Anomalous electron trapping by localized magnetic fields

F. Bentosela,[a,b] R.M. Cavalcanti,[c] P. Exner,[d,e] V.A. Zagrebnov[a,b]

a) Centre de Physique Théorique, C.N.R.S., 13288 Marseille Luminy
b) Université de la Mediterranée (Aix–Marseille II), 13288 Marseille Luminy, France
c) Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106-4030, USA
d) Nuclear Physics Institute, Academy of Sciences, 25068 Řež
e) Doppler Institute, Czech Technical University, Břehová 7, 11519 Prague, Czech Republic
bentosela@cpt.univ-mrs.fr, ricardomc@hotmail.com,
exner@ujf.cas.cz, zagrebnov@cpt.univ-mrs.fr

We consider an electron with an anomalous magnetic moment $g > 2$ confined to a plane and interacting with a nonzero magnetic field B perpendicular to the plane. We show that if B has compact support and the magnetic flux in the natural units is $F \geq 0$, the corresponding Pauli Hamiltonian has at least $1 + [F]$ bound states, without making any assumptions about the field profile. Furthermore, in the zero-flux case there is a pair of bound states with opposite spin orientations. Using a Birman-Schwinger technique, we extend the last claim to a weak rotationally symmetric field with $B(r) = \mathcal{O}(r^{-2-\delta})$ correcting thus a recent result. Finally, we show that under mild regularity assumptions the existence can be proved for non-symmetric fields with tails as well.

1 Introduction

Interaction of electrons with a localized magnetic field has been a subject of interest for a long time. It has been observed recently that a magnetic flux tube can bind charged particles with anomalous magnetic moment $g > 2$. An example of such a particle is the electron, which has $g = 2.0023$.

The effect was observed first in simple examples [BV, CFC, Mo] such as a cylindrical tube with a field which is either homogeneous or supported by the tube surface. The same behaviour has been then demonstrated for any rotationally invariant field $B(r)$ with compact support and which does not change sign [CC]. In the next step the symmetry condition was removed and the positivity requirement weakened [BEZ2]. The main aim of the present paper is to complete this process by showing that bound states exist for any (nontrivial) compactly supported field and their number is controlled by the number of flux quanta: the corresponding Pauli Hamiltonian will be shown to have at least $1+[F]$ negative eigenvalues, where F is the value of the flux through the tube in natural units.

This improvement is made possible by a pair of new tools. First of all, the supersymmetry properties of Pauli operators allow us to show that the matrix element of the field appearing in the sufficient condition of [BEZ2] has in fact a definite sign. This trick will be combined with a more sophisticated variational estimate which enables us to treat the integer-flux situation at the same footing as the other cases. In particular, we will be able to demonstrate in this way that for a nonzero B a bound state due to an excess magnetic moment exists even if the flux is zero. More than that, an analogous argument shows that in this situation the field binds electrons with *both* spin orientations.

While the proof of the last result requires a compact support and does not cover fields with tails extending to infinity, it raises a question about a claim made in a recent paper by some of us [BEZ1]. It was said there that a system with a particular rotationally symmetric field induced by an electric current vortex has no bound states for weak currents. This is not correct: the statement is true for higher partial waves only, while the s-wave part has in reality a nontrivial spectrum for any nonzero current.

The error is subtle and — as we hope — instructive: it illustrates well the fine nature of weakly bound states of Schrödinger operators in one and two dimensions. The point is that caution is needed when the coupling is switched off *nonlinearly*: the case in question represents an example of a two-dimensional Schrödinger operator with a potential which has a *positive mean* for any nonzero coupling constant while still having a bound state.

To set things straight we discuss in the last three sections the weak field, zero-flux case in detail by performing the corresponding Birman-Schwinger analysis to second order. For centrally symmetric fields it yields $g > 2$ as a sufficient condition for the existence of bound states, and provides an asymptotic formula for the bound-state energy. We also show that adding some regularity assumptions one can prove in this way the existence of weakly bound states for non-symmetric fields with tails as well.

2 Preliminaries

As we have said we consider a particle of spin 1/2 living in a plane and subject to a non-homogeneous magnetic field B perpendicular to the plane. Here and in the next section we suppose that B has support in a compact region Σ of $\mathbb{R}^2$; later we shall replace this by a suitable decay requirement. No hypotheses are made here about the field profile; we assume just its integrability, $B \in L^1(\Sigma)$. The corresponding vector potential $A = (A_1, A_2)$ lies in the plane and $B = \partial_1 A_2 - \partial_2 A_1$. Throughout the paper we employ natural units, $2m = \hbar = c = e = 1$.

Remark. The assumptions do not include the singular field profile $B(x) = 2\pi F\,\delta(x)$ (a magnetic string). Although it can be regarded as a squeezing limit of L^1 fields, the procedure is non-trivial: as pointed out in [BV] one has to perform at the same time the non-physical limit $g \to 2$ to preserve the existence of bound states in analogy with the coupling constant renormalization for the usual two-dimensional δ interaction [AGHH]. We will not discuss this case here.

The particle dynamics is described by the two–dimensional Pauli Hamiltonian which we write in the standard form [Th],

$$H_P^{(\pm)}(A) = (-i\nabla - A(x))^2 \pm \frac{g}{2}\,B(x) = D^*D + \frac{1}{2}\,(2 \pm g)B(x)\,, \tag{2.1}$$

where $D := (p_1 - A_1) + i(p_2 - A_2)$ and the two signs correspond to the two possible spin orientations. The quantity

$$F := \frac{1}{2\pi}\int_\Sigma B(x)\,d^2x \tag{2.2}$$

is the total flux measured in the natural units $(2\pi)^{-1}$, or the number of flux quanta through Σ. We assume conventionally that $F \geq 0$, *i.e.*, if the mean field is nonzero it points up. In such a case we will be interested primarily in the operator $H_P^{(-)}(A)$ which describes an electron with its magnetic moment parallel to the flux.

Next we have to recall a classical result of Aharonov and Casher [AC, Th] which will be a basic ingredient of our argument in the next section. It says that if $F = N + \varepsilon$, $\varepsilon \in (0, 1]$ for a positive integer N, the operator $H_P^{(-)}(A)$ with nonanomalous moment, $g = 2$, has N zero energy eigenvalues. In the gauge $A_1 = -\partial_2\phi$, $A_2 = \partial_1\phi$, where

$$\phi(x) := \frac{1}{2\pi}\int_\Sigma B(y)\,\ln|x - y|\,d^2y\,, \tag{2.3}$$

the corresponding eigenfunctions are given explicitly by

$$\chi_j(x) = e^{-\phi(x)}\,(x_1 + ix_2)^j, \quad j = 0, 1, \dots, N-1\,. \tag{2.4}$$

It is easy to check that $D\chi_j = 0$ for any nonnegative integer j, but only those functions listed in (2.4) are square-integrable; this follows from the fact that $\chi_j(x) = \mathcal{O}(|x|^{-F+j})$ as $|x| \to \infty$ — *cf.* [AC], [Th, Sec.7.2]. However, the functions χ_j with $j = [F]$ and $j = [F] - 1$ (the latter in the case F is a positive integer; as usual, the symbol $[\cdot]$ denotes the integer part) are zero energy resonances, since they solve the equation $H_P^{(-)}(A)\chi_j = 0$ and do not grow at large distances.

3 Flux tubes

Now we are in position to state our main result about the existence and number of bound states of the operator (2.1).

Theorem 1. If $B \in L^1$ is nonzero and compactly supported, the operator $H_P^{(-)}(A)$ has for $g > 2$ at least $1+[F]$ negative eigenvalues. Moreover, if $F = 0$ then $H_P^{(+)}(A)$ also has a bound state.

Proof: By the minimax principle, it is sufficient for the first claim to find a subspace of dimension $1+[F]$ on which the quadratic form

$$(\psi, H_P^{(-)}(A)\psi) = \int_{\mathbb{R}^2} |(D\psi)(x)|^2\, d^2x - \frac{1}{2}\,(g-2)\int_{\mathbb{R}^2} B(x)|\psi(x)|^2\, d^2x$$

is negative. To construct appropriate trial functions ψ_α we employ the above mentioned zero–energy solutions; specifically, we choose

$$\psi_\alpha(x) = \sum_{j=0}^{[F]} \alpha_j\left(f_{R,\kappa}(r)\chi_j(x) + \varepsilon h_j(x)\right)\,, \tag{3.1}$$

where $h_j \in C_0^2(\Sigma)$ will be specified later and $f_{R,\kappa} : \mathbb{R}_+ \to \mathbb{R}$ is a suitable function such that $f_{R,\kappa}(r) = 1$ for $r := |x| \le R$, with R chosen in such a way that Σ is a subset of $\mathcal{B}_R := \{x :\ |x| \le R\}$. Clearly it is sufficient to consider coefficient vectors $\alpha \in \mathbb{C}^{1+[F]}$ with $|\alpha| = 1$.

It is straightforward to compute the value of the energy form; with a later purpose on mind we write it as

$$\begin{aligned}(\psi_\alpha, (D^*D+\mu B)\psi_\alpha) \;=\; & \sum_{j,k=0}^{[F]} \bar\alpha_j\alpha_k \left\{\int_{\mathbb{R}^2} \left|f'_{R,\kappa}(r)\right|^2 (\bar\chi_j\chi_k)(x)\, d^2x\right. \\ &+\; \varepsilon^2\int_\Sigma (\overline{Dh_j})(x)(Dh_k)(x)\, d^2x + \mu\left[\int_\Sigma (B\bar\chi_j\chi_k)(x)\, d^2x\right. \\ &+\; \varepsilon\int_\Sigma ((\bar h_j\chi_k + \bar\chi_j h_k)B)(x)\, d^2x + \left.\left.\varepsilon^2\int_\Sigma (B\bar h_j h_k)(x)\, d^2x\right]\right\}\end{aligned} \tag{3.2}$$

with $\mu = -\frac{1}{2}\,(g-2)$. We have employed here the property $D\chi_j = 0$ of the AC functions and the fact that h_j and $f'_{R,\kappa}$ have by assumption disjoint supports: we have $D\Sigma_j\alpha_j f_{R,\kappa}\chi_j = 0$ inside Σ so $D\psi_\alpha = \varepsilon\Sigma_j\alpha_j Dh_j$ there, while outside Σ we have $h_j = 0$, so $D\psi_\alpha = Df_{R,\kappa}\Sigma_j\alpha_j\chi_j = \Sigma_j\alpha_j\chi_j(p_1+ip_2)f_{R,\kappa} = \chi_\alpha(-ix_1+x_2)r^{-1}f'_{R,\kappa}$.

As a warm-up, suppose first that $F - j > 1$ holds for all nonzero coefficients α_j. Then the corresponding $\chi_j \in L^2$ and we can use the simplest choice $f_{R,\kappa} = 1$ and $\varepsilon = 0$ in (3.1) obtaining

$$(\psi_\alpha, H_P^{(-)}(A)\psi_\alpha) = -\frac{1}{2}\,(g-2)\int_\Sigma B(x)|\psi_\alpha(x)|^2\, d^2x\,. \tag{3.3}$$

Suppose that $(\psi_\alpha, B\psi_\alpha) \le 0$. Since $D^*D\psi_\alpha = 0$, this would imply the inequality $(\psi_\alpha, (D^*D+2B)\psi_\alpha) \le 0$, but the operator in parentheses equals DD^* giving thus

$\|D^*\psi_\alpha\|^2 \le 0$. This is possible only if $D^*\psi_\alpha = 0$, which is false, because otherwise we would have $2B\psi_\alpha = (DD^* - D^*D)\psi_\alpha = 0$ or $B(x)\psi_\alpha = 0$ almost everywhere. Since ψ_α is a product of a positive function $e^{-\phi(x)}$ and a polynomial in $x_1 + ix_2$, it has at most $[F] - 1$ zeros — recall that we are assuming $j < F - 1$ — and we arrive at $B(x) = 0$ a.e. which contradicts the assumption. Consequently, the r.h.s. of (3.3) is negative for $g > 2$.

If the linear combination includes α_j with $0 \le F - j \le 1$, the situation is more complicated. Since the corresponding AC functions are no longer L^2, we have to modify the trial function at large distances, but gently enough to make the positive energy contribution from the tails small. We achieve that by choosing

$$f_{R,\kappa}(r) := \min\left\{1,\, \frac{K_0(\kappa r)}{K_0(\kappa R)}\right\}, \tag{3.4}$$

where K_0 is the Macdonald function and the parameter κ will be specified later. Since K_0 is strictly decreasing, the corresponding ψ_α will not be smooth at $r = R$ but it remains continuous, hence it is an admissible trial function. To estimate the first term at the r.h.s. of (3.2), let us compute

$$\begin{aligned} K_0(\kappa R)^2 \int_{\mathbb{R}^2} |f'_{R,\kappa}(r)|^2\, d^2x &= 2\pi \int_{\kappa R}^{\infty} K_1(t)^2\, t\, dt \\ &= \pi\left[\kappa^2 R^2 K_1'(\kappa R)^2 - \left(\kappa^2 R^2 + 1\right) K_1(\kappa R)^2\right], \end{aligned}$$

cf. [AS, 9.6.26], [PBM, 1.12.3.2]. Using $-K_1'(\xi) = K_0(\xi) + \xi^{-1}K_1(\xi)$ in combination with the asymptotic expressions $K_0(\xi) = -\ln\xi + \mathcal{O}(1)$, $K_1(\xi) = \xi^{-1} + \mathcal{O}(\ln\xi)$ for $\xi \to 0$, we find that

$$\left\|f'_{R,\kappa}\right\|^2_{L^2(\mathbb{R}^2)} < -\frac{C}{\ln(\kappa R)} \tag{3.5}$$

holds for a positive constant C and κR small enough. This makes it possible to estimate the first term at the r.h.s. of (3.2) using the fact that the functions χ_j are bounded outside $\mathcal{B}_R$; recall that $\chi_j(x) = \mathcal{O}(|x|^{-F+j})$ at large distances and $F - j \ge 0$.

We will show that $\int_\Sigma B(x)|\sum_j \alpha_j\chi_j(x)|^2\, d^2x > 0$ also holds in this situation again by assuming the opposite. Indeed, let us set $h_j := h\chi_j$ with a real-valued $h \in C_0^2(\Sigma)$ in (3.1); then the fourth term at the r.h.s. of (3.2) acquires the form

$$2\varepsilon \int_\Sigma \left|\sum_{j=0}^{[F]} \alpha_j\chi_j(x)\right|^2 h(x)B(x)\, d^2x\,.$$

Since B is nonzero, it is possible to choose h so that the integral is strictly negative. Taking ε positive and small enough the sum of the last four terms at the r.h.s. of (3.2) with $\mu = 2$ can be made negative, since the the linear term (in ε) prevails over the quadratic ones and the third term is supposedly nonpositive. The first term,

$$\int_{\mathbb{R}^2} \left|f'_{R,\kappa}(r)\right|^2 \left|\sum_{j=0}^{[F]} \alpha_j\chi_j(x)\right|^2 d^2x\,,$$

is positive, but the χ_j's are bounded outside $\mathcal{B}_R$ and $|\alpha| = 1$, so it can be made sufficiently small by a suitable choice of κ. Using again the supersymmetry property, $D^*D + 2B = DD^*$, we arrive at the absurd conclusion that $\|D^*\psi_\alpha\|^2 < 0$.

Hence we can take finally the trial functions (3.1) with $f_{R,\kappa}$ given by (3.4) and $\varepsilon = 0$, which yields the estimate

$$(\psi_\alpha, H_P^{(-)}(A)\psi_\alpha) < -\frac{C}{\ln(\kappa R)} \max_{0 \le j \le [F]} \|\chi_j\|_\infty^2 - \frac{1}{2}\,(g-2) \min_{|\alpha|=1} \int_\Sigma B(x)|\psi_\alpha(x)|^2\, d^2x\,. \tag{3.6}$$

The second term at the r.h.s. is strictly negative if $g > 2$, since $\int_\Sigma B(x)\,|\psi_\alpha(x)|^2\,d^2x > 0$ for any α in a compact set (surface of the hypersphere $|\alpha| = 1$), and it dominates the sum for κ small enough.

To conclude the proof of the first claim, one has to check that the trial functions (3.1) span indeed a subspace of dimension $1 + [F]$. This follows readily from the linear independence of $\psi_j := f_{R,\kappa}\chi_j$, $j = 0, \ldots, [F]$; recall that the χ_j's are linearly independent and coincide with ψ_j at least in the set $\mathcal{B}_R$.

If $F = 0$, the function $\tilde{\chi}_0(x) := e^{\phi(x)}$ which solves $D^*\tilde{\chi}_0 = 0$ is also bounded at large distances and we can apply the analogous argument to the operator $H_P^{(+)}(A) = DD^* + \frac{1}{2}\,(g-2)B$. Using a properly chosen function $\tilde{\psi}_0 = f_{R,\kappa}\tilde{\chi}_0 + \varepsilon h$, we can show that $\int_\Sigma B(x)|\tilde{\chi}_0(x)|^2\,d^2x < 0$, so $(\tilde{\psi}_0, H_P^{(+)}(A)\tilde{\psi}_0) < 0$ for $g > 2$ and small κ; hence $H_P^{(+)}(A)$ has a bound state as well. $\square$

Remarks. (a) The argument fails only if $B = 0$, since then $\chi_0(x) = \tilde{\chi}_0(x) = 1$ and the matrix elements $\langle B\rangle_{\chi_0}$ and $\langle B\rangle_{\tilde{\chi}_0}$ are zero.
(b) Instead of the tail modification (3.4) a simpler one — $f_R(r) := f(r/R)$, where $f \in C_0^\infty(\mathbb{R}_+)$ is such that $f(u) = 0$ for $u \ge 2$ — was employed in [BEZ2]. The kinetic energy is in this case estimated by

$$\frac{1}{R^2}\int_{\mathbb{R}^2} \left|f'\left(\frac{r}{R}\right)\right|^2 \left|\sum_{j=0}^{[F]} \alpha_j\chi_j(x)\right|^2 d^2x \le C\|f'\|_\infty^2 R^{-2(F-[F])}$$

for a positive C. It is clear that one can handle in this way the whole problem except for the case F integer.

4 Weakly bound states in two dimensions

Schrödinger operators in dimension one and two can have bound states for arbitrarily weak potentials, so the behaviour of the ground state in these cases is of particular interest. The corresponding asymptotic formulae, known already to Landau and Lifshitz [LL], were analyzed rigorously in [Si, BGS, Kl]. If we make a digression to the subject here, it is because we want to call attention to interesting aspects of the case when the interaction is switched off in a nonlinear way.

It is sufficient, of course, to describe peculiarities of the nonlinear case. Consider thus a two-dimensional Schrödinger operator family

$$H(\lambda)=-\Delta+V(\lambda,x) \tag{4.1}$$

on $L^2(I\!R^2)$ with λ belonging to an interval $[0,\lambda_0]$, where the potentials satisfy

$$V(\lambda,x)=\lambda V_1(x)+\lambda^2 V_2(x)+W(\lambda,x) \tag{4.2}$$

with

$$|W(\lambda,x)|\le\lambda^3 V_3(x) \tag{4.3}$$

and

$$V_j\in L^{1+\delta}(I\!R^2)\cap L(I\!R^2,(1+|x|^\delta)\,d^2x)\,,\quad j=1,2,3\,, \tag{4.4}$$

for some $\delta>0$. By the Birman-Schwinger principle, (4.1) has an eigenvalue $\epsilon(\lambda)=-\kappa^2$ iff the integral operator K_κ with the kernel

$$K_\kappa^\lambda(x,y)=|V(\lambda,x)|^{1/2}\,R_0(\kappa;x,y)\,V(\lambda,y)^{1/2}$$

(where $V^{1/2}:=|V|^{1/2}\,\mathrm{sign}\,V$) has an eigenvalue -1; here

$$R_0(\kappa;x,y)=\frac{1}{2\pi}\,K_0(\kappa|x-y|) \tag{4.5}$$

is the kernel of the free resolvent $(-\Delta+\kappa^2)^{-1}$. A standard trick is then to split the operator under consideration into two parts, $K_\kappa^\lambda=L_\kappa^\lambda+M_\kappa^\lambda$, where the former is rank-one with the kernel $L_\kappa^\lambda(x,y)=-\frac{1}{2\pi}\,|V(\lambda,x)|^{1/2}\,\ln\kappa\,V(\lambda,y)^{1/2}$, while the latter is regular as $\kappa\to 0+$ and, in this limit, has kernel

$$M_0^\lambda(x,y)=-\,\frac{1}{2\pi}\,|V(\lambda,x)|^{1/2}\left\{\gamma+\ln\frac{|x-y|}{2}\right\}V(\lambda,y)^{1/2}\,,$$

where γ is the Euler constant. Now we employ the identity

$$(I+K_\kappa^\lambda)^{-1}=\left[I+(I+M_\kappa^\lambda)^{-1}L_\kappa^\lambda\right]^{-1}(I+M_\kappa^\lambda)^{-1}\,,$$

where the existence of the inverses at the r.h.s. for sufficiently small λ follows from the assumptions made about the potential in the same way as in [Si]. The spectral problem is thus reduced to finding a singularity of the square bracket, which leads to an implicit equation. If we put $u:=(\ln\kappa)^{-1}$, it can be written as

$$u-\frac{1}{2\pi}\int V(\lambda,x)^{1/2}\,(I+M_\kappa^\lambda)^{-1}(x,y)\,|V(\lambda,y)|^{1/2}\,d^2x\,d^2y=0\,, \tag{4.6}$$

and used to derive the Taylor expansion of the function $\lambda\mapsto u(\lambda)$; a weakly bound state with the eigenvalue $\epsilon(\lambda)=-e^{2/u(\lambda)}$ exists iff $u(\lambda)<0$ around the origin.

In the linear case, $V=\lambda V_1$, we get from here the usual expansion

$$u(\lambda)=\frac{\lambda}{2\pi}\int V_1(x)\,d^2x+\frac{\lambda^2}{4\pi^2}\int V_1(x)\left\{\gamma+\ln\frac{|x-y|}{2}\right\}V_1(y)\,d^2x\,d^2y+\mathcal{O}(\lambda^3)\,, \tag{4.7}$$

which shows that a bound state exists iff $\int V_1(x)\, d^2x \le 0$ (the second term is negative if the potential V_1 is nontrivial and has zero mean [Si]).

For a potential family (4.2) nonlinear in λ the sign of $\int V_1(x)\, d^2x$ is again decisive. An interesting situation arises, however, if the linear part has zero mean,

$$\int V_1(x)\, d^2x = 0\,. \tag{4.8}$$

Replacing λV_1 with (4.2) in (4.6) and expanding in powers of λ we find that

$$u(\lambda) = \lambda^2 \left\{ \frac{1}{2\pi} \int V_2(x)\, d^2x + \frac{1}{4\pi^2} \int V_1(x)\, \ln|x-y|\, V_1(y)\, d^2x\, d^2y \right\} + \mathcal{O}(\lambda^3) \tag{4.9}$$

holds in this case (the term with $\gamma - \ln 2$ splits into a product of one-dimensional integrals and vanishes too). We arrive at the following conclusion.

Proposition 1. An operator family (4.1) with the potential satisfying (4.2)–(4.4) and (4.8) has a weakly bound state provided the leading coefficient in (4.9) is negative. If it is positive, no bound state exists for small λ.

The formula (4.9) yields also the asymptotic behaviour of the corresponding eigenvalue $\epsilon(\lambda) = -e^{2/u(\lambda)}$. We will not inquire about the critical case when the second-order coefficient also vanishes.

5 The centrally symmetric case

Let us return now to the Pauli operator (2.1) and consider the situation when the field is centrally symmetric, so the vector potential can be chosen in the symmetric gauge, $\vec{A}(x) = \lambda A(r)\vec{e}_\varphi$, with $A(r) = r^{-1}\int_0^r B(r')\, r'\, dr'$. We have introduced the positive parameter λ in order to discuss how the spectral properties depend on the field strength. We can perform a partial-wave decomposition and replace (2.1) by the family of operators

$$H_\ell^{(\pm)}(\lambda) = -\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + V_\ell^{(\pm)}(\lambda, r)\,, \qquad V_\ell^{(\pm)}(\lambda, r) := \left(\lambda A(r) + \frac{\ell}{r}\right)^2 \pm \frac{\lambda}{2} g B(r) \tag{5.1}$$

on $L^2(I\!R^+, r\, dr)$. In [BEZ1] these operators were used to discuss the behaviour of an electron in the magnetic field induced by a localized rotating electric current. We need not insist on that here, assuming only that the field is locally integrable with $B(r) = \mathcal{O}(r^{-2-\delta})$ as $r \to \infty$. However, we will be interested primarily in the situation typical for current-induced magnetic fields, in which the field has zero mean (*i.e.*, $F = 0$) since the flux lines are closed in $I\!R^3$.

It is shown in [BEZ1] under stronger assumptions — involving a smoothness and a faster decay of the field — that each orbital Hamiltonian $H_\ell^{(-)}(\lambda)$ has a bound state for λ large enough, the critical values for emergence of these states being, of course, ℓ-dependent. This result relies only on the behaviour of $V_\ell^{(-)}(\lambda, r)$ around the origin and is thus independent of the fact that $F=0$, the important point being that $g>2$ so the ground-state energy of the harmonic oscillator obtained in the limit $\lambda\to\infty$ is negative — *cf.* [BEZ1].

The "spin-up" Pauli operator $H_P^{(+)}(\lambda)$ may exhibit a less intuitive behaviour as suggested by Theorem 1. If $F=0$ for a *compactly supported* field, then $H_P^{(+)}(\lambda)$ has also a bound state for any $\lambda>0$. Recall that Theorem 1 says nothing about the size of Σ, it may be quite large. Inspecting the shape of the effective potentials $V_\ell^{(\pm)}(\lambda,r)$ for the two cases we see that the states with different spin orientations are supported in different regions: "spin-down" states in the vicinity of the origin (out of the centrifugal barrier for $\ell\neq 0$), while the "spin-up" state at large distances where (for an arbitrary but fixed λ) the magnetic field term dominates slightly over the quadratic one in $V_0^{(+)}(\lambda,r)$ creating a shallow potential well.

Let us examine the *weak-coupling behaviour* in the case of a vanishing total flux, $F=0$, in the field with a tail, $B(r)=\mathcal{O}(r^{-2-\delta})$; no smoothness assumption is made. If $\ell\neq 0$, the first term in $V_\ell^{(\pm)}(\lambda,r)$ is bounded below by $\lambda v(r)$ for a suitably chosen positive function v of compact support (the simplest choice is $v(r)=c\,\Theta(r_0-r)$ for appropriate c and r_0). Since the second term does not contribute to $\int_0^\infty V_\ell^{(\pm)}(\lambda,r)\,r\,dr$ which determines the linear part of the weak–coupling behaviour, it follows from (4.7) and the minimax principle that the discrete spectrum of $H_\ell^{(\pm)}(\lambda)$ is empty for λ small enough, when the centrifugal barrier prevents binding.

The interesting case is, of course, the s-wave part, where the effective potential acquires the form (4.2) with the last term absent and

$$V_1(r)=\pm\frac{g}{2}\,B(r)\,,\qquad V_2(r)=A(r)^2\,. \tag{5.2}$$

In view of the assumption about the field, $r\mapsto A(r)$ is absolutely continuous and $\mathcal{O}(r^{-1-\delta})$, so the condition (4.4) is satisfied. It remains to evaluate the second integral in (4.9): we have

$$\begin{aligned}&\frac{1}{4\pi^2}\int_{I\!R^2\times I\!R^2} V_1(x)\,\ln|x-y|\,V_1(y)\,d^2x\,d^2y\\ &\quad=\frac{g^2}{4}\,\frac{1}{2\pi}\int_0^\infty dr\,rB(r)\int_0^\infty dr'\,r'B(r')\int_0^{2\pi}\ln\left[r^2+r'^2-2rr'\cos\varphi\right]^{1/2}d\varphi\,.\end{aligned}$$

By [GR, 4.224.9] the last integral equals $2\pi\ln\max(r,r')$; we substitute this into the formula and integrate repeatedly by parts using $rB(r)=(rA(r))'$. This yields

$$-\frac{g^2}{4}\int_0^\infty\left(\int_r^\infty A(r')\,dr'\right)B(r)\,r\,dr=-\frac{g^2}{4}\int_0^\infty dr\,A(r)^2\,r\,dr\,.$$

We arrive thus at the following conclusion.

Proposition 2. Let a spherically symmetric magnetic field B be locally integrable with $B(r)=\mathcal{O}(r^{-2-\delta})$ and vanishing flux, $F=0$. Then each of the operators $H_0^{(\pm)}(\lambda)$ with $g>2$ has a negative eigenvalue for λ small enough.

Remarks. (a) The relation (4.9) yields also the asymptotic behaviour of the bound state energy,

$$\epsilon^{(\pm)}(\lambda) \approx -\exp\left\{ -\left(\frac{\lambda^2}{8}\,(g^2-4) \int_0^\infty A(r)^2\, r\, dr \right)^{-1} \right\} \tag{5.3}$$

as $\lambda \to 0$ with the usual meaning of $\approx$ (*cf.* [Si]). The leading term is thus the same for both spin orientations; however, since $g \neq 2$, the second theorem of [AC] does not apply and the degeneracy may be lifted in the next order.

(b) Notice that the argument of the previous section cannot be applied to compactly supported fields with a nonzero flux, since the corresponding vector potential has then a too slow decay, $A(r) = \mathcal{O}(r^{-1})$, and consequently $V_1 \notin L(I\!R^2, (1+|x|^\delta)d^2x)$. One may ask whether the asymptotics is nevertheless $\epsilon(\lambda) \approx \exp\left(-\frac{4}{\lambda F g}\right)$ as it follows from a formal application of (4.7). The example worked out in [CFC] leads to the conclusion that it is not the case — see Eq. (17) of that paper. The question about the asymptotic behaviour thus remains open.

(c) Another open question is whether the bound state of $H_0^{(\pm)}(\lambda)$ survives generally for λ large if the field is not compactly supported.

6 Non-symmetric weak coupling revisited

By different means, the result of the previous section complements the zero-flux part of Theorem 1 in the weak-coupling case. While imposing the symmetry requirement, it relaxes the assumptions on the field decay. Here we want to show that the above argument can be carried through for non-symmetric fields as well under mild regularity assumptions; the price we shall pay is to have a weaker form of the asymptotic formula (5.3) only. Specifically, suppose that

$$|B(x)| \le C_1 \langle x \rangle^{-2-\delta}\,, \qquad \int_{I\!R^2} \frac{|B(y)|}{|x-y|}\, d^2y \le C_2 \langle x \rangle^{-1-\delta}\,, \tag{6.1}$$

where $\langle x \rangle := \sqrt{1+r^2}$. Then we have the following result.

Proposition 3. Let a magnetic field B satisfy the conditions (6.1) for some $C_1, C_2, \delta > 0$, and let $F = 0$. Then each of the operators $H_P^{(\pm)}(\lambda)$ with $g > 2$ has a negative eigenvalue for λ small enough.

Proof is based on two observations. The first one concerns the "mixed" term $2iA\cdot\nabla$ in the Hamiltonian; we shall show that it does not contribute to the energy form for real-valued functions (the other "mixed" term, $i\nabla \cdot A$, vanishes in the gauge we have been adopting). More specifically, take a real-valued $\psi \in C_0^2(I\!R^2)$, *i.e.*, twice differentiable with compact support. For the sake of brevity, we write the vector potential components as $A_i = -\epsilon_{ij}\partial_j \phi$, where ϵ is the two-dimensional Levi-Civita tensor, and employ the convention of summation over repeated indices; then

$$\begin{aligned}
(\psi, A\cdot\nabla\psi) &= -\int_{I\!R^2} \psi(x)\,\epsilon_{ij}\,(\partial_j\phi)(x)\,(\partial_i\psi)(x)\,d^2x \\
&= -\frac{1}{2}\lim_{R\to\infty}\int_{\mathcal{B}_R} \epsilon_{ij}\,(\partial_j\phi)(x)\,(\partial_i\psi^2)(x)\,d^2x \\
&= -\frac{1}{2}\lim_{R\to\infty}\int_{\mathcal{B}_R} \epsilon_{ij}\,\left\{(\partial_j(\phi\,\partial_i\psi^2))(x) - \phi(x)\,(\partial_i\partial_j\psi^2)\right\}\,d^2x \\
&= \frac{1}{2}\lim_{R\to\infty}\oint_{\partial\mathcal{B}_R} \phi(x)\,(\nabla\psi^2)(x)\cdot d\vec{\ell}(x) \;=\; 0\,.
\end{aligned}$$

The third line is obtained from the second using integration by parts. Its second term vanishes because $\partial_j\partial_i\psi^2$ is symmetric with respect to the interchange of indices and is contracted with the anti-symmetric symbol ϵ_{ij}. The remaining term is rewritten by means of the Stokes theorem and vanishes in the limit since $\nabla\psi^2$ has compact support.

The second observation is that the relation between the two integrals which we found in the proof of Proposition 2 by explicit computation in polar coordinates is valid generally. To see this, let us rewrite $\int A(x)^2\,d^2x$ by means of the first Green identity:

$$\begin{aligned}
\int_{I\!R^2} A(x)^2 d^2x &= \lim_{R\to\infty}\int_{\mathcal{B}_R} (\nabla\phi(x))^2\,d^2x \\
&= \lim_{R\to\infty}\int_{\mathcal{B}_R} \left\{(\nabla\cdot(\phi\nabla\phi))(x) - \phi(x)\,(\nabla^2\phi)(x)\right\}\,d^2x \qquad (6.2)\\
&= \lim_{R\to\infty}\oint_{\partial\mathcal{B}_R} \phi(x)\,(\nabla\phi)(x)\cdot d\vec{\sigma}(x) - \lim_{R\to\infty}\int_{\mathcal{B}_R} \phi(x)B(x)\,d^2x\,;
\end{aligned}$$

in the second integral we have used $\triangle\phi = B$ and the first one was rewritten by means of Gauss theorem. Our aim is now to use the conditions (6.1) to demonstrate that the first integral at the r.h.s. vanishes in the limit. The decay hypothesis about the field yields

$$\begin{aligned}
|\phi(x)| &\le \frac{1}{2\pi}\int_{|y-x|\le 1} |B(y)|\,|\ln|x-y||\,d^2y + \frac{1}{2\pi}\int_{|y-x|\ge 1} |B(y)|\,\ln|x-y|\,d^2y \\
&\le \frac{C_1}{2\pi}\int_{|z|\le 1} |\ln|z||\,d^2z + \frac{C_1}{2\pi}\int_{|z|\ge 1} \langle x-z\rangle^{-2-\delta}\,\ln|z|\,d^2z\,.
\end{aligned}$$

Denote $\ln_+ u := \max(0, \ln u)$. Then for any $\eta > 0$ there is a $K_\eta > 0$ such that

$$\ln_+|z| < K_\eta\,\langle x-z\rangle^\eta\,(1+\ln_+|x|)\;;$$

this follows from the fact that

$$\limsup_{\zeta,\xi\to\infty} \frac{\ln_+\zeta}{(1+|\xi-\zeta|^2)^{\eta/2}\,(1+\ln_+\xi)} \le 1$$

in the first quadrant, and the function under the limit is continuous there. The second one of the above integrals is thus estimated by

$$\frac{C_1 K_\eta}{2\pi}\,(1+\ln_+|x|)\int_{I\!R^2}\langle y\rangle^{-2-\delta+\eta}\,d^2y\,.$$

For $\eta<\delta$ the last integral is convergent; hence there is a $C'>0$ such that $|\phi(x)|\le C'\ln|x|$ as $|x|\to\infty$. The second one of the conditions (6.1) implies

$$|A(x)|=|(\nabla\phi)(x)|\le\frac{C_2}{2\pi}\langle x\rangle^{-1-\delta}\,,$$

so the first integral at the r.h.s. of (6.2) is $o(R^{-\delta'})$ for any $\delta'<\delta$ and vanishes in the limit that we have set out to prove. Substituting (2.3) for ϕ in the second one we arrive finally at the identity

$$\int_{I\!R^2}A(x)^2\,d^2x=-\,\frac{1}{2\pi}\int_{I\!R^2\times I\!R^2}B(x)\,\ln|x-y|\,B(y)\,d^2x\,d^2y\,; \tag{6.3}$$

the conditions (6.1) ensure that both integrals exist. Now it is easy to conclude the proof. We have

$$\begin{aligned}\inf\sigma\left(H_P^{(\pm)}(\lambda)\right)&=\inf\left\{\left(\psi,H_P^{(\pm)}(\lambda)\psi\right):\;\psi\in D\left(H_P^{(\pm)}(\lambda)\right)\right\}\\&\le\inf\left\{\left(\psi,H_P^{(\pm)}(\lambda)\psi\right):\;\psi\in C_0^2(I\!R^2)\,,\;\psi=\bar\psi\right\}\\&=\inf\sigma\left(\widetilde{H}_P^{(\pm)}(\lambda)\right),\end{aligned} \tag{6.4}$$

where

$$\widetilde{H}_P^{(\pm)}(\lambda):=-\Delta+\lambda^2A(x)^2\pm\frac{\lambda g}{2}B(x)\,.$$

The last equality in (6.4) is due to the fact that $C_0^2(I\!R^2)$ is a core of $\widetilde{H}_P^{(\pm)}(\lambda)$. It is now sufficient to apply Proposition 1 to the operator $\widetilde{H}_P^{(\pm)}(\lambda)$ and to employ the identity (6.3). □

Remark. In view of the estimate used in the proof, the relation (5.3) is now replaced by the asymptotic inequality

$$\epsilon^{(\pm)}(\lambda)\;\lesssim\;-\;\exp\left\{-\left(\frac{\lambda^2}{16\pi}\,(g^2-4)\int_{I\!R^2}A(x)^2\,d^2x\right)^{-1}\right\}; \tag{6.5}$$

the question whether the r.h.s. is still the lower bound remains open.

Acknowledgments

The authors thank for the hospitality extended to them at the institutes where parts of the work were done: P.E. to CPT Luminy and V.Z. to NPI Řež. The research has been partially supported by GACR and GAAS under the contracts 202/96/0218 and 1048801. R.M.C. was supported by CNPq and NSF under grant No. PHY94-07194.

References

[AC] Y. Aharonov, A. Casher: Ground state of a spin–1/2 charged particle in a two–dimensional magnetic field, *Phys. Rev.* **A19** (1979), 2641–2642.

[AS] M.S. Abramowitz, I.A. Stegun, eds.: *Handbook of Mathematical Functions*, Dover, New York 1965.

[AGHH] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden: *Solvable Models in Quantum Mechanics*, Springer, Heidelberg 1988.

[BEZ1] F. Bentosela, P. Exner, V.A. Zagrebnov: Electron trapping by a current vortex, *J. Phys.* **A31** (1998), L305–311.

[BEZ2] F. Bentosela, P. Exner, V.A. Zagrebnov: Anomalous electron trapping by magnetic flux tubes and electric current vortices, in *Mathematical Results in Quantum Mechanics* (J. Dittrich et al., eds), *Operator Theory: Advances and Applications*, Birkhäuser, Basel 1999; pp. 191–196.

[BGS] R. Blanckenbecler, M.L. Goldberger, B. Simon: The bound states of weakly coupled long-range one-dimensional quantum Hamiltonians, *Ann. Phys. (N.Y.)* **108** (1977), 89–78.

[BV] M. Bordag, S. Voropaev: Charged particle with magnetic moment in the Aharonov-Bohm potential, *J. Phys.* **A26** (1993), 7637–7649.

[CC] R.M. Cavalcanti, C.A.A. de Carvalho: Bound states of a spin–1/2 charged particle in a magnetic flux tube, *J. Phys.* **A31** (1998), 7061–7063.

[CFC] R.M. Cavalcanti, E.S. Fraga, C.A.A. de Carvalho: Electron localization by a magnetic vortex, *Phys. Rev.* **B56** (1997), 9243–9246.

[GR] I.S. Gradshtein, I.M. Ryzhik: *Tables of Integral, Series, and Products*, Academic Press, New York 1980.

[Kl] M. Klaus: On the bound state of Schrödinger operators in one dimension, *Ann. Phys. (N.Y.)* **108** (1977), 288–300.

[LL] L.D. Landau, E.M. Lifshitz: *Quantum Mechanics: Non-Relativistic Theory*, Pergamon, Oxford 1965.

[Mo] A. Moroz: Single–particle density of states, bound states, phase–shift flip, and a resonance in the presence of an Aharonov–Bohm potential, *Phys. Rev.* **A53** (1996), 669–694.

[PBM] A.P. Prudnikov, Yu.O. Brychkov, O.I. Marichev: *Integrals and Series, II. Special Functions*, Gordon and Breach, New York 1986.

[Si] B. Simon: The bound state of weakly coupled Schrödinger operators in one and two dimensions, *Ann. Phys. (N.Y.)* **97** (1976), 279–288.

[Th] B. Thaller: *The Dirac equation*, Springer, Berlin 1992.