e to the A, in a New Way

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Abstract

Apparently new expressions are given for the exponential of a hermitian matrix, A, in the $2 \times 2, 3 \times 3$, and 4×4 cases. Replacing A by iA these are explicit formulas for the Fourier transform of e^{iA}.

The support of the Fourier transform of e^{iA}, A hermitian, has been long known [1], [2] and [3]. We exhibit explicit expressions for traceless hermitian matrices of sizes $2\times2, 3\times3$, and 4×4. In d-dimensions, where A act s on $\mathbb{C}^d$, we employ matrices, W, hermitian projections of rank one. Thus W may be written as $W_{ij} = v_i \ \bar{v}_j$, where v_i is a unit vector in $\mathbb{C}^d$, and $\mathrm{Tr}(AW) =< v, Av >$. $\int d\Omega$ denotes a normalized integral over all suc h W defined by integrating the associated v_i over a unit sphere in $\mathbb{C}^d$ with the normalized unitary- invariant measure. The expressions we give are not unique, among similar forms. It is clear one can derive such expressions in any number of dime nsions. We leave to others the task of creating an appropriate theory.

$2-d$ case

$$e^A = \int d\Omega e^{\mathrm{Tr}(AW)} \left[I + 4(W - \frac{1}{2}I) + \ \mathrm{Tr}(AW)I + 2\ \mathrm{Tr}(AW)(W - \frac{1}{2}I) \right]$$

$3-d$ case

$$e^A = \int d\Omega e^{\mathrm{Tr}(AW)} \Big[I + 9(W - \frac{1}{3}I) - \ \mathrm{Tr}(AW)I + 9\ \mathrm{Tr}(AW)W$$

$$-\frac{1}{2}\ A + \left[\frac{3}{2}(\mathrm{Tr}(AW))^2 - \frac{1}{4}\ \ \mathrm{Tr}(A^2) \right] W \Big].$$

$4-d$ case

$$e^A = \int d\Omega e^{\mathrm{Tr}(AW)} \Big[I + \frac{52}{3}(W - \frac{1}{4}I) - \frac{8}{3}\ \mathrm{Tr}(AW)I + \frac{68}{3}\ \mathrm{Tr}(AW)W - A$$

$$+\frac{1}{6} \left[-4\ \mathrm{Tr}(A^2) + 46(\mathrm{Tr}(AW))^2 + 2A\ \mathrm{Tr}(AW) \right] W$$

$$-\frac{1}{6} \left[-\frac{1}{2}\ \mathrm{Tr}(A^2)I + 3(\mathrm{Tr}(AW))^2 I + A^2 + 2A\ \mathrm{Tr}(AW) \right]$$

$$+\frac{2}{3}(\mathrm{Tr}(AW))^3 W - \frac{1}{18}\ \mathrm{Tr}(A^3)W - \frac{1}{6}\ \mathrm{Tr}(A^2)\ \mathrm{Tr}(AW)W \Big].$$

These expressions were derived using such trickery and chicanery as will likewise not be useful to one proving a general theory: extracting leading and subleading asymptotic behaviors as one eigenvalue of A approaches ∞, finding the simple ratio nal numbers involved in the "angular integrals" of W by numerical integration. The expressions were likewise checked by numerical integration.

REFERENCES

[**1**] E. Nelson, *Operants: A functional calculus for non-commuting operators*, Functional Analysis and Related Fields, Proceedings of a conference in honour of Professor Marshal Stone (Univ. of Chicago, May 1968) (F.E. Browder, ed.), Springer-Verlag, Berlin, Heidelberg, and New York, 1970, pp. 172-187. MR 54:978.

[**2**] B. Jefferies, "The Weyl Calculus for Hermitian Matrices", *Proc. A.M.S.* **124** (96) p. 121-128.

[**3**] M.E. Taylor "Functions of Several Self-Adjoint Operators", *Proc. A.M.S.* **19** (1968), 91-98. MR **36**:3149.