# A Simple Model of Concentrated Nonlinearity [1]

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**Abstract**

We study the nonlinear Schrödinger equation in dimension one with a nonlinearity concentrated in a point. We give results on local and global existence of the solution and find conditions for the blow-up. In the critical case an explicit blow-up solution is constructed.

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Recent studies in Solid State Physics concerning resonant tunnelling, interactions with impurities etc. have motivated the analysis of the nonlinear Schrödinger equation (NLSE) with a nonlinearity concentrated in a (small) region of space. Such equation exhibits interesting qualitative properties which have been discussed in the physical literature ([J-LPS], [MA], [BKB]). Some rigorous mathematical results are given in [N], where the existence of steady states in the case of an open system with smooth short-range nonlinearities is proved.

In this note we approach what we consider the simplest case of concentrated nonlinearity, i.e. the one-dimensional Schrödinger equation with nonlinearity concentrated in a fixed point.

In spite of its simplicity, such model seems to be not completely trivial (for instance it produces blow-up solutions).

We consider this work as the first step in a line of research devoted to the mathematical analysis of the Schrödinger equation in dimension $d$ with a nonlinearity concentrated on a set of lower dimension.

The paper is organized as follows.

First we formulate the nonlinear evolution problem and give a local existence theorem.

The second step is the analysis of the conservation laws and the proof of a global existence theorem.

Finally we give conditions for the blow-up of the solution and, in the critical case, we explicitely construct a class of blow-up solutions.

We start with the formulation of the problem. For any fixed real function $\alpha(t)$, we denote by $H_{\alpha(t)}$ the Schrödinger operator with point interaction of strength $\alpha(t)$ placed at the origin. It is well known ([AGH-KH]) that such operator can be defined for all t as a self-adjoint operator in $L^2(I\!R)$. In particular, its form domain is $H^1(I\!R)$ and the operator domain consists of functions $u \in H^2(I\!R\backslash\{0\})$ satisfying the boundary condition at the origin $u'(0^+) - u'(0^-) = \alpha(t)u(0)$. Moreover the linear Schrödinger equation with time-dependent hamiltonian $H_{\alpha(t)}$ can also be correctly defined and it admits a unique solution for $\alpha(t)$ smooth.

We define the nonlinear evolution problem by replacing $\alpha(t)$ with a function of the value at the origin of the solution itself. More precisely, for $\gamma \in I\!R$, $\sigma > 0$, we consider

$$i\frac{\partial \psi_t}{\partial t} = H_{\alpha(t)}\psi_t, \quad \alpha(t) = \gamma|\psi_t(0)|^{2\sigma}, \quad \psi_{t=0} = \psi_0 \tag{1}$$

For the sake of simplicity we have considered in (1) a power law nonlinearity; more general nonlinearities can also be introduced.

Problem (1) corresponds to an evolution generated by the laplacian in $I\!R$ with a nonlinear boundary condition at the origin, namely

$$\psi_t'(0^+) - \psi_t'(0^-) = \gamma|\psi_t(0)|^{2\sigma}\psi_t(0) \tag{2}$$

It is convenient to reformulate problem (1) in integral form. One obtains

$$\psi_t(x) + i\gamma \int_0^t ds U\left(t-s, |x|\right) |\psi_s(0)|^{2\sigma} \psi_s(0) = \left(U(t)\psi_0\right)(x) \tag{3}$$

where $U(t)$ denotes the free Schrödinger unitary group in dimension one. From (3) it is clear that the solution is enterely determined by its value at the origin. Evaluating equation (3) in $x = 0$ one has a closed equation for $\psi_t(0)$

$$\psi_t(0) + i\gamma \int_0^t ds \frac{1}{\sqrt{4\pi i(t-s)}} |\psi_s(0)|^{2\sigma} \psi_s(0) = \left(U(t)\psi_0\right)(0) \tag{4}$$

Equation (4) is a nonlinear Abel integral equation, involving only the time variable. The search for the solution of (3) is then reduced to the solution of (4).
As for the standard NLSE (see e.g. [GV]), the natural space where we study the equation (3) is $H^1(\mathbb{R})$, i.e. the space of finite energy. For technical reasons, in what follows we choose an initial datum in $H^2(\mathbb{R})$ but we are convinced that all the results remain valid under the more natural assumption $\psi_0 \in H^1(\mathbb{R})$.

*Proposition 1.* For any $\psi_0 \in H^2(\mathbb{R})$, there exists $\bar{t} > 0$ s.t. problem (3) has a unique solution $\psi_t \in H^1(\mathbb{R})$, $t \in [0, \bar{t})$.

*Sketch of the proof.* The r.h.s. of (4) is clearly continuous in $t$. Then, using the contraction principle, equation (4) can be uniquely solved for small times in the space of continuous functions ([M]). Substituting the solution $\psi_t(0)$ in (3), one easily sees that $\psi_t \in L^2(\mathbb{R})$. Now, exploiting the smoothing properties of the Abel operator ([GoVe]) and the regularity of $\psi_0$, one can prove that $\psi_t(0)$ is smooth enough to ensure $\psi_t \in H^1(\mathbb{R})$.

One of the main tools for the analysis of nonlinear evolution problems are conservation laws. In our case there are three conserved quantities: $L^2$-norm (denoted by $\|\cdot\|$), energy and parity.

*Proposition 2.* For any $\psi_0 \in H^2(\mathbb{R})$ we have

$$\|\psi_t\| = \|\psi_0\| \tag{5}$$

$$E(\psi_t) \equiv \int_{\mathbb{R}} dx |\psi_t'(x)|^2 + \frac{\gamma}{\sigma+1} |\psi_t(0)|^{2\sigma+2} = E(\psi_0) \tag{6}$$

$$\psi_0(x) = \pm\psi_0(-x) \text{ implies } \psi_t(x) = \pm\psi_t(-x) \tag{7}$$

*Proof.* We give some details for (6). The proof of (5) is similar while (7) is trivial. We define $q(t) \equiv |\psi_t(0)|^{2\sigma}\psi_t(0)$. The hypotesis $\psi_0 \in H^2(\mathbb{R})$ implies $q \in H^1((0,t))$ for all $t < \bar{t}$.

$$\|\psi_t'\|^2 = \int_{\mathbb{R}} dk\, k^2 \,|\psi_t(k)|^2$$

$$
\begin{aligned}
= & \int_{I\!R} dk\, k^2\, |\psi_0(k)|^2 - \sqrt{\frac{2}{\pi}}\, i\, \gamma\, Re \int_{I\!R} dk\, k^2 \overline{\tilde{\psi}_0(k)} \int_0^t ds\, e^{ik^2 s}\, q(s) \\
& + \frac{\gamma^2}{2\pi} \int_{I\!R} dk\, k^2 \int_0^t ds \int_0^t ds'\, e^{ik^2(s'-s)} \overline{q(s)} q(s') \qquad (8)
\end{aligned}
$$

where $\tilde{\psi}$ denotes the Fourier transform of $\psi$.
After some calculations one obtains
$Re \int_{I\!R} dk\, k^2 \overline{\tilde{\psi}_0(k)} \int_0^t ds\, e^{ik^2 s}\, q(s)$

$$
\begin{aligned}
= & \frac{\sqrt{\pi}}{\sqrt{2}\,(\sigma+1)} \left( |\psi_t(0)|^{2\sigma+2} - |\psi_0(0)|^{2\sigma+2} \right) \\
& + \frac{\gamma}{2} \int_0^t \frac{ds}{\sqrt{t-s}} \left[ Re(q(t)\overline{q(s)}) + Im(q(t)\overline{q(s)}) \right] \\
& - \frac{\gamma}{2} \int_{0 \le s' \le s \le t} \frac{ds\, ds'}{\sqrt{s-s'}} \left[ Re(\dot{q}(s)\, \overline{q(s')}) + Im(\dot{q}(s)\, \overline{q(s')}) \right] \qquad (9)
\end{aligned}
$$

and
$\int_{I\!R} dk\, k^2 \int_0^t ds \int_0^t ds'\, e^{ik^2(s'-s)} \overline{q(s)} q(s')$

$$
\begin{aligned}
= & \sqrt{2\pi} \int_0^t \frac{ds}{\sqrt{t-s}} \left[ Re(q(t)\overline{q(s)}) + Im(q(t)\overline{q(s)}) \right] \\
& - \sqrt{2\pi} \int_{0 \le s' \le s \le t} \frac{ds\, ds'}{\sqrt{s-s'}} \left[ Re(\dot{q}(s)\, \overline{q(s')}) + Im(\dot{q}(s)\, \overline{q(s')}) \right] \qquad (10)
\end{aligned}
$$

Then, from (8), (9) and (10), conservation of energy is proved.

*Remark 1.* Since $\psi_0^{odd}(0) = 0$, from (3) one has $\psi_t^{odd} = U(t)\psi_0^{odd}$ and $\psi_t^{even}$ is given by equation (3), where $\psi_0$ is replaced by $\psi_0^{even}$ (here we have denoted by $f^{even}$ and $f^{odd}$ respectively the even and the odd part of the function $f$). One can therefore restrict oneself to even initial data.

Using the conservation laws and the Sobolev inequality we easily obtain the following global existence theorem.

*Proposition 3.* Assume $\gamma > 0$ (case a)) or $\gamma < 0$, $\sigma < 1$ (case b)). Then for any $\psi_0 \in H^2(I\!R)$ and any $T > 0$ problem (3) has a unique solution $\psi_t \in H^1(I\!R)$, for $t \in [0, T]$.

*Proof.* In case a) the conservation of the energy directly provides an a priori estimate of $\|\psi_t'\|$, which implies that the solution is global in time.
In case b) we apply the Sobolev inequality ([A]) and the conservation of the $L^2$-norm to estimate the "potential" term in the energy

$$|\psi_t(0)|^{2\sigma+2} \leq \|\psi_0\|^{\sigma+1}\|\psi_t'\|^{\sigma+1} \tag{11}$$

Combining (6) and (11) we have

$$E(\psi_0) \geq \|\psi_t'\|^2 \left(1 - \frac{|\gamma|}{\sigma+1}\|\psi_0\|^{\sigma+1}\|\psi_t'\|^{\sigma-1}\right) \tag{12}$$

Inequality (12) implies that $\|\psi_t'\|$ cannot diverge in finite time and this implies global existence.

Let us consider the problem of the existence of blow-up solutions of (3).
Following the usual definition, we say that $\psi_t$ is a blow-up solution if there exists $t_0 < \infty$ s.t. $\|\psi_t'\| \to \infty$ for $t \to t_0$.
Due to the conservation of the energy, it is evident that $\psi_t$ is a blow-up solution if and only if $|\psi_t(0)|$ diverges for $t \to t_0$. This means that, in our model, a blow-up solution $\psi_t$ can be equivalently defined by the condition $\|\psi_t\|_{L^\infty} \to \infty$ for $t \to t_0$.
Using standard arguments, i.e. the computation of the second time-derivative of the moment of inertia and the uncertainty principle ([RR]), we can prove existence of blow-up solutions.

*Proposition 4.* Assume $\gamma < 0$ and $\sigma \geq 1$. Then for any $\psi_0 \in H^2(I\!R)$ s.t. $\psi_0$ is an even function, $\|x\psi_0\| < \infty$ and $E(\psi_0) < 0$ the solution $\psi_t$ is a blow-up solution.

*Proof.* Let us define the moment of inertia (with respect to the origin) of a solution of (3)

$$I(t) \equiv \int_{I\!R} dx x^2 |\psi_t(x)|^2 \tag{13}$$

An explicit calculation yields

$$\dot{I}(t) = 4\, Im \int_{I\!R} dx \overline{\psi_t(x)} x \psi_t'(x) \tag{14}$$

$$\ddot{I}(t) = 8E(\psi_0) - 4|\gamma| \left(1 - \frac{2}{\sigma+1}\right) |\psi_t(0)|^{2\sigma+2} \tag{15}$$

Since $\ddot{I}(t) < 0$, there exists $t_0 < \infty$ s.t. $I(t) \to 0$ for $t \to t_0$. Applying the uncertainty principle

$$\|\psi_0\|^2 \leq 2\, \|x\psi_t\|\, \|\psi_t'\| \tag{16}$$

we conclude that $\psi_t$ is a blow-up solution.

*Remark 2.* The initial datum in the above proposition is chosen to be even. For an arbitrary initial datum the conclusion of the theorem still holds if the condition $E(\psi_0) < 0$ is replaced by $E(\psi_0^{even}) < 0$.

¿From propositions 3, 4 it is clear that the critical exponent of the nonlinearity is $\sigma = 1$.
The critical case is of special interest because of the existence of an additional symmetry, the so-called pseudo-conformal invariance. This symmetry can be expressed as follows: if $u_t(x)$ is a solution then

$$v_\tau(y) \equiv \frac{e^{-i\frac{|y|^2}{4(t_0-\tau)}}}{[a(t_0-\tau)]^{\frac{1}{2}}}\, u_{\frac{1}{a^2(t_0-\tau)}}\left(\frac{y}{a(t_0-\tau)}\right) \quad a, t_0 > 0 \tag{17}$$

is again a solution. Moreover, a stationary solution of the problem, i.e. a solution of the form $Q(x)e^{i\lambda t}$, is easily computed

$$u_t(x) \equiv e^{-\frac{|x|}{2}+i\frac{t}{4}} \tag{18}$$

Combining (17) and (18), we explicitly construct the blow-up solution

$$\psi_t(x) = \frac{1}{\sqrt{a(t_0-t)}} e^{-\frac{|x|}{2a(t_0-t)}-i\frac{|x|^2}{4(t_0-t)}+i\frac{1}{4a^2(t_0-t)}} \tag{19}$$

An advantage to have the explicit solution (19) is that one can directly check the properties of the solution near the blow-up (non existence of strong $L^2$-limit, $L^2$-concentration phenomenon, local behaviour, etc.).
On the other hand, the standard NLSE with critical power nonlinearity is also invariant under the pseudo-conformal transformation (17). This fact suggests that the behaviour of a solution that blows up at a single point can be well approximated by our solution (19) near the blow-up time. In this sense our model would exhibit a universal behaviour near the blow-up.
The euristic motivation is the following. In the standard NLSE (critical case) the "nonlinear potential" is $|\psi_t(x)|^4$ and, near the blow-up, it behaves like the square of a $\delta$-function (see e.g. [Me]).
In our case the "potential term" is $|\psi_t(0)|^2\delta(x)$ and it exhibits the same behaviour along the solution (19) for $t \to t_0$, since the square of the modulus of (19) is a $\delta$-function approximation for $t \to t_0$.
In conclusion we point out that the natural generalization of the model discussed here is the case of NLSE with nonlinearities concentrated in $n$ fixed points in dimensions one, two and three.
The two and three dimensional cases require some care in their formulation since the space of finite energy is strictly larger than $H^1$. This means that natural instruments like the Sobolev inequalities and the uncertainty principle cannot be directly used. Moreover, in the $n$ points case the analysis of the blow-up solutions

becomes more subtle because of the lack of conservation of parity (in dimension one) and angular momentum (in dimension two and three). We plan to approach these questions in a further work.

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