mathrm{Im}\omega < 0\} \cap \{\mathrm{Re}\omega > 0\}$$

and the interval of the real axis $[\rho, 0]$ as the actual integration path. The value of the integral along this curve is half the value of the full integral (5.5).

The problem of finding the total number of resonances within a disc of radius ρ is much more burdensome from the numerical point of view than the calculation of a specific resonance, especially when we are given a proper initial approximation. Indeed, dealing with a large contour, we often have to refine the mesh at it. We do not know in advance how many resonances we expect to find inside the disc and have no information about their location either. Therefore, the estimation of the number of resonances for different radii up to $\rho = 30$ turned out to be a formidable numerical problem.

Table 4 gives results for problems (**N**) and (**D**). The domain G is one of those considered in Section 5: a waveguide with a disc of radius $R = 0.5$ shifted by $\delta = 0.25$. In our calculations we took $a = 0.9$. The number k_0 (see (5.2)) was chosen to vary as we moved along the path: $k_0(\omega) = \left[\frac{2}{\pi}\sqrt{\mathrm{Re}(\omega^2)}\right] + 3$, where $[z]$ stands for the integer part of z. Thus, in formula (2.7) we disregarded all the terms which are sufficiently small at $x = a$. Note that when we consider real $\omega \geq \frac{\pi}{2}$ the oscillating terms appearing in the series cannot be neglected.

Table 1. Values of ω_δ. $R = 0.3,\ \omega_0 = 1.5048,\ a = 0.675;$ $R = 0.5,\ \omega_0 = 1.3913,\ a = 0.9$

δ	R=0.3	R=0.5
0.050		$1.3946 + 0.2 \cdot 10^{-3}i$
0.067	$1.5099 + 10^{-4}i$	$1.3958 + 0.4 \cdot 10^{-3}i$
0.100	$1.5102 + 10^{-4}i$	$1.3995 + 0.8 \cdot 10^{-3}i$
0.125	$1.5118 + 2 \cdot 10^{-4}i$	$1.4034 + 1.4 \cdot 10^{-3}i$
0.167	$1.5153 + 3 \cdot 10^{-4}i$	$1.4117 + 2.6 \cdot 10^{-3}i$
0.200	$1.5188 + 5 \cdot 10^{-4}i$	$1.4196 + 3.9 \cdot 10^{-3}i$
0.250	$1.5249 + 7 \cdot 10^{-4}i$	$1.4349 + 6.3 \cdot 10^{-3}i$

Table 2. Values of ω_δ. $R = 0.7,\ \omega_0 = 1.3229,\ a = 1.0;$ $R = 0.9,\ \omega_0 = 1.3518,\ a = 1.3$

δ	R=0.7	R=0.9
0.050	$1.3271 + 1.0 \cdot 10^{-3}i$	$1.3570 + 3.1 \cdot 10^{-3}i$
0.067	$1.3291 + 1.8 \cdot 10^{-3}i$	$1.3612 + 6.6 \cdot 10^{-3}i$
0.100	$1.3349 + 4.1 \cdot 10^{-3}i$	$1.3691 + 1.28 \cdot 10^{-2}i$
0.125	$1.3415 + 6.4 \cdot 10^{-3}i$	$1.3710 + 1.61 \cdot 10^{-2}i$
0.167	$1.3538 + 1.12 \cdot 10^{-2}i$	$1.3918 + 3.31 \cdot 10^{-2}i$
0.200	$1.3662 + 1.61 \cdot 10^{-2}i$	$1.4023 + 5.01 \cdot 10^{-2}i$
0.250	$1.3790 + 2.46 \cdot 10^{-2}i$	$1.4098 + 6.12 \cdot 10^{-2}i$

Table 3. Values of $(|\alpha_+|^2 + |\alpha_-|^2)\delta^2$

δ	$R = 0.3$	$R = 0.5$	$R = 0.7$
0.050		$0.2 \cdot 10^{-3}$	$0.9 \cdot 10^{-3}$
0.067	$0.05 \cdot 10^{-3}$	$0.4 \cdot 10^{-3}$	$1.7 \cdot 10^{-3}$
0.100	$0.12 \cdot 10^{-3}$	$0.9 \cdot 10^{-3}$	$3.8 \cdot 10^{-3}$
0.125	$0.19 \cdot 10^{-3}$	$1.4 \cdot 10^{-3}$	$5.9 \cdot 10^{-3}$
0.167	$0.33 \cdot 10^{-3}$	$2.6 \cdot 10^{-3}$	$10.4 \cdot 10^{-3}$
0.200	$0.47 \cdot 10^{-3}$	$3.7 \cdot 10^{-3}$	$15.0 \cdot 10^{-3}$
0.250	$0.76 \cdot 10^{-3}$	$5.8 \cdot 10^{-3}$	$23.5 \cdot 10^{-3}$

Table 4. Distribution of resonances for problems (**N**) and (**D**)

ρ	3	5	10	20	30
(**N**)	2	6	16	54	70
(**D**)	0	2	12	34	42

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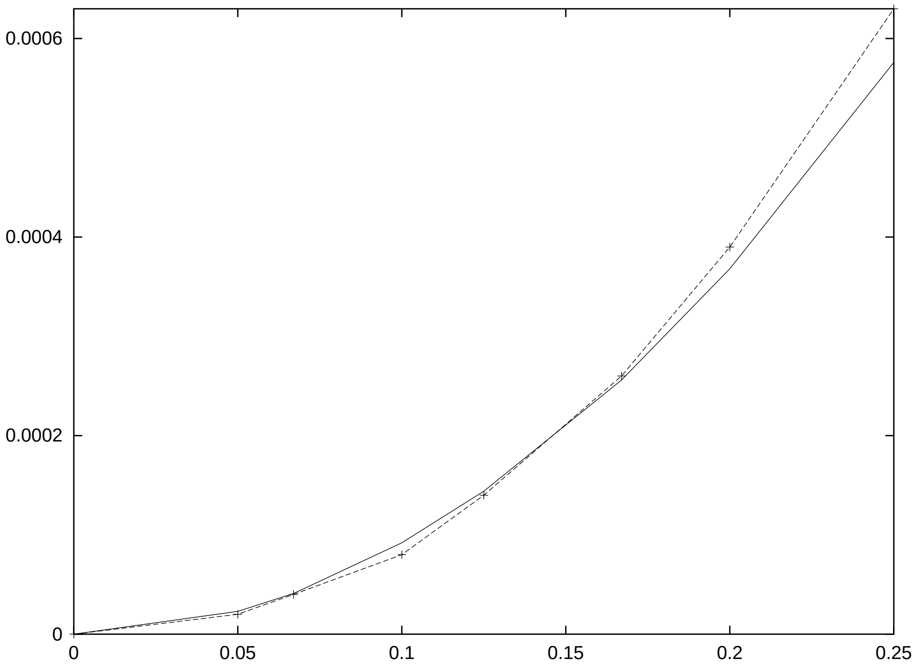

Figure 2. Dependence of $\mathrm{Im}\omega_\delta$ on δ. Dotted line relates to Table 1, solid line — to Table 3; $R = 0.5$

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